Two Weighted Norm Inequalities for Riesz Potentials and Uniform L^p–Weighted Sobolev Inequalities

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0. Introduction. Let $-\Delta + V$ be the time independent Schrödinger operator in $\mathbf{R}^n$. S. Chanillo and E. Sawyer have proved (see [C–S]) that this operator has "the strong unique continuation property" if the potential V satisfies

$$\limsup_{r\to 0}\sup_{x\in K} r^2 \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} |V(x)|^t \, dx \right)^{1/t} = 0$$

for some $t > \frac{n-1}{2}$ and for any compact set K. For a general introduction, further references, and historical comments on unique continuation and related questions we refer to [K].

A main ingredient in the proof of this result is the following weighted inequality:

$$\int I_\alpha(f)(x)^2 u(x)\, dx \le \int f(x)^2 u(x)^{-1}\, dx, \qquad f \ge 0,$$

where I_α is the Riesz potential operator defined for locally integrable functions by

$$I_\alpha(f)(x) = \int \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \qquad 0 < \alpha < n.$$

Chanillo and Sawyer have shown that if the weight u is in the Muckenhoupt class A_2 (see [GC–RF], Ch. 4) and satisfies

$$r^\alpha \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)^t \, dy \right)^{1/t} \le C, \qquad x \in \mathbf{R}^n,\ r > 0 \tag{0.2}$$

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for some $t > 1$, then (0.1) holds. This condition was introduced by Feferman and Phong ([F]) in their study of the eigenvalues of $-\Delta + V$.

One of the consequences of our main result (Theorem 1.2) is that if the weight u belongs to the largest class of Muckenhoupt, i.e. to the A_∞ class, then (0.1) holds if and only if u satisfies

$$r^\alpha \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \le C, \qquad x \in \mathbf{R}^n,\ r > 0. \tag{0.3}$$

By Hölder's inequality, this condition is weaker than (0.2).

We apply our main theorem to obtain uniform L^p–weighted Sobolev inequalities (see [C–R] for related results). We then relate these to a question of C. Fefferman [F]. He asks for conditions on the weight u which imply

$$\int |f(x)|^2 u(x)\, dx \le C \int |\nabla f(x)|^2\, dx, \qquad f \in C_0^\infty, \tag{0.4}$$

for some constant C. By taking the Fourier transform, (0.4) is easily seen to be equivalent to

$$\int |I_1(f)(x)|^2 u(x)\, dx \le C \int |f(x)|^2\, dx, \qquad f \in C_0^\infty.$$

Corollary 1.6 states that if we start with $u \in A_\infty$, then (0.4) holds if and only if

$$r^2 \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \le C, \qquad x \in \mathbf{R}^n,\ r > 0.$$

This result was obtained in [C–W–W], and also independently by E. Sawyer, using different arguments.

We would like to remark that Theorem 1.2 follows from an analogous result, Theorem 1.1, for the fractional maximal operator M_α, which was already obtained in [S1]. The approach here is somewhat simpler, and combines some ideas from [S3] and [J] (see also [GC–RF], Ch. 4), and can be extended to more general measures (see Remark 2 in Section 2).

We also note that the results we establish in this paper have analogues for more general convolution operators whose kernels are nonnegative and satisfy certain weak size conditions, cf. [P] and [J–P–W].

The paper is organized as follows. In the first section we state our main results and the conclusions that follow from them. In the second section we give the proof of our main theorem. Also, we outline how the result can be sharpened. In the third and last section we provide explicit examples of weights in the classes that come out in our results.

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1. Statement of results and consequences. We shall start by recalling some definitions and notations.

The letter C will be used to denote a positive constant, not necessarily the same at each occurrence. Q will always be a cibe with sides parallel to the axes. λQ denotes the cube concentric with Q and having sildelength λ times that of Q.

For $0 < \alpha < n$ we let I_α be the *Riesz potential operator* defined for locally integrable functions by

$$I_\alpha(f)(x) = \int \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Also, for $0 \leq \alpha < n$, we let M_α be the *fractional maximal operator* defined for locally integrable functions f in $\mathbf{R}^n$ by

$$M_\alpha(f)(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy.$$

M_α^d denotes the dyadic fractional maximal operator.

A *weight* u is a non-negative, locally integrable function. $u(A)$ stands for $\int_A u(y)\, dy$.

We adopt the convention to write $T : L^p(d\mu) \to L^p(d\nu)$ if T is bounded from $L^p(d\mu)$ to $L^p(d\nu)$, $\|Tf\|_{L^p(d\nu)} \leq c\|f\|_{L^p(d\mu)}$.

We say that the weight u delongs to the A_∞ class of Muckenhoupt if it satisfies the following property:

(1.1) there are constants c, $\delta > 0$ so that, for each cube Q

$$\frac{u(E)}{u(Q)} \leq c\left(\frac{|E|}{|Q|}\right)^\delta$$

whenever E is a measurable subset of Q.

We also denote by A_∞^d the dyadic analog. for more information about weights see [GC–RF].

Our first main result is:

Theorem 1.1. *Let $1 < p \le q < \infty$ and let (u,v) be a couple of weights such that $\sigma = v^{-1/(p-1)}$ belongs to A_∞^d. Then the following statements are equivalent:*

(1.2) $M_\alpha^d : L^p(v) \to L^q(u)$;

(1.3) *there exists $K > 0$ such that for all dyadic cubes Q*

$$\left(\frac{1}{|Q|^{1-\alpha/n}}\right)^p u(Q)^{p/q}\sigma(Q)^{p-1} \le K.$$

In [M–W], Muckenhoupt and Wheeded show that if $0 < p < \infty$ and $u \in A_\infty$, then

$$\int I_\alpha f(x)^p u(x)\,dx \cong \int M_\alpha f(x)^p u(x)\,dx, \qquad f \in C_0^\infty.$$

See also [Sc], Theorem 2.1. By a standard argument we get

$$\int I_\alpha f(x)^p u(x)\,dx \cong \int M_\alpha^d f(x)^p u(x)\,dx, \qquad f \in C_0^\infty.$$

This, together with Theorem 1.1, gives our main result.

Theorem 1.2. *Let $1 < p \le q < \infty$ and let (u,v) be a couple opf weights such that $u \in A_\infty$, and $\sigma = v^{-1/(p-1)} \in A_\infty^d$. Then the following statements are equivalent:*

(1.4) $I_\alpha : L^p(v) \to L^q(u)$;

(1.5) *there exists $K > 0$ so that, for all dyadic cubes Q*

$$\left(\frac{1}{|Q|^{1-\alpha/n}}\right)^p u(Q)^{p/q}\sigma(Q)^{p-1} \le K.$$

As a consequence we obtain the following corollary which contains Chanillo's and Sawyer's result mentioned in the introduction.

Corollary 1.3. *Let u be a weight such that $u \in A_\infty$. Then the following statements are equivalent:*

(1.6) $I_\alpha : L^2\left(\frac{1}{u}\right) \to L^2(u)$;

(1.7) *there exists $K > 0$ so that, for all dyadic cubes Q*

$$\frac{1}{|Q|^{1-\alpha/n}} u(Q) \le K.$$

Clearly, (1.7) is equivalent to (0.3).

The next corollary gives the weighted Sobolev inequalities also mentioned above.

Corollary 1.4. *Let (u,v) be a couple of weights such that $u \in A_\infty$, and $\sigma = v^{-1/(p-1)} \in A_\infty^d$. Then for $1 < p < \infty$ and $n > 2$ the following statements are equivalent:*

(1.8) *there exists a constant C so that*

$$\|g\|_{L^p(u)} \le C\|\Delta g\|_{L^p(v)}\,;$$

(1.9) *there exists $K > 0$ so that, for all dyadic cubes Q*

$$|Q|^{2p/n}\left(\frac{1}{|Q|}\int_Q u(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q v(y)^{-1/(p-1)}\,dy\right)^{p-1} \le K\,.$$

This follows from Theorem 1.1 by using that I_2 is the inverse operator of $-\Delta$; see [St], Ch. 4.

We single out the cases and $v = u^{-1}$ and $v = 1$ for $p = 2$, since they are especially interesting.

Corollary 1.5. *Let u be a weight such that $u \in A_\infty$, and let $n > 2$. Then the following statements are equivalent:*

(1.10) *there exists C so that*

$$\|f\|_{L^2(u)} \le C\|\Delta f\|_{L^2(1/u)}\,, \qquad f \in C_0^\infty\,,$$

(1.11) *there exists $K > 0$ so that, for all dyadic cubes Q*

$$|Q|^{2/n}\frac{1}{|Q|}\int_Q u(y)\,dy \le K\,.$$

Corollary 1.6. *Let u be a weight such that $u \in A_\infty$, and let $n > 1$. Then the following statements are equivalent:*

(1.12) *there exists C so that*

$$\|f\|_{L^2(u)} \le C\|\nabla f\|_{L^2}\,, \qquad f \in C_0^\infty\,;$$

(1.13) *there exists $K > 0$ so that, for all dyadic cubes Q*

$$|Q|^{2/n}\frac{1}{|Q|}\int_Q u(y)\,dy \le K\,.$$

Indeed, by taking Fourier transform we see that (1.12) is equivalent to

$$\int |I_1 f(x)|^2 u(x)\,dx \le C\int |f(x)|^2\,dx\,, \qquad f \in C_0^\infty\,,$$

and we can apply Theorem 1.2 with $\sigma = 1 \in A_\infty$, $p = 2$ and $\alpha = 1$.

2. Proof of Theorem 1.1. (1.3) follows easily from (1.2) by testing with $f = v^{-1/(p-1)}\chi_Q$ and using the definition of M_α.

To prove the converse, let us fix a constant $a > 2^n$. For each integer k we define the sets S_k and D_k by

$$S_k = \{x \in \mathbf{R}^n : a^k < M_\alpha^d(f)(x) \le a^{k+1}\}.$$

$$D_k = \{x \in \mathbf{R}^n : a^k < M_\alpha^d(f)(x)\}.$$

Let $\{Q_{k,j}\}$ denote the maximal non–overlapping dyadic cubes satisfying

$$a^k < \frac{1}{|Q_{k,j}|^{1-\alpha/n}} \int_{Q_{k,j}} f(y)\,dy$$

so that

$$D_k = \bigcup_j Q_{k,j}.$$

Notice that if $Q'_{k,j}$ is the "father" cube of $Q_{k,j}$, i.e., the unique dyadic cube containing $Q_{k,j}$, we have

$$\begin{aligned} a^k &< \frac{1}{|Q_{k,j}|^{1-\alpha/n}} \int_{Q_{k,j}} f(y)\,dy \\ &\le \frac{2^{n-\alpha}}{|Q'_{k,j}|^{1-\alpha/n}} \int_{Q'_{k,j}} f(y)\,dy \le 2^n a^k. \end{aligned}$$

We also notice that $D_{k+1} \subset D_k$. Hence, by setting

$$E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1},$$

we obtain a disjoint family so that $D_k = \bigcup E_{k,j}$. We claim that

$$|Q_{k,j}| < \beta |E_{k,j}|, \tag{2.1}$$

where $\beta > 1$ depends only on the dimension n.

To prove the claim we estimate the portion of $Q_{k,j}$ that is covered by Q_{k+1}. We have

$$\begin{aligned} \frac{|Q \cap D_{k+1}|}{|Q_{k,j}|} &= \sum_i \frac{|Q_{k,j} \cap Q_{k+1,i}|}{|Q_{k,j}|} \\ &= \sum_{Q_{k+1,i} \subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} \quad \le \end{aligned}$$

$$\leq \sum_{Q_{k+1,i}\subset Q_{k,j}} \frac{1}{|Q_{k,j}|}\frac{|Q_{k+1,i}|^{\alpha/n}}{a^{k+1}}\int_{Q_{k+1,i}} f(y)\,dy$$

$$\leq \frac{1}{|Q_{k,j}|}\frac{|Q_{k,j}|^{\alpha/n}}{a^{k+1}}\int_{Q_{k,j}\cap\cup_i Q_{k+1,i}} f(y)\,dy$$

$$\leq \frac{2^n}{a} < 1$$

by our choice of a. This estimate yields

$$\frac{|E_{k,j}|}{|Q_{k,j}|} > 1 - \frac{2^n}{a} = \beta^{-1} > 0$$

and the claim follows.

Now, we can write

$$(M_\alpha^d f)^p = \sum_k \chi_{S_k}(M_\alpha^d f)^p \leq a^p \sum_k a^{kp}\chi_{D_k}$$

$$\leq a^p \sum_{k,j}\left(\frac{1}{|Q_{k,j}|^{1-\alpha/n}}\int_{Q_{k,j}} f(y)\,dy\right)^p \chi_{Q_{k,j}}$$

$$= a^p \sum_{k,j} h_{k,j}^p,$$

where

$$h_{k,j} = \frac{1}{|Q_{k,j}|^{1-\alpha/n}}\int_{Q_{k,j}} f(y)\,dy\chi_{Q_{k,j}}.$$

Since $L^p(\mu)$, $1 \leq p < \infty$, is a Banach space we have the following chain of inequalities:

$$\left(\int M_\alpha^d(f)(x)^q u(x)\,dx\right)^{p/q} = \|M_\alpha^d f^p\|_{L^{q/p}(u)}$$

$$\leq C\Big\|\sum_{k,j} h_{k,j}^p\Big\|_{L^{q/p}(u)}$$

$$\leq C\sum_{k,j}\|h_{k,j}^p\|_{L^{q/p}(u)}$$

$$\leq C\sum_{k,j}\left(\int h_{k,j}(x)^q u(x)\,dx\right)^{p/q} \qquad \leq$$

$$\leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1-\alpha/n}} \int_{Q_{k,j}} f(y)\, dy \right)^q u(Q_{k,j})^{p/q}$$

$$\leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1-\alpha/n}} \int_{Q_{k,j}} f(y)\, dy \right)^p u(Q_{k,j})^{p/q}.$$

Rewriting this inequality and using (1.3) we get

$$\leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1-\alpha/n}} \right)^p \sigma(Q_{k,j})^{p-1} u(Q_{k,j})^{p/q}$$

$$\left(\frac{1}{\sigma(Q_{k,j})} \int_{Q_{k,j}} f(y)\sigma(y)^{-1}\sigma(y)\, dy \right)^p \sigma(Q_{k,j})$$

$$\leq KC \sum_{k,j} \left(\frac{1}{\sigma(Q_{k,j})} \int_{Q_{k,j}} f(y)\sigma(y)^{-1}\sigma(y)\, dy \right)^p \sigma(Q_{k,j}).$$

Now, since $\sigma \in A_\infty^d$, it follows from our claim above that

$$\sigma(Q_{k,j}) \leq \rho\sigma(E_{k,j}) \tag{2.3}$$

for each k,j and for some $\rho \geq 0$, where $\{E_{k,j}\}$ is a disjoint family. Hence we can continue our chain of inequalities with

$$\rho C \sum_{k,j} \left(\frac{1}{\sigma(Q_{k,j})} \int_{Q_{k,j}} f(y)\sigma(y)^{-1}\sigma(y)\, dy \right)^p \sigma(E_{k,j})$$

$$\leq C \sum_{k,j} \int_{E_{k,j}} M_\sigma^d (f\sigma^{-1})(x)^p \sigma(x)\, dx$$

$$\leq C \int M_\sigma^d (f\sigma^{-1})(x)^p \sigma(x)\, dx$$

$$\leq C \int f(x)^p v(x)\, dx,$$

where M_σ^d stands for the weighted dyadic Hardy–Littlewood maximal operator, that is,

$$M_\sigma^d f(x) = \sup_{x\in Q} \frac{1}{\sigma(Q)} \int_Q |f(y)|\sigma(y)\,dy\,.$$

where the supremum is taken over the dyadic cubes. Here we have used the well–known fact that, for any regular positive measure μ and any p, $1 < p < \infty$, it holds

$$M_\mu^d : L^p(\mu) \to L^p(\mu)\,.$$

The proof of the theorem is complete. □

Remark 1. If we want to extend (1.2) to M_α, with $\sigma \in A_\infty$ and Condition (1.3) satisfied for all cubes, we may argue as follows. We have

$$M_\alpha^{2^k} f(x) \le C \frac{1}{|Q_{2^{k+2}}(0)|} \int_{Q_{2^{k+2}}(0)} (\tau_{-t} \circ M_\alpha^d \circ \tau_t) f(x)\,dt\,, \quad x \in \mathbf{R}^n\,, \tag{2.4}$$

where $\tau_t g(x) = g(x-t)$, $Q_r(0)$ is the cube centered at the origin with sidelength r and $M_\alpha^{2^k}$ is defined as M_α, with the supremum taken only over cubes with sidelength less than 2^k. This inequality, for the Hardy–Littlewood maximal oeprator, was obtained by Fefferman and Stein (see [GC–RF], p. 431). For M_α it is due to Sawyer (see [Sa]). Now, since the class A_∞ and the condition (1.3) are invariant under translation, we can easily use Theorem 1.1 to deduce that $\tau_{-t} \circ M_\alpha^d \circ \tau_t$ is bounded from $L^p(v)$ to $L^q(u)$, uniformly in $t \in \mathbf{R}^n$. Finally, (2.4), Minkowski's integral inequality and the monotone convergence theorem yield the boundedness of M_α.

We may also supply a different proof which is better adapted to the setting of the spaces of homogeneous type. Indeed, by using a straightforward modification of the classical Calderón–Zygmund decomposition, related to the operator M_α, we can obtain the following. Let $E_t = \{x \in \mathbf{R}^n : M_\alpha f(x) > t\}$ for each $t > 0$. There is a family of non–overlapping maximal dyadic cubes $\{Q_j\}$, which satisfy

$$E_{t/4} \subset \bigcup_j 3Q_j \tag{2.5}$$

and

$$\frac{t}{4^n} < \frac{1}{|Q_j|^{1-\alpha/n}} \int_{Q_j} |f(y)|\,dy \le \frac{t}{2^n} \tag{2.6}$$

for each integer j. A similar argument as in (2.2) would yield (2.1) and then (2.3). The proof of the theorem can now be completed in (essentially) the same way as the proof of Theorem 1.1. we note that if we assume that u is doubling and σ is only in A_∞^d, then this proof shows that $M_\alpha : L^p(v) \to L^q(u)$ is equivalent to the condition (1.3) restricted to all dyadic cubes Q.

Remark 2. Theorem 1.1 can be sharpened as follows. We say that a weight ω satisfies the B_β condition with $0 < \beta < \infty$ if

$$\frac{\omega(Q')}{\omega(Q)} \le C \left(\frac{|Q'|}{|Q|}\right)^\beta$$

for all pairs of cubes $Q' \subset Q$. The dyadic case B^d_β is defined similarly. We note that the Lebesgue measure satisfies the B_1 condition.

We also say that the weight σ is in $A_\infty(\omega)$, for a weight ω, if there exist C, $\varepsilon > 0$ with

$$\frac{\sigma(E)}{\sigma(Q)} \le C \left(\frac{\omega(E)}{\omega(Q)}\right)^\varepsilon$$

whenever E is a subset of a cube Q. Notice that if $\omega \equiv 1$, we recover A_∞. Now, let us suppose that $\sigma \in A^d_\infty(\omega)$ in Theorem 1.1 with ω satisfying the $B_{1-\alpha/n}$ condition (the Newtonian potential is in $B_{1-2/n}$). Then, a similar argument as in (2.2), with ω instead of the Lebesgue measure, yields (2.1) for ω. Then (2.3) holds and the rest of the proof goes through without change.

3. Some examples of nice weights. From Theorem 1.1 and Theorem 1.2 it is clear that it is interesting to get a better understanding of the class of weights

$$A_{p,\alpha} = \left\{(u,v) : |Q|^{\alpha p/n} \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C \text{ for every } Q\right\}$$

and, more specifically, of the smaller classes

$$B_{p,\alpha} = A_{p,\alpha} \cap \{v : v^{-1/(p-1)} \in A_\infty\}$$

and

$$C_{p,\alpha} = B_{p,\alpha} \cap \{u : u \in A_\infty\}.$$

We also define the class

$$D_\alpha = \left\{u \in A_\infty : \frac{|Q|^{\alpha/n}}{|Q|}\int_Q u(x)\,dx \le C \text{ for every } Q\right\},$$

which is the one that is relevant in Corollary 1.2.

In particular, we would like to have explicit examples of couples of weights in these classes. We shall show that there is in fact a lot of them. To this end, we use the following extension of the Coifman–Rochberg theorem (see [GC–RF], p. 158), which was already observed in [S2], p. 113.

Lemma 3.1. *Let $0 \le \alpha < n$ and $0 < \gamma < \frac{n}{n-\alpha}$. If μ is a positive Borel measure with $M_\alpha\mu(x) < \infty$ (a.e. $x \in \mathbf{R}^n$), then the function $\left(M_\alpha\mu(x)\right)^\gamma$ is an A_1 weight with a constant depending on γ, α, and n.*

In the following discussion we assume that $0 < \alpha < n$, and that $1 < p < \infty$. Also, p' denotes the dual of p, $\frac{1}{p} + \frac{1}{p'} = 1$. The weights u and v may be assumed in a "nice" class if necessary, σ stands for $\sigma = v^{-1/(p-1)}$.

Id we take a weight v, and we assume that $\alpha p' < n$, then the couple of weights $\left((M_{\alpha p'}\sigma)^{1-p}, v\right)$ is in the $A_{p,\alpha}$ class. Indeed,

$$|Q|^{\alpha p/n}\left(\frac{1}{|Q|}\int_Q (M_{\alpha p'}\sigma)^{1-p}\,dx\right)\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{p-1}$$

$$\le |Q|^{\alpha p/n}\frac{1}{|Q|}\int_Q\left(\frac{|Q|^{\alpha p'/n}}{|Q|}\int_Q \sigma(y)\,dy\right)^{1-p}dx\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{p-1} = 1.$$

The restriction $\alpha p' < n$ ensures that $M_{\alpha p'}\sigma$ is finite almost everywhere if σ is in some L^q class. Furthermore, if σ is assumed to be in A_∞, then $\left((M_{\alpha p'}\sigma)^{1-p}, v\right) \in B_{p,\alpha}$. Also, we notice that by lemma 3.1, $M_{\alpha p'}\sigma$ is in A_1, and this implies that $(M_{\alpha p'}\sigma)^{1-p}$ is in $A_p \subset A_\infty$. Hence $\left((M_{\alpha p'}\sigma)^{1-p}, v\right) \in C_{p,\alpha}$.

Suppose we start with a weight u and assume $\alpha p < n$. If we take s so that $\frac{\alpha p}{n} < s \le 1$, then for $\beta = \frac{\alpha p}{s}$

$$|Q|^{\alpha p/n}\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q M_\beta u(x)^{-s/(p-1)}\,dx\right)^{p-1}$$

$$\le |Q|^{\alpha p/n}\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int\left(\frac{|Q|^{\alpha p/sn}}{|Q|}\int_Q u(y)\,dy\right)^{-s/(p-1)}dx\right)^{p-1}$$

$$= \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)^{1-s}.$$

If $s = 1$ we get that $(u, M_\beta u) \in A_{p,\alpha}$. We point out that this result can be obtained directly from the following inequality due to Sawyer (see, for instance, [A]):

$$\int M_\alpha f(x)^p u(x)\,dx \le C\int f(x)^p M_{\alpha p}u(x)\,dx.$$

On the other hand, if $s < 1$, we also have that $\left(u, (M_\beta u)^s v\right) \in A_{p,\alpha}$ if we assume u to be bounded.

Now, by Lemma 1.1, $M_\beta u \in A_1$ and, consequently, $(M_\beta u)^s \in A_1$. This means that $(M_{\alpha p} u)^{-s/(p-1)} \in A_{p'} \subset A_\infty$, and, therefore, $\big(u,\ (M_\beta u)^s\big) \in B_{p,\alpha}$. Furthermore, if $u \in A_\infty$, then $\big(u,\ (M_\beta u)^s\big) \in C_{p,\alpha}$.

So far we have always had a restriction on the indices α, p and n. In the first case we needed $\alpha p' < n$, and in the second one $\alpha p < n$. With the next example we get rid of these restrictions. Let us take a number s so that $\frac{n}{n+\alpha p} < s < \frac{n}{\alpha p}$. Then we claim that $\big(M(u^{-1})^{-1},\ M_\beta(u^s)^{1/s}\big)$ is in $B_{p,\alpha}$, where $\beta = \alpha p s < n$, and M is the usual Hardy–Littlewood maximal operator. Indeed, as above we get

$$
\begin{aligned}
&|Q|^{\alpha p/n}\left(\frac{1}{|Q|}\int_Q M(u^{-1})(x)^{-1}\,dx\right)\left(\frac{1}{|Q|}\int_{Q_\beta}(u^s)(x)^{-1/s(p-1)}\,dx\right)^{p-1}\\
&\le |Q|^{\alpha p/n}\left(\frac{1}{|Q|}\int_Q u(y)^{-1}\,dy\right)^{-1}\left(\frac{|Q|^{\beta/n}}{|Q|}\int_Q u(y)^s\,dy\right)^{-1/s}\\
&= |Q|^{(\alpha p-\beta/s)/n}\left(\frac{1}{|Q|}\int_Q u(x)^{-1}\,dx\right)^{-1}\left(\frac{1}{|Q|}\int_Q u(x)^s\,dx\right)^{1/s} \le 1.
\end{aligned}
$$

Here we used that, for any $s > 0$, we have

$$
1 \le \left(\frac{1}{|Q|}\int_Q u(x)^{-1}\,dx\right)\left(\frac{1}{|Q|}\int_Q u(x)^s\,dx\right)^{1/s},
$$

which follows from Hölder's inequality.

Notice that by the choice of s we have that $\beta < n$ and that $\frac{1}{s} < \frac{n}{n-\beta}$. The last inequality implies that $M_\beta(u^s)^{1/s}$ is an A_1 weight by Lemma 3.1. Hence, $\big(M(u^{-1})^{-1},\ M_\beta(u^s)^{1/s}\big)$ is in $B_{p,\alpha}$.

We finally study the class D_α. Let us pick $v \in A_1 \cap L^q$ for some $1 < q < \infty$ with $M_\alpha v$ finite almost everywhere (there are many such v's). Using Lemma 1.1 with $\gamma = 1$ we see that $M_\alpha v$ is an A_1 weight and, hence, $u = v(M_\alpha v)^{1-2} = v(M_\alpha v)^{-1} \in A_2 \subset A_\infty$. Furthermore,

$$
\begin{aligned}
\frac{|Q|^{\alpha/n}}{|Q|}\int_Q u(x)\,dx &\le \frac{|Q|^{\alpha/n}}{|Q|}\int_Q v(x)\left(\frac{|Q|^{\alpha/n}}{|Q|}\int_Q v(y)\,dy\right)^{-1}dx\\
&= 1.
\end{aligned}
$$

Consequently, we find that $u \in D_\alpha$.

Remark. We want to point out that by using similar arguments as above, it is possible to prove that the class

$$B_{p,q,\alpha} = A_{p,q,\alpha} \cap \{v : v^{-1/(p-1)} \in A_\infty\},$$

where

$$A_{p,q,\alpha} = \left\{(u,v) : \left(\frac{1}{|Q|^{1-\alpha/n}}\right)^p u(Q)^{p/q}\sigma(Q)^{p-1} \leq K \text{ for each } Q\right\}$$

is not empty for the case $p \neq q$.

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