Conjugacy in Garside Groups III: Periodic braids

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Abstract

An element in Artin's braid group B_n is said to be periodic if some power of it lies in the center of B_n. In this paper we prove that all previously known algorithms for solving the conjugacy search problem in B_n are exponential in the braid index n for the special case of periodic braids. We overcome this difficulty by putting to work several known isomorphisms between Garside structures in the braid group B_n and other Garside groups. This allows us to obtain a polynomial solution to the original problem in the spirit of the previously known algorithms.

This paper is the third in a series of papers by the same authors about the conjugacy problem in Garside groups. They have a unified goal: the development of a polynomial algorithm for the conjugacy decision and search problems in B_n, which generalizes to other Garside groups whenever possible. It is our hope that the methods introduced here will allow the generalization of the results in this paper to all Artin-Tits groups of spherical type.

1 Introduction

Given a group, a solution to the *conjugacy decision problem* is an algorithm that determines whether two given elements are conjugate or not. On the other hand, a solution to the *conjugacy search problem* is an algorithm that finds a conjugating element for a given pair of conjugate elements. In §1.4 of [6] we presented a project to find a polynomial solution to the conjugacy decision problem and the conjugacy search problem in the particular case of Artin's braid group, that is, the Artin-Tits group of type $\mathbf{A}_{n-1}$, with its classical or *Artin* presentation [1]:

$$(1) \qquad B_n^A : \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } \ |i-j|>1, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } \ |i-j|=1. \end{array} \right\rangle .$$

One of the steps in the mentioned project asks for a polynomial solution to the above conjugacy problems for special type of elements in the braid groups, called *periodic braids*. This is achieved in the present paper. More precisely, if we denote by $|w|$ the letter length of a word w in $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, we will prove:

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Theorem 1. *Let w_X and w_Y be two words in the generators $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, representing two braids $X, Y \in B_n^A$, and let $l = \max\{|w_X|, |w_Y|\}$. Then there is an algorithm of complexity $O(l^3 n^2 \log n)$ which does the following.*

(1) *It determines whether X and Y are periodic.*

(2) *If yes, it determines whether they are conjugate.*

(3) *If yes, it finds a braid $C \in B_n^A$ such that $Y = C^{-1}XC$.*

Here is a guide to this paper. In Section 2, we will review what is known and explain why steps (1) and (2) of Theorem 1 follow easily from the work in [17, 24, 22]. On the other hand, in Section 3 we show that the previously known solutions to the conjugacy search problem in the Artin-Tits group of type $\mathbf{A}_{n-1}$ present unexpected difficultites, which result in exponential complexity for periodic braids. Thus they do not meet the requirements of Theorem 1.

A new idea allows us to overcome the difficulty. We have shown that the approach using the classical Garside structure does not work. The new idea is to put to work the other known Garside structure on the braid groups and in addition to consider a certain subgroup of the braid group that arises in the course of our work, and use two known Garside structures on it. This is accomplished in Section 4, where we give a solution to the conjugacy search problem for periodic braids which has the stated polynomial complexity. Section 4 divides naturally into two subsections, according to whether a given periodic braid is conjugate to a power of δ or ε, two braids that are defined in Section 2 below. The proof in the two cases are treated in Sections 4.1 and 4.2 respectively. Finally, in Section 5 we compare actual running times of the algorithms developed in Section 4 to the ones of the best previously known algorithm.

Remark 2. We learned from D. Bessis that he has characterized the conjugacy classes of periodic elements for all Artin-Tits groups of spherical type. We hope that this characterization will allow the generalization of both the techniques and the results of this paper to all other Artin-Tits groups of spherical type.

Acknowledgements: We are grateful to D. Bessis for useful discussions about his work in [2] and his forthcoming results, to J. Michel for pointing out that our Corollaries 12 and 15 were known to specialists in Coxeter groups, and also to H. Morton for showing us the algorithm in [26].

2 Known results imply steps (1) and (2) of Theorem 1

Our work begins with a review of known results. Garside groups were introduced by Dehornoy and Paris in [15]. The main examples of Garside groups are Artin-Tits groups of spherical type, in particular, Artin braid groups. In this paper we will use two known Garside structures in the Artin-Tits group of type $\mathbf{A}_{n-1}$, and also one Garside structure in the Artin-Tits group of type $\mathbf{B}_m$.

Although we refer to [6] for a detailed description of Garside structures, we recall here that such a structure in a group G is given by a lattice order on its elements, together with a distinguished element of G, called the *Garside element*, which is usually denoted by Δ. This partial order and this element Δ must satisfy several suitable conditions [6].

The classical Garside structure in the braid groups is related to the presentation (1). The *positive* braids are those which can be written as a word in $\sigma_1, \dots, \sigma_{n-1}$ (not using their inverses). The lattice order is defined by saying that $X \preccurlyeq Y$ if $X^{-1}Y$ is a positive braid (we will say that X is a *prefix* of Y). There are special elements called *simple braids* which are those positive braids in which any two strands cross *at most* once. The Garside element Δ is the positive braid in which any two strands cross *exactly* once, that is, $\Delta = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1)$. It is also called the *half twist*, since its geometrical representation corresponds to a half twist of the n strands. For every braid $X \in B_n^A$, given as a word of letter length l, there exists a *left normal form*, which is a unique way to decompose the braid as $X = \Delta^p x_1 \cdots x_r$, where p is maximal and each x_i is a simple braid, namely the maximal simple prefix of $x_i \cdots x_r$. This left normal form can be computed in time $O(l^2 n \log n)$ [19].

Artin proved in [1] that the center of B_n^A is infinite cyclic and generated by the *full twist* $\Delta^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ of the braid strands. If the braid group is regarded as the mapping class group of the n-times punctured disc $\mathbb{D}_n^2$, then Δ^2 is a Dehn twist about a curve which lies in a collar neighborhood of the boundary $\partial\mathbb{D}_n^2$ and is parallel to it. An element $X \in B_n^A$ is said to be *periodic* if some power of X is a power of Δ^2.

Periodic braids can be thought of as *rotations* of the disc. Indeed, there is a classical result by Eilenberg [17] and Kérékjártó[24] (see also [12]) showing that an automorphism of the disc which is a root of the identity (a periodic automorphism) is conjugate to a rotation. Since a finite order mapping class can always be realized by a finite order homeomorphism [23], this implies that a periodic braid is conjugate to a rotation. It is not difficult to see that a braid can be represented by a rotation of $\mathbb{D}^2$ if and only if it is conjugate to a power of one of the two braids represented in Figure 1, that is, $\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ and $\varepsilon = \sigma_1(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$. (If we need to specify the number of strands, we will write $\delta = \delta_n$ and $\varepsilon = \varepsilon_n$.)

Remark 3. The braid ε defined in Figure 1 has a fixed strand, namely strand 2. There are, to be sure, braids which are conjugate to ε in which the fixed strand is the first one or the last one, seemingly more natural choices. However, ε is a simple braid, and (as we shall prove in Proposition 13 below) there is no simple braid which is conjugate to ε and which fixes either the first or the last strand. This is why we decided to use ε, which fixes the second strand, as a representative of its conjugacy class. And this is also the reason why, in Section 4.2 below, we identify the Artin-Tits group of type $\mathbf{B}_{n-1}$ with the subgroup of the n-strand braid group formed by those braids which fix the *second* strand, a choice that will surely seem awkward to specialists.

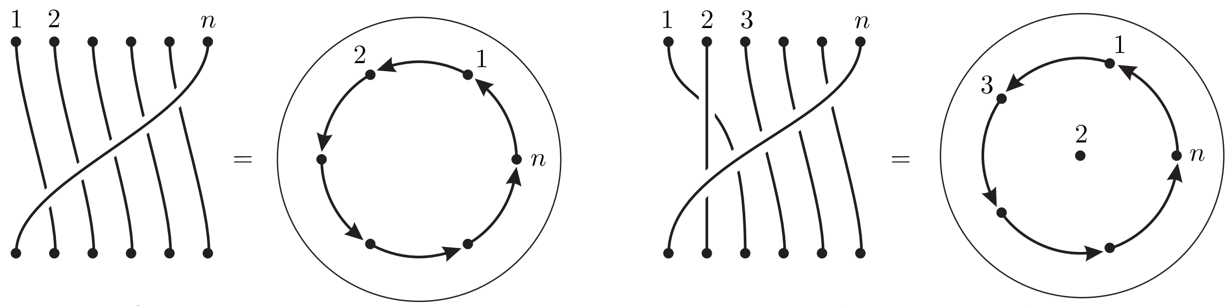
The theorem of Eilenberg and Kérékjártó can then be restated as follows.

Theorem 4. [17, 24] *A braid X is periodic if and only if it is conjugate to a power of either δ or ε.*

Notice that $\delta^n = \varepsilon^{n-1} = \Delta^2$. Since Δ^2 belongs to the center of B_n^A, this immediately gives an efficient algorithm to check whether a braid is periodic.

Corollary 5. *A braid $X \in B_n^A$ is periodic if and only if either X^{n-1} or X^n is a power of Δ^2.*

Proof. We only need to prove that the condition is necessary. Suppose that X is periodic. By Theorem 4, X is conjugate to a power of either δ or ε. In the first case, $X = C^{-1}\delta^k C$ for



$\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ $\qquad$ $\varepsilon = \sigma_1(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$

Figure 1: The periodic elements δ and ε.

some $C \in B_n^A$. Then $X^n = C^{-1}\delta^{kn}C = C^{-1}\Delta^{2k}C = \Delta^{2k}$, where the last equality holds since Δ^2 is central. In the second case, $X = C^{-1}\varepsilon^k C$, so that $X^{n-1} = C^{-1}\varepsilon^{k(n-1)}C = C^{-1}\Delta^{2k}C = \Delta^{2k}$. □

After this result, one can determine whether X is periodic, and also find the power of δ or ε which is conjugate to X, by the following algorithm.

Algorithm A.

Input: A word w in Artin generators and their inverses representing a braid $X \in B_n^A$.

1. Compute the left normal form of X^{n-1}.
2. If it is equal to Δ^{2k}, return *'X is periodic and conjugate to ε^k'*.
3. Compute the left normal form of X^n.
4. If it is equal to Δ^{2k}, return *'X is periodic and conjugate to δ^k'*.
5. Return *'X is not periodic'*.

Proposition 6. *The complexity of Algorithm A is $O(l^2n^3\log n)$, where l is the letter length of w.*

Proof. Algorithm A computes two normal forms of words whose lengths are at most nl. By [19], these computations have complexity $O((nl)^2 n\log n)$, and the result follows. □

We remark that if one knows, a priori, that the braid X is periodic, then one can determine the power of δ or ε which is conjugate to X by a faster method: Observe that the exponent sum of a braid X, written as a word in the generators $\sigma_1,\dots,\sigma_{n-1}$ and their inverses is well defined, since the relations in (1) are homogeneous. The exponent sum is furthermore invariant under conjugacy, hence every conjugate of δ^k has exponent sum $k(n-1)$, whereas every conjugate of ε^k has exponent sum kn. Moreover, the exponent sum determines the conjugacy class of a periodic braid:

Lemma 7. (Proposition 4.2 of [22]) *Let X be a periodic braid. Then X is conjugate to δ^k (resp. ε^k) if and only if X has exponent sum $k(n-1)$ (resp. kn).*

Computing the exponent sum of a word of length l has complexity $O(l)$. Hence, once it is known that two given braids are periodic, the conjugacy decision problem takes linear time.

3 Known algorithms are not efficient for periodic braids

We have already determined all conjugacy classes of periodic braids, and we have seen that the conjugacy decision problem for these braids can be solved very fast. It is then natural to wonder whether this is also true for the conjugacy search problem. The first natural question is: Are the existing algorithms for the conjugacy search problem efficient for periodic braids?

The best known algorithm to solve the conjugacy decision problem and also the conjugacy search problem in braid groups (and in every Garside group) is the one in [21], which consists of computing the *ultra summit set* of a braid, defined as follows. Denote by τ the inner automorphism that is defined by conjugation by Δ. Given $Y \in B_n^A$ whose left normal form is $\Delta^p y_1 \cdots y_r$, we define its *canonical length* as $\ell(Y) = r$, and call the conjugates $\mathbf{c}(Y) = \Delta^p y_2 \cdots y_r \tau^{-p}(y_1)$ and $\mathbf{d}(Y) = \Delta^p \tau^p(y_r) y_2 \cdots y_{r-1}$ of Y its *cycling* respectively its *decycling*. For every $X \in B_n^A$, the ultra summit set $USS(X)$ is the set of conjugates Y of X such that $\ell(Y)$ is minimal and $\mathbf{c}^t(Y) = Y$ for some $t \geq 1$. It is explained in [21] how the computation of $USS(X)$ solves the conjugacy decision and search problems in Garside groups.

The complexity of the conjugacy search algorithm given in [21] is proportional to the size of $USS(X)$, so if one is interested in complexity, it is essential to know how large the ultra summit sets of periodic braids are. If they turned out to be small, the algorithm in [21] would be efficient, but we will see in this section that the sizes of ultra summit sets of periodic braids are in general exponential in n.

More precisely, it was shown by Coxeter in 1934 [13, Theorem 11], that in any finite Coxeter group, any two elements which are the product of all standard generators, in arbitrary order, are conjugate. Applied to our case, one sees that the elements of $USS(\delta)$ are in bijection with the elements of the above kind, in the symmetric group Σ_n. One can count the number of different elements, and it follows that $\#(USS(\delta)) = 2^{n-2}$. The same result is shown in [9, Chapter V, §6. Proposition 1], in the more general case in which the Coxeter group is defined by a tree, and also in [29, Lemma 3.2] and in [26, Theorem 2]. Moreover, it can be seen from the proof in [9] that any two elements in $USS(\delta)$ are conjugate by a sequence of special conjugations, that we denote *partial cyclings* in [6].

Concerning the elements in $USS(\varepsilon)$, in [16, Proposition 9.1] it is shown that any two such elements are conjugate by a sequence of partial cyclings. It also follows from [16] that every element in $USS(\varepsilon)$ is represented by a word of length n, which is the product of all $n-1$ generators, in some order, with one of the generators repeated. One can also count the number of different elements of this kind, to obtain that $\#(USS(\varepsilon)) = (n-2)2^{n-3}$.

The above arguments show that the sizes of $USS(\delta)$ and $USS(\varepsilon)$ are exponential with respect to the number of strands, hence the algorithm in [21] is not polynomial for conjugates of these braids. In this paper we shall study $USS(\delta)$ and $USS(\varepsilon)$ in a new way. More precisely, in Corollaries 12 and 15 we will show that $\#(USS(\delta)) = 2^{n-2}$ and $\#(USS(\varepsilon)) = (n-2)2^{n-3}$ just by looking at the permutations induced by their elements. This will also provide a fast solution to the conjugacy search problem in the particular cases of conjugates of δ or ε.

Once shown that the algorithm in [21] is not polynomial, in general, for periodic braids, in Section 4 we will give a procedure to solve the conjugacy search problem for all periodic braids in polynomial time.

Let us then study the ultra summit sets of δ and ε. First, we recall that the factors in a left normal form are *simple braids*, which are in bijection with the elements of the symmetric group Σ_n. More precisely, every braid X, being a mapping class group of the n-times punctured disc, determines a permutation π_X of the n punctures. Conversely, there is exactly one simple braid for each permutation. We will then determine simple elements by their permutations, written as a product of disjoint cycles. For instance, the permutation associated to δ is $\pi_\delta = (1\ 2\ \cdots\ n)$, and the permutation associated to ε is $\pi_\varepsilon = (2)(1\ 3\ 4\ \cdots\ n)$.

Remark 8. Although we described braids as mapping classes, we will not adopt the usual convention for compositions of maps. We consider braids as acting on the punctures from the right. This means that the braid $\sigma_1\sigma_2$ first swaps the punctures in positions 1 and 2, and then the punctures in positions 2 and 3. Hence $\pi_{\sigma_1\sigma_2} = (132)$.

Remark 9. The permutation associated to a simple braid s determines the pairs of strands that cross in s. More precisely, two strands i and j $(i < j)$ cross in s if and only if the induced permutation reverses their order, that is, if $\pi_s(i) > \pi_s(j)$.

For simplicity of notation let us define, for $1 \leq i < j \leq n$, the braids $\sigma_{[i\to j]} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-1}$ and $\sigma_{[j\to i]} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_i$. Notice that $\sigma_{[k\to l]}$ (no matter which subindex is bigger) is the shortest positive braid sending the puncture k to the puncture l.

Let us characterize the elements in $USS(\delta)$.

Proposition 10. *An element $s \in B_n^A$ belongs to $USS(\delta)$ if and only if it is simple and its permutation π_s is a cycle of the form:*

$$\pi_s = (1\ u_1\ u_2\cdots u_r\ n\ d_t\ d_{t-1}\cdots d_1),$$

for some $u_1 < u_2 < \cdots < u_r$ and some $d_t > d_{t-1} > \cdots > d_1$, with $r, t \geq 0$ and $r + t + 2 = n$. Moreover, in this case $\alpha^{-1}s\alpha = \delta$, where

$$\alpha = \sigma_{[d_1\to 1]}\,\sigma_{[d_2\to 1]}\cdots\sigma_{[d_t\to 1]}.$$

Proof. First notice that, since δ is simple, all elements in $USS(\delta)$ are simple, so that by the definition of a simple element they can be characterized by their permutations. Actually, $USS(\delta)$ is the set of simple conjugates of δ. Notice also that π_δ is a single cycle of length n. Since conjugation of braids in B_n^A implies conjugation of their corresponding permutations, it follows that the elements in $USS(\delta)$, which are conjugates of δ, are simple elements determined by a cycle of length n. Moreover, if $s \in USS(\delta)$ then $s^n = \Delta^2$, which is a positive braid in which any two strands cross exactly twice.

Let $s \in USS(\delta)$. Its permutation can be written as $\pi_s = (1\ u_1\ u_2\ \cdots\ u_r\ n\ d_t\ d_{t-1}\ \cdots\ d_1)$, where $r, t \geq 0$ and $r + t + 2 = n$. We must show that $u_1 < \cdots < u_r$ and $d_t > \cdots > d_1$. See in Figure 2 an example of two simple braids whose permutations are cycles of length n, so the permutations are conjugate in the symmetric group, but one of the braids satisfies the above inequalities and the other one does not.

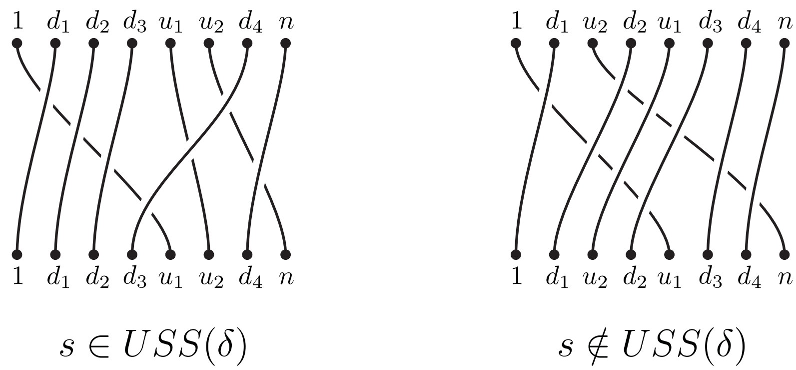


Figure 2: Two simple braids in B_8^A whose permutations are cycles of length 8. By Proposition 10, the first one is conjugate to δ and the second one is not. Notice that the exponent sum of the second one (i.e. the number of crossings or the letter length, in this case) is 9, while the exponent sum of conjugates of $\delta \in B_8^A$ is 7.

Suppose that $u_i > u_{i+1}$ for some i, where $1 \leq i < r$, and consider the strands 1 and u_1. We will see that these two strands cross more than twice in s^n. Indeed, one has $1 < u_1$, but in s^i these strands end at u_i and u_{i+1}, respectively. Since $u_i > u_{i+1}$, this means that they have crossed at least once in s^i. Now in s^r these two strands end at u_r and n, respectively, and since u_r is necessarily less than n, they have crossed again. Next, in s^{r+1} they end at n and d_t (or n and 1 if there are no d_j's), so they have crossed one more time. This means that in s^{r+1} the strands 1 and u_1 cross at least three times, showing that s^n cannot be equal to Δ^2, a contradiction. Therefore $u_1 < \cdots < u_r$. Similarly, if we had $d_{i+1} < d_i$ for some i, then strands n and d_t would cross more than twice in s^n, which is impossible. Therefore $d_t > \cdots > d_1$.

Conversely, suppose that s is simple and $\pi_s = (1\ u_1\ u_2\ \cdots\ u_r\ n\ d_t\ d_{t-1}\ \cdots\ d_1)$ for some $u_1 < \cdots < u_r$ and $d_t > \cdots > d_1$. We will show that s is conjugate to δ in a constructive way, by finding a conjugating element. First notice that if $t = 0$ then $\pi_s = (1\ 2\ \cdots\ n) = \pi_\delta$. Since simple elements are determined by their permutations, this means that $s = \delta$. Hence we can assume that $t > 0$. Denote $k = d_1$. One has

$$\pi_s = (\underline{1\ 2\ \cdots\ k-1}\ u_{k-1}\ \cdots\ u_r\ n\ d_t\ \cdots\ d_2\ \underline{k}\ ).$$

A schematic picture of the first k strands of s can be seen in Figure 3. We will conjugate s by $\sigma_{[k\to 1]}$, so we consider $s' = \sigma_{[k\to 1]}^{-1}\, s\, \sigma_{[k\to 1]}$. Recall that two strands i and j $(i < j)$ cross in s if and only if $\pi_s(i) > \pi_s(j)$. Then we can easily check that the strand of s ending at k (that is, the strand d_2 if $t > 1$ or the strand n if $t = 1$) does not cross the strands ending at $1, 2, \ldots, k-1$ (that is, the strands $k, 1, 2, \ldots, k-2$, respectively). This implies that $s\,\sigma_{[k\to 1]}$ is a simple braid. Moreover, one can also check that the strand k of s (thus the strand k of $s\,\sigma_{[k\to 1]}$) crosses the strands $k-1, k-2, \ldots, 1$, hence $s' = \sigma_{[k\to 1]}^{-1}\, s\, \sigma_{[k\to 1]}$ is a simple braid. Since the permutation associated to $\sigma_{[k\to 1]}$ is $(1\ 2\ \cdots\ k)$, it follows that

$$\pi_{s'} = (\underline{1\ 2\ \cdots\ k-1\ k}\ u_{k-1}\ \cdots\ u_r\ n\ d_t\ \cdots\ d_2).$$

We can continue this process, by recurrence on t, conjugating by elements of the form $\sigma_{[d_i\to 1]}$ and obtaining new simple conjugates of s whose permutations have more indices between 1 and n at each step, until we get the permutation $(1\ 2\ \cdots\ n)$, that is, until we obtain δ. In this way we have shown that if s is a simple element with the permutation given in the statement, then $\alpha^{-1}s\alpha = \delta$, where

$$\alpha = \sigma_{[d_1 \to 1]}\,\sigma_{[d_2 \to 1]} \cdots \sigma_{[d_t \to 1]}.$$

Therefore, we have determined the elements in $USS(\delta)$ in terms of their permutations. ☐

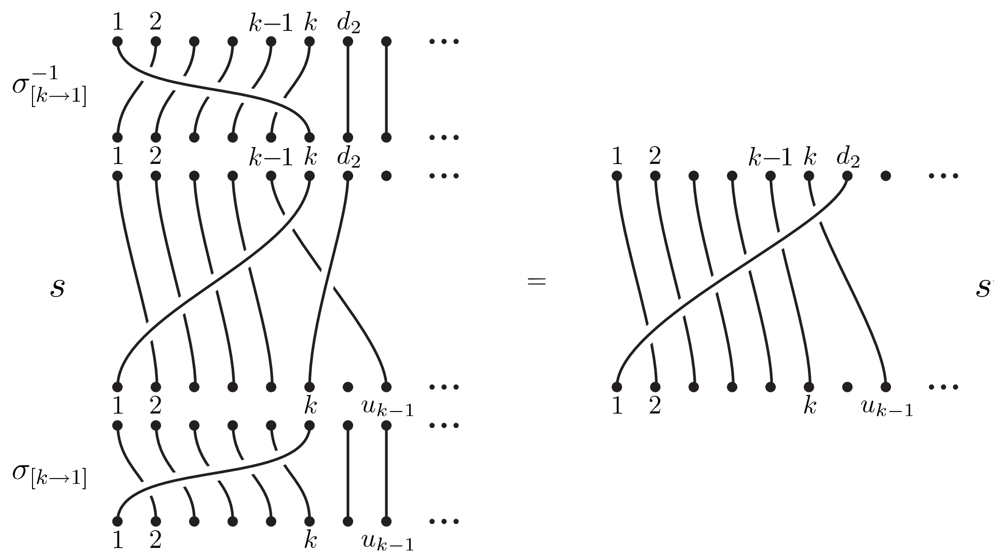


Figure 3: Conjugating s to s'.

Remark 11. The above element α is simple, hence all elements in $USS(\delta)$ are conjugate to δ by a simple element.

Corollary 12. *If* $\delta = \sigma_{n-1} \cdots \sigma_1 \in B_n^A$ *then* $\#(USS(\delta)) = 2^{n-2}$.

Proof. The elements in $USS(\delta)$ are characterized by the permutation given in the above result, which is itself characterized by the sequence $1 < u_1 < \cdots < u_r < n$. The number of possible sequences is equal to the number of subsets of $\{2, \ldots, n-1\}$ which is precisely 2^{n-2}. ☐

Now let us do the same for $USS(\varepsilon)$.

Proposition 13. *An element* $s \in B_n^A$ *belongs to* $USS(\varepsilon)$ *if and only if it is simple and*

$$\pi_s = (a)(1\ u_1\ u_2\ \cdots\ u_r\ n\ d_t\ d_{t-1}\ \cdots\ d_1),$$

for some $u_1 < u_2 < \cdots < u_r$ *and some* $d_t > d_{t-1} > \cdots > d_1$*, with* $r, t \geq 0$ *and* $r + t + 3 = n$*. Notice that* $a \neq 1, n$*. Moreover, in this case one has* $\beta^{-1}s\beta = \varepsilon$*, where*

$$\beta = \sigma_{[d_1 \to 1]}\,\sigma_{[d_2 \to 1]} \cdots \sigma_{[d_t \to 1]}\,\sigma_{[b \to 2]}$$

and $b = a + t - \max\{i\,:\,d_i < a\}$*,* *Proof.* Since ε is simple, the elements of $USS(\varepsilon)$ are precisely the simple conjugates of ε; in particular, $USS(\varepsilon)$ consists of simple elements whose permutation is the product of a cycle of length 1 (a fixed point) and a cycle of length $n-1$. Moreover, if $s \in USS(\varepsilon)$ then $s^{n-1} = \Delta^2$, where any two strands cross exactly twice.

Let $s \in USS(\varepsilon)$, and let $\pi_s = (a)(x_1 \ \cdots \ x_{n-1})$. If $a = 1$ then the first strand of s does not cross any other strand. This means that we can write s as a word in Artin generators in which the letter σ_1 does not appear. But in that case every power of s would satisfy the same property. In particular, the first strand of $s^{n-1} = \Delta^2$ would not cross any other strand, a contradiction. Hence $a \neq 1$. In the same way one shows that $a \neq n$. Therefore the permutation induced by s can be written as

$$\pi_s = (a)(1 \ u_1 \ u_2 \ \cdots \ u_r \ n \ d_t \ d_{t-1} \ \cdots \ d_1).$$

We can show that $u_1 < \cdots < u_r$ and that $d_t > \cdots > d_1$, using the same proof as in Proposition 10. In Figure 4 we can see an example of two braids whose permutations are cycles of length $n-1$. The first one satisfies the above inequalities and the second one does not.

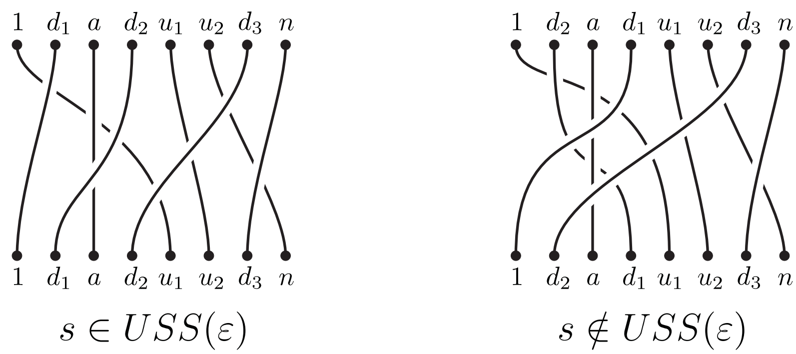


Figure 4: Two simple braids in B_8^A whose permutations are cycles of length 7. By Proposition 13, the first one is conjugate to ε and the second one is not. As in Figure 2, the exponent sums of the two braids differ; the exponent sum of second one is 12, while the exponent sum of conjugates of $\varepsilon \in B_8^A$ is 8.

Now let s be a simple element such that

$$\pi_s = (a)(1 \ u_1 \ u_2 \ \cdots \ u_r \ n \ d_t \ d_{t-1} \ \cdots \ d_1)$$

for some $u_1 < u_2 < \cdots < u_r$, some $d_t > d_{t-1} > \cdots > d_1$ and some $a \neq 1$ or n.

Suppose that $t > 0$. Similarly to the proof of Proposition 10, we will conjugate s by $\sigma_{[d_1 \to 1]}$, and this will reduce the index t. Let $k = d_1$. If $a > k$ one has

$$\pi_s = (a)(\underline{1 \ 2 \ \cdots \ k-1} \ u_{k-1} \ \cdots \ u_r \ \ n \ d_t \ \cdots \ d_2 \underline{\ k\ }),$$

otherwise

$$\pi_s = (a)(\underline{1 \ 2 \ \cdots \ a-1 \ a+1 \ \cdots \ k-1} \ u_{k-2} \ \cdots \ u_r \ \ n \ d_t \ \cdots \ d_2 \underline{\ k\ }).$$

The picture in the former case is the same as in Figure 3, while the latter case is represented in Figure 5. In either case, the strand of s that ends at k (that is, d_2 if $t > 1$ or n if $t = 1$)

does not cross the strands that end at $1, 2, \ldots, k-1$ (that is $k, 1, 2, \cdots, k-2$, where one of them could possibly be equal to a). Therefore $s\,\sigma_{[k\to 1]}$ is simple. At the same time, the strand k of s crosses the strands $k-1, k-2, \ldots, 1$ (where one of them could be equal to a). Hence $s' = \sigma_{[k\to 1]}^{-1} s\, \sigma_{[k\to 1]}$ is simple. Depending on whether $a > k$ or not, one has either

$$\pi_{s'} = (a)(\underline{1\ 2\ \cdots\ k-1\ k}\ u_{k-1}\ \cdots\ u_r\ \ n\ d_t\ \cdots\ d_2),$$

or

$$\pi_{s'} = (a+1)(\underline{1\ 2\ \cdots\ a\ a+2\ \cdots\ k}\ u_{k-2}\ \cdots\ u_r\ \ n\ d_t\ \cdots\ d_2).$$

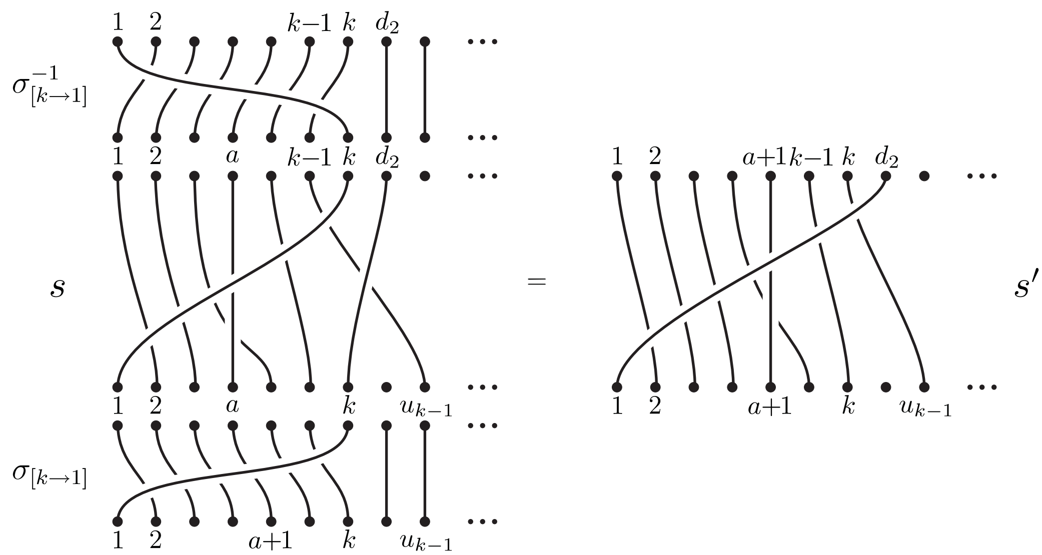


Figure 5: Conjugating s to s' when $a < k$.

We can continue this process, increasing the number of indices between 1 and n. Notice that the index a increases at some step i if and only if $d_i > a$, and in that case we will also have $d_{i+1}, \ldots, d_t > a$, so once the index a increases, it continues increasing at every further step of the procedure. Eventually, one obtains a simple element $s_0 = \alpha^{-1} s \alpha$ such that

$$\alpha = \sigma_{[d_1\to 1]}\sigma_{[d_2\to 1]}\cdots\sigma_{[d_t\to 1]}$$

and

$$\pi_{s_0} = (b)(1\ 2\ \cdots\ b-1\ b+1\ \cdots\ n),$$

where $b = a + t - \max\{i\ :\ d_i < a\}$.

If $b = 2$ we already have $s_0 = \varepsilon$. Otherwise we will conjugate s_0 by $\sigma_{[b\to 2]}$. Notice that the strand of s_0 that ends at b (that is, strand b itself) does not cross the strands ending at $b-1, b-2, \ldots, 2$. Hence $s_0\sigma_{[b\to 2]}$ is simple. Next, the strand b of $s_0\sigma_{[b\to 2]}$ crosses the strands $b-1, b-2, \ldots, 2$, hence $\sigma_{[b\to 2]}^{-1} s_0 \sigma_{[b\to 2]}$ is simple, and its permutation is equal to $(2)(1\ 3\ 4\ \cdots\ n)$, hence this simple braid is equal to ε. Therefore, if we define

$$\beta = \sigma_{[d_1\to 1]}\,\sigma_{[d_2\to 1]}\cdots\sigma_{[d_t\to 1]}\,\sigma_{[b\to 2]}$$

where $b = a + t - \max\{i\ :\ d_i < a\}$, then $\beta^{-1} s \beta = \varepsilon$. □

Remark 14. The element β defined above is not necessarily simple, but in the worst case it is the product of two simple elements, $\sigma_{[d_1\to 1]}\,\sigma_{[d_2\to 1]}\cdots\sigma_{[d_t\to 1]}$ and $\sigma_{[b\to 2]}$. Hence, every element in $USS(\varepsilon)$ is connected to ε by a conjugating element of canonical length at most 2.

Corollary 15. *If* $\varepsilon = \sigma_1(\sigma_{n-1}\cdots\sigma_1) \in B_n^A$ *then* $\#(USS(\varepsilon)) = (n-2)2^{n-3}$.

Proof. The elements in $USS(\varepsilon)$ are characterized by the permutation given in Proposition 13, which is itself characterized by the sequence $1 < u_1 < \cdots < u_r < n$ and the number a. Since $a \neq 1, n$, one has $n-2$ choices for the index a. And for every choice of a, the number of possible sequences is equal to the number of subsets of $\{2, \dots, a-1, a+1, \dots, n-1\}$, which is precisely 2^{n-3}. Hence the total number of choices is $(n-2)2^{n-3}$. □

Notice that the results in this section not only characterize the elements in $USS(\delta)$ and $USS(\varepsilon)$ by their permutations, determining the sizes of these two sets, but also find conjugating elements from any given element in $USS(\delta)$ (resp. $USS(\varepsilon)$) to δ (resp. ε). This fact, together with the known algorithm for obtaining for any braid X a conjugate Y of X whose canonical length is minimal [18], which for periodic X implies $Y \in USS(X)$, provides a solution to the conjugacy search problem for conjugates of δ or ε. Moreover, this algorithm has complexity $O(l^3 n^3 \log n)$, where l is the letter length in Artin generators of the input braid. But this algorithm is not easily generalized to other periodic braids (conjugates of *powers* of δ or ε), so in the next section we will present an alternative approach that solves the conjugacy search problem for every periodic braid, using other Garside structures and other groups (namely Artin-Tits groups of type **B**).

Remark 16. In [26] there is a simple algorithm which finds a conjugating element from any braid in $USS(\delta)$ to δ. It is also an efficient algorithm, and very easy to implement, but Proposition 10 directly provides a conjugating element and at the same time characterizes the elements in $USS(\delta)$, in such a way that we can count them all.

Remark 17. We end this section by remarking that, in practical computations for small n, the sizes of $USS(\delta^k)$ and $USS(\varepsilon^k)$ for different values of k are in most cases much bigger than the sizes of $USS(\delta)$ and $USS(\varepsilon)$, respectively. Hence the usual algorithm in [21] is not efficient in general for periodic braids. We also notice that the algorithm in [26] can be generalized to ε, but it does not generalize to powers of δ or ε. Hence the algorithm in the next section is, to our knowledge, the first efficient algorithm to solve the conjugacy search problem for periodic braids.

4 Proof of Theorem 1

In this section we will complete the proof of Theorem 1 by developing a polynomial algorithm to solve the conjugacy search problem for periodic braids.

Suppose that we are given two braids $X, Y \in B_n^A$. Using Algorithm A, we may assume that X and Y are periodic, and that they are conjugate to the same power of δ or ε (otherwise we would stop and return a negative answer for steps (1) or (2) in Theorem 1). We can also assume that we know the specific power of δ (resp. ε) which is conjugate to X and Y, say δ^k (resp. ε^k). Clearly, we just need an algorithm that finds a conjugating element from X to δ^k

(resp. ε^k), since the same algorithm can be applied to Y and we would immediately obtain a conjugating element from X to Y.

Therefore, we will suppose that we are given a braid $X \in B_n^A$ as a word of length l in $\sigma_1, \dots, \sigma_{n-1}$ and their inverses, and that X is conjugate to δ^k respectively ε^k for some $k \neq 0$. We will describe algorithms finding a conjugating element from X to δ^k or ε^k, whose complexities are polynomial in n and l. The two cases are treated separately, in Sections 4.1 and 4.2 below.

4.1 Solving the conjugacy search problem for conjugates of δ^k

We remind the reader that in [4], Birman, Ko and Lee investigated a then-new presentation for the braid groups:

$$B_n^B : \left\langle a_{ts}, \quad 1 \leq s < t \leq n \;\middle|\; \begin{array}{lll} a_{ts}a_{rq} = a_{rq}a_{ts} & \text{if} & (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{if} & 1 \leq r < s < t \leq n. \end{array} \right\rangle . \tag{2}$$

The elements a_{ts} are called *band generators* or *Birman-Ko-Lee generators*. The left sketch in Figure 6 shows one way to think of the generator a_{ts}. A different way is shown on the right, where we consider $\mathbb{D}_n^2$ to be the disc in $\mathbb{C}$ centered at the origin with radius 2, the n punctures being the n-th roots of unity $\zeta_k = e^{2k\pi i/n}$ for $k = 1, \dots, n$. Then a_{ts} is the braid that swaps the punctures ζ_s and ζ_t as shown in the right hand side of Figure 6.

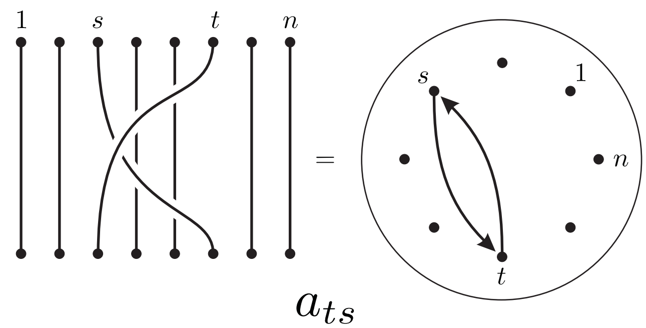


Figure 6: The band generator a_{ts}.

In will be convenient to think of B_n^A and B_n^B as defining distinct groups. The relation between them is then given by the isomorphism $\Phi : B_n^B \to B_n^A$:

$$\begin{array}{rcl} \Phi(a_{i,i+1}) & = & \sigma_i, \quad 1 \leq i \leq n-1 \\ \Phi(a_{ts}) & = & (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}), \quad 1 \leq s < t-1 \leq n. \end{array} \tag{3}$$

The inverse automorphism sends σ_i to $a_{i,i+1}$.

The reason we wish to think of these two groups as being distinct, is because we need to distinguish the Garside structure on B_n^A [20] from that on B_n^B, introduced in [4]. When we say that $X \in B_n^A$ (resp. $X \in B_n^B$) is written in left normal form, our notation is intended to mean that we are using the Garside structure associated to the presentation (1) (resp. (2)). The key point here (which we will generalize when we treat the case of braids conjugate to ε, is that the Garside element for B_n^B is precisely our periodic braid δ. It is shown in [4] that with respect to the Garside structure introduced in [4], the left normal form of a braid in B_n^B, given as a word of length l in the band generators and their inverses, can be computed in time $O(l^2 n)$. We will solve the conjugacy search problem for braid conjugate to δ by making use of the algorithm in [21], using the Garside structure on B_n^B. This will enable us to bypass the difficulty which was uncovered in Corollary 15.

It will be important for our purposes to describe the simple elements in the Garside structure on B_n^B. These simple elements are known to be in bijection with the *non-crossing partitions* of the n-th roots of unity $\mathcal{R} = \{\zeta_1, \dots, \zeta_n\}$ [4, 2]. Non-crossing partitions can be defined as follows: Given a partition $\wp$ of $\mathcal{R}$, every part of $\wp$ with d elements ($d \geq 2$) gives rise to a unique convex polygon joining the d punctures (if $d = 2$ the polygon is just a segment). The partition $\wp$ is said to be non-crossing if these polygons are pairwise disjoint. Each polygon determines a braid which corresponds to a rotation of its d vertices in the counterclockwise sense, and that we will call a *polygonal braid*. Disjoint polygons determine commuting polygonal braids. The simple element corresponding to a non-crossing partition $\wp$ is the product of the (mutually commuting) polygonal braids determined by $\wp$, as is shown in Figure 7. Hence, each simple element of B_n^B is a product of at most $n/2$ polygonal braids. Notice also that the polygonal braid corresponding to the part $\{\zeta_{i_1}, \zeta_{i_2}, \cdots, \zeta_{i_k}\}$, with $i_1 < i_2 < \cdots < i_k$, is precisely $a_{i_k, i_{k-1}}\, a_{i_{k-1}, i_{k-2}} \cdots a_{i_2, i_1}$. The element δ is the polygonal braid corresponding to the whole set $\{\zeta_1, \dots, \zeta_n\}$.

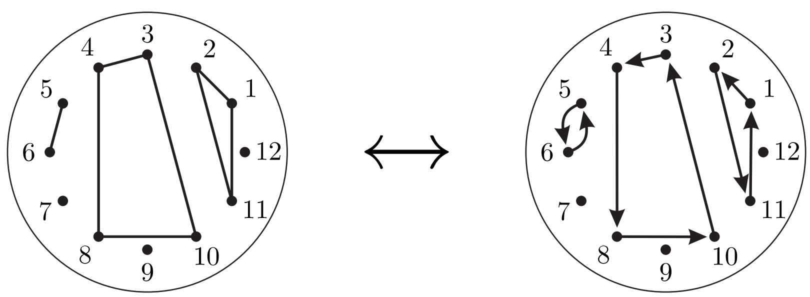


Figure 7: A simple element of B_{12}^B, which is a product of three polygonal braids. It is the braid $(a_{11,2}a_{2,1})(a_{10,8}a_{8,4}a_{4,3})(a_{6,5})$, where the three factors (the polygonal braids) commute.

Before stating and proving the main result of this section, we need a lemma that will improve the estimation of the complexity of our algorithms for periodic braids. It is the following:

Lemma 18. *If a nontrivial periodic braid $X \in B_n^A$ is given as a word of length l in the Artin generators and their inverses, then $l \geq n - 1$.*

Proof. Suppose that $l < n - 1$. Then the exponent sum of X is an integer m with $0 \leq |m| < n - 1$. By Lemma 7, the exponent sum of a periodic braid is a multiple of either $n - 1$ or n. It follows that $m = 0$, so X is conjugate to $\delta^0 = 1$. But in this case X is trivial, a contradiction. □

We can finally show our main result for conjugates of powers of δ.

Proposition 19. *Let $X \in B_n^A$ be given as a word of length l in the Artin generators $\sigma_1, \dots, \sigma_{n-1}$ and their inverses. If X is conjugate to δ^k for $k \neq 0$, there exists an algorithm of complexity $O(l^3 n^2)$ that finds a conjugating element $C \in B_n^A$ such that $C^{-1}XC = \delta^k$.*

Proof. We are given a word $X \in B_n^A$:

$$X = \sigma_{\mu_1}^{\epsilon_1} \sigma_{\mu_2}^{\epsilon_2} \cdots \sigma_{\mu_l}^{\epsilon_l}.$$

It is very simple to rewrite X as a word in the band generators, because $\Phi^{-1}(\sigma_i) = a_{i+1,i}$ for each $i = 1 \ldots, n-1$. So we have:

$$\Phi^{-1}(X) = a_{\mu_1+1,\mu_1}^{\epsilon_1} a_{\mu_2+1,\mu_2}^{\epsilon_2} \cdots a_{\mu_l+1,\mu_l}^{\epsilon_l}.$$

We can then apply iterated cycling and decycling to $X \in B_n^B$, in order to obtain a conjugate $X' \in B_n^B$ of minimal canonical length, together with a conjugating element. It is shown in [5] that we need to apply at most $|\delta|\, l$ cyclings and decyclings, this means at most $|\delta|\, l$ computations of normal forms, where $|\delta|$ is the letter length of δ written as a positive word in the band generators. Since $|\delta| = n-1$, it follows that we can obtain $X' \in B_n^B$ of minimal canonical length, and a conjugating element from X to X', in time $O(l^3 n^2)$.

But X is conjugate to δ^k, which is a power of the Garside element of B_n^B, so $\ell(\delta^k) = 0$. This means that $USS(X) = \{\delta^k\}$, and more precisely $X' = \delta^k$. Hence, we have found a conjugating element $C \in B_n^B$ from X to δ^k in time $O(l^3 n^2)$. As this conjugating element is given in terms of band generators, the last step consists of translating C to Artin generators.

Recall that a cycling (resp. a decycling) consists of a conjugation by a simple element (resp. by the inverse of a simple element). So C is a product of at most $(n-1)l$ simple elements (or inverses) in B_n^B. In the Birman-Ko-Lee structure, the letter length of a simple element is at most $n-1$, so $C \in B_n^B$ has letter length at most $(n-1)^2 l$. Since each band generator is equal to a word in Artin generators of length at most $2n-3$, this means that one can translate C to Artin generators, via the isomorphism Φ, in time $O(n^3 l)$.

Therefore, the conjugacy search problem for conjugates of δ^k, given as words in Artin generators, can be solved in time $O(l^3 n^2 + l n^3)$. By Lemma 18, one has $l \geq n-1$, so that $l n^3 \leq l(l+1) n^2 < l^3 n^2$ (we can assume $l > 1$). Hence this complexity is equal to $O(l^3 n^2)$. □

The algorithm described in the proof of Proposition 19 is the following.

Algorithm B:

Input: A word w in Artin generators and their inverses representing $X \in B_n^A$ conjugate to δ^k.

Output: $C \in B_n^A$ such that $C^{-1} X C = \delta^k$.

1. Translate w to a word w' in band generators using the rule $\sigma_i \to a_{i+1,i}$.
2. Apply iterated cyclings and decyclings in B_n^B to w' until δ^k is obtained. Let $C' \in B_n^B$ be the product of all the conjugating elements in this process.
3. Translate C' to a word $C \in B_n^A$, using the rule

$$a_{ts} \to (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}).$$

4. Return C.

By Proposition 19, Algorithm B has complexity $O(l^3 n^2)$.

4.2 Solving the conjugacy search problem for conjugates of ε^k

Our final task is to learn how to find the conjugating element in the case when X is conjugate to ε^k. The methods will be identical to those used in case of conjugates of δ^k: We will begin with $X \in B_n^A$, i.e. X will be given as a word in the generators of B_n^A and their inverses. Using Algorithm A we will have verified that X is periodic and conjugate to a known power of ε. Our task will be to find the conjugating element. We will prove that there is also a suitable Garside group, with a known Garside structure, whose Garside element is ε. This group, however, is not the braid group, rather it is a subgroup of the braid group that we will denote $P_{n,2}$. The subgroup is formed by the braids whose corresponding permutation preserves the second puncture. It is well known that $P_{n,2}$ is a Garside group, since it is isomorphic to the Artin-Tits group of type $\mathbf{B}_{n-1}$ [14]. Nevertheless, we won't use the classical Garside structure on the Artin-Tits group $\mathcal{A}(\mathbf{B}_{n-1})$, but the *dual* Garside structure defined in [2]. This explains why we shall start, in Section 4.2.1, by describing the groups, embeddings and Garside structures that we will need to use in our algorithm. We then put them to work in Section 4.2.2

4.2.1 Braids fixing one puncture, Artin-Tits groups of type B and symmetric braids

We shall now describe the five groups we are interested in, with their corresponding Garside structures. The first two groups are well known, they are just B_n^A and B_{2n-2}^B.

Next, let us consider the subgroup $P_{n,2} \subset B_n^A$, consisting of braids that fix the second puncture. That is, $P_{n,2} = \{X \in B_n^A : \pi_X(2) = 2\}$. We will not consider right now a Garside structure on $P_{n,2}$, but we remark that $\varepsilon \in P_{n,2}$.

Now let Sym_{2n-2} be the centralizer of δ^{n-1} in B_{2n-2}^B, where we write δ for δ_{2n-2}. In other words, if we represent the $2n-2$ punctures of $\mathbb{D}_{2n-2}^2$ as the $(2n-2)$-nd roots of unity, the elements of Sym_{2n-2} are precisely the braids which are invariant under a rotation of 180 degrees. This is why they are called *symmetric braids.*

Finally, consider the Artin-Tits group $\mathcal{A}(\mathbf{B}_{n-1})$, whose presentation is

$$\mathcal{A}(\mathbf{B}_{n-1}) = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j = s_j s_i & \text{if } |i-j| > 1 \\ s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \text{ and } i, j \neq 1 \\ s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 & \end{array} \right\rangle.$$

We shall now recall from the literature that the last three groups we just considered are isomorphic, that is, one has the following situation:

$$\begin{array}{ccccc} B_n^A & & & & B_{2n-2}^B \\ \cup & & & & \cup \\ P_{n,2} & \cong & \mathcal{A}(\mathbf{B}_{n-1}) & \cong & Sym_{2n-2} \end{array} \tag{4}$$

Moreover, it can be deduced from [28] that the restriction of the Garside structure of B_{2n-2}^B determines a Garside structure in Sym_{2n-2}. Via the above isomorphisms, this induces Garside structures in $\mathcal{A}(\mathbf{B}_{n-1})$ and in $P_{n,2}$. We shall see that the Garside element of the latter is precisely ε, and this will help us to solve the conjugacy search problem for conjugates of ε^k.

Let us then study in detail the mentioned isomorphisms.

Lemma 20. *The map $\rho: \mathcal{A}(\mathbf{B}_{n-1}) \to P_{n,2}$ given by $\rho(s_1) = \sigma_1^2$, $\rho(s_2) = \sigma_1\sigma_2\sigma_1^{-1}$ and $\rho(s_i) = \sigma_i$ for $i > 2$, is an isomorphism.*

Proof. Proposition 5.1 in [14] provides an isomorphism $\rho_0: \mathcal{A}(\mathbf{B}_{n-1}) \to P_{n,1}$, where $P_{n,1}$ is the subgroup of B_n^A consisting of braids which fix the first puncture. This isomorphism is given by $\rho_0(s_1) = \sigma_1^2$ and $\rho_0(s_i) = \sigma_i$ for $i > 1$, and it was already known to specialists, prior to [14]. Now we just need to notice that the inner automorphism $\varphi: B_n^A \to B_n^A$ given by $\varphi(X) = \sigma_1 X \sigma_1^{-1}$ sends $P_{n,1}$ isomorphically to $P_{n,2}$, and that $\varphi_{|P_{n,1}} \circ \rho_0 = \rho$. □

Remark 21. It is well known [14] that $P_{n,2}$ (hence $\mathcal{A}(\mathbf{B}_{n-1})$) can be identified with the braid group of the open annulus $\mathbb{D}^2\backslash\{0\}$ on $n-1$ strands. Indeed, an element $X \in P_{n,2}$ fixes the second puncture, so it can be isotoped to a braid whose second strand in $\mathbb{D}^2 \times [0,1]$ is a straight line, say $\{0\} \times [0,1]$. This second strand can be considered to be a hole of $\mathbb{D}^2$, so X can be regarded as a braid on $n-1$ strands of $\mathbb{D}^2\backslash\{0\}$.

In order to avoid confusion, we will represent elements in $P_{n,2} \in B_n^A$ in the usual way, as they are represented at the bottom of Figure 8, while elements of $\mathcal{A}(\mathbf{B}_{n-1})$ will be represented in the Birman-Ko-Lee style, as braids on $\mathbb{D}^2\backslash\{0\}$ whose base points are the $(n-1)$-st roots of unity, as we can see at the top of Figure 8.

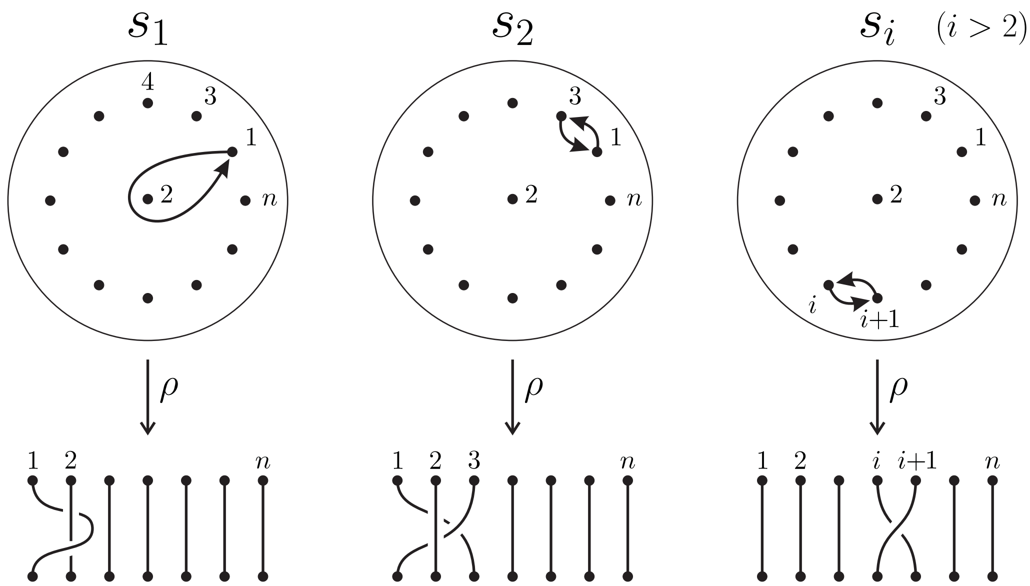


Figure 8: The generators of $\mathcal{A}(\mathbf{B}_{n-1})$, represented as braids on $\mathbb{D}^2\backslash\{0\}$, and their images under the isomorphism $\rho: \mathcal{A}(\mathbf{B}_{n-1}) \to P_{n,2}$.

Lemma 22. *The map $\theta': \mathcal{A}(\mathbf{B}_{n-1}) \to Sym_{2n-2}$ given by $\theta'(s_1) = a_{n,1}$ and $\theta'(s_i) = a_{i,i-1}\, a_{i+n-1,i+n-2}$ for $i > 1$, is an isomorphism.*

Proof. In [10], Brieskorn showed that an Artin-Tits group of finite type is the fundamental group of the regular orbit space of its corresponding Coxeter group, acting as a finite real reflection group on a complex space. In particular, since the Coxeter group associated to $\mathcal{A}(\mathbf{B}_{n-1})$ is $W = \Sigma_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$, where the symmetric group acts by permuting coordinates (that is, W is the signed permutation group), and its corresponding hyperplane arrangement is $x_1x_2\cdots x_{n-1}\prod_{i\neq j}(x_i-x_j)(x_i+x_j)$, it follows that $\mathcal{A}(\mathbf{B}_{n-1})=\pi_1(X_{\mathbf{B}_{n-1}}/W)$, where

$$X_{\mathbf{B}_{n-1}}=\{(x_1,\ldots,x_{n-1})\in\mathbb{C}^{n-1}\mid x_i\neq\pm x_j \text{ for } i\neq j;\ x_i\neq 0 \text{ for all } i\}.$$

A good way to describe the space $X_{\mathbf{B}_{n-1}}$ is as the set of $(n-1)$-tuples of pairs

$$((x_1,-x_1),(x_2,-x_2),\ldots,(x_{n-1},-x_{n-1})),$$

where each $x_i\in\mathbb{C}$, any two pairs are distinct, and $x_i\neq 0$ for all i. Considering the action of W, all the above pairs and $(n-1)$-tuples can be regarded as unordered. Hence $X_{\mathbf{B}_{n-1}}/W$ is the configuration space of $2n-2$ disjoint and undistinguishable points in $\mathbb{C}$, whose configuration is invariant under multiplication by -1. We can choose as a base point of this space the $(2n-2)$-nd roots of unity. Hence, an element of its fundamental group is represented by a braid which is invariant under a rotation by 180 degrees, that is, by a symmetric braid in B^B_{2n-2}.

It is important to note that two symmetric braids represent the same element in $\pi_1(X_{\mathbf{B}_{n-1}}/W)$ if and only if they are isotopic *through symmetric braids*, hence one cannot say a priori that two symmetric braids that are isotopic in B^B_{2n-2} represent the same element of $\pi_1(X_{\mathbf{B}_{n-1}}/W)$. Fortunately, it is shown in [3] that two symmetric braids are isotopic in B^B_{2n-2} if and only if they are isotopic through symmetric braids. That is, it is shown that $\mathcal{A}(\mathbf{B}_{n-1})=\pi_1(X_{\mathbf{B}_{n-1}}/W)\cong Sym_{2n-2}$.

Moreover, from the work in [3] one obtains an isomorphism $\theta:\ Sym_{2n-2}\rightarrow\mathcal{A}(\mathbf{B}_{n-1})$, where elements of Sym_{2n-2} are symmetric braids based on the $(2n-2)$-nd roots of unity, and the elements of $\mathcal{A}(\mathbf{B}_{n-1})$ are considered as braids on the annulus $\mathbb{D}^2\backslash\{0\}$ based on the $(n-1)$-st roots of unity. The isomorphism θ can be easily described geometrically, since it just identifies antipodal points in $\mathbb{C}$. That is, it sends $z\in\mathbb{C}\backslash\{0\}$ to $z^2/|z|$. This corresponds to a two-sheeted covering map of $\mathbb{C}\backslash\{0\}$, and since no strand of a symmetric braid touches the axis $\{0\}\times[0,1]$, this map is well defined.

In Figure 9 we can see that $\theta(a_{n,1})=s_1$ and that $\theta(a_{i,i-1}\,a_{i+n-1,i+n-2})=s_i$ for $i>1$, where in the picture one has $\zeta_k=e^{2k\pi i/(2n-2)}$ and $\xi_k=e^{2k\pi i/(n-1)}$. Therefore $\theta'=\theta^{-1}$, so it is an isomorphism. □

By Lemmas 20 and 22 we know that $P_{n,2}\cong\mathcal{A}(\mathbf{B}_{n-1})\cong Sym_{2n-2}$, and we also know how to transform any word in the generators $s_1,\ldots,s_{n-1}$ of $\mathcal{A}(\mathbf{B}_{n-1})$ and their inverses, into a word in either the Artin generators of $P_{n,2}$ or the band generators of Sym_{2n-2}, via the isomorphisms ρ and $\theta'=\theta^{-1}$.

$$\begin{array}{ccccc} B^A_n & & & & B^B_{2n-2} \\ \cup & & & & \cup \\ P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \stackrel{\theta'}{\longrightarrow} & Sym_{2n-2}. \end{array}$$

But in our algorithm we will need to translate any word in the Artin generators of B^A_n, representing an element of $P_{n,2}$, to a word in the band generators of Sym_{2n-2}, and vice versa. Hence, we need the following results.

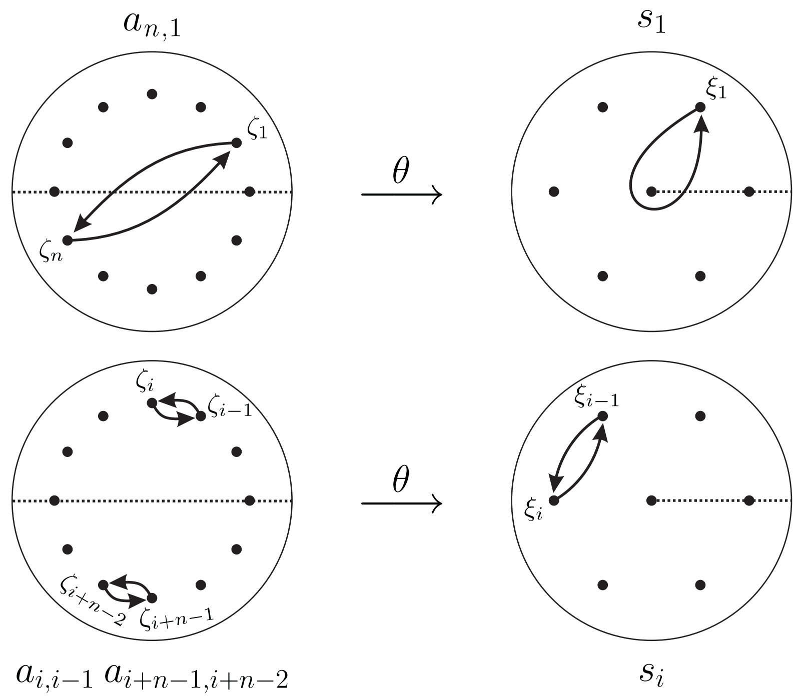


Figure 9: The map θ transforms the symmetric braids on the left hand side to the generators of $\mathcal{A}(\mathbf{B}_{n-1})$ on the right hand side.

Lemma 23. *Let $X \in P_{n,2} \subset B_n^A$ be given as a word of length l in the Artin generators and their inverses, $X = \sigma_{\mu_1}^{\epsilon_1}\sigma_{\mu_2}^{\epsilon_2}\cdots\sigma_{\mu_l}^{\epsilon_l}$. For $i = 0,\dots,l$, let $X_i = \sigma_{\mu_1}^{\epsilon_1}\sigma_{\mu_2}^{\epsilon_2}\cdots\sigma_{\mu_i}^{\epsilon_i}$ and let $k_i = \pi_{X_i}(2)$, that is, the final position of the second strand of X_i. Then one obtains a word in the band generators and their inverses representing $\theta'(\rho^{-1}(X)) \in Sym_{2n-2}$, by replacing each letter $\sigma_{\mu_i}^{\epsilon_i}$ using the following rules:*

$$\sigma_{\mu_i} \longrightarrow \begin{cases} a_{\mu_i+1,\mu_i}\, a_{\mu_i+n,\mu_i+n-1} & \text{if} \quad \mu_i < k_{i-1}-1, \\ 1 & \text{if} \quad \mu_i = k_{i-1}-1, \\ a_{\mu_i+n-1,\mu_i} & \text{if} \quad \mu_i = k_{i-1}, \\ a_{\mu_i,\mu_i-1}\, a_{\mu_i+n-1,\mu_i+n-2} & \text{if} \quad \mu_i > k_{i-1}, \end{cases}$$

and

$$\sigma_{\mu_i}^{-1} \longrightarrow \begin{cases} a_{\mu_i+n,\mu_i+n-1}^{-1}\, a_{\mu_i+1,\mu_i}^{-1} & \text{if} \quad \mu_i < k_{i-1}-1, \\ a_{\mu_i+n-1,\mu_i}^{-1} & \text{if} \quad \mu_i = k_{i-1}-1, \\ 1 & \text{if} \quad \mu_i = k_{i-1}, \\ a_{\mu_i+n-1,\mu_i+n-2}^{-1}\, a_{\mu_i,\mu_i-1}^{-1} & \text{if} \quad \mu_i > k_{i-1}. \end{cases}$$

Moreover, this algorithm has complexity $O(l)$, and produces a word of length at most $2l$.

Proof. Recall that we are given a braid $X \in B_n^A$ that fixes the second puncture, that is, $X \in P_{n,2}$, written as a word in the Artin generators of B_n^A and their inverses. We want to write $\rho^{-1}(X)$ as a word in the generators $s_1, \ldots, s_{n-1}$ and their inverses, and then $\theta'(\rho^{-1}(X))$ as a word in the band generators of B^B_{2n-2}.

The first problem is that X is not given as a word in the generators of $P_{n,2}$, but in the generators of B^A_n. We will then use the Reidemeister-Schreier method (see Section 2.3 of [27]) to decompose X as a product of elements in $P_{n,2}$. In order to do this, notice that $P_{n,2}$ is a subgroup of B^A_n of index n. The right coset of a braid Z depends on where it sends the second puncture. If $\pi_Z(2) = k$, we denote by R_k a representative of the right coset $P_{n,2}\, Z \in P_{n,2}\backslash B^A_n$. For technical reasons, we will choose as coset representatives the elements $R_1 = \sigma_1$, $R_2 = 1$ and $R_k = \sigma^{-1}_{[k\to 2]} = \sigma_2^{-1} \cdots \sigma_{k-1}^{-1}$ if $k > 2$.

Then, for $i = 0, \ldots, l$, we define $\overline{X_i} = R_{k_i}$. That is, $\overline{X_i}$ is the chosen representative of $P_{n,2}\, X_i \in P_{n,2}\backslash B^A_n$. Note that $\overline{X_0} = \overline{X_l} = R_2 = 1$.

By the Reidemeister-Schreier method, one has

$$X = \prod_{i=1}^{l} \left(\overline{X_{i-1}}\, \sigma_{\mu_i}^{\epsilon_i}\, \overline{X_i}^{\,-1}\right) = \prod_{i=1}^{l} \left(R_{k_{i-1}} \sigma_{\mu_i}^{\epsilon_i} R_{k_i}^{-1}\right),$$

where each of the above l factors belongs to $P_{n,2}$. Notice that $k_i = k_{i-1}$, unless either $\mu_i = k_{i-1}$ (in which case $k_i = k_{i-1}+1$) or $\mu_i = k_{i-1}-1$ (and then $k_i = k_{i-1}-1$). One can check that, depending on μ_i and k_{i-1}, each of the above factors can be written in terms of the Artin generators and their inverses as follows. If $\epsilon_i = 1$, one has:

$$(R_{k_{i-1}}\sigma_{\mu_i}R_{k_i}^{-1}) = \begin{cases} \sigma_2^{-1}\sigma_1\sigma_2 & \text{if } \quad 1 = \mu_i < k_{i-1}-1, \\ \sigma_{\mu_i+1} & \text{if } \quad 1 \neq \mu_i < k_{i-1}-1, \\ 1 & \text{if } \quad \mu_i = k_{i-1}-1, \\ \sigma_1^2 & \text{if } \quad 1 = \mu_i = k_{i-1}, \\ (\sigma_2^{-1}\sigma_3^{-1}\cdots\sigma_{\mu_i-1}^{-1})\sigma_{\mu_i}(\sigma_{\mu_i}\sigma_{\mu_i-1}\cdots\sigma_2) & \text{if } \quad 1 \neq \mu_i = k_{i-1}, \\ \sigma_1\sigma_2\sigma_1^{-1} & \text{if } \quad 2 = \mu_i > k_{i-1}, \\ \sigma_{\mu_i} & \text{if } \quad 2 \neq \mu_i > k_{i-1}. \end{cases}$$

If $\epsilon_i = -1$, one obtains the inverses of the above, in the following way:

$$(R_{k_{i-1}}\sigma_{\mu_i}^{-1}R_{k_i}^{-1}) = \begin{cases} \sigma_2^{-1}\sigma_1^{-1}\sigma_2 & \text{if } \quad 1 = \mu_i < k_{i-1}-1, \\ \sigma_{\mu_i+1}^{-1} & \text{if } \quad 1 \neq \mu_i < k_{i-1}-1, \\ \sigma_1^{-2} & \text{if } \quad 1 = \mu_i = k_{i-1}-1, \\ (\sigma_2^{-1}\sigma_3^{-1}\cdots\sigma_{\mu_i}^{-1})\sigma_{\mu_i}^{-1}(\sigma_{\mu_i-1}\cdots\sigma_2) & \text{if } \quad 1 \neq \mu_i = k_{i-1}-1, \\ 1 & \text{if } \quad \mu_i = k_{i-1}, \\ \sigma_1\sigma_2^{-1}\sigma_1^{-1} & \text{if } \quad 2 = \mu_i > k_{i-1}, \\ \sigma_{\mu_i}^{-1} & \text{if } \quad 2 \neq \mu_i > k_{i-1}. \end{cases}$$

Now we need to apply ρ^{-1} to each factor $(R_{k_{i-1}}\sigma_{\mu_i}^{\epsilon_i}R_{k_i}^{-1})$, and write the image in in terms of the generators $s_1, \ldots, s_{n-1}$ and their inverses. Recall that $\rho(s_1) = \sigma_1^2$, $\rho(s_2) = \sigma_1\sigma_2\sigma_1^{-1} =$

$\sigma_2^{-1}\sigma_1\sigma_2$ and $\rho(s_i) = \sigma_i$ for $i > 2$. Notice also that $\rho(s_2 s_1 s_2^{-1}) = \sigma_2^2$, and that if $\mu_i > 2$ one has

$$\begin{aligned}\rho\left((s_{\mu_i} s_{\mu_i-1}\cdots s_3\, s_2)\, s_1\, (s_2^{-1} s_3^{-1}\cdots s_{\mu_i}^{-1})\right) &= (\sigma_{\mu_i}\sigma_{\mu_i-1}\cdots\sigma_3)\sigma_2^2(\sigma_3^{-1}\cdots\sigma_{\mu_i}^{-1}) \\ &= (\sigma_2^{-1}\cdots\sigma_{\mu_i-1}^{-1})\sigma_{\mu_i}(\sigma_{\mu_i}\cdots\sigma_2).\end{aligned}$$

Therefore, if $\epsilon_i = 1$, one has:

$$\rho^{-1}(R_{k_{i-1}}\sigma_{\mu_i}R_{k_i}^{-1}) = \begin{cases} s_{\mu_i+1} & \text{if} \quad \mu_i < k_{i-1}-1, \\ 1 & \text{if} \quad \mu_i = k_{i-1}-1, \\ (s_{\mu_i}s_{\mu_i-1}\cdots s_2)\, s_1\, (s_2^{-1}s_3^{-1}\cdots s_{\mu_i}^{-1}) & \text{if} \quad \mu_i = k_{i-1}, \\ s_{\mu_i} & \text{if} \quad \mu_i > k_{i-1}, \end{cases}$$

and if $\epsilon_i = -1$, one obtains:

$$\rho^{-1}(R_{k_{i-1}}\sigma_{\mu_i}^{-1}R_{k_i}^{-1}) = \begin{cases} s_{\mu_i+1}^{-1} & \text{if} \quad \mu_i < k_{i-1}-1, \\ (s_{\mu_i}s_{\mu_i-1}\cdots s_2)\, s_1^{-1}\, (s_2^{-1}s_3^{-1}\cdots s_{\mu_i}^{-1}) & \text{if} \quad \mu_i = k_{i-1}-1, \\ 1 & \text{if} \quad \mu_i = k_{i-1}, \\ s_{\mu_i}^{-1} & \text{if} \quad \mu_i > k_{i-1}. \end{cases}$$

Finally, we need to apply θ' to the above factors. Notice that there are only two kinds of elements to consider. The first one is s_i, with $i > 1$, which by definition is mapped to $\theta'(s_i) = a_{i,i-1}\, a_{i+n-1,i+n-2}$. The elements of the second kind are those of the form $(s_i s_{i-1}\cdots s_2)\, s_1\, (s_2^{-1}s_3^{-1}\cdots s_i^{-1})$, for $i = 1,\ldots,n-1$. One can use the Birman-Ko-Lee presentation to show that the image under θ' of this element is precisely $a_{i+n-1,i}$, but it is easier to show it geometrically, since the element $(s_i s_{i-1}\cdots s_2)\, s_1\, (s_2^{-1}s_3^{-1}\cdots s_i^{-1})$ is precisely the one in the right hand side of Figure 10, in which the puncture corresponding to the $(n-1)$-st root of unity ξ_i makes a loop around the origin. It is then easy to lift such a path via θ^{-1}, obtaining the braid $a_{i+n-1,i}$. Since $\theta^{-1} = \theta'$, one has $\theta'\left((s_i s_{i-1}\cdots s_2)\, s_1\, (s_2^{-1}s_3^{-1}\cdots s_i^{-1})\right) = a_{i+n-1,i}$, as we wanted to show.

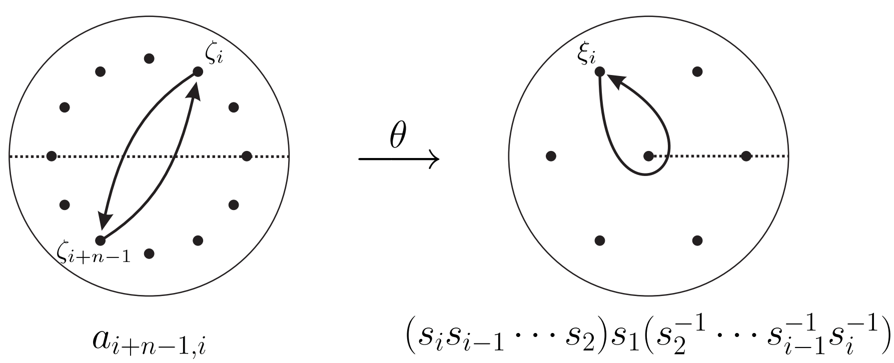


Figure 10: The image under θ of $a_{i+n-1,i}$.

One can finally transform the word $X = \sigma_{\mu_1}^{\epsilon_1}\cdots\sigma_{\mu_l}^{\epsilon_l}$ to a word representing $\theta'(\rho^{-1}(X))$, if one replaces each $\sigma_{\mu_i}^{\epsilon_i}$ by $\theta'(\rho^{-1}(R_{k_{i-1}}\sigma_{\mu_i}^{\epsilon_i}R_{k_i}^{-1}))$. By the above discussion, the formulae in the statement hold.

It remains to notice that the numbers μ_i and k_i, for $i = 1,\ldots,l$ can be obtained in time $O(l)$, and that the procedure given by the statement replaces each letter of X by at most two letters of $\theta'(\rho^{-1}(X))$. Hence the length of the obtained word is at most $2l$, and the whole procedure has complexity $O(l)$. □

Now we also need to know how to translate an element $Y \in Sym_{2n-2}$, given as a word in the band generators of B^B_{2n-2} and their inverses, to a word representing $\rho(\theta(Y)) \in P_{n,2} \subset B^A_n$. We first need a preparatory result:

Lemma 24. *If $Y \in Sym_{2n-2}$ is given as a word of length l in the band generators of B^B_{2n-2} and their inverses, then one can compute in time $O(l^2 n)$ a word $\delta^t p_1 p_2 \cdots p_k$ representing Y, such that each $p_i \in Sym_{2n-2}$ is either a symmetric polygonal braid Σ_P, or the product of two commuting polygonal braids $\Sigma_{P_1}\Sigma_{P_2}$ such that a rotation of 180 degrees permutes Σ_{P_1} and Σ_{P_2}. Moreover, $|t| \leq l$ and $k \leq ln/2$.*

Proof. The way to obtain the word $p_1 \cdots p_k$ is just the computation of the left normal form of Y in B^B_{2n-2}. It is shown in [28] that the set of *symmetric* non-crossing partitions of the $(2n-2)$-nd roots of unity (the symmetric simple elements in B^B_{2n-2}) is a sublattice of the whole lattice of non-crossing partitions. This implies that the Garside structure of B^B_{2n-2} restricts to a Garside structure on Sym_{2n-2}. Therefore, since $\delta \in Sym_{2n-2}$, the greatest common divisor of Y and any power of δ is also symmetric, and hence every factor in the left normal form of Y is symmetric.

By [4], the left normal form of Y can be computed in time $O(l^2 n)$. Once that it is computed, each non-δ factor is the product of mutually commuting polygonal braids, and the union of these polygons must be symmetric. Hence, each of these polygons is either symmetric, or it belongs of a pair of polygons which are permuted by a rotation of 180 degrees, so the result follows.

Finally, notice that the left normal form of Y has the form $\delta^t y_1 \cdots y_s$ with $|t| \leq l$ and $s \leq l$. Now every y_i contains at most one symmetric polygonal braid, namely the one containing the origin. The remaining polygonal braids of y_i come in pairs. The symmetric polygonal braid, if it exists, involves at least two punctures, and each pair of polygonal braids involves at least 4 punctures. Hence y_i can be decomposed into a product of at most $1 + (2n-4)/4 = n/2$ factors of the form p_j. Since $s \leq l$, one finally obtains $k \leq ln/2$, as we wanted to show. ☐

Lemma 25. *Let $Y \in Sym_{2n-2}$ be given as a word of length l in the band generators and their inverses, and let $Y = \delta^t p_1 \cdots p_k$ be the decomposition given in Lemma 24. Then one obtains a word in the Artin generators and their inverses representing $\rho(\theta(Y))$ as follows.*

1. *Each $\delta \in B^B_{2n-2}$ should be replaced by $\rho(\theta(\delta)) = \varepsilon \in B^A_n$.*
2. *If p_i is the product of two polygonal braids $\Sigma_{P_1}\Sigma_{P_2}$, where the vertices of the polygons are $\{\zeta_{i_1}, \ldots, \zeta_{i_d}\}$ and $\{-\zeta_{i_1}, \ldots, -\zeta_{i_d}\}$ respectively, let $k \in \{0, \ldots, n-2\}$ be such that $\{\zeta_{i_1+k}, \ldots, \zeta_{i_d+k}\} = \{\zeta_{j_1}, \ldots, \zeta_{j_d}\}$ with $1 \leq j_1 < \cdots < j_d < n$. Then p_i should be replaced by*

$$\rho(\theta(\Sigma_{P_1}\Sigma_{P_2})) = \varepsilon^k \sigma_1 \left(\prod_{\substack{i=j_1+1 \\ (i \neq j_k \ \forall k)}}^{j_d-1} \sigma_i^{-1} \right) (\sigma_{j_d}\sigma_{j_d-1} \cdots \sigma_{j_1+1})\, \sigma_1^{-1} \varepsilon^{-k}.$$

3. *If p_i is a symmetric polygonal braid Σ_P, and the vertices of the polygon P are*

$$\{\zeta_{j_1}, \ldots, \zeta_{j_d}, -\zeta_{j_1}, \ldots, -\zeta_{j_d}\},$$

with $1 \leq j_1 < \cdots < j_d < n$, *then* p_i *should be replaced by*

$$\rho(\theta(\Sigma_P)) = \sigma_1 \left(\prod_{\substack{i=j_1+1 \\ (i \neq j_k \ \forall k)}}^{j_d-1} \sigma_i^{-1} \right) (\sigma_{j_d}\sigma_{j_d-1}\cdots\sigma_1)\sigma_1(\sigma_2^{-1}\cdots\sigma_{j_1}^{-1})\,\sigma_1^{-1}.$$

Proof. Consider the element $\alpha = s_{n-1}s_{n-2}\cdots s_1 \in \mathcal{A}(\mathbf{B}_{n-1})$. It is represented in the central picture of Figure 11. On the one hand, by Lemma 20 one has:

$$\rho(\alpha) = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_3\,(\sigma_1\sigma_2\sigma_1^{-1})\,\sigma_1^2 = \sigma_1(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1) = \varepsilon.$$

On the other hand, Lemma 22 together with presentation (2) tell us that

$$\begin{aligned} \theta'(\alpha) &= (a_{n-1,n-2}\,a_{2n-2,2n-3})(a_{n-2,n-3}\,a_{2n-3,2n-4})\cdots(a_{2,1}\,a_{n+1,n})\,a_{n,1} \\ &= (a_{2n-2,2n-3}\,a_{2n-3,2n-4}\cdots a_{n+1,n})(a_{n-1,n-2}\,a_{n-2,n-3}\cdots a_{2,1})\,a_{n,1} \\ &= (a_{2n-2,2n-3}\,a_{2n-3,2n-4}\cdots a_{n+1,n})\,a_{n,n-1}\,(a_{n-1,n-2}\,a_{n-2,n-3}\cdots a_{2,1}) = \delta. \end{aligned}$$

Therefore, since $\theta' = \theta^{-1}$, one has $\rho(\theta(\delta)) = \rho(\alpha) = \varepsilon$ and the first case holds.

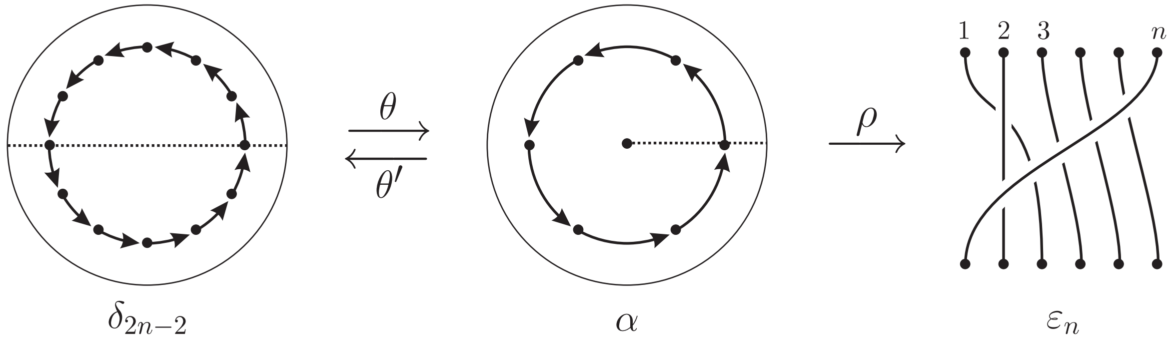


Figure 11: A geometric interpretation of $\rho(\theta(\delta)) = \varepsilon$.

Now suppose that p_i is the product of two polygonal braids $\Sigma_{P_1}\Sigma_{P_2}$, where the vertices of the polygons are $\{\zeta_{i_1},\ldots,\zeta_{i_d}\}$ and $\{-\zeta_{i_1},\ldots,-\zeta_{i_d}\}$. Notice that conjugation by δ in B_{2n-2}^B rotates the base points, increasing each index by one. Therefore, since P_1 and P_2 belong to a non-crossing partition, there exists some $k \in \{0,\ldots,n-2\}$ such that the rotation induced by δ^k transforms $\{P_1,P_2\}$ into $\{P_1',P_2'\}$, where the vertices of P_1' belong to $\{\zeta_1,\ldots,\zeta_{n-1}\}$. Then $\Sigma_{P_1}\Sigma_{P_2} = \delta^k\Sigma_{P_1'}\Sigma_{P_2'}\delta^{-k}$. Since $\rho(\theta(\delta)) = \varepsilon$, in order to compute $\rho(\theta(\Sigma_{P_1}\Sigma_{P_2}))$ it suffices to know the value of $\rho(\theta(\Sigma_{P_1'}\Sigma_{P_2'}))$. See an example in Figure 12.

Let $\zeta_{j_1},\ldots,\zeta_{j_d}$ be the vertices of P_1' in increasing order, as in the statement. For simplicity of notation, denote $j^* = j+n-1$ for $j = 1,\ldots,n-1$. The computation goes as follows:

$$\begin{aligned} \Sigma_{P_1'}\Sigma_{P_2'} &= (a_{j_d,j_{d-1}}\,a_{j_{d-1},j_{d-2}}\cdots a_{j_2,j_1})(a_{j_d^*,j_{d-1}^*}\,a_{j_{d-1}^*,j_{d-2}^*}\cdots a_{j_2^*,j_1^*}) \\ &= (a_{j_d,j_{d-1}}\,a_{j_d^*,j_{d-1}^*})\cdots(a_{j_2,j_1}\,a_{j_2^*,j_1^*}) \\ &= \prod_{i=d}^{2}(a_{j_i,j_{i-1}}\,a_{j_i^*,j_{i-1}^*}), \end{aligned}$$

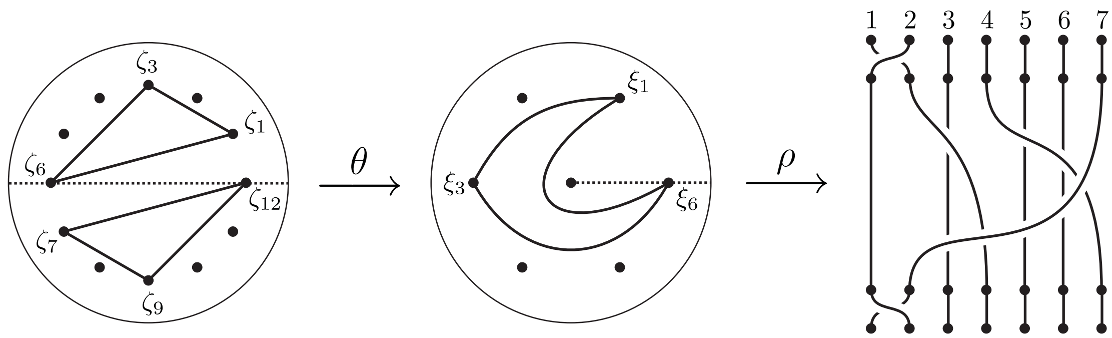


Figure 12: Translating pairs of symmetric polygonal braids in B^B_{2n-2} to Artin generators in B^A_n.

where the index i decreases from d to 2.

Now one can check using Lemma 22 and presentation 2, or just by drawing the corresponding pictures, that for $1 \leq u < v < n$ one has $\theta'((s_{u+1}^{-1}s_{u+2}^{-1}\cdots s_{v-1}^{-1})(s_v s_{v-1}\cdots s_{u+1})) = a_{v,u}a_{v^*,u^*}$. Hence, since $\theta' = \theta^{-1}$, one obtains:

$$\theta(\Sigma_{P_1'}\Sigma_{P_2'}) = \prod_{i=d}^{2}(s_{j_{i-1}+1}^{-1}s_{j_{i-1}+2}^{-1}\cdots s_{j_i-1}^{-1})(s_{j_i}s_{j_i-1}\cdots s_{j_{i-1}+1}).$$

Notice that s_i commutes with s_j if $|i-j|>1$, hence all positive letters in the above formula can be collected to the right (the only exception would appear if j_{i-1} and j_i are consecutive for some i, but in that case the corresponding negative factor is empty). It follows that:

$$\theta(\Sigma_{P_1'}\Sigma_{P_2'}) = \left(\prod_{i=d}^{2}(s_{j_{i-1}+1}^{-1}s_{j_{i-1}+2}^{-1}\cdots s_{j_i-1}^{-1})\right)(s_{j_d}s_{j_d-1}\cdots s_{j_1+1}).$$

Also, the $d-1$ factors made by negative letters commute with each other, so one finally obtains:

$$\theta(\Sigma_{P_1'}\Sigma_{P_2'}) = \left(\prod_{i=2}^{d}(s_{j_{i-1}+1}^{-1}s_{j_{i-1}+2}^{-1}\cdots s_{j_i-1}^{-1})\right)(s_{j_d}s_{j_d-1}\cdots s_{j_1+1}).$$

$$= \left(\prod_{\substack{i=j_1+1\\(i\neq j_k\ \forall k)}}^{j_d-1} s_i^{-1}\right)(s_{j_d}s_{j_d-1}\cdots s_{j_1+1}).$$

Now we must apply ρ to the above element. Notice that all indices are greater than 1, so this will replace s_2 by $\sigma_1\sigma_2\sigma_1^{-1}$ and s_i by σ_i for $i>2$. This is equivalent to replacing s_i by $\sigma_1\sigma_i\sigma_1^{-1}$ for every $i>1$. Hence, applying ρ reduces to replacing each s_i by σ_i, and then conjugating the whole element by σ_1^{-1}. That is,

$$\rho(\theta(\Sigma_{P_1'}\Sigma_{P_2'})) = \sigma_1\left(\prod_{\substack{i=j_1+1\\(i\neq j_k\ \forall k)}}^{j_d-1}\sigma_i^{-1}\right)(\sigma_{j_d}\sigma_{j_d-1}\cdots\sigma_{j_1+1})\,\sigma_1^{-1},$$

and $\rho(\theta(\Sigma_{P_1}\Sigma_{P_2}))$ is precisely as we stated.

It remains to show the third case, in which p_i is a single symmetric polygonal braid Σ_P, where the vertices of P are $\{\zeta_{j_1}, \cdots \zeta_{j_d}, -\zeta_{j_1}, \cdots -\zeta_{j_d}\} = \{\zeta_{j_1}, \cdots \zeta_{j_d}, \zeta_{j_1+n-1}, \cdots \zeta_{j_d+n-1}\}$. An example can be seen in Figure 13.

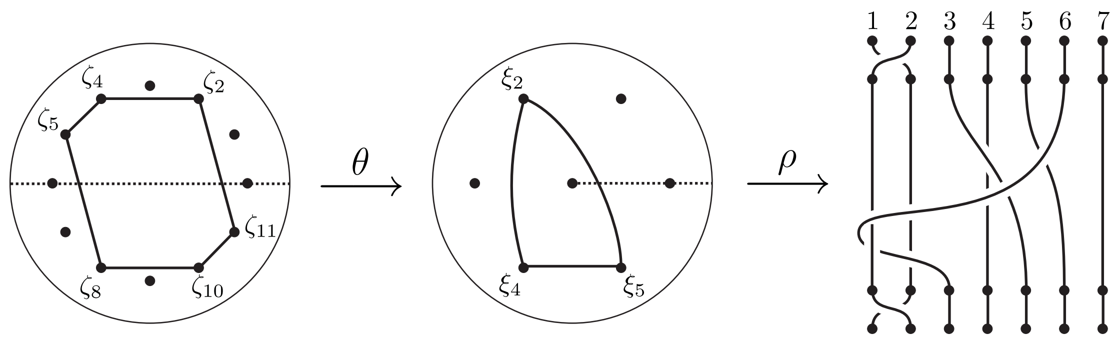


Figure 13: Translating a single symmetric polygonal braid in B^B_{2n-2} to Artin generators in B^A_n.

Recall that $j^* = j + n - 1$ for $j = 1, \dots, n-1$. In this case one has

$$\begin{aligned}\Sigma_P &= (a_{j_d^*,j_{d-1}^*}\, a_{j_{d-1}^*,j_{d-2}^*} \cdots a_{j_2^*,j_1^*})\, a_{j_1^*,j_d}\, (a_{j_d,j_{d-1}}\, a_{j_{d-1},j_{d-2}} \cdots a_{j_2,j_1}) \\ &= (a_{j_d^*,j_{d-1}^*}\, a_{j_{d-1}^*,j_{d-2}^*} \cdots a_{j_2^*,j_1^*})\, (a_{j_d,j_{d-1}}\, a_{j_{d-1},j_{d-2}} \cdots a_{j_2,j_1})\, a_{j_1^*,j_1}.\end{aligned}$$

One can apply the reasoning of the previous step to the first two factors, so it only remains to compute $\rho(\theta(a_{j_1^*,j_1}))$. This is done by noticing that

$$\begin{aligned}a_{j_1^*,j_1} &= (a_{j_1,j_1-1}\, a_{j_1^*,j_1^*-1})(a_{j_1-1,j_1-2}\, a_{j_1^*-1,j_1^*-2}) \cdots (a_{2,1}\, a_{n+1,n}) \cdot a_{n,1} \cdot \\ &\quad \cdot (a_{2,1}^{-1}\, a_{n+1,n}^{-1}) \cdots (a_{j_1-1,j_1-2}^{-1}\, a_{j_1^*-1,j_1^*-2}^{-1})(a_{j_1,j_1-1}^{-1}\, a_{j_1^*,j_1^*-1}^{-1}),\end{aligned}$$

which yields

$$\theta(a_{j_1,j_1^*}) = (\theta')^{-1}(a_{j_1,j_1^*}) = (s_{j_1} \cdots s_2)\, s_1\, (s_2^{-1} \cdots s_{j_1}^{-1}).$$

Since applying ρ reduces to replacing s_1 by σ_1^2, then s_i by σ_i for $i > 1$, and then conjugating everything by σ_1^{-1}, one obtains:

$$\rho(\theta(a_{j_1,j_1^*})) = \sigma_1 (\sigma_{j_1} \cdots \sigma_2)\, \sigma_1^2\, (\sigma_2^{-1} \cdots \sigma_{j_1}^{-1}) \sigma_1^{-1}.$$

Therefore

$$\rho(\theta(\Sigma_P)) = \sigma_1 \left(\prod_{\substack{i=j_1+1 \\ (i \neq j_k\ \forall k)}}^{j_d-1} \sigma_i^{-1} \right) (\sigma_{j_d}\sigma_{j_d-1} \cdots \sigma_{j_1+1})(\sigma_{j_1} \cdots \sigma_2)\, \sigma_1^2\, (\sigma_2^{-1} \cdots \sigma_{j_1}^{-1}) \sigma_1^{-1},$$

which is precisely the formula in the statement, so the proof is finished. □

4.2.2 Using symmetric braids to solve the conjugacy search problem

Recall that we are given $X \in B_n^A$ as a word in the Artin generators $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, and we know that X is conjugate to ε^k for some $k \neq 0$. This means that the permutation π_X consists of the k-th power of a cycle of length $n-1$, that is $\pi_X = (a)(b_1 \ \cdots \ b_{n-1})^k$, where $a \neq b_i$ for every i.

The easy case happens when k is a multiple of $n-1$, say $k = (n-1)t$. Then $\varepsilon^k = \Delta^{2t}$, so X is conjugate to a power of Δ^2. But since Δ^2 is a central element, this implies that $X = \Delta^{2t}$. Hence $X = \varepsilon^k$ and we are done.

We can then assume that k is not a multiple of $n-1$. This means that the only puncture which is fixed by X is the a-th one. If we denote $C_1 = \sigma_{[a\to 2]}$, it clearly follows that $Y = C_1^{-1}XC_1$ fixes the second strand, that is, $Y \in P_{n,2}$. Notice also that $\varepsilon \in P_{n,2}$, so $\varepsilon^k \in P_{n,2}$. This means that Y and ε^k are two elements in $P_{n,2}$ which are conjugate in B_n^A. Fortunately, they are also conjugate in $P_{n,2}$, as it is shown in the following result.

Lemma 26. *If $Y, Z \in P_{n,2}$ are conjugate braids whose permutations have a single fixed point (namely 2), then for every conjugating element $C \in B_n^A$ such that $C^{-1}YC = Z$, one has $C \in P_{n,2}$.*

Proof. Let $j = \pi_C(2)$. If $j \neq 2$, then $\pi_{YC}(2) = \pi_C(\pi_Y(2)) = \pi_C(2) = j$, while $\pi_{CZ}(2) = \pi_Z(\pi_C(2)) = \pi_Z(j) \neq j$ (since the only fixed point of π_Z is 2, and $j \neq 2$). This contradicts the assumption $YC = CZ$, so we must have $\pi_C(2) = 2$, that is $C \in P_{n,2}$. □

As a consequence, every conjugating element from Y to ε^k, when k is not a multiple of $n-1$, must belong to $P_{n,2}$. Therefore, finding a conjugating element from Y to ε^k in B_n^A reduces to solving the conjugacy search problem in $P_{n,2}$ for conjugates of ε^k.

Our strategy consists of applying $\theta' \circ \rho^{-1}$, solving the resulting problem in Sym_{2n-2}, and then mapping the solution back to $P_{n,2}$ using $\rho \circ \theta$. Recall from Lemma 25 that $\rho(\theta(\delta)) = \varepsilon$, hence $\theta'(\rho^{-1}(\varepsilon)) = \delta \in Sym_{2n-2}$. Therefore we must solve the conjugacy search problem in Sym_{2n-2} for $\theta'(\rho^{-1}(Y))$ and δ^k.

Recall that, as a consequence of [28], the group Sym_{2n-2} has a Garside structure which is the restriction of the Birman-Ko-Lee structure of B_{2n-2}^B. The Garside element of this structure is hence δ, so the conjugacy search problem for powers of $\delta \in Sym_{2n-2}$ can be solved very fast, by applying iterated cyclings and decyclings. But one does not need to care about the Garside structure of Sym_{2n-2}, since one can directly work with the Garside structure of B_{2n-2}^B, as it is shown in the following result.

Lemma 27. *Let $Z \in Sym_{2n-2} \subset B_{2n-2}^B$ be given as a word of length l in the band generators and their inverses. Suppose that Z is conjugate to δ^k for some $k \neq 0$. Then by applying at most $(2n-3)l$ cyclings and decyclings to Z, using the Garside structure of B_{2n-2}^B, one conjugates Z to δ^k and the conjugating element that is obtained belongs to Sym_{2n-2}.*

Proof. By [5], by applying at most $(2n-3)l$ cyclings and decyclings to Z one obtains an element which has minimal canonical length. Since Z is conjugate to δ^k, and δ is the Garside element of B_{2n-2}^B, it follows that the resulting element is precisely δ^k. Hence one obtains $C \in B_{2n-2}^B$ such that $C^{-1}ZC = \delta^k$.

Now recall that if a braid in B_{2n-2}^B is symmetric, then every factor in its left normal form is also symmetric. Hence the conjugating elements in all cyclings and decyclings applied above are symmetric braids, so $C \in Sym_{2n-2}$, as we wanted to show. $\square$

This finally gives us the algorithm to solve the conjugacy search problem for conjugates of ε^k.

Algorithm C:

Input: A word w in Artin generators and their inverses representing $X \in B_n^A$ conjugate to ε^k.

Output: $C \in B_n^A$ such that $C^{-1}XC = \varepsilon^k$.

1. If k is a multiple of $n-1$, return $C = 1$.
2. Compute a, the only puncture fixed by π_X. Let $Y = \sigma_{[a\to 2]}^{-1} X \sigma_{[a\to 2]} \in P_{n,2}$.
3. Using Lemma 23, compute $Z = \theta'(\rho^{-1}(Y))$.
4. Apply iterated cycling and decycling to $Z \in B_{2n-2}^B$ until δ^k is obtained. Let $C_0 \in Sym_{2n-2}$ be the conjugating element.
5. Using Lemma 25, compute $C_1 = \rho(\theta(C_0))$.
6. Return $C = \sigma_{[a\to 2]} C_1$.

Proposition 28. *Algorithm C has complexity $O(l^3n^2)$.*

Proof. The number a in step 2 can be computed in time $O(ln)$, and the letter length of the word Y is at most $2(n-2)+l$, hence, by Lemma 23, the word Z is obtained in time $O(n+l)$, and its letter length is at most $4(n-2)+2l$, that is, $O(n+l)$ as well. By Lemma 18, $O(n+l) = O(l)$, so the letter length of Z in band generators is $O(l)$.

By Lemma 27, one just needs to apply $O(nl)$ cyclings and decyclings to Z in step 4, each computation taking time $O(l^2n)$ since it is equivalent to computing a left normal form of a word of length $O(l)$. Hence, step 4 takes time $O(l^3n^2)$, and it is the most time-consuming step of the algorithm. The conjugating element $C_0 \in Sym_{2n-2}$ consists of at most $O(ln)$ simple factors.

Notice that C_0 is already given as a product of symmetric simple elements. Hence one can directly apply the formulae in Lemma 25, to compute $C_1 = \rho(\theta(C_0))$. Since there are $O(ln)$ factors, and each one is replaced by at most $n/2$ words of letter length $O(3n+2n^2) = O(n^2)$, it follows that step 5 takes time $O(ln^4) = O(l^3n^2)$, hence the whole algorithm has complexity $O(l^3n^2)$ as we wanted to show. $\square$

4.3 The complete algorithm

We are finally ready to prove Theorem 1 by giving an algorithm which solves step (3) in the statement of Theorem 1 in time $O(l^3n^2 \log n)$.

Algorithm D

Input: Two words w_X, w_Y in Artin generators and their inverses representing two braids $X, Y \in B_n^A$.

Output: 'Fail' if either X or Y is not periodic, or if they are not conjugate. Otherwise, an element $C \in B_n^A$ such that $C^{-1}XC = Y$.

1. Apply Algorithm A to w_X and w_Y.
2. If either X or Y is not periodic return 'Fail'. If X and Y are not conjugate to the same power of δ or ε, return 'Fail'.
3. If X and Y are conjugate to δ^k for some k, apply Algorithm B to X and Y to find $C_1, C_2 \in B_n^A$ such that $C_1^{-1}XC_1 = \delta^k = C_2^{-1}YC_2$. Return $C = C_1C_2^{-1}$.
4. If X and Y are conjugate to ε^k for some k, apply Algorithm C to X and Y to find $C_1, C_2 \in B_n^A$ such that $C_1^{-1}XC_1 = \varepsilon^k = C_2^{-1}YC_2$. Return $C = C_1C_2^{-1}$.

Proposition 29. *Algorithm D has complexity* $O(l^3n^2 \log n)$*, where* $l = \max\{|w_X|, |w_Y|\}$.

Proof. By Proposition 6, Algorithm A has complexity $O(l^2n^3 \log n)$. By Proposition 19, the complexity of Algorithm B is $O(l^3n^2)$, which is the same complexity as that of Algorithm C, by Proposition 28. Therefore, Algorithm D has complexity $O(l^2n^3 \log n + l^3n^2)$. By Lemma 18, this complexity is equivalent to $O(l^3n^2 \log n)$, as we wanted to show. □

5 Timing results

In this section we present and analyze running times for the conjugacy search for periodic elements in Artin braid groups; we compare the established algorithm based on computing ultra summit sets [21] to the algorithms developed in this paper.

For several values of the parameters n, k and c, tests in B_n were conducted as follows.

1. For $i = 1, \dots, 100$, we construct a pseudo-random element $z_i \in B_n^A$ as the product of c randomly chosen simple elements.
2. We compute the samples $\{(\delta^k)^{z_i} : i = 1, \dots, 100\}$ and $\{(\varepsilon^k)^{z_i} : i = 1, \dots, 100\}$; each element is stored in left normal form.
3. For each element x in a sample we compute an element conjugating x to δ^k or ε^k, respectively.

Step 3 was performed separately for each sample, first using the algorithm from [21], in the sequel referred to as Algorithm U, and then again using Algorithm B or Algorithm C. Only the total time for this step was measured for each case. A memory limit of 512 MB and a time limit of 250 s were applied for each test.

All computations were performed on a Linux PC with a 2.4 GHz Pentium 4 CPU, 533 MHz system bus and 1.5 GB of RAM using the computational algebra system MAGMA [8]. An implementation of Algorithm U written in C is part of the MAGMA kernel; Algorithms B and C were implemented in the MAGMA language.

Remark: One technical aspect of the implementation of Algorithms B and C needs to be mentioned briefly to explain the observed behavior.

As Algorithms B and C involve mapping a given word, generator by generator, to another Garside group, a naive implementation of these algorithms will react very sensitively to the word length of the given element x.

Note, however, that a conjugate y of x having minimal canonical length with respect to the usual Garside structure, together with a conjugating element, can be computed by iterated application of cycling and decycling in time $O(\ell^3 n^3 \log n)$, where ℓ is the number of simple factors of x.[1] Note further that if x is periodic, the canonical length of y as above is at most 1. Moreover, powers of Δ^2 can be discarded for the purpose of computing conjugating elements, as Δ^2 is central in B_n. The techniques from Algorithms B and C are then applied to the resulting element whose length in terms of Artin generators is bounded by n^2.

While this does not improve the complexity bounds, it significantly reduces computation times, especially for large values of the parameter c above, and is critical for the cross-over points between Algorithm U on the one hand and Algorithms B and C on the other hand.

We finally remark that in the special case that the minimal canonical length of conjugates of x is 0, that is, in the case that x is conjugate to a power of Δ, its ultra summit set has cardinality 1 and we do not have to use Algorithms B and C, as a conjugating element can be obtained directly, just by iterated application of cycling and decycling.

The main results can be summarized as follows; see Tables 1 and 2.

1. Time (and memory) requirements of Algorithm U increase rapidly with increasing value of n. With the exception of elements which are conjugate to a power of Δ, conjugacy search using Algorithm U fails for $n \gtrsim 15$.

   In the light of the exponential growth of $USS(\delta)$ and $USS(\varepsilon)$ established in Corollaries 12 and 15, this had to be expected.

2. In contrast to this, the computation times for Algorithms B and C grow much more slowly with increasing value of n. The data is consistent with a polynomial growth; a regression analysis for fixed values of the parameters k and c suggests that average times are proportional to n^{e_n}, where the value $e_n \approx 2.2$ is suggested by a regression analysis.[2]

[1] Note that ℓ, unlike the letter length l, is not bounded below by n for periodic braids.

[2] Note that for fixed values of k and c the word length l is not fixed but grows at least linearly in n; cf. Lemma 18. Hence this value of e_n does not contradict the complexity bounds from Propositions 19 and 28.

Table 1: Total execution times of Algorithms U, B and C for all 100 elements of a sample for $c = 10$ and various values of n and k. Where no value is given, either the memory limit of 512 MB or the time limit of 250 s was exceeded.

k	1						2					
n	5	7	10	15	20	50	5	7	10	15	20	50
U[δ]	0.03	0.12	1.56	88.14	—	—	0.02	0.38	22.15	—	—	—
B	0.02	0.04	0.07	0.16	0.34	3.56	0.02	0.03	0.06	0.16	0.29	2.75
U[ε]	0.03	0.19	4.05	—	—	—	0.02	0.16	64.22	—	—	—
C	0.05	0.12	0.23	0.53	0.97	6.92	0.01	0.10	0.25	0.52	1.01	6.95

k	3					4				6		
n	7	10	15	20	50	10	15	20	50	15	20	50
U[δ]	0.05	58.81	—	—	—	3.86	—	—	—	—	—	—
B	0.04	0.08	0.12	0.34	2.79	0.06	0.16	0.23	2.37	0.10	0.29	2.39
U[ε]	0.02	9.59	—	—	—	0.45	—	—	—	—	—	—
C	0.01	0.22	0.60	1.03	7.02	0.22	0.57	1.07	7.02	0.53	1.09	7.22

k	7			8		9		10		11	12
n	15	20	50	20	50	20	50	20	50	50	50
U[δ]	6.17	—	—	—	—	—	—	0.16	—	—	—
B	0.12	0.33	3.04	0.18	2.60	0.23	3.03	0.03	1.71	3.18	2.83
U[ε]	0.09	—	—	—	—	130.34	—	67.69	—	—	—
C	0.02	1.06	7.86	1.02	7.68	0.95	7.84	0.73	7.96	8.23	8.26

Table 2: Total execution times of Algorithms U, B and C for all 100 elements of a sample for $c = 250$ and various values of n and k. Where no value is given, either the memory limit of 512 MB or the time limit of 250 s was exceeded.

k	1						2					
n	5	7	10	15	20	50	5	7	10	15	20	50
U[δ]	0.16	0.40	2.05	85.20	—	—	0.15	0.65	20.42	—	—	—
B	0.16	0.32	0.67	1.21	1.83	8.24	0.16	0.33	0.66	1.22	1.76	6.79
U[ε]	0.16	0.49	4.32	—	—	—	0.14	0.42	59.76	—	—	—
C	0.19	0.40	0.83	1.51	2.37	10.75	0.14	0.38	0.86	1.57	2.42	10.69

k	3					4				6		
n	7	10	15	20	50	10	15	20	50	15	20	50
U[δ]	0.33	56.14	—	—	—	4.36	—	—	—	—	—	—
B	0.31	0.69	1.14	1.81	6.86	0.65	1.22	1.66	6.26	1.11	1.76	6.36
U[ε]	0.31	9.64	—	—	—	0.99	—	—	—	—	—	—
C	0.29	0.83	1.59	2.47	11.06	0.85	1.60	2.55	10.85	1.57	2.52	11.19

k	7			8		9		10		11	12
n	15	20	50	20	50	20	50	20	50	50	50
U[δ]	7.72	—	—	—	—	—	—	1.44	—	—	—
B	1.15	1.89	6.79	1.62	6.37	1.70	6.80	1.41	5.23	7.00	6.53
U[ε]	1.04	—	—	—	—	162.83	—	90.88	—	—	—
C	0.99	2.50	11.57	2.49	11.47	2.43	11.55	2.21	11.70	12.51	11.93

In particular, solving the conjugacy search problem for periodic elements using Algorithm D is is not significantly harder than other operations in with braids, that is, it is feasible whenever the parameter values permit any computations at all.

3. The computation times of Algorithm U depend in a very sensitive way on the value of k, whereas the running times of Algorithms B and C, with the exception of elements which are conjugate to a power of Δ and are treated differently, show relatively little dependency on k.

4. Average running times for all algorithms appear to be sub-linear in c for fixed values of the parameters n and k.

   For Algorithm U, the effect of c becomes negligible for $n \gtrsim 10$. This is no surprise as the value of c only affects the initial computation of a conjugate with minimal canonical length; the time used in this step of the computation is only relevant if the ultra summit set is small.

5. Using the implementations as explained above, the cross-over point between Algorithm U and Algorithm B was $n \approx 5$, whereas the cross-over point between Algorithm U and Algorithm C was $n \approx 7$; the latter corresponds to the cross-over point between Algorithm U and Algorithm D for the implementations used in our tests.

   We remark that the fact that Algorithms B and C were implemented in the MAGMA language (which is partly an interpreter language) incurs some overhead compared to the C implementation of Algorithm U. This overhead is probably not significant for Algorithm B, as its implementation is quite simple.[3] However, for Algorithm C the overhead can be expected to be significant, as its implementation is rather complex.[4] This difference can be assumed to be the main cause for the different cross-over points, whence a cross-over point of $n \approx 5$ for comparable implementations of Algorithms U and D seems likely.

Remark 30. After this paper was accepted for publication, and as we were preparing this final copy for the publisher, we learned that E-K Lee and S.J. Lee had posted on the arXiv their own solution to the same problem [25]. They reference our work and suggest some small improvements in it.

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[3]Uses 20 lines of MAGMA code. As MAGMA provides a kernel function computing ultra summit sets with respect to the Birman-Ko-Lee presentation, no low level operations had to be written in the MAGMA language.

[4]Uses 200 lines of MAGMA code. Many low level operations had to be written in the MAGMA language.

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