SIAM J. CONTROL OPTIM.
Vol. 53, No. 4, pp. 2319–2349

SMOOTHNESS PROPERTIES FOR THE OPTIMAL MIXTURE OF TWO ISOTROPIC MATERIALS: THE COMPLIANCE AND EIGENVALUE PROBLEMS*

JUAN CASADO-DÍAZ†

Abstract. The present paper is devoted to obtaining some smoothness results for the solution of two classical control problems relative to the optimal mixture of two isotropic materials. In the first one, the goal is to maximize the energy. In the second one, we want to minimize the first eigenvalue of the corresponding elliptic operator. At least for the first problem it is well known that it does not have a solution in general. Thus, we deal with a relaxed formulation. One of the applications of our results is in fact the nonexistence of a solution for the unrelaxed problem. In this sense, we improve a classical nonexistence result by Murat and Tartar for the maximization of the energy which was obtained assuming the solution smooth. We also get a counterexample to the existence of a solution for the eigenvalue problem which, to our knowledge, was an open problem.

Key words. optimal design, smooth solution, relaxation, nonexistence

AMS subject classifications. 49J20, 35B65

DOI. 10.1137/140971087

1. Introduction.

A very classical problem in optimal design, which we will refer as the compliance problem, consists in mixing two isotropic elastic materials in the cross-section of a beam in order to minimize the torsion. This can be modeled as follows: Assume the beam defined as $\Omega \times (0,L)$ with $\Omega \subset \mathbb{R}^2$ open and bounded and the elastic materials given through their corresponding Lamé's constants (λ_1, μ_1), (λ_2, μ_2). They are homogeneously distributed in the direction of the axis of the beam in two sets $\omega \times (0,L)$ and $(\Omega \setminus \omega) \times (0,L)$ with $\omega \subset \Omega$ measurable. In the basis $\{x_3 = 0\}$, the beam is not rotated, while in $\{x_3 = L\}$ it is rotated with small angle a. If the volume and surface forces are neglected, the deformation of the beam $v = (v_1, v_2, v_3)$ is the solution of the elasticity system

$$\begin{cases} -\operatorname{div} \sigma^* = 0 \ \text{ in } \Omega \times (0,L), \\ \sigma^* \nu = 0 \ \text{ on } \partial\Omega \times (0,L), \qquad (\sigma^*\nu)_3 = 0 \ \text{ on } \Omega \times \{0, L\}, \\ v_1 = v_2 = 0 \ \text{ on } \Omega \times \{0\}, \qquad v_1 = -aLx_2, \ \ v_2 = aLx_1 \ \text{ on } \Omega \times \{L\}, \end{cases}$$

where ν denotes the unitary outward vector and the stress tensor σ^* is given by

$$\sigma^* = \big(\lambda_1 \chi_{\omega \times (0,L)} + \lambda_2 \chi_{(\Omega\setminus\omega)\times(0,L)}\big)\operatorname{tr}(e(v)) I + 2\big(\mu_1 \chi_{\omega\times(0,L)} + \mu_2 \chi_{(\Omega\setminus\omega)\times(0,L)}\big) e(v).$$

It can be proved (see, e.g., [11] for $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$) that the components of σ^* satisfy

$$\sigma^*_{11} = \sigma^*_{22} = \sigma^*_{33} = \sigma^*_{12} = 0,$$

*Received by the editors June 2, 2014; accepted for publication (in revised form) May 27, 2015; published electronically August 6, 2015. This work has been partially supported by the projects MTM 2011-24457 of the Ministerio de Ciencia e Innovación of Spain and FQM-309 of the Junta de Andalucía.
http://www.siam.org/journals/sicon/53-4/97108.html
†Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Sevilla 41012, Spain (jcasadod@us.es).

and, assuming Ω simply connected,

$$\sigma^*_{13} = 2a\partial_2 u,\ \sigma^*_{23} = -2a\partial_1 u \quad \text{in } \Omega\times(0,L)$$

with u, solution of the Dirichlet problem

$$\left\{\begin{array}{l} -\mathrm{div}\big((\mu_1^{-1}\chi_\omega + \mu_2^{-1}\chi_{\Omega\setminus\omega})\nabla u\big) = 1 \ \text{ in } \Omega,\\ u = 0 \ \text{ on } \partial\Omega. \end{array}\right. \tag{1.1}$$

Our aim is to choose ω such that the energy required to carry out the torsion is maximal. This is equivalent to maximizing the potential energy given by

$$\int_{\Omega\times(0,L)} \sigma^* : e(v)\, dx = 2a^2 L \int_\Omega (\mu_1^{-1}\chi_\omega + \mu_2^{-1}\chi_{\Omega\setminus\omega})|\nabla u|^2 dx_1 dx_2.$$

Assuming $\mu_1 > \mu_2$, the solution is trivial and given by $\omega = \Omega$. The interesting problem comes when for economic reasons, the quantity of the best material is limited and then the choice $\omega = \Omega$ is not possible. Another classical application of the same mathematical formulation is the optimal arrangement of two viscous fluids moving parallel to the axis of a pipe (Poiseuille flow) in order to maximize the flux.

Using the characterization of (1.1) as a minimum problem and denoting $\alpha = \mu_1^{-1}$, $\beta = \mu_2^{-1}$, the problem can be also modeled as

$$\min\left\{ \int_\Omega \Big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})|\nabla u|^2 - 2u\Big) dx : \quad u \in H^1_0(\Omega), \quad |\omega| \le \kappa \right\}. \tag{1.2}$$

This problem has been studied in several papers. It is known that it does not have a solution in general (see, e.g., [19], [20] for nonexistence results in optimal design). Then it is usual to work with a relaxed formulation which can be obtained by using the homogenization theory (see, e.g., [2], [21], [24], [26]). For (1.2), it is shown in [22] that it consists in replacing the mixture $\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}$ by the harmonic mean of α and β with respective proportions θ and $1-\theta$, where $\theta \in L^\infty(\Omega;[0,1])$ represents the density of the material α in the homogenized mixture. Thus, instead of (1.2), we have

$$\min\left\{ \int_\Omega \left(\left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} |\nabla u|^2 - 2u \right) dx : \quad u \in H^1_0(\Omega),\ \int_\Omega \theta\, dx \le \kappa \right\}. \tag{1.3}$$

Although the solution can be not unique in general, it has been shown in [22] that for every solution (u,θ), the density flux

$$\sigma = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} \nabla u \tag{1.4}$$

is unique and there exists $\mu > 0$ such that

$$\{x\in\Omega:\ |\sigma| > \mu\} \subset \omega \subset \{x \in \Omega :\ |\sigma| \ge \mu\}.$$

Therefore, if the measure of the set $\{|\sigma| = \mu\}$ has null measure, we get the existence and uniqueness of a solution for the unrelaxed problem. Moreover, the interface of the corresponding solution is the level curve $\{|\sigma| = \mu\}$. Assuming it is smooth, it can also be shown that it is a level curve for the state function u. However, these assumptions do not usually hold. Namely, for Ω simply connected, the following interesting result has been proved in [22]:

$$\text{if (1.3) has a solution } (u,\omega) \text{ with } \omega \text{ smooth, } \Longrightarrow \Omega \text{ is a circle.} \tag{1.5}$$

A numerical study of (1.3) is carried out in [15] (see also [16]) by using a different relaxation. It can be obtained from (1.3) by directly computing the minimum in θ for every $u \in H_0^1(\Omega)$.

Our aim in the present paper is to obtain some smoothness results for the solutions of (1.3) or more exactly for a generalized problem where Ω is a bounded open set in $\mathbb{R}^N$ with $N \geq 2$ and where the right-hand side 1 in (1.1) is replaced by an arbitrary $f \in H^{-1}(\Omega)$. Similarly to (1.3), this provides

$$\min \left\{ \int_\Omega \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1} |\nabla u|^2 dx - 2\langle f, u\rangle : \quad u \in H_0^1(\Omega), \ \int_\Omega \theta \, dx \leq \kappa \right\}. \tag{1.6}$$

Our results are based on the relaxed formulation given in [15]. They mainly refer to the function σ given by (1.4), which we recall is unique. Assuming $\Omega \in C^{1,1}$, we prove (local smoothness is also obtained)

$$f \in W^{-1,p}(\Omega), \quad p > 1 \Longrightarrow \sigma \in L^p(\Omega)^N, \tag{1.7}$$

$$f \in L^p(\Omega), \ p > N \Longrightarrow \sigma \in L^\infty(\Omega)^N, \tag{1.8}$$

$$f \in W^{1,1}(\Omega) \cap L^2(\Omega) \Longrightarrow \begin{cases} \sigma \in H^1(\Omega)^N, \\ \partial_i \theta \, \sigma_j - \partial_j \theta \, \sigma_i \in L^2(\Omega), \quad 1 \leq i,j \leq N. \end{cases} \tag{1.9}$$

We observe that (1.7) and (1.8) are equivalent to $u \in W^{1,p}(\Omega)$, $p > 1$, and $u \in W^{1,\infty}(\Omega)$, respectively. The assertion $u \in W^{1,\infty}_{loc}(\Omega)$ has been previously obtained in [16] as an application of the results in [6]. Thus, the main novelty of the above theorem refers to the boundary estimates and especially to (1.9), which is the main result of the paper. On the one hand, it shows that σ is once derivable. On the other hand, it shows that the density function θ is derivable in the orthogonal directions to σ or equivalently to ∇u because these two vectors are parallel, i.e., θ is derivable in the direction of the level sets of u.

In a later work, we want to use the above result to estimate the error in the numerical computation of the solution of (1.6). We refer to [4] for estimates in the numerical study of some optimal design problems for two-phase materials in dimension one. In the present paper, we observe that (1.9) has important consequences with respect to the existence of a solution for the unrelaxed problem, i.e., where θ is a characteristic function. In such a case, the derivative in the orthogonal directions to σ can only vanish. We will show that this is very restrictive and allows us to improve (1.5) by eliminating the strong restriction ω smooth.

In [3], we have studied a problem related to (1.6), the energy problem, where instead of maximizing the energy we want to minimize it. Sometimes this is also called the compliance problem, playing the displacement the role of the torsion in our case. The smoothness results we got are in some sense dual of the obtained in the present paper. While here it is σ, defined by (1.4), which is unique and once derivable, for the energy problem it is the state function u which is unique and twice derivable. As we will see in the proof of Theorem 3.1 below, to obtain our smoothness result, we must deal with a linear elliptic problem where the matrix is bounded but not uniformly elliptic (see (3.21) below). For the energy problem, we deal with a problem where the matrix is uniformly elliptic but not bounded. We mention that although in the present paper our smoothness results are local, in [3] we are only able to obtain global regularity.

Another classical problem in the optimal design of two-phase materials is choosing a measurable subset $\omega \subset \Omega$ such that the first eigenvalue of the operator

$$u \in H^1_0(\Omega) \mapsto -\mathrm{div}\Big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\Big) \in H^{-1}(\Omega)$$

becomes minimal. It models, for example, the optimal distribution of two materials in heat conduction in order to obtain the most insulated one.

For dimension one, the existence and characterization of a solution has been obtained in [17]. For arbitrary dimension, assuming existence and regularity, some optimality conditions have been obtained in [9, 10]. The results in this paper are devoted not only to the first eigenvalue but to an arbitrary one and refer to minimization and maximization.

A more detailed study of the problem has been carried out when Ω is a ball. In this case, the results in [1] show that there exists a solution and that the optimal set ω is an union of annuli. From some numerical computations, it was conjectured in [8] that the optimal solution is in fact obtained by taking the bad material β in a concentric ball to Ω and the good material α in the annulus around this ball. However, taking α close to β an asymptotic calculus has shown that the result is more involved and that other annuli can appear ([7], [14], [18]). In [7], the authors also give some numerical results for domains different from a ball and β close to α.

In the present paper, we show that the problem of minimizing the first eigenvalue is in fact very related to the previous one. For the relaxed formulation, it consists in solving (1.6) for an arbitrary $f \in L^2(\Omega)$ with norm smaller or equal than 1 and then minimizing in f. Thus, the smoothness results obtained for the previous problem also apply for this one. As an application, we give a counterexample to the existence of a solution for the unrelaxed eigenvalue problem which, to our knowledge, was an open question. Namely, we show that although a solution always exists for a circle, this is not true for a rectangle or an ellipse.

2. Preliminary results for the compliance problem. In this section, we introduce the compliance problem and recall some well known results about it ([2], [6], [15], [16], [22]).

We consider a bounded open set $\Omega \subset \mathbb{R}^N$ and two positive constants α, $\beta > 0$ with $\alpha < \beta$. Then, for a distribution $\tilde f \in H^{-1}(\Omega)$ and a constant $\kappa \in (0, |\Omega|)$, we consider the control problem

$$\text{(2.1)} \qquad \begin{cases} \max\left\{\displaystyle\int_\Omega \big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\,|\nabla u_\omega|^2 dx\right\}, \\ \omega \subset \Omega \ \text{ measurable}, \quad |\omega| \le \kappa, \\ -\mathrm{div}\Big(\big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\,\nabla u_\omega\Big) = \tilde f \ \text{ in } \Omega, \qquad u_\omega \in H^1_0(\Omega). \end{cases}$$

A different formulation can be obtained as follows: Choosing u_ω as a test function in the state equation in (2.1), we deduce

$$\int_\Omega \big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\,|\nabla u_\omega|^2 dx = \langle \tilde f, u_\omega\rangle$$

and thus

$$\int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right)|\nabla u_\omega|^2 dx = -\left(\int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right)|\nabla u_\omega|^2 dx - 2\langle \tilde f, u_\omega\rangle\right).$$

Combining this equality with the classical characterization of u_ω as the solution of a minimum problem, we get

$$\begin{aligned}\int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right)|\nabla u_\omega|^2 dx \\ = -\min_{u\in H_0^1(\Omega)}\left\{\int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right)|\nabla u|^2 dx - 2\langle \tilde f, u\rangle\right\}.\end{aligned} \tag{2.2}$$

Therefore, the control problem (2.1) is equivalent to

$$\begin{cases} \min\left\{\int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right)|\nabla u|^2 dx - 2\langle \tilde f, u\rangle\right\}, \\ \omega\subset\Omega \ \text{ measurable}, \quad |\omega|\le\kappa, \quad u\in H_0^1(\Omega). \end{cases} \tag{2.3}$$

Remark 1. From (2.3), it is clear that if we eliminate the volume restriction $|\omega|\le\kappa$, then the solution of problem (2.1) is given by $\omega=\Omega$. This restriction means that although the material α is better than the material β, it is also more expensive and thus we want to use only a certain quantity κ of such material.

It is known that problem (2.1) has no solution in general and thus it is necessary to introduce a relaxation. Following [22], this can be obtained by replacing in (2.1) the mixtures of materials of the form

$$\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}, \quad \omega\subset\Omega, \text{ measurable}, \tag{2.4}$$

by the most general ones

$$\left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}, \quad \theta\in L^\infty(\Omega;[0,1]). \tag{2.5}$$

These new mixtures are obtained from the previous ones by using a rank-one laminate in the direction of the gradient of α and β with respective proportions θ and $1-\theta$. Introducing

$$c = \frac{\beta-\alpha}{\alpha}, \quad f = \frac{1}{\beta}\tilde f, \tag{2.6}$$

we then get the following relaxed formulation for (2.1) or (2.3):

$$\begin{cases} \max \int_\Omega \frac{|\nabla u_\theta|^2}{1+c\,\theta} dx, \\ \theta\in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta\, dx\le\kappa, \\ -\text{div}\,\frac{\nabla u_\theta}{1+c\,\theta} = f \ \text{ in } \Omega, \qquad u_\theta\in H_0^1(\Omega). \end{cases} \tag{2.7}$$

$$\begin{cases} \min\left\{\int_\Omega \frac{|\nabla u|^2}{1+c\,\theta} dx - 2\langle f, u\rangle\right\}, \\ \theta\in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta\, dx\le\kappa, \quad u\in H_0^1(\Omega). \end{cases} \tag{2.8}$$

The solution θ of (2.7) is not unique in general, but reasoning similarly to [22] we can prove that the product $\nabla u_\theta/(1+c\theta)$ is independent of the solution θ chosen. This is given by the following theorem

THEOREM 2.1. *There exists a unique function $\hat\sigma \in L^2(\Omega)^N$ such that for every $\hat\theta$ solution of* (2.7), *we have*

$$\hat\sigma = \frac{\nabla u_{\hat\theta}}{1+c\,\hat\theta}. \tag{2.9}$$

This function $\hat\sigma$ is characterized as the unique solution of

$$\min_{\substack{\sigma\in L^2(\Omega)^N\\ -\operatorname{div}\sigma=f}} \max_{\substack{\int_\Omega \theta dx\le\kappa\\ 0\le\theta\le 1}} \int_\Omega (1+c\,\theta)|\sigma|^2dx. \tag{2.10}$$

Moreover, a function $\hat\theta$ is a solution of (2.7) *if and only if it is a solution of*

$$\max_{\substack{\int_\Omega \theta dx\le\kappa\\ 0\le\theta\le 1}} \min_{\substack{\sigma\in L^2(\Omega)^N\\ -\operatorname{div}\sigma=f}} \int_\Omega (1+c\,\theta)|\sigma|^2dx, \tag{2.11}$$

and in this case the minimum in σ for such $\hat\theta$ is given by $\hat\sigma$.

Proof. By duality (see, e.g., [13, Chapter 1, section 7]), the solution u_θ of the state equation in (2.7) is such that $\sigma_\theta := \nabla u_\theta/(1+c\,\theta)$ is the unique solution of

$$\min_{\substack{\sigma\in L^2(\Omega)^N\\ -\operatorname{div}\sigma=f}} \int_\Omega (1+c\,\theta)|\sigma|^2dx.$$

Thus, we have

$$\max_{\substack{\int_\Omega \theta dx\le\kappa\\ 0\le\theta\le 1}} \int_\Omega \frac{|\nabla u_\theta|^2}{1+c\,\theta}dx = \max_{\substack{\int_\Omega \theta dx\le\kappa\\ 0\le\theta\le 1}} \min_{\substack{\sigma\in L^2(\Omega)^N\\ -\operatorname{div}\sigma=f}} \int_\Omega (1+c\,\theta)|\sigma|^2dx. \tag{2.12}$$

This proves that $\hat\theta$ is a solution of (2.7) if and only if it is a solution of (2.11).

Applying the min-max theorem, the right-hand side of (2.12) also agrees with (2.10). Moreover, θ is a solution of (2.12) and σ is a solution of (2.10) if and only if (θ,σ) is a saddle point.

Since for θ fixed, the problem

$$\min_{\substack{\sigma\in L^2(\Omega)^N\\ -\operatorname{div}\sigma=f}} \int_\Omega (1+c\,\theta)|\sigma|^2dx$$

has as unique solution σ_θ, we then deduce that σ_θ must be a solution of (2.10), but this problem has a unique solution because as maximum (not just a supremum) of a family of strictly convex functionals, the functional

$$\sigma\in L^2(\Omega)^N \to \max_{\substack{\int_\Omega \theta dx\le\kappa\\ 0\le\theta\le 1}} \int_\Omega (1+c\,\theta)|\sigma|^2dx$$

is strictly convex. ∎

A simple application of the Kuhn–Tucker theorem allows us to compute the maximum in θ in (2.11) for $\sigma=\hat\sigma$. This proves the following.

THEOREM 2.2. *Define $\hat\sigma\in L^2(\Omega)^N$ by Theorem* 2.1 *and $\hat\mu$ by*

$$\hat\mu = \min\big\{\mu\ge 0 : |\{x\in\Omega :\ |\hat\sigma(x)|>\mu\}|\le\kappa\big\}. \tag{2.13}$$

Then, if $\hat\theta$ is a solution of (2.7), *it satisfies*

$$\hat\theta(x) = \begin{cases} 1 & \textit{if } |\hat\sigma(x)| > \hat\mu, \\ 0 & \textit{if } |\hat\sigma(x)| < \hat\mu. \end{cases} \tag{2.14}$$

Moreover, if $\hat\mu > 0$, then

$$\int_\Omega \hat\theta\, dx = \kappa. \tag{2.15}$$

Remark 2. The constant $\hat\mu$ given by (2.13) is zero if and only if the solution of problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.16}$$

satisfies

$$\Big|\big\{x \in \Omega :\ |\nabla u(x)| = 0\big\}\Big| \geq |\Omega| - \kappa. \tag{2.17}$$

In such case every function $\hat\theta \in L^\infty(\Omega; [0,1])$ such that $\hat\theta = 1$ in $\{x \in \Omega :\ |\nabla u(x)| > 0\}$ and has an integral less or equal than κ is a solution of (2.7).

Taking into account (2.14), equality (2.9), and $-\operatorname{div}\hat\sigma = f$ in Ω, we get Theorem 2.3 below. It is related to another relaxation formulation for problem (2.1), which can be found in [15]. It can also be obtained from (2.8) computing the minimum in θ for every $u \in H^1_0(\Omega)$.

THEOREM 2.3. *For $\hat\mu$ given by* (2.13), *we define the positive convex function $F \in W^{2,\infty}(0,+\infty)$ by*

$$F(s) = \begin{cases} s^2 & \textit{if } 0 \leq s < \hat\mu, \\ 2\hat\mu\, s - \hat\mu^2 & \textit{if } \hat\mu \leq s \leq (1+c)\hat\mu, \\ \dfrac{s^2}{(1+c)} + c\hat\mu^2 & \textit{if } s > (1+c)\hat\mu. \end{cases} \tag{2.18}$$

Then, if $\hat\theta \in L^\infty(\Omega; [0,1])$ is a solution of (2.7), *the corresponding function $u_{\hat\theta}$ is a solution of*

$$\min_{u \in H^1_0(\Omega)} \left\{ \int_\Omega F(|\nabla u|) dx - 2\langle f, u\rangle \right\}. \tag{2.19}$$

Moreover, if $\hat\mu > 0$, then every solution $\hat\theta$ of (2.7) *can be obtained from the corresponding state function $u_{\hat\theta}$ by*

$$\hat\theta(x) = \begin{cases} 0 & \textit{if } 0 \leq |\nabla u_{\hat\theta}| < \hat\mu, \\ \dfrac{1}{c}\left(\dfrac{|\nabla u_{\hat\theta}|}{\hat\mu} - 1\right) & \textit{if } \hat\mu \leq |\nabla u_{\hat\theta}| \leq (1+c)\hat\mu, \\ 1 & \textit{if } |\nabla u_{\hat\theta}| > (1+c)\hat\mu. \end{cases} \tag{2.20}$$

Remark 3. For a solution u of (2.19), we define θ by (2.20) with $u_{\hat\theta}$ replaced by u. Since F in Theorem (2.3) is not strictly convex, uniqueness for problem (2.19) can fail and thus, if θ is a solution of (2.7), we can have $u \neq u_{\hat\theta}$, $\theta \neq \hat\theta$. However, taking into account that F is strictly convex outside the interval $[\hat\mu, (1+c)\hat\mu]$ and that the function $x \in \mathbb{R}^N \mapsto |x| \in \mathbb{R}$ satisfies

$$\text{if } x, y \in \mathbb{R}^N, x \neq y,\ |x+y| = |x| + |y| \Rightarrow \exists \lambda \geq 0 \ \text{ such that } y = \lambda x,$$

we get

$$\nabla u = \frac{1 + c\,\theta}{1 + c\,\hat\theta}\, \nabla u_{\hat\theta} \ \text{ a.e. in } \Omega. \tag{2.21}$$

The function $\theta \in L^\infty(\Omega, ; [0,1])$ is not necessarily a solution of (2.7) because its integral can be strictly greater than κ, but by (2.9) we have

$$\hat\sigma = \frac{\nabla u}{1 + c\theta} \ \text{ a.e. in } \Omega$$

with $\hat\sigma$ given by (2.2).

3. Some smoothness results for the compliance theorem. In this section, we get some smoothness properties for the solutions of the relaxed problem (2.7). They mainly refer to the function $\hat\sigma$ defined by Theorem 2.1, which we recall is unique. As we will see later, it has several applications relative to the nonexistence of a solution for the unrelaxed problem (2.1).

THEOREM 3.1. *We consider an open set $U \subset \mathbb{R}^N$ such that $U \cap \Omega$ is of class $C^{1,1}$ and define $\hat\sigma$ by Theorem 2.1. Then we have the following:*

- *For every $p \in [2, \infty)$ and every open set $O \Subset U$, there exists $C > 0$, depending on p and O, such that if f belongs to $W^{-1,p}(U \cap \Omega)$, then $\hat\sigma$ belongs to $L^p(O \cap \Omega)^N$ and*

$$\|\hat\sigma\|_{L^p(O\cap\Omega)^N} \leq C \left(\|f\|_{W^{-1,p}(U\cap\Omega)} + \|f\|_{H^{-1}(\Omega)} \right). \tag{3.1}$$

- *For every $p > N$ and every open set $O \Subset U$, there exists $C > 0$, depending on p and O, such that if f belongs to $L^p(U \cap \Omega)$, then $\hat\sigma$ belongs to $L^\infty(O \cap \Omega)^N$ and*

$$\|\hat\sigma\|_{L^\infty(O\cap\Omega)^N} \leq C \left(\|f\|_{L^p(U\cap\Omega)} + \|f\|_{H^{-1}(\Omega)} \right). \tag{3.2}$$

- *For every open set $O \Subset U$, there exists $C > 0$, depending on O, such that if f belongs to $W^{1,1}(U \cap \Omega) \cap L^2(U \cap \Omega)$, then $\hat\sigma$ belongs to $H^1(O \cap \Omega)^N$ and*

$$\|\hat\sigma\|_{H^1(O\cap\Omega)^N} \leq C \left(1 + \|f\|_{L^2(U\cap\Omega)} + \|f\|_{W^{1,1}(U\cap\Omega)} + \|f\|_{H^{-1}(\Omega)} \right). \tag{3.3}$$

  Moreover, the tangential component of $\hat\sigma$ vanishes on $U \cap \partial\Omega$ and every solution $\hat\theta$ of (2.7) satisfies that $\partial_i\hat\theta\hat\sigma_j - \partial_j\hat\theta\hat\sigma_i$ belongs to $L^2(O \cap \Omega)$, $1 \leq i,j \leq N$ with

$$\begin{aligned} &\|\partial_i\hat\theta\hat\sigma_j - \partial_j\hat\theta\hat\sigma_i\|_{L^2(O\cap\Omega)^N} \\ &\quad \leq C \left(1 + \|f\|_{L^2(U\cap\Omega)} + \|f\|_{W^{1,1}(U\cap\Omega)} + \|f\|_{H^{-1}(\Omega)} \right), \ 1 \leq i,j \leq N. \end{aligned} \tag{3.4}$$

- *If f belongs to $W^{1,1}(U \cap \Omega) \cap L^2(U \cap \Omega)$ and there exists a solution $\hat\theta$ of (2.7) taking only the values 0 and 1, then*

$$\partial_i\hat\theta\hat\sigma_j - \partial_j\hat\theta\hat\sigma_i = 0 \ \text{ in } U \cap \Omega, \quad 1 \leq i,j \leq N, \tag{3.5}$$

$$\operatorname{curl}(\hat\sigma) = 0 \ \text{ in } U \cap \Omega. \tag{3.6}$$

Theorem 3.1 clearly implies the following.

COROLLARY 3.2. *Assume $\Omega \in C^{1,1}$ and define $\hat{\sigma}$ by Theorem 2.1. Then we have the following:*

- *For every $p \in [1, \infty)$, there exists $C > 0$, depending on p, such that if f belongs to $W^{-1,p}(\Omega)$, then $\hat{\sigma}$ belongs to $L^p(\Omega)^N$ and*

$$\|\hat{\sigma}\|_{L^p(\Omega)^N} \le C\|f\|_{W^{-1,p}(\Omega)}. \tag{3.7}$$

- *For every $p > N$, there exists $C > 0$ such that if f belongs to $L^p(\Omega)$, then $\hat{\sigma}$ belongs to $L^\infty(\Omega)^N$ and*

$$\|\hat{\sigma}\|_{L^\infty(\Omega)^N} \le C\|f\|_{L^p(\Omega)}. \tag{3.8}$$

- *There exists $C > 0$ such that if f belongs to $W^{1,1}(\Omega) \cap L^2(\Omega)$, then $\hat{\sigma}$ belongs $H^1(\Omega)^N$ and*

$$\|\hat{\sigma}\|_{H^1(\Omega)^N} \le C\left(1 + \|f\|_{L^2(\Omega)} + \|f\|_{W^{1,1}(\Omega)}\right). \tag{3.9}$$

Moreover, the tangential component of $\hat{\sigma}$ vanishes on $\partial\Omega$.

Remark 4. Observe that (3.1) and (3.2) prove that if $\hat{\theta}$ is a solution of (2.7) and f belongs to $W^{-1,p}(U \cap \Omega)$, $p > 2$ or f belongs to $L^p(U \cap \Omega)$, $p > N$, then the corresponding state function $u_{\hat{\theta}}$ is in $W^{1,p}(O \cap \Omega)$ and $W^{1,\infty}(O \cap \Omega)$, respectively.

Remark 5. Since (3.5), (3.6) refer to interior points in Ω, they hold without any smoothness assumptions on U.

Remark 6. Taking into account that if $(\hat{\theta}, u_{\hat{\theta}})$ is a solution of (2.7), then $\hat{\sigma}$ is proportional to $\nabla u_{\hat{\theta}}$, estimate (3.4) shows that $\hat{\theta}$ is smooth in the directions of the level sets of $u_{\hat{\theta}}$.

Remark 7. If problem (2.1) has a solution $(\hat{\omega}, u_{\hat{\omega}})$ with $\hat{\omega}$ smooth (for example, $C^{0,1}$), then, using that

$$\nabla \chi_{\hat{\omega}} = \nu\, H_{N-1} \llcorner \partial\omega$$

with ν the unitary outward normal to $\hat{\omega}$ on $\partial\hat{\omega}$ and H_{N-1} the $N-1$ Hausdorff measure, we get that equality (3.5) with $\hat{\theta} = \chi_{\hat{\omega}}$ equivalent to $\hat{\sigma}$ and then $\nabla u_{\hat{\omega}}$ parallel to ν on $\partial\hat{\omega}$. This proves that u is constant on the connected components of $\partial\hat{\omega}$, which is a classical optimality condition for the smooth solutions of problem (2.1).

Remark 8. If $\Omega \in C^{1,1}$ is a simply connected open set with connected boundary, $f \in L^2(\Omega) \cap W^{1,1}(\Omega)$, and there exists a solution $(\hat{\omega}, u_{\hat{\omega}})$ of (2.1), then we deduce from (3.6) and the tangential component of $\hat{\sigma}$ vanishing on $U \cap \partial\Omega$ (see the third assertion in Theorem 3.1) the existence of $w \in H^1(\Omega)$ such that $\hat{\sigma} = \nabla w$ with ∇w normal on $\partial\Omega$. Since we have assumed $\partial\Omega$ connected and know that $-\operatorname{div}\hat{\sigma} = f$ in Ω, we get that w can be chosen as the unique solution of the Dirichlet problem

$$-\Delta w = f \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega. \tag{3.10}$$

This remark has also been carried out in [22], but assuming that ω is an open set of class C^2.

The condition $\hat{\sigma} = \nabla w$ with w a solution of (3.10) is too restrictive and allows us to show that problem (2.1) does not have a solution in general. For the minimization of the torsion in a beam (see the introduction), the following interesting result has been proved in [22]: If Ω is the interior of a smooth Jordan curve in $\mathbb{R}^2$, then problem (1.1) has a smooth solution if and only if Ω is a circle. Here, we extend the result to any dimension and (most importantly) eliminate the strong restriction ω smooth. The result is given by the following theorem, which we prove below.

THEOREM 3.3. *Assume that* $\Omega \in C^{1,1}$ *is a simply connected open set in* $\mathbb{R}^N$ *with connected boundary such that problem* (2.1) *with* $f = 1$ *has a solution. Then* Ω *is a ball.*

Proof of Theorem 3.1. Along the proof, we fix a solution $\hat\theta$ of (2.7) and denote by $u = u_{\hat\theta}$ the corresponding state function, solution of

$$-\text{div}\,\frac{\nabla u}{1+c\hat\theta} = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{3.11}$$

Step 1. From Theorem 2.3, we know that u satisfies

$$\begin{cases} -\text{div}\left(\dfrac{F'(|\nabla u|)}{2|\nabla u|}\nabla u\right) = f \ \text{ in } U\cap\Omega, \\ u = 0 \ \text{ on } \partial\Omega\cap U, \end{cases} \tag{3.12}$$

which, using that $F'(s) = 2s/(1+c)$ for $s \geq (1+c)\hat\mu$, can also be written as

$$\begin{cases} -\dfrac{1}{1+c}\Delta u = \text{div}\left(\left(\dfrac{F'(|\nabla u|)}{2|\nabla u|} - \dfrac{1}{1+c}\right)\nabla u \chi_{\{|\nabla u|<(1+c)\hat\mu\}}\right) + f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Observing that the first term on the right-hand side is in $W^{-1,\infty}(U\cap\Omega)$, we then deduce (3.1) from the classical smoothness results for the Poisson equation and

$$\hat\sigma = \frac{F'(|\nabla u|)}{2|\nabla u|}\nabla u \ \text{ in } \Omega. \tag{3.13}$$

Step 2. For F given by (2.18) and $\varepsilon > 0$, small enough, we consider a sequence F_ε of nonnegative convex functions of class C^3 in $[0,+\infty)$, such that

$$F_\varepsilon(s) \to F(s), \quad F'_\varepsilon(s) \geq \frac{s}{1+c}, \quad F''_\varepsilon(s) \geq \varepsilon, \quad \forall\, s \geq 0,$$
$$F_\varepsilon(s) = F(s)\ \forall\, s \geq (1+c)\hat\mu, \quad \|F''_\varepsilon\|_{L^\infty(0,\infty)} \ \text{ bounded.}$$

Assuming that the restriction of f to $U\cap\Omega$ is in a certain space $X \supset L^2(U\cap\Omega)$ such that $C^\infty(\overline{U\cap\Omega})$ is dense in X, we take $f_\varepsilon \in C^\infty(\overline{U\cap\Omega})$ such that

$$f_\varepsilon \to f \ \text{ in } X. \tag{3.14}$$

Then we define $u_\varepsilon \in H^1(U\cap\Omega)$ as the unique solution of

$$\min_{v-u\in H^1_0(U\cap\Omega)}\left(\int_{U\cap\Omega}\left(F_\varepsilon(|\nabla v|) + |v-u|^2\right)dx - 2\int_{U\cap\Omega} f_\varepsilon v\,dx\right), \tag{3.15}$$

or equivalently, as the unique solution of

$$-\text{div}\left(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{2|\nabla u_\varepsilon|}\nabla u_\varepsilon\right) + u_\varepsilon - u = f_\varepsilon \ \text{ in } U\cap\Omega, \quad u_\varepsilon = u \ \text{ on } \partial(U\cap\Omega). \tag{3.16}$$

Using $F'_\varepsilon(s)/s \geq 1/(1+c)$ for every $s \in [0,+\infty)$, $\|F''_\varepsilon\|_{L^\infty(0,\infty)}$ bounded, and f_ε bounded in $H^{-1}(\Omega)$, we get u_ε bounded in $H^1(U\cap\Omega)$ and

$$\sigma_\varepsilon = \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{2|\nabla u_\varepsilon|}\nabla u_\varepsilon \tag{3.17}$$

bounded in $L^2(U\cap\Omega)^N$. Therefore, extracting a subsequence if necessary, we can assume that u_ε converges weakly in $H^1(U\cap\Omega)$ to some function $\breve{u}$ and that σ_ε converges weakly in $L^2(U\cap\Omega)^N$ to some function $\breve{\sigma}$. From (3.16), these functions are related by

$$-\mathrm{div}\,\breve{\sigma}+\breve{u}-u=f \quad \text{in } U\cap\Omega. \tag{3.18}$$

In order to characterize $\breve{\sigma}$ and $\breve{u}$, we apply the Minity trick. First, we observe that taking $u_\varepsilon-u$ as test function in (3.16) and $\breve{u}-u$ as test function in (3.18), we deduce

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{U\cap\Omega}\sigma_\varepsilon\cdot\nabla u_\varepsilon\,dx\\
&\quad=\lim_{\varepsilon\to0}\left(\int_{U\cap\Omega}f_\varepsilon(u_\varepsilon-u)dx-\int_{U\cap\Omega}|u_\varepsilon-u|^2dx+\int_{U\cap\Omega}\sigma_\varepsilon\cdot\nabla u\,dx\right)\\
&\quad=\left(\int_{U\cap\Omega}f(\breve{u}-u)dx-\int_{U\cap\Omega}|\breve{u}-u|^2dx+\int_{U\cap\Omega}\breve{\sigma}\cdot\nabla u^0\,dx\right)=\int_{U\cap\Omega}\breve{\sigma}\cdot\nabla\breve{u}\,dx.
\end{aligned}$$

This allows us to pass to the limit in

$$0\le\int_{U\cap\Omega}\left(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{2|\nabla u_\varepsilon|}\nabla u_\varepsilon-\frac{F'_\varepsilon(|\nabla\breve{u}+t\Phi|)}{2|\nabla\breve{u}+t\Phi|}(\nabla\breve{u}+t\Phi)\right)\cdot(\nabla(u_\varepsilon-\breve{u})-t\Phi)dx$$

for every $\Phi\in L^2(U\cap\Omega)^N$ and every $t\in\mathbb{R}$ to deduce

$$0\le-t\int_{U\cap\Omega}\left(\breve{\sigma}-\frac{F'(|\nabla\breve{u}+t\Phi|)}{2|\nabla\breve{u}+t\Phi|}(\nabla\breve{u}+t\Phi)\right)\cdot\Phi\,dx\quad\forall\,\Phi\in L^2(U\cap\Omega)^N,\ \forall\,t\in\mathbb{R},$$

and then that

$$\breve{\sigma}=\frac{F'(|\nabla\breve{u}|)}{2|\nabla\breve{u}|}\nabla\breve{u}\quad\text{in } U\cap\Omega. \tag{3.19}$$

Therefore, the function $\breve{u}$ satisfies

$$\begin{cases}-\mathrm{div}\left(\dfrac{F'(|\nabla\breve{u}|)}{2|\nabla\breve{u}|}\nabla\breve{u}\right)+\breve{u}-u=f\quad\text{in } U\cap\Omega,\\ \breve{u}=u\quad\text{on } \partial(U\cap\Omega).\end{cases}$$

By convexity, this means that $\breve{u}$ is a solution of

$$\min_{v-u\in H_0^1(U\cap\Omega)}\left(\int_{U\cap\Omega}\left(F(|\nabla v|)+|v-u|^2\right)dx-2\int_{U\cap\Omega}fv\,dx\right).$$

On the other hand, since u is a solution of (2.19), it is also a solution of

$$\min_{v-u\in H_0^1(U\cap\Omega)}\left(\int_{U\cap\Omega}F(|\nabla v|)dx-2\int_{U\cap\Omega}fv\,dx\right).$$

Thus, we have

$$\begin{aligned}
\int_{U\cap\Omega}F(|\nabla u|)dx-2\int_{U\cap\Omega}fu\,dx&\le\int_{U\cap\Omega}F(|\nabla\breve{u}|)\,dx-2\int_{U\cap\Omega}f\breve{u}\,dx\\
&\le\int_{U\cap\Omega}\left(F(|\nabla\breve{u}|)+|\breve{u}-u|^2\right)dx-2\int_{U\cap\Omega}f\breve{u}\,dx\\
&\le\int_{U\cap\Omega}F(|\nabla u|)dx-2\int_{U\cap\Omega}fu\,dx,
\end{aligned}$$

and then

$$\int_{U\cap\Omega}|\check{u}-u|^2dx=0,$$

i.e., $\check{u}=u$ a.e. in $U\cap\Omega$, while from (3.13) and (3.19), we also deduce that $\check{\sigma}=\hat{\sigma}$.

Reasoning as in Step 1 for problem (3.16), we also observe that

$$\|\sigma_\varepsilon\|_{L^q(O\cap\Omega)}\leq C\big(\|f_\varepsilon\|_{W^{-1,q}(U\cap\Omega)}+\|f\|_{H^{-1}(\Omega)}\big)\quad\forall\, O\Subset U \text{ open},\ q>2. \tag{3.20}$$

Step 3. We assume that $U\cap\Omega$ is of class $C^{2,\gamma}$ for some $\gamma\in(0,1)$. Then, applying Theorem 15.11 in [12] to problem (3.16), we have that u_ε is in $C^{2,\gamma}(U\cap\Omega)$. Thus, we can derive with respect to x_i in (3.16) $1\leq i\leq N$ to deduce that $\partial_i u_\varepsilon$ satisfies

$$-\text{div}\,(M_\varepsilon\nabla(\partial_i u_\varepsilon))+\partial_i(u_\varepsilon-u)=\partial_i f_\varepsilon\quad\text{in } U\cap\Omega \tag{3.21}$$

with

$$2M_\varepsilon=F_\varepsilon''(|\nabla u_\varepsilon|)\frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}+\frac{F_\varepsilon'(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\left(I-\frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right). \tag{3.22}$$

Using the existence of $C>0$ such that $0\leq F_\varepsilon''\leq C$ and $0\leq F_\varepsilon'\leq Cs$ in $[0,+\infty)$, we have that M_ε satisfies (remark that M_ε is barely elliptic by a constant ε)

$$|M_\varepsilon(x)\xi|^2\leq\frac{C}{2}M_\varepsilon(x)\xi\cdot\xi\leq\frac{C}{2}|\xi|^2\quad\forall\,\xi\in\mathbb{R}^N\ \forall\,x\in\overline{U\cap\Omega}. \tag{3.23}$$

Moreover, we observe that

$$\partial_i\sigma_\varepsilon=\partial_i\left(\frac{F_\varepsilon'(|\nabla u_\varepsilon|)}{2|\nabla u_\varepsilon|}\nabla u_\varepsilon\right)=M_\varepsilon\nabla(\partial_i u_\varepsilon)\quad\text{in } U\cap\Omega. \tag{3.24}$$

In order to estimate $\nabla\partial_i u_\varepsilon$ from (3.21), we will also need some boundary conditions. Given a point $\bar{x}\in U\cap\partial\Omega$, we can consider a ball $B(\bar{x},r)\subset U$ and functions $\tau^i=(\tau^i_1,\dots,\tau^i_N)\in C^{1,\gamma}(B(x_0,r))$, $1\leq i\leq N$, providing an orthonormal basis of $\mathbb{R}^N$ for every $x\in B(\bar{x},r)$ and such that τ^N agrees with the unitary outward normal vector to Ω on $B(\bar{x},r)\cap\partial\Omega$.

We define the functions $v^j_\varepsilon\in C^{1,\gamma}(\Omega)$, $1\leq j\leq N$, by

$$v^j_\varepsilon=\nabla u_\varepsilon\cdot\tau^j,\quad 1\leq j\leq N. \tag{3.25}$$

From (3.21), we deduce that v^j_ε satisfies

$$\begin{aligned}-\text{div}\,\big(M_\varepsilon\nabla v^j_\varepsilon\big)&=\nabla(f-u_\varepsilon+u)\cdot\tau^j-\sum_{i=1}^N M_\varepsilon\nabla\partial_i u_\varepsilon\cdot\nabla\tau^j_i\\&\quad-\sum_{i=1}^N\text{div}\left(M_\varepsilon\nabla\tau^j_i\,\partial_i u_\varepsilon\right)\quad\text{in } B(\bar{x},r)\cap\Omega.\end{aligned} \tag{3.26}$$

Since $u_\varepsilon=0$ on $\partial\Omega$ implies that the tangential derivative of u_ε vanishes on $B(\bar{x},r)\cap\partial\Omega$, we conclude with the following Dirichlet boundary conditions for v^j_ε with $1\leq j\leq N-1$:

$$v^j_\varepsilon=0\quad\text{on } B(\bar{x},r)\cap\partial\Omega,\quad 1\leq j\leq N-1. \tag{3.27}$$

It remains to obtain a boundary condition for v_ε^N. Developing the first term in (3.16), we get

$$-\left(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\right)\Delta u_\varepsilon - \left(F''_\varepsilon(|\nabla u_\varepsilon|) - \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\right) D^2 u_\varepsilon \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\cdot\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} + 2(u_\varepsilon - u) = 2f_\varepsilon$$

in $B(\bar{x}, r)\cap\overline{\Omega}$, or equivalently

$$-M_\varepsilon : D^2 u_\varepsilon = f_\varepsilon - (u_\varepsilon - u) \text{ in } B(\bar{x}, r)\cap\overline{\Omega}. \tag{3.28}$$

Now, we use that by definition (3.25) of v_ε^j, we have

$$\nabla u_\varepsilon = \sum_{l=1}^{N} v_\varepsilon^l \tau^l \text{ in } B(\bar{x}, r)\cap\overline{\Omega}. \tag{3.29}$$

Deriving this expression and using that for $1 \le l \le N-1$ the functions v_ε^l and then their tangential derivatives vanish on $B(\bar{x}, r)\cap\partial\Omega$, we deduce

$$D^2 u_\varepsilon = \tau^N \otimes \nabla v_\varepsilon^N + v_\varepsilon^N D\tau^N + \sum_{l=1}^{N-1}(\nabla v_\varepsilon^l \cdot \tau^N)\,\tau^l \otimes \tau^N \text{ on } B(\bar{x}, r)\cap\partial\Omega.$$

Substituting this expression in (3.28) and using that $u_\varepsilon = u = 0$ on $B(\bar{x}, r)\cap\partial\Omega$, we then get

$$M_\varepsilon \nabla v_\varepsilon^N \cdot \tau^N = -f_\varepsilon - M_\varepsilon : D\tau^N v_\varepsilon^N - \sum_{l=1}^{N-1} M_\varepsilon \tau^N \cdot \tau^l\,(\nabla v_\varepsilon^l \cdot \tau^N) \text{ on } B(\bar{x}, r)\cap\partial\Omega.$$

In the last term of this equality, we use again that u_ε vanishes on $\partial\Omega$, which implies that ∇u_ε is proportional to τ^N on $B(\bar{x}, r)\cap\partial\Omega$ and then, from the expression (3.22) of M_ε, that $M_\varepsilon\tau^N$ is also proportional to τ^N. This shows

$$M_\varepsilon \tau^N \cdot \tau^l = 0 \text{ on } U\cap\partial\Omega,\ 1 \le l \le N-1.$$

Using also that $|\tau^N| = 1$ implies $D\tau^N\,\tau^N = 0$, expression (3.22) of M_ε, and ∇u_ε parallel to τ^N on $B(\bar{x}, r)\cap\partial\Omega$, we finally get the Neumann boundary condition for v_ε^N

$$M_\varepsilon \nabla v_\varepsilon^N \cdot \tau^N = -f_\varepsilon - \frac{F'_\varepsilon(|v_\varepsilon^N|)}{2}\,\mathrm{sgn}(v_\varepsilon^N)\,\mathrm{div}\,\tau^N \text{ on } B(\bar{x}, r)\cap\partial\Omega. \tag{3.30}$$

Step 4. Let us prove that for $f \in L^p(\Omega)$, $p > N$, the function ∇u and then σ is in $L^\infty(O\cap\Omega)^N$.

By (3.21) and (3.24), for every $\varphi \in C_c^\infty(U)$ with $0 \le \varphi \le 1$, the function $\partial_i u_\varepsilon \varphi$, $1 \le i \le N$, satisfies

$$-\mathrm{div}(M_\varepsilon \nabla(\partial_i u_\varepsilon \varphi)) = \varphi\partial_i(f_\varepsilon - u_\varepsilon + u) - \mathrm{div}\,(M_\varepsilon \nabla\varphi\,\partial_i u_\varepsilon) - \tfrac{1}{2}\partial_i\sigma_\varepsilon \cdot \nabla\varphi \text{ in } U. \tag{3.31}$$

By (3.20), the sequence σ_ε and then ∇u_ε is bounded in $L^q_{loc}(U)^N$ for every $q < +\infty$. Then the right-hand side of (3.31), which we denote as h_ε, is bounded in $W^{-1,p}_{loc}(U)$. Multiplying (3.31) by $(\partial_i u_\varepsilon\,\varphi - k)^+$ with $k > (1+c)\hat{\mu}$ and observing that $\partial_i u_\varepsilon\,\varphi \ge (1+c)\hat{\mu}$ implies that $(1+c)M_\varepsilon = I$, we get

$$\int_{\{\partial_i u_\varepsilon\varphi \ge k\}} |\nabla(\partial_i u_\varepsilon \varphi)|^2 dx \le C\left(\int_{\{\partial_i u_\varepsilon\varphi \ge k\}} |\nabla(\partial_i u_\varepsilon\varphi)|^{p'} dx\right)^{\frac{2}{p'}}.$$

This inequality allows us to repeat the classical Stampacchia's reasoning to estimate the solution of an elliptic equation in $L^\infty(U)$ (see Theorem 4.1 in [25]) to prove the existence of $C>0$, depending on φ such that

$$\partial_i u_\varepsilon \varphi \leq C\left(\|u_\varepsilon\|_{H^1(U\cap\Omega)} + \|f_\varepsilon\|_{L^p(U\cap\Omega)} + \|u\|_{L^p(U\cap\Omega)}\right) \quad \text{a.e. in } U.$$

A similar reasoning using as a test function $(\partial_i u_\varepsilon\, \varphi + k)^-$ with $k < (1+c)\hat{\mu}$ also provides a lower bound for $\partial_i u_\varepsilon \varphi$ and then proves

$$\|\partial_i u_\varepsilon \varphi\|_{L^\infty(U\cap\Omega)} \leq C\left(\|u_\varepsilon\|_{H^1(U\cap\Omega)} + \|f_\varepsilon\|_{L^p(U\cap\Omega)} + \|u\|_{L^p(U\cap\Omega)}\right).$$

In order to obtain boundary estimates, we reason analogously, using (3.26), (3.27), and (3.30). Therefore, we have proved that for every open set O strictly contained in U, there exists $C>0$, depending on the distance of O to $\partial\Omega$ and of the norm in $W^{1,\infty}$ of the functions τ^i in Step 3, such that

$$\|\partial_i u_\varepsilon\|_{L^\infty(O\cap\Omega)} \leq C\left(\|u_\varepsilon\|_{H^1(U\cap\Omega)} + \|f_\varepsilon\|_{L^p(U\cap\Omega)} + \|f\|_{L^p(\Omega)^N}\right), \quad 1\leq i\leq N.$$

Taking in Step 2 $X = L^p(U\cap\Omega)$ with $p>N$, we can then pass to the limit in this equality to conclude with (3.2) for $U\cap\Omega$ of class $C^{2,\gamma}$ with $\gamma\in(0,1)$. Observe that the dependence of C with respect to the smoothness of $\partial(U\cap\Omega)$ is throughout the norm in $W^{1,\infty}$ of the functions τ^i in Step 3. Thus, regularizing the boundary, we can show that the result holds true just assuming $U\cap\Omega$ of class $C^{1,1}$.

Step 5. For $\varphi\in C_c^\infty(U\cap\Omega)$, $\varphi\geq 0$, and $1\leq i\leq N$, we take $\partial_i u_\varepsilon \varphi^2$ as a test function in (3.22). This gives

$$\begin{aligned}&\int_{U\cap\Omega} M_\varepsilon \nabla(\partial_i u_\varepsilon)\nabla(\partial_i u_\varepsilon)\,\varphi^2 dx + 2\int_{U\cap\Omega} M_\varepsilon\nabla(\partial_i u_\varepsilon)\nabla\varphi\,\partial_i u_\varepsilon \varphi\,dx\\ &\quad = \int_{U\cap\Omega}\partial_i f_\varepsilon \partial_i u_\varepsilon \varphi^2 dx - \int_{U\cap\Omega}\partial_i(u_\varepsilon - u)\partial_i u_\varepsilon\varphi^2 dx.\end{aligned} \tag{3.32}$$

We introduce the truncated function T by

$$T(s) = \begin{cases} -(1+c)\hat{\mu} & \text{if } s < -(1+c)\hat{\mu},\\ s & \text{if } -(1+c)\hat{\mu}\leq s\leq (1+c)\hat{\mu},\\ (1+c)\hat{\mu} & \text{if } (1+c)\hat{\mu} < s.\end{cases}$$

Then the first term on the right-hand side of this equality can be estimated by using

$$\begin{aligned}\int_{U\cap\Omega}\partial_i f_\varepsilon \partial_i u_\varepsilon\varphi^2 dx &= \int_{U\cap\Omega}\partial_i f_\varepsilon T(\partial_i u_\varepsilon)\varphi^2 dx + \int_{U\cap\Omega}\partial_i f_\varepsilon(\partial_i u_\varepsilon - T(\partial_i u_\varepsilon))\varphi^2 dx\\ &= \int_{U\cap\Omega}\partial_i f_\varepsilon T(\partial_i u_\varepsilon)\varphi^2 dx - \int_{\{|\partial_i u_\varepsilon|>(1+c)\hat{\mu}\}} f_\varepsilon \partial^2_{ii} u_\varepsilon \varphi^2 dx\\ &\quad - 2\int_{U\cap\Omega} f_\varepsilon(\partial_i u_\varepsilon - T(\partial_i u_\varepsilon))\partial_i\varphi\,\varphi\,dx.\end{aligned}$$

The first and third terms on the right-hand side of this equality satisfy

$$\left|\int_{U\cap\Omega}\partial_i f_\varepsilon T(\partial_i u_\varepsilon)\varphi^2 dx\right| \leq C\|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)},$$
$$\left|\int_{U\cap\Omega} f_\varepsilon(\partial_i u_\varepsilon - T(\partial_i u_\varepsilon))\partial_i\varphi\,\varphi\,dx\right| \leq C\|f_\varepsilon\|_{L^2(U\cap\Omega)}\|u_\varepsilon\|_{H^1(U\cap\Omega)},$$

while for the second term on the right-hand side, using that $(1+c)M_\varepsilon = I$ a.e. in $\{|\nabla u_\varepsilon| > (1+c)\hat{\mu}\}$, we get

$$\left|\int_{\{|\partial_i u_\varepsilon|>(1+c)\hat{\mu}\}} f_\varepsilon \partial^2_{ii} u_\varepsilon \varphi^2 dx\right| \leq C\|f_\varepsilon\|_{L^2(U\cap\Omega)}\|M_\varepsilon \nabla(\partial_i u_\varepsilon)\varphi\|_{L^2(U\cap\Omega)}.$$

Taking into account these estimates in (3.32) and applying Young's inequality, we have then proved

$$\begin{aligned}(3.33)\qquad &\int_{U\cap\Omega} M_\varepsilon \nabla(\partial_i u_\varepsilon)\nabla(\partial_i u_\varepsilon)\,\varphi^2 dx\\ &\leq C\left(\|u_\varepsilon\|^2_{H^1(U\cap\Omega)} + \|u\|^2_{H^1(\Omega)} + \|f_\varepsilon\|^2_{L^2(U\cap\Omega)} + \|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}\right)\end{aligned}$$

for a constant C depending on φ. Using (3.24) and (3.23), we then conclude that, for $1 \leq i \leq N$,

$$\begin{aligned}(3.34)\qquad &\int_{U\cap\Omega} |\partial_i \sigma_\varepsilon|^2\,\varphi^2 dx = \int_{U\cap\Omega} |M_\varepsilon \nabla(\partial_i u_\varepsilon)|^2\,\varphi^2 dx\\ &\leq C\left(\|u_\varepsilon\|^2_{H^1(U\cap\Omega)} + \|u\|^2_{H^1(\Omega)} + \|f_\varepsilon\|^2_{L^2(U\cap\Omega)} + \|f_\varepsilon\|_{W^{1,1,}(U\cap\Omega)}\right).\end{aligned}$$

Using then $X = W^{1,1}(U\cap\Omega)\cap L^2(U\cap\Omega)$ in Step 2, we can pass to the limit in (3.34) when ε tends to zero to deduce for $1 \leq i \leq N$

$$(3.35)\qquad \int_{U\cap\Omega} |\partial_i \sigma|^2\,\varphi^2 dx \leq C\left(\|u\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(U\cap\Omega)} + \|f\|_{W^{1,1}(U\cap\Omega)}\right)$$

for every $\varphi \in C^\infty_c(U)$, where the constant C depends on φ.

In order to obtain the corresponding boundary estimates, we consider a point $\bar{x} \in U \cap \partial\Omega$ a ball $B(\bar{x}, r)$ and functions τ^i as in Step 3. Defining the functions v^j_ε by (3.25) and taking into account (3.26), (3.27), we can reason as above to prove similarly to (3.33)

$$\begin{aligned}(3.36)\qquad &\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon \nabla v^j_\varepsilon \cdot \nabla v^j_\varepsilon\,\varphi^2\,dx\\ &\leq C\left(\|u\|^2_{H^1(\Omega)} + \|u_\varepsilon\|^2_{H^1(U\cap\Omega)} + \|f_\varepsilon\|^2_{L^2(U\cap\Omega)} + \|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}\right)\end{aligned}$$

for $1 \leq j \leq N-1$ and $\varphi \in C^\infty_c(B(\bar{x}, r))$.

The estimate for the normal derivative v^N_ε of u_ε is a little more difficult. For $\varphi \in C^\infty_c(B(\bar{x}, r))$, we take $v^N_\varepsilon \varphi^2$ as test function in (3.26). By (3.30) and $2M_\varepsilon \tau^N =$

$F''_\varepsilon(|v_\varepsilon^N|)\tau^N$ on $B(\bar{x},r)\cap\partial\Omega$, we get

$$\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla v_\varepsilon^N\cdot\nabla v_\varepsilon^N\,\varphi^2\,dx+2\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla v_\varepsilon^N\cdot\nabla\varphi\,\varphi\,v_\varepsilon^N\,dx+\int_{B(\bar{x},r)\cap\partial\Omega} f_\varepsilon\,v_\varepsilon^N\varphi^2ds$$
$$+\int_{B(\bar{x},r)\cap\partial\Omega}\frac{F'_\varepsilon(|v_\varepsilon^N|)}{2}\,|v_\varepsilon^N|\,\mathrm{div}\,\tau^N\varphi^2ds=\int_{B(\bar{x},r)\cap\Omega}\nabla f_\varepsilon\cdot\tau^N\,v_\varepsilon^N\varphi^2dx$$
$$-\int_{B(\bar{x},r)\cap\Omega}\nabla(u_\varepsilon-u)\cdot\tau^N\,v_\varepsilon^N\varphi^2dx-\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega}M_\varepsilon\nabla(\partial_iu_\varepsilon)\nabla\tau_i^j\,v_\varepsilon^N\varphi^2dx$$
$$+\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega}M_\varepsilon\nabla\tau_i^N\cdot\nabla v_\varepsilon^N\,\partial_iu_\varepsilon\,\varphi^2dx+2\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega}M_\varepsilon\nabla\tau_i^N\cdot\nabla\varphi\,v_\varepsilon^N\,\partial_iu_\varepsilon\,\varphi\,dx$$
$$-\sum_{i=1}^N\int_{B(\bar{x},r)\cap\partial\Omega}\frac{F''_\varepsilon(|v_\varepsilon^N|)}{2}\nabla\tau_i^N\cdot\tau^N\partial_iu_\varepsilon\,v_\varepsilon^N\varphi^2dx.$$

Using in the first term on the right-hand side of this equality the decomposition

$$\int_{B(\bar{x},r)\cap\Omega}\nabla f_\varepsilon\cdot\tau^N\,v_\varepsilon^N\varphi^2dx$$
$$=\int_{B(\bar{x},r)\cap\Omega}\nabla f_\varepsilon\cdot\tau^N\,T(v_\varepsilon^N)\varphi^2dx+\int_{B(\bar{x},r)\cap\Omega}\nabla f_\varepsilon\cdot\tau^N\,(v_\varepsilon^N-T(v_\varepsilon^N))\varphi^2dx$$
$$=\int_{B(\bar{x},r)\cap\Omega}\nabla f_\varepsilon\cdot\tau^N\,T(v_\varepsilon^N)\varphi^2dx-\int_{B(\bar{x},r)\cap\Omega}f_\varepsilon\,\mathrm{div}(\tau^N\varphi^2)\,(v_\varepsilon^N-T(v_\varepsilon^N))dx$$
$$-\int_{\{|v_\varepsilon^N|>(1+c)\hat{\mu}\}}f_\varepsilon\nabla v_\varepsilon^N\cdot\tau^N\varphi^2dx+\int_{B(\bar{x},r)\cap\partial\Omega}f_\varepsilon\,(v_\varepsilon^N-T(v_\varepsilon^N))\varphi^2ds,$$

then Young's inequality and then $(1+c)M_\varepsilon=I$ a.e. on $\{|\nabla u_\varepsilon|>(1+c)\hat{\mu}\}\supset\{|\nabla u_\varepsilon|\geq|v_\varepsilon^N|\}\cap\{|v_\varepsilon^N|>(1+c)\hat{\mu}\}$, we deduce that for every $\delta>0$, there exists a constant $C_\delta>0$, such that

$$\int_{B(\bar{x},r)\cap\Omega}M_\varepsilon\nabla v_\varepsilon^N\cdot\nabla v_\varepsilon^N\,\varphi^2\,dx\leq\delta\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega}M_\varepsilon\nabla(\partial_iu_\varepsilon)\cdot\nabla(\partial_iu_\varepsilon)\,\varphi^2\,dx$$
$$+C_\delta\Bigg(\|u\|^2_{H^1(\Omega)}+\|u_\varepsilon\|^2_{H^1(U\cap\Omega)}+\|f_\varepsilon\|^2_{L^2(U\cap\Omega)}$$
$$+\|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}+\int_{B(\bar{x},r)\cap\partial\Omega}|v_\varepsilon^N|^2\varphi^2ds\Bigg).$$

The last term can be estimated by using that the embedding of $H^1(B(\bar{x},r)\cap\Omega)$ into $L^2(B(\bar{x},r)\cap\partial\Omega)$ is compact, which implies the existence of a positive constant, still denoted by C_δ such that

$$\int_{B(\bar{x},r)\cap\partial\Omega}|w|^2dx$$
$$\leq\delta\int_{B(\bar{x},r)\cap\Omega}|\nabla w|^2dx+C_\delta\int_{B(\bar{x},r)\cap\Omega}|w|^2dx\quad\forall\,w\in H^1(B(\bar{x},r)\cap\Omega),$$

and then (for a different C_δ)

$$\begin{aligned}\int_{B(\bar{x},r)\cap\partial\Omega}|v_\varepsilon^N|^2\varphi^2 ds &= \int_{B(\bar{x},r)\cap\partial\Omega}|T(v_\varepsilon^N)|^2\varphi^2 ds + \int_{B(\bar{x},r)\cap\partial\Omega}|v_\varepsilon^N - T(v_\varepsilon^N)|^2\varphi^2 ds\\ &\le C_\delta\left(1+\|u_\varepsilon\|^2_{H^1(U)}\right) + \frac{\delta}{1+c}\int_{\{|v_\varepsilon^N|>(1+c)\hat{\mu}\}}|\nabla v_\varepsilon^N|^2\varphi^2 dx\\ &\le C_\delta\left(1+\|u_\varepsilon\|^2_{H^1(U\cap\Omega)}\right) + \delta\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla v_\varepsilon^N\cdot\nabla v_\varepsilon^N\varphi^2 dx.\end{aligned}$$

Therefore, we have proved that for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned}\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla v_\varepsilon^N\cdot\nabla v_\varepsilon^N\,\varphi^2\,dx &\le \delta\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla(\partial_i u_\varepsilon)\cdot\nabla(\partial_i u_\varepsilon)\,\varphi^2\,dx\\ &\quad + C_\delta\Big(1+\|u\|^2_{H^1(\Omega)}+\|u_\varepsilon\|^2_{H^1(U\cap\Omega)}+\|f_\varepsilon\|^2_{L^2(U\cap\Omega)}\\ &\qquad + \|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}\Big).\end{aligned}$$

Combining this inequality with (3.36) and taking into account that the definition of the functions v_ε^j also implies

$$\begin{aligned}\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega} &M_\varepsilon\nabla(\partial_i u_\varepsilon)\cdot\nabla(\partial_i u_\varepsilon)\,\varphi^2\,dx\\ &\le C\sum_{j=1}^N\int_{B(\bar{x},r)\cap\Omega} M_\varepsilon\nabla v_\varepsilon^j\cdot\nabla v_\varepsilon^j\,\varphi^2\,dx + C\|u_\varepsilon\|^2_{L^2(U\cap\Omega)},\end{aligned}$$

we conclude with the inequality

$$\begin{aligned}\sum_{i=1}^N\int_{B(\bar{x},r)\cap\Omega} &M_\varepsilon\nabla(\partial_i u_\varepsilon)\cdot\nabla(\partial_i u_\varepsilon)\,\varphi^2\,dx\\ &\le C\left(1+\|u\|^2_{H^1(\Omega)}+\|u_\varepsilon\|^2_{H^1(U\cap\Omega)}+\|f_\varepsilon\|^2_{L^2(U\cap\Omega)}+\|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}\right).\end{aligned}$$

Taking into account (3.24) and (3.23), this also implies

$$\begin{aligned}\int_{B(\bar{x},r)\cap\Omega}&|D\sigma_\varepsilon|^2\varphi^2\,dx\\ &\le C\left(1+\|u\|^2_{H^1(\Omega)}+\|u_\varepsilon\|^2_{H^1(U\cap\Omega)}+\|f_\varepsilon\|^2_{L^2(U\cap\Omega)}+\|f_\varepsilon\|_{W^{1,1}(U\cap\Omega)}\right).\end{aligned}\tag{3.37}$$

Taking $X = W^{1,1}(U\cap\Omega)\cap L^2(U\cap\Omega)$ in Step 2, we can then pass to the limit to prove (3.3) for $U\cap\Omega$ of class $C^{2,\gamma}$. The case $U\cap\Omega$ of class $C^{1,1}$ follows as in Step 3 by remarking that the constant C in (3.3) only depends on the smoothness of $U\cap\partial\Omega$ throughout the norm in $W^{1,\infty}$ of the functions τ^i.

Since the gradient of u_ε is parallel to the outward normal of Ω on $U\cap\partial\Omega$, we also get that the tangential components of $\hat{\sigma}$ vanish on $U\cap\partial\Omega$.

To prove (3.4), we use that $(1+c\hat{\theta})\hat{\sigma} = \nabla u$ and that $\hat{\sigma}$ is in $H^1(O\cap\Omega)$ for every open set O strictly contained in U. This gives

$$0 = \partial_i((1+c\hat{\theta})\hat{\sigma}_j) - \partial_j((1+c\hat{\theta})\hat{\sigma}_i) = c(\partial_i\hat{\theta}\sigma_j - \partial_j\hat{\theta}\hat{\sigma}_i) + (1+c\hat{\theta})(\partial_i\hat{\sigma}_j - \partial_i\hat{\sigma}_i),\tag{3.38}$$

and then from (3.3) we get (3.4).

Step 6. Let us now prove that if $\hat\theta$ only takes the values 0 and 1 and f belongs to $W^{1,1}(U\cap\Omega)\cap L^2(U\cap\Omega)$, then (3.5) and (3.6) hold.

We define $\hat\theta_\varepsilon$ and θ_ε by

$$1+c\,\hat\theta_\varepsilon=\frac{2|\nabla u_\varepsilon|}{F'_\varepsilon(|\nabla u_\varepsilon|)},\quad \theta_\varepsilon=\max\{0,\min\{\hat\theta_\varepsilon,1\}\} \tag{3.39}$$

and observe that definition (2.18) of F and the uniform convergence of F'_ε to F' prove

$$\hat\theta_\varepsilon-\theta_\varepsilon\to 0 \ \text{ in } L^\infty(U\cap\Omega) \tag{3.40}$$

and the existence of $\check\theta\in L^\infty(\Omega)$, such that up to a subsequence

$$\hat\theta_\varepsilon,\theta_\varepsilon\rightharpoonup\check\theta \ \text{ in } L^\infty(\Omega) \ \text{ weak-}*\,.$$

In order to characterize $\check\theta$, we observe that

$$(1+c\hat\theta_\varepsilon)\sigma_\varepsilon=\nabla u_\varepsilon. \tag{3.41}$$

Using then that u_ε converges weakly to u in $H^1(U\cap\Omega)$ and σ_ε converges weakly to $\hat\sigma$ in $H^1_{loc}(U\cap\Omega)$ and then strongly in $L^2_{loc}(U\cap\Omega)$, we get

$$(1+c\check\theta)\hat\sigma=\nabla u \ \text{ in } \Omega,$$

which proves

$$\check\theta=\hat\theta \ \text{ a.e. in } \{\hat\sigma\neq 0\}.$$

Since $\hat\theta$ only take the values 0 and 1, and θ_ε is compressed between 0 and 1, this allows us to show

$$\int_{\{\hat\sigma\neq 0\}}|\theta_\varepsilon-\hat\theta|dx=\int_{\{\hat\sigma\neq0,\ \hat\theta=0\}}\theta_\varepsilon dx+\int_{\{\hat\sigma\neq0,\ \hat\theta=1\}}(1-\theta_\varepsilon)dx\to 0. \tag{3.42}$$

On the other hand, we observe that $\hat\sigma_\varepsilon$ converges strongly to zero in $L^2(\{\hat\sigma=0\})$, which combined with $|\nabla u_\varepsilon|\le(1+c)|\hat\sigma_\varepsilon|$ also proves that ∇u_ε converges strongly to zero in $L^2(\{\hat\sigma=0\})$, and then by definition (2.18) of F we get

$$\theta_\varepsilon\to\begin{cases}1 & \text{if } \hat\mu=0,\\ 0 & \text{if } \hat\mu>0\end{cases}\quad \text{in } L^1(\{\hat\sigma=0\}). \tag{3.43}$$

Now, we observe that analogously to (3.38), equality (3.41) proves

$$\partial_i\hat\theta_\varepsilon\sigma_{\varepsilon,j}-\partial_j\hat\theta_\varepsilon\sigma_{\varepsilon,i}=\frac{1+c\hat\theta_\varepsilon}{c}(\partial_i\sigma_{\varepsilon,j}-\partial_j\sigma_{\varepsilon,i}),\quad 1\le i,j\le N,$$

where using (3.40), (3.42), (3.43), and (3.38) we can pass to the limit in ε to deduce

$$\partial_i\hat\theta_\varepsilon\sigma_{\varepsilon,j}-\partial_j\hat\theta_\varepsilon\sigma_{\varepsilon,i}\rightharpoonup\partial_i\hat\theta\hat\sigma_j-\partial_j\hat\theta\hat\sigma_i \ \text{ in } L^2(O\cap\Omega)\quad\forall\, O\Subset U,\quad 1\le i,j\le N.$$

Using again (3.40), (3.42), and (3.43), we also have

$$\hat\theta^k_\varepsilon(\partial_i\hat\theta_\varepsilon\sigma_{\varepsilon,j}-\partial_j\hat\theta_\varepsilon\sigma_{\varepsilon,i})\rightharpoonup\hat\theta^k(\partial_i\hat\theta\hat\sigma_j-\partial_j\hat\theta\hat\sigma_i) \ \text{ in } L^2(O\cap\Omega)\quad\forall\, O\Subset U,\quad 1\le i,j\le N,$$

for every $k \in \mathbb{N}$, but on the other hand, we have

$$\hat{\theta}_\varepsilon^k(\partial_i\hat{\theta}_\varepsilon\sigma_{\varepsilon,j} - \partial_j\hat{\theta}_\varepsilon\sigma_{\varepsilon,i}) = \frac{1}{k+1}\big(\partial_i(\hat{\theta}_\varepsilon^{k+1}\sigma_{\varepsilon,j}) - \partial_j(\hat{\theta}_\varepsilon^{k+1}\sigma_{\varepsilon,i}) - \hat{\theta}_\varepsilon^{k+1}(\partial_i\sigma_{\varepsilon,j} - \partial_j\sigma_{\varepsilon,i})\big)$$
$$\to \frac{1}{k+1}\big(\partial_i(\hat{\theta}^{k+1}\hat{\sigma}_j) - \partial_j(\hat{\theta}^{k+1}\hat{\sigma}_i) - \hat{\theta}^{k+1}(\partial_i\hat{\sigma}_j - \partial_j\hat{\sigma}_i)\big),$$

in $H^{-1}(O\cap\Omega)$, for every $O \Subset U$ open. Then, taking into account that $\hat{\theta}$ takes only the values 0 and 1, we get

$$\hat{\theta}(\partial_i\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_i) = \frac{1}{k+1}\big(\partial_i(\hat{\theta}\hat{\sigma}_j) - \partial_j(\hat{\theta}\hat{\sigma}_i) - \hat{\theta}(\partial_i\hat{\sigma}_j - \partial_j\hat{\sigma}_i)\big) \text{ a.e. in } U\cap\Omega\ \forall\, k\geq 1.$$

Taking k converging to infinity, this proves

$$(\partial_i\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_i)\chi_{\{\hat{\theta}=1\}} = 0.$$

Similarly, we can show

$$(1-\hat{\theta})(\partial_i\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_i) = -\frac{1}{k+1}\big(\partial_i((1-\hat{\theta})\hat{\sigma}_j) - \partial_j((1-\hat{\theta})\hat{\sigma}_i) - (1-\hat{\theta})(\partial_i\hat{\sigma}_j - \partial_j\hat{\sigma}_i)\big)$$

a.e. in $U\cap\Omega$ for every $k\geq 1$ and then

$$(\partial_i\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_i)\chi_{\{\hat{\theta}=0\}} = 0.$$

This proves (3.5), which combined with (3.38) also proves (3.6). ◻

Proof of Theorem 3.3. Assume that (ω, u_ω) is a solution of (2.1). From Remark 8, we know that

$$(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u_\omega = \nabla w \text{ in } \Omega \tag{3.44}$$

with w the unique solution of

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

Defining $\hat{\mu}$ by (2.13), statement (2.14) with $\hat{\theta} = \chi_\omega$ implies

$$\{x\in\Omega:\ |\nabla w(x)| > \hat{\mu}\} \subset \omega \subset \{x\in\Omega:\ |\nabla w(x)| \geq \hat{\mu}\}. \tag{3.46}$$

On the other hand, taking into account that $\Delta w = 0$ a.e. in $\{\nabla w = 0\}$, we get that $|\nabla w| > 0$ a.e. in Ω, and then we can apply Remark 2 and (2.15) to get

$$|\omega| = \kappa. \tag{3.47}$$

Step 1. Let us prove the existence of a point $x_0\in\Omega$, such that

$$|\nabla w(x_0)| = \hat{\mu}, \qquad \nabla|\nabla w|^2(x_0) \neq 0. \tag{3.48}$$

We consider a connected component O of the set

$$\{x\in\Omega:\ |\nabla w(x)| < \hat{\mu}\}$$

(for example, the connected component corresponding to a point where w attains its maximum). Since by (3.46) and (3.47), O does not agree with Ω, we have that $\partial O\cap\Omega$ is not empty and so we can take a point x^* in $\partial O\cap\Omega$. Now, we choose $\tilde{x}_0\in O$ sufficiently close to x^* to have

$$|\tilde{x}_0-x^*|<\operatorname{dist}(\tilde{x}_0,\partial\Omega),$$

and we take

$$r=\operatorname{dist}(\tilde{x}_0,\partial O)\leq|\tilde{x}_0-x^*|<\operatorname{dist}(\tilde{x}_0,\partial\Omega)$$

and $x_0\in\partial O$ such that

$$|\tilde{x}_0-x_0|=r.$$

Note that x_0 is not in $\partial\Omega$ since $r<\operatorname{dist}(\tilde{x}_0,\partial\Omega)$. By the definition of O and $x_0\in\partial O$, we have that the first equality in (3.48) holds and

$$x_0\in\partial B(\tilde{x}_0,r),\qquad |\nabla w|^2<|\nabla w(x_0)|^2\quad\text{in } B(\tilde{x}_0,r).$$

This inequality combined with

$$-\Delta|\nabla w|^2=-2|D^2w|^2\leq 0 \text{ in } \Omega,$$

which is a consequence of (3.45), allows us to use the Hopf's lemma to deduce that the normal derivative to O of $|\nabla w|^2$ at x_0 is strictly positive, and then that x_0 satisfies the second assertion in (3.48).

Step 2. Since from (3.45) w is analytic in Ω, we can apply the implicit function theorem to deduce the existence of a neighborhood $U\subset\Omega$ of x_0 such that (use (3.46))

$$U\cap\partial\omega=U\cap\{x\in\Omega:\ |\nabla w|=\hat{\mu}\}$$

is a connected analytic manifold of dimension $N-1$. From Remark 7, we also have that ∇w is parallel to the normal on $\partial\omega$ and then that the tangential derivative of w in the connected variety $U\cap\partial\omega$ vanishes. Since $w>0$ in Ω, we deduce the existence of $a>0$ such that

$$w=a\quad\text{on } U\cap\partial\omega,$$

i.e., the $N-1$ variety $U\cap\partial\omega$ satisfies

$$U\cap\partial\omega=U\cap\Big\{x\in\Omega:\quad w(x)=a,\quad|\nabla w(x)|=\hat{\mu}\Big\}.$$

We define the analytic manifold $\tilde{M}$ by

$$\tilde{M}=\left\{x\in\Omega:\ w(x)=a,\quad|\nabla w(x)|>\frac{\hat{\mu}}{2}\right\}$$

and M as the interior of the set

$$\{x\in\Omega:\ w(x)=a,\quad|\nabla w(x)|=\hat{\mu}\}$$

with respect to $\tilde{M}$. Remark that M is not empty because the $N-1$ variety $U\cap\partial\omega$ is contained in M and then M is also an analytic manifold of dimension $N-1$. Let us prove that M is closed.

We consider a sequence $x_n \in M$ which converges to $\bar{x} \in \bar{\Omega}$. Then $\bar{x}$ belongs to the analytic manifold $\tilde{M}$, and so there exists a ball B of center the origin in $\mathbb{R}^N$ and an analytic injective function $\Phi : B \to \mathbb{R}^N$ with $\Phi(B)$ open, such that

$$\Phi(0) = \bar{x}, \quad \Phi(B \cap \{y = (y_1, \dots, y_N) : \ y_N = 0\}) = \tilde{M} \cap \Phi(B).$$

The function $\Psi : B \cap \{y = (y_1, \dots, y_N) : \ y_N = 0\} \to \mathbb{R}$ defined by

$$\Psi(y_1, \dots, y_{N-1}) = |\nabla w(\Phi(y_1, \dots, y_{N-1}, 0))|^2$$

is then analytic. On the other hand, since $x_n \in M$ converges to $\bar{x}$, M is open with respect to $\tilde{M}$, and $|\nabla w| = \hat{\mu}$ on M, there exists a ball $B' \subset B$ such that $\Psi = \hat{\mu}$ on B'. The analyticity of Ψ then proves that $\Psi = \hat{\mu}$ on $B \cap \{y_N = 0\}$ or equivalently that $|\nabla w| = \hat{\mu}$ on $\tilde{M} \cap \Phi(B)$, which combined with $\Phi(B)$ open proves that $\bar{x}$ belongs to M.

Since $w = a$ and $|\nabla w| = \hat{\mu}$ in M, we also have that ∇w is a nonvanishing normal vector on M and thus M is orientable.

Step 3. We consider a connected component M^* of M. Then M^* is a connected compact orientable manifold of dimension $N-1$ contained in Ω. By the Jordan–Brower theorem, it is then the boundary of an open set $\Theta \subset \Omega$. On this point, we follow the ideas in [22]. We have proved that w satisfies

$$\begin{cases} -\Delta w = 1 \ \text{ on } \Theta, \\ w = c \ \text{ on } \partial\Theta, \quad \dfrac{\partial w}{\partial \nu} \ \text{ constant on } \partial\Theta. \end{cases} \tag{3.49}$$

Since we also know that $M^* = \partial\Theta$ is analytic, we can apply Serrin's theorem (see [23]) to deduce that Θ is a ball $B(z_0, R)$ and that w solution of (3.49) satisfies

$$w(x) = \frac{1}{2N}\left(R^2 - |x - z_0|^2\right) + c \ \text{ in } B(z_0, R). \tag{3.50}$$

Since w is analytic in Ω, we have that (3.50) is valid not only in $B(z_0, R)$ but in the whole of Ω, and then, using that $w = 0$ on $\partial\Omega$, we get that Ω agrees with the ball $B(z_0, \sqrt{R^2 + 2cN})$. ∎

4. Applications to the minimization of the first eigenvalue. In the present section, we show how the results obtained previously for problem (2.1) or its relaxed version (2.7) can be applied to the minimization of the first eigenvalue corresponding to the operator

$$u \in H_0^1(\Omega) \mapsto -\text{div}\big((\alpha\chi_\omega + \beta(1 - \chi_\omega))\nabla u\big) \in H^{-1}(\Omega),$$

under the restriction $|\omega| \le k$, i.e., to the control problem

$$\begin{cases} \min \displaystyle\int_\Omega \big(\alpha\chi_\omega + \beta(1 - \chi_\omega)\big)|\nabla u|^2 dx, \\ \omega \subset \Omega, \ \text{measurable}, \quad |\omega| \le \kappa, \quad u \in H_0^1(\Omega), \quad \displaystyle\int_\Omega |u|^2 dx = 1, \end{cases} \tag{4.1}$$

where as in the previous sections Ω is a bounded open set of $\mathbb{R}^N$, $0 < \alpha < \beta$, and $0 < \kappa < |\Omega|$. As for the compliance problem, it is not clear that this problem has a solution, and thus it is necessary to introduce a relaxation which is given by

$$\text{(4.2)}\qquad \begin{cases} \min \displaystyle\int_\Omega \frac{|\nabla u|^2}{1+c\,\theta}dx, \\ \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta\,dx \leq \kappa, \quad u \in H^1_0(\Omega), \quad \displaystyle\int_\Omega |u|^2 dx = 1, \end{cases}$$

with c defined by (2.6).

The relationship between problems (4.2) and (2.3) is a consequence of the following result (see [2], [5]).

LEMMA 4.1. *For $A \in L^\infty(\Omega)^{N\times N}$ symmetric and uniformly elliptic, the first eigenvalue*

$$\text{(4.3)}\qquad \lambda_1(A) = \min_{\substack{u\in H^1_0(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \int_\Omega A\nabla u\cdot\nabla u\,dx$$

of the operator $u \in H^1_0(\Omega) \mapsto -\operatorname{div}(A\nabla u) \in H^{-1}(\Omega)$ is characterized by

$$\text{(4.4)}\qquad \frac{1}{\lambda_1(A)} = \max_{\|f\|_{L^2(\Omega)}=1}\left\{\int_\Omega A\nabla u\cdot\nabla u\,dx:\ -\operatorname{div}(A\nabla u) = f \ \text{ in } \Omega,\ u \in H^1_0(\Omega)\right\}.$$

Moreover, the maximum in (4.4) is attained in a certain f if and only if f is an eigenfunction relative to $\lambda_1(A)$.

Proof. Let f be in $L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} = 1$ and define $u \in H^1_0(\Omega)$ as the solution of

$$\text{(4.5)}\qquad -\operatorname{div}(A\nabla u) = f \ \text{ in } \Omega, \quad u \in H^1_0(\Omega).$$

We observe that definition (4.3) of $\lambda_1(A)$ implies

$$\text{(4.6)}\qquad \int_\Omega A\nabla u\cdot\nabla u\,dx = \int_\Omega fu\,dx \leq \|u\|_{L^2(\Omega)} \leq \left(\frac{1}{\lambda_1(A)}\int_\Omega A\nabla u\cdot\nabla u\,dx\right)^{\frac{1}{2}},$$

and then the arbitrariness of f proves

$$\max_{\|f\|_{L^2(\Omega)}=1}\left\{\int_\Omega A\nabla u\cdot\nabla u\,dx:\ -\operatorname{div}(A\nabla u) = f \ \text{ in } \Omega,\ u \in H^1_0(\Omega)\right\} \leq \frac{1}{\lambda_1(A)}.$$

In order to prove the contrary inequality, we take $f \in H^1_0(\Omega)$ as an eigenfunction relative to $\lambda_1(A)$ of the unitary norm in $L^2(\Omega)$. Then the solution u of problem (4.5) is given by $u = f/\lambda_1(A)$ and satisfies

$$\int_\Omega A\nabla u\cdot\nabla u\,dx = \int_\Omega fu\,dx = \frac{1}{\lambda_1(A)}.$$

To finish the proof, it only remains to show that if the maximum in (4.4) is attained in a certain f, then f is an eigenfunction relative to $\lambda_1(A)$. For this purpose, we obseve that for such f, the inequalities in (4.6) are in fact equalities. Using then

$$\|f\|_{L^2(\Omega)} = 1, \quad \int_\Omega fu\,dx = \|u\|_{L^2(\Omega)},$$

we deduce the existence of $t > 0$ such that $u = tf$. Since we also know

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega, \qquad \|u\|_{L^2(\Omega)} = \left(\frac{1}{\lambda_1(A)}\int_\Omega A\nabla u\cdot\nabla u\,dx\right)^{\frac{1}{2}},$$

we deduce that $t = 1/\lambda_1(A)$, which finishes the proof. □

From Lemma 4.1, problem (4.2) is equivalent to

$$\max_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega\theta dx\le\kappa}} \max_{\|f\|_{L^2(\Omega)}=1}\left\{\int_\Omega \frac{|\nabla u|^2}{1+c\,\theta}dx:\ -\mathrm{div}\left(\frac{\nabla u}{1+c\theta}\right) = f \text{ in } \Omega,\ u\in H^1_0(\Omega)\right\}$$

or changing the order in the maximum problems to

$$\max_{\|f\|_{L^2(\Omega)}=1} \max_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega\theta dx\le\kappa}}\left\{\int_\Omega \frac{|\nabla u|^2}{1+c\,\theta}dx:\ -\mathrm{div}\left(\frac{\nabla u}{1+c\theta}\right) = f \text{ in } \Omega,\ u\in H^1_0(\Omega)\right\}, \tag{4.7}$$

i.e., it consists of solving the compliance problem for every $f \in L^2(\Omega)$ with unitary norm and then taking the maximum in f.

As an consequence of this equivalence and Theorem 3.1, we get the following.

THEOREM 4.2. *We consider an open set $U \subset \mathbb{R}^N$ such that $U \cap \Omega$ is of class $C^{1,1}$, and consider a pair $(\hat\theta, \hat u)$ solution of* (4.2). *Then, for every open set $O \Subset U$, we have the following:*

- *The function $\hat u$ belongs to $W^{1,\infty}(O)$.*
- *The function*

$$\hat\sigma = \frac{\nabla \hat u}{1 + c\,\hat\theta}$$

belongs to $H^1(O)^N$, and its tangential component vanishes on $U \cap \partial\Omega$.
- *For every $i, j \in \{1, \dots, N\}$, we have*

$$\partial_i\hat\theta\hat\sigma_j - \partial_j\hat\theta\hat\sigma_i \in L^2(O).$$

- *If $\hat\theta$ only takes the values* 0 *and* 1, *then*

$$\partial_i\hat\theta\hat\sigma_j - \partial_j\hat\theta\hat\sigma_i = 0 \quad \text{in } U\cap\Omega, \quad 1 \le i, j \le N, \tag{4.8}$$

$$\mathrm{curl}(\hat\sigma) = 0 \quad \text{in } U\cap\Omega. \tag{4.9}$$

Proof. By Lemma 4.1, we know that $(\hat u, \hat\theta)$ is also a solution of problem (2.8) with

$$f = \frac{\hat u}{\|\hat u\|_{L^2(\Omega)}}. \tag{4.10}$$

Then the result follows from Theorem 3.1, just proving that $\hat u$ is smooth enough (we just need $\hat u \in L^p(O)$, $p > N$) for every open set $O \Subset U \cap \Omega$.

Since $\hat u$ belongs to $H^1_0(\Omega)$, the Sobolev imbedding theorem implies

$$\hat u \in \begin{cases} W^{-1,\infty}(\Omega) & \text{if } N \le 4,\\ W^{-1,\frac{2N}{N-4}}(\Omega) & \text{if } N > 4.\end{cases}$$

Thus, by Theorem 3.1 with f given by (4.10) (see also Remark 4) we conclude that for every open set $O \Subset U \cap \Omega$, we have

$$\hat{u} \in \begin{cases} L^\infty(O) & \text{if } N < 6, \\ L^p(O)\ \forall\, p \in [0,\infty) & \text{if } N = 6, \\ L^{\frac{2N}{N-6}}(O) & \text{if } N > 6. \end{cases}$$

This proves smoothness for f given by (4.10) and allows us to improve the smoothness for $\hat{u}$ using again Theorem 3.1. Thus a bootstrap argument finishes the proof. ∎

From the last assertion in Theorem 4.2, we can now obtain a counterexample to the existence of solution for problem (4.1). This is given by the following result.

THEOREM 4.3. *Take $\Omega = (-\pi,\pi)\times(-2\pi,2\pi)^{N-1}$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, problem (4.1) with $\kappa = |\Omega| - \varepsilon$ has no solution.*

Proof. We reason by contradiction.

Step 1. Assume that problem (4.1) with $\kappa = |\Omega| - \varepsilon$ has a solution $(\omega_\varepsilon, u_\varepsilon)$, where we can take u_ε strictly positive in Ω. By Remark 2 and (2.15), we have that

$$|\omega_\varepsilon| = |\Omega| - \varepsilon. \tag{4.11}$$

Moreover, taking into account (4.9) and Ω simply connected, we deduce the existence of $w_\varepsilon \in H^1(O)$ such that

$$\big(\alpha\chi_{\omega_\varepsilon} + \beta(1-\chi_{\omega_\varepsilon})\big)\nabla u_\varepsilon = \nabla w_\varepsilon, \quad -\Delta w_\varepsilon = \lambda_{1,\varepsilon} u_\varepsilon \ \text{ in } \Omega \tag{4.12}$$

with $\lambda_{1,\varepsilon}$ the minimum value of (4.1). Since ∇w_ε is normal to each side of $\partial\Omega$, we have that w_ε is constant in each side of $\partial\Omega$, and then, since it is in $H^1(\Omega)$, it must be constant on $\partial\Omega$. Thus, we can take w_ε as the solution of

$$w_\varepsilon = 0 \ \text{ on } \partial\Omega. \tag{4.13}$$

Step 2. We define

$$\lambda_{1,0} = \alpha \min_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \int_\Omega |\nabla u|^2 dx = \alpha\frac{N+3}{16} \tag{4.14}$$

and

$$u_0 = \sqrt{\frac{2}{\pi^N}}\cos\left(\frac{y_1}{2}\right)\prod_{j=2}^N \cos\left(\frac{y_j}{4}\right), \tag{4.15}$$

a unique solution of

$$-\Delta u_0 = \lambda_{1,0} u_0 \ \text{ in } \Omega, \quad u_0 \in H_0^1(\Omega), \quad u_0 \ge 0 \ \text{ in } \Omega, \quad \|u_0\|_{L^2(\Omega)} = 1.$$

Then, observing that

$$\lambda_{1,\varepsilon} = \min_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^2(\Omega)}=1\\ |\Omega\setminus\omega|=\varepsilon}} \left(\alpha\int_\Omega |\nabla u|^2 dx + (\beta-\alpha)\int_{\Omega\setminus\omega}|\nabla u|^2 dx\right),$$

we deduce that for every $\varepsilon > 0$, we have

$$\lambda_{1,0} \leq \lambda_{1,\varepsilon} \leq \lambda_{1,0} + (\beta - \alpha) \inf_{|\Omega\setminus\omega|=\varepsilon} \int_{\Omega\setminus\omega} |\nabla u_0|^2 dx,$$

which proves

$$\lim_{\varepsilon\to 0} \lambda_{1,\varepsilon} = \lambda_{1,0}. \tag{4.16}$$

From (4.12), (4.13), (4.16), u_ε nonnegative, and $\|u_\varepsilon\|_{L^2(\Omega)} = 1$, we get that u_ε is bounded in $H_0^1(\Omega)$, which combined with (4.11), (4.16), and

$$-\alpha\Delta u_\varepsilon = \lambda_{1,\varepsilon} u_\varepsilon + (\beta - \alpha)\mathrm{div}\big(\chi_{\Omega\setminus\omega_\varepsilon} \nabla u_\varepsilon\big) \ \text{ in } \Omega$$

implies that u_ε converges weakly in $H_0^1(\Omega)$ to the function u_0. Using (4.12), Ω Lipschitz, and a bootstrap argument, we conclude that

$$w_\varepsilon \to u_0 \ \text{ in } H_0^1(\Omega) \cap C^0(\bar{\Omega}), \tag{4.17}$$

$$w_\varepsilon \to u_0 \ \text{ in } C^{2,\gamma}(\overline{O}) \quad \forall\, O \Subset \Omega, \ \text{open}\ \forall\, \gamma \in (0,1). \tag{4.18}$$

Step 3. We have

$$D^2 u_0(0) = -\sqrt{\frac{2}{\pi^N}}\,\mathrm{diag}\left(\frac{1}{4}, \frac{1}{16}, \dots, \frac{1}{16}\right). \tag{4.19}$$

In particular, $D^2 u_0(0)$ is not singular, and thus there exists $\delta > 0$ such that

$$\det(D^2 u_0) \neq 0 \ \text{ in } \overline{B}(0,\delta). \tag{4.20}$$

By (4.15) and (4.20), we have

$$\det(D^2 w_\varepsilon) \neq 0 \ \text{ in } \overline{B}(0,\delta) \quad \text{ for } \varepsilon > 0 \text{ small enough.} \tag{4.21}$$

Now, we recall that by Theorem 2.2 and the definition of w_ε in (4.12), there exists $\mu_\varepsilon > 0$ such that

$$\{x \in \Omega : |\nabla w_\varepsilon(x)| > \mu_\varepsilon\} \subset \omega_\varepsilon \subset \{x \in \Omega : |\nabla w_\varepsilon(x)| \geq \mu_\varepsilon\}. \tag{4.22}$$

Since $|\omega_\varepsilon|$ tends to $|\Omega|$, we have that μ_ε converges to zero.

Step 4. We take $x_\varepsilon \in \Omega$ such that

$$w_\varepsilon(x_\varepsilon) = \max_{\overline{\Omega}} w_\varepsilon,$$

and we observe that (4.17) and u_0 attaining its maximum at zero imply

$$x_\varepsilon \to 0. \tag{4.23}$$

We denote by O_ε the connected component of the open set $\{x \in \Omega : |\nabla w_\varepsilon| < \mu_\varepsilon\}$ containing x_ε. Taking $\varepsilon > 0$ small enough to have $x_\varepsilon \in B(0,\delta)$ and such that (use (4.18))

$$\min_{\partial B(0,\delta)} |\nabla w_\varepsilon| > \mu_\varepsilon,$$

we get that $\bar{O}_\varepsilon$ is contained in $B(0,\delta)$, and then from (4.21) and $|\nabla w_\varepsilon| = \mu_\varepsilon > 0$ on ∂O_ε we deduce that

$$\nabla(|\nabla w_\varepsilon|^2) = D^2 w_\varepsilon \nabla w_\varepsilon \neq 0 \ \text{ in } \partial O_\varepsilon,$$

which allows us to use the implicit function theorem to prove that O_ε is of class C^1

and that, up to a set of null measure,

$$(\Omega \setminus \omega_\varepsilon) \cap B(0,\delta) = \{x \in B(0,\delta) : \ |\nabla w_\varepsilon| < \mu_\varepsilon\}.$$

From Remark 7 and $|\nabla w_\varepsilon| = \mu_\varepsilon$ on ∂O_ε, we also know that

$$w_\varepsilon, \ \frac{\partial w_\varepsilon}{\partial \nu} \ \text{ are constant on each connected component of } \partial O_\varepsilon. \tag{4.24}$$

In particular, since w_ε is a C^2 function, this proves that ∂O_ε is not only C^1 but C^2.

Step 5. Let us now show that O_ε is star-shaped with respect to x_ε and then that ∂O_ε is connected. For this purpose, let us estimate

$$\frac{d}{dr}|\nabla w_\varepsilon(x_\varepsilon + ry)|^2 = 2D^2 w_\varepsilon(x_\varepsilon + ry)\nabla w_\varepsilon(x_\varepsilon + ry)\cdot y$$

for every $y \in \mathbb{R}^N$ with $|y| = 1$ and every $r > 0$ such that $x_\varepsilon + ry \in B(0,\delta)$.

Fixing $\gamma \in (0,1)$ and using (4.18), we have

$$|D^2 w_\varepsilon(x_\varepsilon + ry) - D^2 w_\varepsilon(x_\varepsilon)| \le C r^\gamma. \tag{4.25}$$

Using then that $\nabla w_\varepsilon(x_\varepsilon) = 0$, we can use a Taylor expansion to prove the existence of $t \in (0,1)$ (depending on r, y and ε) such that

$$|\nabla w_\varepsilon(x_\varepsilon + ry) - rD^2 w_\varepsilon(x_\varepsilon)y| \le r|D^2 w_\varepsilon(x_\varepsilon + try)y - D^2 w_\varepsilon(x_\varepsilon)y| \le Cr^{1+\gamma}. \tag{4.26}$$

From (4.25) and (4.26), we deduce

$$\left|\frac{d}{dr}|\nabla w_\varepsilon(x_\varepsilon + ry)|^2 - 2rD^2 u_0(0)^2 y\cdot y\right| \le 2r|D^2 w_\varepsilon(x_\varepsilon) - D^2 u_0(0)| + Cr^{1+\gamma},$$

and then, thanks to (4.19), we deduce the existence of a constant $\gamma > 0$ such that for $\varepsilon, r > 0$ small enough, we have

$$\frac{d}{dr}|\nabla w_\varepsilon(x_\varepsilon + ry)|^2 \ge \gamma\, r,$$

which proves that for every unitary vector $y \in \mathbb{R}^N$, the equation $|\nabla w_\varepsilon(x_\varepsilon + ry)| = \mu_\varepsilon$ (which is equivalent to $x_\varepsilon + ry \in \partial O_\varepsilon$) has at most a solution $r > 0$ such that $x_\varepsilon + ry \in B(0,\delta)$. Since x_ε belongs to $O_\varepsilon \subset B(0,\delta)$, we then have that this equation has in fact a unique solution and then that O_ε is star-shaped.

Step 6. From (4.12) and $O_\varepsilon \subset \Omega \setminus \Omega_\varepsilon$, we get that

$$\beta \nabla u_\varepsilon = \nabla w_\varepsilon \ \text{ in } O_\varepsilon,$$

which combined with O_ε connected shows the existence of $c_\varepsilon \in \mathbb{R}$ such that

$$u_\varepsilon = \frac{1}{\beta} w_\varepsilon + c_\varepsilon.$$

Then, taking into account (4.24) and the second assertion in (4.12) we get that w_ε is a solution of

$$\begin{cases} -\Delta w_\varepsilon = \lambda_{1,\varepsilon}\left(\dfrac{1}{\beta} w_\varepsilon + c_\varepsilon\right) \ \text{ in } O_\varepsilon, \\[2ex] w_\varepsilon, \ \dfrac{\partial w_\varepsilon}{\partial \nu} \ \text{ constant on } \partial O_\varepsilon. \end{cases}$$

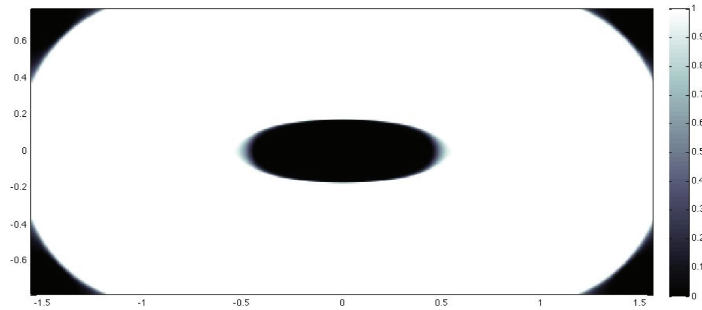

FIG. 1. $\varepsilon = 0.1\,|\Omega|$.

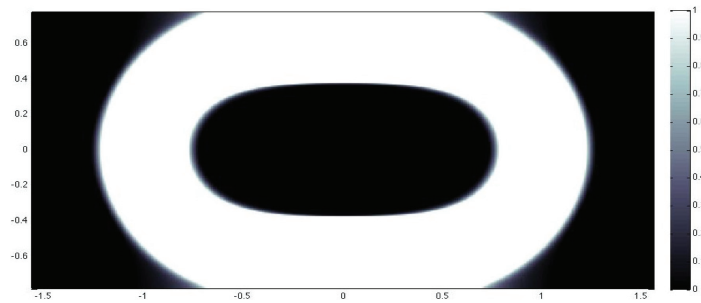

FIG. 2. $\varepsilon = 0.5\,|\Omega|$.

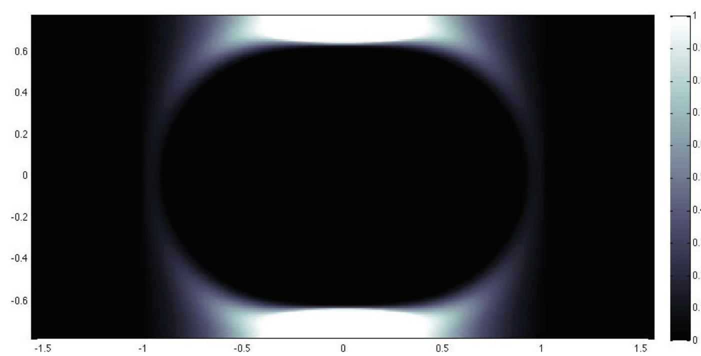

FIG. 3. $\varepsilon = 0.9\,|\Omega|$.

Using that O_ε is C^2 and connected, we can then apply Serrin's theorem [23] to deduce that O_ε is a ball and w_ε is radial. Since we also know that ∇w_ε vanishes at x_ε, we get that x_ε is the center of the ball O_ε and then that $D^2 w_\varepsilon$ is a scalar matrix at x_ε. Therefore, passing to the limit in ε by (4.18), we conclude that $D^2 u_0$ is a scalar matrix in contradiction with (4.19). ∎

To illustrate the example given in Theorem 4.3, we have introduced Figures 1, 2, 3. They correspond to the numerical solution of problem (4.2) for Ω given as in Theorem 4.3 with $N = 2$, $\alpha = 1$, $\beta = 5$ and $\varepsilon = 0.1\,|\Omega|$, $\varepsilon = 0.5\,|\Omega|$, $\varepsilon = 0.9\,|\Omega|$, respectively. White color corresponds to the good material α, black color corresponds to the bad material β, and grey colors refer to homogenization mixtures. Although Theorem 4.3 refers to ε small, we have always found homogenized zones, and in fact they become especially signifiant for ε large, i.e., when we only dispose of a little quantity of the good material α.

To better appreciate the homogenized zones, the figures have been obtained by using a large precision where Ω is decomposed in 250,000 triangles. The algorithm used consists in finding the corresponding eigenvalue function u for a given choice of θ and then constructing a new function θ by solving the minimum in the first line of (2.8) for u fixed. The calculus has been carried out using MATLAB. However, we remark that we have not proved the convergence of the method. We do not know if there is uniqueness for the optimal solution, but using different initializations we have always obtained the same result.

Remark 9. The unique properties of Ω that we have used in the proof of Theorem 4.3 are that Ω is a simply connected sufficiently smooth open set with connected boundary, that the positive eigenfunction u_0 corresponding to the first eigenvalue of the Laplace operator with homogeneous Dirichlet condition at the boundary has a unique maximum point x_0, and that

$$D^2u_0(x_0) \text{ is nonsingular and not a scalar matrix.} \tag{4.27}$$

This allows us to extend the result to some other choices of Ω. In particular, using the following result, it applies to an ellipse in $\mathbb{R}^2$ which is not a circle. (We recall that the unrelaxed problem always has a solution if Ω is a circle [1].) This provides a counterexample to the existence of a solution for problem (4.1), for which Ω is very smooth.

PROPOSITION 4.4. *For $0 < b < a$, we define Ω as the ellipse in $\mathbb{R}^2$ given by*

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\},$$

and we denote by λ_1 the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.28}$$

Then, if u is a positive eigenfunction corresponding to λ_1, we have that zero is the unique point where u attaints its maximum and

$$D^2u(0,0) = \operatorname{diag}(d_1, d_2) \quad \text{with } 0 > d_1 > d_2. \tag{4.29}$$

Proof. From the symmetry properties of Ω and the dimension of the space of eigenfunctions relative to λ_1 equals to one, it is clear that u has the following symmetry properties:

$$u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2) \quad \forall\, (x_1, x_2) \in \Omega. \tag{4.30}$$

In particular,

$$\partial_1 u(0, x_2) = 0 \ \ \forall\, x_2 \in [-b, b], \qquad \partial_2 u(x_1, 0) = 0 \ \ \forall\, x_1 \in [-a, a], \tag{4.31}$$

and thus

$$\partial^2_{12} u(0,0) = 0. \tag{4.32}$$

Now, we observe that thanks to the Hopf lemma, there exists $h \in C^\infty(\partial\Omega)$ strictly positive such that

$$\frac{\partial u}{\partial \nu} = -h \ \text{ on } \partial\Omega,$$

while the Dirichlet condition $u = 0$ on $\partial\Omega$ implies ($\nu = (\nu_1, \nu_2)$)

$$-\partial_1 u\, \nu_2 + \partial_2 u\, \nu_1 = 0 \ \text{ on } \partial\Omega,$$

and then

$$\partial_1 u = -h\,\nu_1, \quad \partial_2 u = -h\,\nu_2 \ \text{ on } \partial\Omega. \tag{4.33}$$

From (4.31), (4.33), and the definition of u, we have that $\partial_1 u$ satisfies

$$\begin{cases} -\Delta\partial_1 u = \lambda_1 \partial_1 u \ \text{ in } \Omega \cap \{x_1 > 0\}, \\ \partial_1 u = 0 \ \text{ on } \{x_1 = 0\}, \quad \partial_1 u < 0 \ \text{ on } \partial\Omega \cap \{x_1 > 0\}, \end{cases}$$

which combined with

$$\hat{\lambda}_1 \stackrel{\text{def}}{:=} \min_{v \in H^1_0(\Omega \cap \{x_1 > 0\})} \frac{\int_{\Omega\cap\{x_1>0\}} |\nabla v|^2 dx}{\int_{\Omega\cap\{x_1>0\}} |v|^2 dx} > \min_{v \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega |v|^2 dx} = \lambda_1 \tag{4.34}$$

allows us to use the strong maximum principle and the Holpf lemma to deduce

$$\partial_1 u < 0 \ \text{ in } \Omega \cap \{x_1 > 0\}, \qquad \partial^2_{11} u(0, x_2) < 0, \quad \forall\, x_2 \in (-b, b). \tag{4.35}$$

Analogously, we can prove

$$\partial_2 u < 0 \ \text{ in } \Omega \cap \{x_2 > 0\}, \qquad \partial^2_{22} u(x_1, 0) < 0, \quad \forall\, x_1 \in (-a, a). \tag{4.36}$$

From (4.30), (4.31), (4.35), and (4.36), we conclude that zero is the unique point where u attains its maximum.

We introduce

$$\mathcal{O} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \quad 0 < |x_2| < x_1,\ \frac{x_2^2}{a^2} + \frac{x_1^2}{b^2} < 1 \right\}$$

and $z : \mathcal{O} \mapsto \mathbb{R}$ by (observe that $\mathcal{O} \subset \Omega$)

$$z(x_1, x_2) = u(x_1, x_2) - u(x_2, x_1) \quad \forall\, (x_1, x_2) \in \mathcal{O}.$$

Thanks to the definition of u, (4.30), $u > 0$ in Ω, and $u = 0$ on $\partial\Omega$, the function z satisfies

$$\begin{cases} -\Delta z = \lambda_1 z \ \text{ in } \mathcal{O}, \\ z = 0 \ \text{ on } \partial\mathcal{O} \cap \{|x_2| = x_1\}, \quad z > 0 \ \text{ on } \partial\mathcal{O} \setminus \{|x_2| = x_1\}, \end{cases} \tag{4.37}$$

but reasoning as in (4.34), we get that the first eigenvalue of the Laplace operator with Dirichlet conditions on $\mathcal{O}$ is strictly smaller than λ_1, and then we can apply the strong maximum principle to (4.37) to deduce

$$z > 0 \ \text{ in } \mathcal{O}. \tag{4.38}$$

Now, identifying $\mathbb{R}^2$ with the complex field $\mathbb{C}$, we take

$$\tilde{\mathcal{O}} = \left\{(x_1, x_2)^2 : \ (x_1, x_2) \in \mathcal{O}\right\},$$

and we define $\tilde{z} : \tilde{\mathcal{O}} \mapsto \mathbb{R}$ by

$$\tilde{z}(x_1, x_2) = z\left(\sqrt{(x_1, x_2)}\right) \quad \forall\, (x_1, x_2) \in \mathcal{O}^2,$$

i.e., using polar coordinates,

$$\tilde{z}(x_1, x_2) = u\left(\sqrt{r}\, \cos\left(\frac{\rho}{2}\right), \sqrt{r}\, \sin\left(\frac{\rho}{2}\right)\right) - u\left(\sqrt{r}\, \sin\left(\frac{\rho}{2}\right), \sqrt{r}\, \cos\left(\frac{\rho}{2}\right)\right)$$

with

$$x_1 = r\cos\rho, \quad x_2 = r\sin\rho$$

for

$$-\pi < \rho < \pi, \quad r\left(\frac{1+\cos\rho}{a^2} + \frac{1-\cos\rho}{b^2}\right) < 2.$$

The function $\tilde{z}$ satisfies

$$\begin{cases} -\Delta\tilde{z} = \dfrac{\lambda_1}{4|x|}\tilde{z} > 0 \ \text{ in } \tilde{\mathcal{O}}, \\ \tilde{z} = 0 \ \text{ on } \partial\tilde{\mathcal{O}} \cap \{x_1 = 0\}, \quad \tilde{z} > 0 \ \text{ on } \partial\tilde{\mathcal{O}} \setminus \{x_1 = 0\}, \end{cases}$$

and it is in $C^\infty(\overline{\tilde{\mathcal{O}}})$ thanks to $z \in C^\infty(\overline{\mathcal{O}})$, $\nabla z(0,0) = 0$. So, we can apply the Hopf lemma to deduce

$$\partial_1\tilde{z} > 0 \ \text{ on } \partial\tilde{\mathcal{O}} \cap \{x_1 = 0\},$$

which, thanks to $\nabla u(0,0) = 0$, shows in particular

$$\begin{aligned} 0 < \partial_1\tilde{z}(0,0) &= \lim_{h\to 0^+} \frac{\tilde{z}(h,0) - \tilde{z}(0,0)}{h} = \lim_{h\to 0^+} \frac{z(\sqrt{h},0) - z(0,0)}{h} \\ &= \lim_{h\to 0^+} \frac{u(\sqrt{h},0) - u(0,\sqrt{h})}{h} = \frac{\partial^2_{11}u(0,0) - \partial^2_{22}u(0,0)}{2}. \end{aligned}$$

Combining this inequality with (4.35) and (4.32), we then deduce (4.29). ∎

Acknowledgment. I thank professor F. Murat for interesting discussions concerning this paper.

REFERENCES

[1] A. Alvino, G. Trombetti, and P.L. Lions, *On optimization problems with prescribed rearrangements*, Nonlinear Anal., 13 (1989), pp. 185–220.

[2] G. Allaire, *Shape Optimization by the Homogenization Method*, Appl. Math. Sci. 146, Springer-Verlag, Berlin, 2002.

[3] J. Casado-Díaz. *Some smoothness results for the optimal design of a two-composite material which minimizes the energy*, Calc. Var. Partial Differential Equations, 53 (2015), pp. 649–673.

[4] J. Casado-Díaz, C. Castro, M. Luna-Laynez, and E. Zuazua, *Numerical approximation of a one-dimensional elliptic optimal design problem*, Multiscale Model. Simul., 9 (2011), pp. 1181–1561.

[5] A. Cherkaev and E. Cherkaeva, *Stable optimal design for uncertain loading conditions*, in Homogenization: In Memory of Serguei Kozlov, V. Berdichevsky, V. Jikov, and G. Papanicolau, eds., Ser. Adv. Math. Appl. Sci. 50, World Scientific, Singapore, 1999, pp. 193–213.

[6] M. Chipot and L. Evans, *Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations*, Proc. Roy. Soc. Edinburgh Sect. A, 102 (1986), pp. 291–303.

[7] C. Conca, A. Laurain, and R. Mahadevan, *Minimization of the ground state for two phase conductors in low contrast regime*, SIAM J. Appl. Math., 72 (2012), pp. 1238–1259.

[8] C. Conca, R. Mahadevan, and L. Sanz. *An extremal eigenvalue problem for a two-phase conductor in a ball*, Appl. Math. Optim., 60 (2009), pp. 173–184.

[9] S. Cox and J.R. McLaughlin, *Extremal eigenvalue problems for composite membranes* I, Appl. Math. Optim., 22 (1990), pp. 153–167.

[10] S. Cox and J.R. McLaughlin, *Extremal eigenvalue problems for composite membranes* II, Appl. Math. Optim., 22 (1990), pp. 169–187.

[11] G. Duvaut, *Mécanique des Milieux Continus*, Masson, Paris, 1990.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[13] J.L. Lions, *Some Methods in the Mathematical Analysis of Systems and Their Control*, Science Press, Beijing, 1981.

[14] A. Mohammadi and M. Yousefnezhad, *Optimal ground state energy of two phase conductors*, Electron. J. Differential Equations, 171 (2014), pp. 1–8.

[15] J. Goodman, R.V. Kohn, and L. Reyna, *Numerical study of a relaxed variational problem for optimal design*, Comput. Methods Appl. Mech. Engrg., 57 (1986), pp. 107–127.

[16] B. Kawohl, J. Stara, and G. Wittum, *Analysis and numerical studies of a problem of shape design*, Arch. Ration. Mech. Anal., 114 (1991), pp. 343–363.

[17] M.G. Krein, *On certain problems on the maximum and minimum of characteristic values and on the Liapunov zones of stability*, Amer. Math. Soc. Transl. Ser. 2, 1 (1955), pp. 163–187.

[18] A. Laurain, *Global minimizer of the ground state for two phase conductors in low contrast regime*, ESAIM Control Optim. Calc. Var. 20, 2 (2014), pp. 362–388.

[19] F. Murat, *Un contre-exemple pour le problème du contrôle dans les coefficients.* C.R.A.S Sci. Paris A, 273 (1971), pp. 708–711.

[20] F. Murat, *Théorèmes de non existence pour des problèmes de contrôle dans les coefficients*, C.R.A.S Sci. Paris A, 274 (1972), pp. 395–398.

[21] F. Murat and L. Tartar, *H-convergence*, in Topics in the Mathematical Modelling of Composite Materials, L. Cherkaev and R.V. Kohn, eds., Progr. Nonlinear Differential Equations Appl. 31, Birkhaüser, Boston, 1998, pp. 21–43 (in English).

[22] F. Murat and L. Tartar, *Calculus of variations and homogenization*, in Topics in the Mathematical Modelling of Composite Materials, L. Cherkaev and R.V. Kohn, eds., Progr. Nonlinear Differential Equations Appl. 31, Birkaüser, Boston, 1998, pp. 139–174 (in English).

[23] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal., 43 (1971), pp. 304–318.

[24] S. Spagnolo, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 22 (1968), pp. 571–597.

[25] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble), 15 (1965), pp. 189–258.

[26] L. Tartar, *The General Theory of Homogenization. A Personalized Introduction*, Springer-Verlag, Berlin, 2009.