Exponential growth of rank jumps for A–hypergeometric systems.

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Abstract

The dimension of the space of holomorphic solutions at nonsingular points (also called the holonomic rank) of a A–hypergeometric system $M_A(\beta)$ is known to be bounded above by $2^{2d}\operatorname{vol}(A)$ [SST00], where d is the rank of the matrix A and $\operatorname{vol}(A)$ is its normalized volume. This bound was thought to be very vast because it is exponential on d. Indeed, all the examples we have found in the literature verify that $\operatorname{rank}(M_A(\beta)) < 2\operatorname{vol}(A)$. We construct here, in a very elementary way, some families of matrices $A_{(d)} \in \mathbb{Z}^{d\times n}$ and parameter vectors $\beta_{(d)} \in \mathbb{C}^d$, $d \geq 2$, such that $\operatorname{rank}(M_{A_{(d)}}(\beta_{(d)})) \geq a^d \operatorname{vol}(A_{(d)})$ for certain $a > 1$.

1 Introduction

Let $A = (a_{ij}) = (a_1\ a_2 \cdots a_n)$ be a full rank matrix with columns $a_j \in \mathbb{Z}^d$ and $d \leq n$. Following Gel'fand, Graev, Kapranov and Zelevinsky (see [GGZ87] and [GZK89]) we can define the A–hypergeometric system with parameter $\beta \in \mathbb{C}^d$ as the left ideal $H_A(\beta)$ of the Weyl algebra $D = \mathbb{C}[x_1,\ldots,x_n]\langle\partial_1,\ldots,\partial_n\rangle$ generated by the following set of differential operators:

$$\Box_u := (\prod_{i:u_i>0} \partial_i^{u_i}) - (\prod_{i:u_i<0} \partial_i^{-u_i}) \quad \text{for all } u \in \mathbb{Z}^n \text{ such that } Au = 0 \tag{1}$$

and

$$E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \quad \text{for } i = 1,\ldots,d \tag{2}$$

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The operators given in (1) generate the so-called toric ideal $I_A \subseteq \mathbb{C}[\partial_1, \dots, \partial_n]$ associated with A and the d operators given in (2) are called the Euler operators associated with the pair (A, β). The hypergeometric D–module associated with the pair (A, β) is the quotient $M_A(\beta) = D/DH_A(\beta)$. It is a holonomic D–module for any pair (A, β) as above (see [GZK89], [Ado94]). In particular, the space of holomorphic solutions of $M_A(\beta)$ at a nonsingular point has finite dimension. This dimension or, equivalently, the holonomic rank of $M_A(\beta)$ equals the normalized volume $\mathrm{vol}_{\mathbb{Z}A}(A)$ of the matrix A (see (3)) when either I_A is Cohen-Macaulay or β is generic (see [GZK89], [Ado94], [SST00]).

The first example of a pair (A, β) for which $\mathrm{rank}(M_A(\beta)) > \mathrm{vol}_{\mathbb{Z}A}(A)$ was described in [ST98] (see Example 2.5). A complete description of the case $d = 2$ appears in [CDD99], revealing that in this case the rank of $M_A(\beta)$ can be only $\mathrm{vol}_{\mathbb{Z}A}(A)$ (the generic value) or $\mathrm{vol}_{\mathbb{Z}A}(A) + 1$ (the exceptional value).

In general it is known that $\mathrm{rank}(M_A(\beta)) \geq \mathrm{vol}_{\mathbb{Z}A}(A)$ for all β [SST00, MMW05]. In fact, it is proved in [MMW05] that the map $\beta \in \mathbb{C}^d \mapsto \mathrm{rank}(M_A(\beta))$ is upper semi–continuous in the Zarisky topology and they also provide an explicit description of the exceptional set

$$\varepsilon(A) = \{\beta \in \mathbb{C}^d : \ \mathrm{rank}(M_A(\beta)) > \mathrm{vol}_{\mathbb{Z}A}(A)\}$$

that turns out to be an affine subspace arrangement with codimension at least 2. Previous descriptions of the exceptional set in particular cases appear in [CDD99, Mat01, Sai02, Mat03].

If for a fixed matrix A we have that $j_A(\beta) = \mathrm{rank}(M_A(\beta)) - \mathrm{vol}_{\mathbb{Z}A}(A) > 0$ then it said that the A–hypergeometric system has a *rank jump* of $j_A(\beta)$ at β or that β is a rank jumping parameter for A.

The paper [MW07] provides the first family of hypergeometric systems with rank jump greater than 2. Indeed, they construct a family of pairs $(A_{(d)}, \beta_{(d)})$ with $A_{(d)} \in \mathbb{Z}^{d\times 2d}$ and $\beta_{(d)} \in \mathbb{C}^d$ such that $j_{A_{(d)}}(\beta_{(d)}) = d - 1$. However, for this family $\mathrm{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = d + 2$ and thus

$$\frac{\mathrm{rank}(M_{A_{(d)}}(\beta_{(d)}))}{\mathrm{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)})} = 2 - \frac{3}{d+2} < 2$$

More recently, in [Ber11] a general combinatorial formula is provided for the rank jump $j_A(\beta)$ of the A–hypergeometric system at a given β. However, the formula is very complicated and, in fact, all the examples included in [Ber11] verify that $\mathrm{rank}(M_A(\beta)) < 2\mathrm{vol}_{\mathbb{Z}A}(A)$ as well. Previous computations of $j_A(\beta)$ in particular cases appear for example in [CDD99], [Sai02], [Oku06].

In the case when the toric ideal is standard homogeneous, the following upper bound for the holonomic rank of a hypergeometric system is proved in [SST00]:

$$\mathrm{rank}(M_A(\beta)) \leq 2^{2d}\mathrm{vol}_{\mathbb{Z}A}(A)$$

However, it is mentioned in [SST00, p. 159] that this upper bound is most likely far from optimal and that it would be desirable to know whether the ratio $\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A)$ can be bounded above by some polynomial function in d. Here we provide a very elementary construction of some families of hypergeometric systems for which the ratio $\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A)$ is exponential on d, giving a negative answer to this last question.

Moreover, for one of the families constructed the dimension of Laurent polynomial solutions is lower than the rank jump (see Remark 2.8). This is in contrast with the general observation in the examples found in the literature (see for example [MW07]).

I am grateful to Christine Berkesch for many helpful conversations about her paper [Ber11].

2 Construction of the examples.

Recall that the normalized volume of a full rank matrix $A \in \mathbb{Z}^{d\times n}$ is given by

$$\text{vol}_{\mathbb{Z}A}(A) = d! \frac{\text{vol}_{\mathbb{R}^d}(\Delta_A)}{[\mathbb{Z}^d : \mathbb{Z}A]} \tag{3}$$

where $[\mathbb{Z}^d : \mathbb{Z}A]$ is the index of the subgroup $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}a_i \subseteq \mathbb{Z}^d$, Δ_A is the convex hull of the columns of A and the origin in $\mathbb{R}^d$ and $\text{vol}_{\mathbb{R}^d}(\Delta_A)$ denotes the Euclidean volume of the polytope Δ_A.

Let us also recall that the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1\times n_1}, A_2 \in \mathbb{Z}^{d_2\times n_2}$ is the following $(d_1 + d_2) \times (n_1 + n_2)$ matrix:

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0_{d_1\times n_2} \\ 0_{d_2\times n_1} & A_2 \end{pmatrix}$$

where $0_{d\times n}$ denotes the $d \times n$ zero matrix.

The following two lemmas are easy to prove.

Lemma 2.1. *If A is the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1\times n_1}, A_2 \in \mathbb{Z}^{d_2\times n_2}$ then $\text{vol}_{\mathbb{Z}A}(A) = \text{vol}_{\mathbb{Z}A_1}(A_1) \cdot \text{vol}_{\mathbb{Z}A_2}(A_2)$.*

Lemma 2.2. *Let $A_i \in \mathbb{Z}^{d_i\times n_i}$ be full rank matrices, $d_i \leq n_i$, and $\beta_{(i)} \in \mathbb{C}^{d_i}$ for $i = 1, 2$. If $A = A_1 \oplus A_2$ and $\beta = (\beta_{(1)}, \beta_{(2)})$ then we have that $H_A(\beta) = DH_{A_1}(\beta_{(1)}) + DH_{A_2}(\beta_{(2)})$ where $H_{A_1}(\beta_{(1)})$ is a left ideal of the Weyl Algebra $D_{A_1} = \mathbb{C}[x_1, \ldots, x_{n_1}]\langle\partial_1, \ldots, \partial_{n_1}\rangle$ and $H_{A_2}(\beta_{(2)})$ is a left ideal of the Weyl Algebra $D_{A_2} = \mathbb{C}[x_{n_1+1}, \ldots, x_{n_1+n_2}]\langle\partial_{n_1+1}, \ldots, \partial_{n_1+n_2}\rangle$ (equivalently, $M_A(\beta)$ is the exterior tensor product of $M_{A_1}(\beta_{(1)})$ and $M_{A_2}(\beta_{(2)})$).*

The following corollary follows from Lemma 2.2 by general properties of the exterior tensor product of holonomic D–modules.

Corollary 2.3. *Under the assumptions of Lemma 2.2 we have:*

i) $\mathrm{rank}(M_A(\beta)) = \mathrm{rank}(M_{A_1}(\beta_{(1)})) \cdot \mathrm{rank}(M_{A_2}(\beta_{(2)}))$.

ii) If Ω_i is a basis for the space of (holomorphic) solutions of the hypergeometric system $M_{A_i}(\beta_{(i)})$ at a point $p_i \in \mathbb{C}^{n_i}$, then the set

$$\Omega = \{f_1(x_1, \ldots, x_{n_1}) \cdot f_2(x_{n_1+1}, \ldots, x_{n_1+n_2}) : f_i \in \Omega_i, i = 1, 2\}$$

is a basis for the space of (holomorphic) solutions of $M_A(\beta)$ at $p = (p_1, p_2) \in \mathbb{C}^{n_1+n_2}$.

In view of Corollary 2.3, we can already give a first type of families of hypergeometric systems for which the rank jump grows exponentially with d.

Theorem 2.4. *Let $A \in \mathbb{Z}^{d\times n}$ and $\beta \in \mathbb{C}^d$ be such that $M_A(\beta)$ has a rank jump, i. e. $\mathrm{rank}(M_A(\beta))/\mathrm{vol}_{\mathbb{Z}A}(A) = q > 1$. Consider for $d_r = rd$ with $r \geq 1$ the matrix $A_r \in \mathbb{Z}^{d_r \times n_r}$ ($n_r = rn$), defined as the direct sum of r copies of A, and the parameter vector $\beta_r = (\beta, \ldots, \beta) \in \mathbb{C}^{d_r}$, defined by r copies of β as well. We have that the family given by (A_r, β_r) satisfies $\mathrm{rank}(M_{A_r}(\beta_r))/\mathrm{vol}_{\mathbb{Z}A_r}(A_r) \geq a^{d_r}$ where $a = \sqrt[d]{q} > 1$.*

In the sequel, we will first consider an example of a family similar to the ones given by Theorem 2.4 and then, we will modify this example in order to exhibit for all $d \geq 2$ a family of A–hypergeometric systems with exponential growth of rank jumps on $d = \mathrm{rank}(A)$ which are not exterior tensor products of smaller hypergeometric systems.

Example 2.5. *For $d = 2$ we will consider the first example of a hypergeometric system with rank jump described in [ST98]. Consider the pair $(A_{(2)}, \beta_{(2)})$ where*

$$A_{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta_{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{4}$$

The toric ideal associated with $A_{(2)}$ is

$$I_{A_{(2)}} = \left(\partial_1\partial_4 - \partial_2\partial_3, \partial_1^2\partial_3 - \partial_2^3, \partial_2\partial_4^2 - \partial_3^3, \partial_1\partial_3^2 - \partial_2^2\partial_4\right)$$

and the Euler operators are $E_1 - \beta_{(2),1} = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - 1$ and $E_2 - \beta_{(2),2} = x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - 2$.

For this example $\mathrm{rank}(M_{A_{(2)}}(\beta)) = \mathrm{vol}_{\mathbb{Z}A_{(2)}}(A_{(2)}) = 4$ for all $\beta \in \mathbb{C}^2 \setminus \{\beta_{(2)}\}$ but $\mathrm{rank}(M_{A_{(2)}}(\beta_{(2)})) = 5$. A basis of the space of solutions of $M_{A_{(2)}}(\beta_{(2)})$ can also be found in [ST98]. Let us point out that this basis consists of the two Laurent polynomials $p_1 = x_2^2/x_1$, $p_2 = x_3^2/x_4$ and other 3 functions that are not Laurent polynomials.

Example 2.6. *For $d = 3$, we will consider the hypergeometric system of the family $\{M_{A_{(d)}}(\beta_{(d)})\}_{d\geq 2}$ described in [MW07]. It is the one associated with the pair*

$$A_{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \text{ and } \beta_{(3)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{5}$$

The volume of $A_{(3)}$ is $d + 2 = 5$ while the rank of $M_{A_{(3)}}(\beta(3))$ is $2d + 1 = 7$.

Example 2.7. *For any $d \geq 4$, let $r, s \in \mathbb{N}$ be such that $2r + 3s = d$. We will choose s as high as possible in order to fix uniques $r, s \in \mathbb{N}$ for each d (in particular $0 \leq r \leq 4$).*

We define $A_{(d)} \in \mathbb{Z}^{d\times 2d}$ to be the direct sum of r copies of the matrix $A_{(2)}$ and s copies of the matrix $A_{(3)}$. By Lemma 2.1 and examples 2.5 and 2.6 we have that $\mathrm{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = 4^r 5^s$.

On the other hand, let $\beta_{(d)} \in \mathbb{C}^d$ be the complex vector with coordinates $\beta_{(d),2i-1} = 1$ and $\beta_{(d),2i} = 2$ for $1 \leq i \leq r$ and $\beta_{(d),2r+3j-2} = 1, \beta_{(d),2r+3j-1} = 0, \beta_{(d),2r+3j} = 2$ for $1 \leq j \leq s$ (i. e., $\beta_{(d)}$ has a copy of $\beta_{(2)}$ for each copy of $A_{(2)}$ and a copy of $\beta_{(3)}$ for each copy of $A_{(3)}$). With this definition of $(A_{(d)}, \beta_{(d)})$ and using Corollary 2.3 and examples 2.5 and 2.6 we have that $\mathrm{rank}(M_{A_{(d)}}(\beta_{(d)})) = 5^r 7^s$. Thus $\mathrm{rank}(M_{A_{(d)}}(\beta(d)))/\mathrm{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = (5/4)^r (7/5)^s \geq (\sqrt{5}/2)^d$.

Remark 2.8. *Example 2.7 also shows that the rank jump $j_A(\beta)$ can be greater than the number of Laurent polynomial solutions of $M_A(\beta)$. Indeed, since the space of Laurent polynomial solutions of $M_{A_{(2)}}(\beta_{(2)})$ has dimension 2 (see [ST98]) and the space of Laurent polynomial solutions of $M_{A_{(3)}}(\beta_{(3)})$ has dimension 4 (see [MW07]) then, by Corollary 2.3, the space of of Laurent polynomial solutions of $M_{A_{(d)}}(\beta(d))$ has dimension $2^r 4^s < j_{A_{(d)}}(\beta_{(d)}) = 5^r 7^s - 4^r 5^s$ for $r, s \geq 1$.*

We are going to modify Example 2.7 in order to get hypergeometric systems that are not exterior tensor products of smaller hypergeometric systems.

Consider the following matrices and parameters:

$$\hat{A}_{(2)} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \hat{\beta}_{(2)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{6}$$

$$\hat{A}_{(3)} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \text{ and parameter } \hat{\beta}_{(3)} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \tag{7}$$

Notice that $\hat{A}_{(2)}$ and $\hat{A}_{(2)}$ are obtained from $A_{(2)}$ and $A_{(3)}$ respectively by multiplying the first row by 2 (this doesn't change the hypergeometric system) and then by adding a first column with its first coordinate equal to 1 and the other coordinates equal to zero. After these modifications we get that $\mathrm{vol}_{\mathbb{Z}\hat{A}_{(2)}}(\hat{A}_{(2)}) = 2 \cdot 4 = 8$ and that $\mathrm{vol}_{\mathbb{Z}\hat{A}_{(3)}}(\hat{A}_{(3)}) = 2 \cdot 5 = 10$. However, since $\hat{\beta}_{(i)}$ is a *hole* in $\mathbb{N}\hat{A}_{(i)}$

(meaning that $\hat{\beta}_{(i)} \notin \mathbb{N}\hat{A}_{(i)}$ but $\hat{\beta}_{(i)} + (\mathbb{N}\hat{A}_{(i)} \setminus \{0\}) \subseteq \mathbb{N}\hat{A}_{(i)}$) we have by Remark 4.14 in [Oku06] that $\operatorname{rank}(M_{\hat{A}_{(i)}}(\hat{\beta}_{(i)})) = \operatorname{vol}_{\mathbb{Z}\hat{A}_{(i)}}(\hat{A}_{(i)}) + (i-1)$, $i = 2, 3$.

The following Lemma follows from the results in [Ber11].

Lemma 2.9. *Let $A \in \mathbb{Z}^{d\times n}$ and $B \in \mathbb{Z}^{d\times m}$ be two matrices verifying that $\mathbb{N}A = \mathbb{N}B$ and $\Delta_A = \Delta_B$ then $\operatorname{rank}(M_A(\beta)) = \operatorname{rank}(M_B(\beta))$ for all $\beta \in \mathbb{C}^d$.*

For $d = 2r + 3s \geq 2$, $r, s \in \mathbb{N}$ (with s as high as possible), let $\hat{\beta}_{(d)} \in \mathbb{C}^d$ be the complex vector that is given by r copies of $\hat{\beta}_{(2)}$ and s copies of $\hat{\beta}_{(3)}$. The new matrix $\hat{A}_{(d)} \in \mathbb{Z}^{d\times(6r+8s-1)}$ is constructed as follows.

Let $a_1, a_2, \ldots, a_{5r+7s} \in \mathbb{Z}^d$ be the columns of the matrix $A_{r,s} = \hat{A}_{(2)} \oplus \overbrace{\cdots}^{r} \oplus \hat{A}_{(2)} \oplus \hat{A}_{(3)} \overbrace{\cdots}^{s} \oplus \hat{A}_{(3)} \in \mathbb{Z}^{d\times(5r+7s)}$.

We will construct a matrix $\hat{A}_{(d)}$ by adding $r + s - 1$ column vectors to the matrix $A_{r,s}$. This vectors will belong to both $\Delta_{A_{r,s}}$ and $\mathbb{N}A_{r,s}$. These conditions guarantee that $\operatorname{vol}_{\mathbb{Z}\hat{A}_{(d)}}(\hat{A}_{(d)}) = \operatorname{vol}_{\mathbb{Z}A_{r,s}}(A_{r,s}) = 8^r 10^s$ and by Lemma 2.9, we will also have that $\operatorname{rank}(M_{\hat{A}_{(d)}}(\beta)) = \operatorname{rank}(M_{A_{r,s}}(\beta))$ for all $\beta \in \mathbb{C}^d$. In particular, for $\beta = \hat{\beta}_{(d)}$, we have $\operatorname{rank}(M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})) = 9^r 12^s$.

If $r \geq 2$ then for $1 \leq i \leq r-1$ we define:

$$a_{5r+7s+i} = a_1 + a_{5i+1} = \frac{1}{2}a_2 + \frac{1}{2}a_{5i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

Notice that $(a_{5r+7s+i})_j$ equals 1 for $j = 1, 2i+1$ and 0 otherwise.

If $r, s \geq 1$ then for $1 \leq i \leq s$ we define

$$a_{5r+7s+r-1+i} = a_1 + a_{5r+7i+1} = \frac{1}{2}a_2 + \frac{1}{2}a_{5r+7i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

If $r = 0$ and $s \geq 1$ then for $1 \leq i \leq s-1$ we define

$$a_{7s+i} = a_1 + a_{7i+1} = \frac{1}{2}a_2 + \frac{1}{2}a_{7i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

Let us define $\hat{A}_d = (a_1\ a_2\ \ldots\ a_{6r+8s-1})$ and recall that $\hat{\beta}_{(d)} \in \mathbb{C}^d$ is given by r copies of $\hat{\beta}_{(2)}$ and s copies of $\hat{\beta}_{(3)}$. The hypergeometric system $M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})$ is not an exterior tensor product of smaller hypergeometric systems and we have proved the following.

Theorem 2.10. *With the notations above we have*

$$\frac{\operatorname{rank}(M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)}))}{\operatorname{vol}_{\mathbb{Z}\hat{A}_{(d)}}(\hat{A}_{(d)})} = (9/8)^r (12/10)^s \geq (\sqrt{9/8})^d$$

Remark 2.11. *Notice that the toric ideal associated with $\hat{A}_{(d)}$ is not homogeneous. However, by Theorem 7.3 in [Ber10], if we consider the associated homogeneous matrix $\hat{A}^h_{(d)}$ (that is obtained by adding to the matrix $\hat{A}_{(d)}$ a first column of zeroes and after that a first row of ones) and the parameter $\hat{\beta}^h_{(d)} = (\beta_0, \hat{\beta}_{(d)})$ with $\beta_0 \in \mathbb{C}$ then the rank of $M_{\hat{A}^h_{(d)}}(\hat{\beta}_{(d)})$ equals the rank of $M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})$ if $\beta_0 \in \mathbb{C}$ is generic. This implies that for particular β_0 (for example $\beta_0 = 0$) the rank of $M_{\hat{A}^h_{(d)}}(\hat{\beta}^h_{(d)})$ will be greater than or equal to the rank of $M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})$ by the upper semi–continuity of the rank [MMW05]. Moreover,* $\mathrm{vol}_{\hat{A}^h_{(d)}}(\hat{A}^h_{(d)}) = \mathrm{vol}_{\hat{A}_{(d)}}(\hat{A}_{(d)})$.

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