Irregular hypergeometric $\mathcal{D}$-modules

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Abstract

We study the irregularity of hypergeometric $\mathcal{D}$-modules $\mathcal{M}_A(\beta)$ via the explicit construction of Gevrey series solutions along coordinate subspaces in $X = \mathbb{C}^n$. As a consequence, we prove that along coordinate hyperplanes the combinatorial characterization of the slopes of $\mathcal{M}_A(\beta)$ given by M. Schulze and U. Walther in [25] still holds for any full rank integer matrix A. We also provide a lower bound for the dimensions of the spaces of Gevrey solutions along coordinate subspaces in terms of volumes of polytopes and prove the equality for very generic parameters. Holomorphic solutions outside the singular locus of $\mathcal{M}_A(\beta)$ can be understood as Gevrey solutions of order one along X at generic points and so they are included as a particular case.

Contents

1 Introduction

This paper is devoted to the study of the irregularity of the GKZ-hypergeometric $\mathcal{D}$-modules. To this end we explicitly construct Gevrey series solutions along coordinate subspaces in $\mathbb{C}^n$. Let us first recall some notions and results about the irregularity in $\mathcal{D}$-module Theory.

Let X be a complex manifold and $\mathcal{D}_X$ the sheaf of linear partial differential operators with coefficients in the sheaf of holomorphic functions $\mathcal{O}_X$.

One fundamental problem in the study of the irregularity of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is the description of its *analytic slopes* along smooth hypersurfaces Y in X (see Z. Mebkhout [19]). An analytic slope is a gap $s > 1$ in the Gevrey filtration $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ of the irregularity complex $\mathrm{Irr}_Y(\mathcal{M})$ (see Definitions 2.4 and 2.5).

Y. Laurent also defined the *algebraic slopes* of $\mathcal{M}$ along a smooth variety Z ([14], [15]) as those real numbers $s > 1$ such that the s-micro-characteristic variety of $\mathcal{M}$ with respect to Z is not homogeneous with respect to the filtration by the order of the differential operators. He proved that the set of slopes of $\mathcal{M}$ along Z is a finite set of rational numbers (see [15]).

When $\mathcal{M}$ is a holonomic $\mathcal{D}$-module and Y is a smooth hypersurface, the Comparison Theorem of the slopes (due to Laurent and Mebkhout [16]) states that the algebraic slopes coincide with the analytic ones. However, as far as we know, the analytic slopes of a holonomic $\mathcal{D}$-module along varieties Z of codimension greater than one are not defined yet in the literature. One problem is that the complexes $\mathrm{Irr}_Z^{(s)}(\mathcal{M})$ and $\mathrm{Irr}_Z(\mathcal{M})$ are constructible but they are not necessarily perverse in such a case (see examples in [18]). The category of perverse sheaves is an abelian category while the category of constructible sheaves is just additive (see [2]).

The description of the Gevrey series solutions of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along a smooth variety Z is another fundamental problem in the study of its irregularity. If Y is a smooth hypersurface the index of any non convergent Gevrey solution of $\mathcal{M}$ along Y is an analytic slope of $\mathcal{M}$ along Y (see Definition 2.5).

From now on we consider the complex manifold $X = \mathbb{C}^n$ and denote $\mathcal{D} := \mathcal{D}_X$. We also will write $\partial_i := \frac{\partial}{\partial x_i}$ for the i-th partial derivative.

Hypergeometric systems were introduced by Gel'fand, Graev, Kapranov and Zelevinsky (see [8] and [9]) and they are associated with a pair (A, β) where A is a full rank $d \times n$ matrix $A = (a_{ij})$ with integer entries $(d \leq n)$ and $\beta \in \mathbb{C}^d$ is a vector of complex parameters. They are left ideals $H_A(\beta)$ of the Weyl algebra $\mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ generated by the following set of differential operators:

$$\square_u := \partial^{u_+} - \partial^{u_-} \qquad \text{for } u \in \mathbb{Z}^n, \ Au = 0 \tag{1}$$

where $u = u_+ - u_-$ and $u_+, u_- \in \mathbb{N}^n$ have disjoint supports, and

$$E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \qquad \text{for } i = 1, \ldots, d \tag{2}$$

The hypergeometric $\mathcal{D}$–module associated with the pair (A, β) is the quotient sheaf $\mathcal{M}_A(\beta) = \mathcal{D}/\mathcal{D}H_A(\beta)$.

The operators given in (1) are called the toric operators associated with A and they generate the so-called toric ideal $I_A \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ associated with A. It is a prime ideal whose zeros variety $\mathcal{V}(I_A) \subseteq \mathbb{C}^n$ is an affine toric variety with Krull dimension d (see for example [26]). The operator E_i is called the i-th Euler operator associated with A for $i = 1, \ldots, d$.

A good introduction for the theory of hypergeometric systems is [24]. These systems are known to be holonomic and their holonomic rank (equivalently, the dimension of the space of holomorphic solutions at nonsingular points) is the normalized volume of the matrix $A = (a_i)_{i=1}^n \in \mathbb{Z}^{d \times n}$ with respect to the lattice $\mathbb{Z}A := \sum_{i=1}^{n} \mathbb{Z}a_i \subseteq \mathbb{Z}^d$ (see Definition 8.1) when either β is generic or I_A is Cohen-Macaulay (see [9], [1]). For results about rank-jumping parameters β see [17], [3] and the references therein. Several authors have studied the holomorphic solutions at nonsingular points of $\mathcal{M}_A(\beta)$ (see [9], [24] and [21]).

A theorem of R. Hotta [13, Ch. II, §6.2, Thm.] assures that when the toric ideal I_A is homogeneous the hypergeometric $\mathcal{D}$-module $\mathcal{M}_A(\beta)$ is regular holonomic. The converse to this theorem was proved by Saito, Sturmfels and Takayama [24, Thm. 2.4.11] when β is generic and by Schulze and Walther [25, Corollary 3.16] when A is a *pointed* matrix such that $\mathbb{Z}A = \mathbb{Z}^d$. A matrix A is said to be pointed if its columns $a_1, \ldots, a_n$ lie in a single open linear half-space of $\mathbb{R}^d$ (equivalently, the associated affine toric variety $\mathcal{V}(I_A)$ passes through the origin). On the other hand, when A is non-pointed then $\mathcal{M}_A(\beta)$ is never regular holonomic: the existence of a toric operator $\partial^u - 1 \in I_A$, $u \in \mathbb{N}^n$, implies that the holonomic rank of some initial ideals of $H_A(\beta)$ is zero and this cannot hold for regular holonomic ideals with positive rank (see [24, Thm. 2.5.1.]).

Let us explain the structure of this paper. In Section 2 we recall some general definitions (Gevrey series, irregularity and analytic slopes of a holonomic $\mathcal{D}$-module) and prove Lemma 2.6 which concerns the slopes of holonomic $\mathcal{D}$-modules and will be used in the sequel.

In Section 3 we consider a simplex σ, i.e., a set $\sigma \subseteq \{1, \ldots, n\}$ such that $A_\sigma = (a_i)_{i \in \sigma}$ is an invertible submatrix of A, and we use the Γ–series introduced in [9] and slightly generalized in [24] to explicitly construct a set of linearly independent Gevrey solutions of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_i = 0 : i \notin \sigma\}$. The cardinal of this set of solutions is the normalized volume of A_σ with respect to the lattice $\mathbb{Z}A$ and we prove that they are Gevrey series of order $s = \max\{|A_\sigma^{-1}a_i| : i \notin \sigma\}$ along the coordinate subspace $Y = \{x_i = 0 : |A_\sigma^{-1}a_i| > 1\} \supseteq Y_\sigma$. Moreover, we also prove that s is their Gevrey index when β is very generic.

In Section 4 we construct for any simplex σ and for all β a set of Gevrey series along Y with index s that are solutions of $\mathcal{M}_A(\beta)$ modulo the sheaf of Gevrey series with lower index. This implies for $s > 1$ that s is a slope of $\mathcal{M}_A(\beta)$ along Y when Y is a hyperplane by Lemma 2.6.

In Section 5 we describe all the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes Y at any point $p \in Y$ (see Theorem 5.10). To this end, and using some ideas of [25], we prove that the s-micro-characteristic varieties with respect to Y of $\mathcal{M}_A(\beta)$ are homogeneous with respect to the order filtration for all $s \geq 1$ but a finite set of candidates s to be algebraic slopes. Then we use the results in Sections 3 and 4 to prove that all the candidates s to be algebraic slopes along hyperplanes occur as the Gevrey index of a Gevrey series solution of $\mathcal{M}_A(\beta)$ modulo convergent series and thus they are analytic slopes. In particular we prove that the set of algebraic slopes of $\mathcal{M}_A(\beta)$ along any coordinate hyperplane is contained in the set of analytic slopes without using the Comparison Theorem of the slopes (due to Laurent and Mebkhout [16]). We use this theorem in the converse direction to prove that there are no more slopes. M. Schulze and U. Walther [25] described in a combinatorial way all the algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces assuming that $\mathbb{Z}A = \mathbb{Z}^d$ and that A is pointed. Previous computations in the particular cases $d = 1$ and $n = d + 1$ of the slopes along coordinate hyperplanes appear in [5], [12] and [11].

In Section 6 we recall the definition of regular triangulation of a matrix and make some remarks that will be used in Section 7.1.

In Section 7.1 we use the Gevrey series constructed in Section 3 and convenient regular triangulations of the matrix A to provide a lower bound for the dimensions of the Gevrey solution spaces. In particular, the lower bound that we obtain for the dimension of the formal solution space of $\mathcal{M}_A(\beta)$ along any coordinate subspace $Y_\tau = \{x_i = 0 : i \notin \tau\}$ at generic points of Y_τ is nothing but the normalized volume of the matrix A_τ with respect to $\mathbb{Z}A$.

In Section 7.2 we prove that this lower bound is actually an equality for very generic parameters $\beta \in \mathbb{C}^d$ and then we have the explicit description of the basis of the corresponding Gevrey solution space. Example 5.12 shows that this condition on the parameters is necessary in general to obtain a basis. This example also points out a special phenomenon: some algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces of codimension greater than one do not appear as the Gevrey index of any formal solution modulo convergent series.

Finally, in Section 8 we assume some conditions ($\mathbb{Z}A = \mathbb{Z}^d$, A is pointed, β is non-rank-jumping and Y is a coordinate hyperplane) in order to use some multiplicity formulas for the s-characteristic cycles of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [25] and general results on the irregularity of holonomic $\mathcal{D}$-modules due to Y. Laurent and Z. Mebkhout [16] to compute the dimension of $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ for generic points $p \in Y$. Then the set of the classes in $\mathcal{Q}_Y(s)$ of the Gevrey solutions that we construct along a hyperplane is a basis for very generic parameters. Moreover, since $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a perverse sheaf on Y by a theorem of Z. Mebkhout [19], we know that for all $i \geq 1$ the i-th cohomology sheaf of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ has support contained in a subvariety of Y with codimension i. This gives the stalk of the cohomology of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at generic points of Y.

This paper is very related with [6] and [7]. In [7] we use deep results in $\mathcal{D}$-module Theory and restriction theorems to reduce the computation of the co-homology sheaves of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ for a pointed one-row matrix A to the case associated with a 1×2 matrix (that we solved by elementary methods in [6]). We also described a basis of the Gevrey solutions in both articles. However, the problem of the combinatorial description of the higher cohomology of the irregularity sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at non generic points of Y for general hypergeometric $\mathcal{D}$-modules seems much more involved since free resolutions are very difficult to compute.

2 Gevrey series and slopes of $\mathcal{D}$-modules

Let $Y \subseteq X = \mathbb{C}^n$ be a smooth analytic subvariety and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ its defining ideal. The formal completion of $\mathcal{O}_X$ along Y is given by

$$\mathcal{O}_{\widehat{X|Y}} := \varprojlim_{k} \mathcal{O}_X/\mathcal{I}_Y^k.$$

In this section, we can assume that locally $Y = Y_\tau = \{x_i = 0 : i \notin \tau\}$ for $\tau \subseteq \{1, \ldots, n\}$ with cardinal $r = \dim_{\mathbb{C}}(Y)$. We will denote $x_\tau := (x_i : i \in \tau)$ and $\overline{\tau} = \{1, \ldots, n\} \setminus \tau$. A germ of $\mathcal{O}_{\widehat{X|Y}}$ at $p \in Y$ has the form

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\overline{\tau}}^\alpha \in \mathcal{O}_{\widehat{X|Y},p} \subseteq \mathbb{C}\{x_\tau - p_\tau\}[[x_{\overline{\tau}}]]$$

where $f_\alpha(x_\tau) \in \mathcal{O}_Y(U)$ for certain nonempty relatively open subset $U \subseteq Y$, $p \in U$. The germs of $\mathcal{O}_{\widehat{X|Y}}$ are called formal series along Y.

Definition 2.1. *A formal series*

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\overline{\tau}}^\alpha \in \mathbb{C}\{x_\tau - p_\tau\}[[x_{\overline{\tau}}]]$$

is said to be Gevrey of multi-order $\mathbf{s} = (s_i)_{i \notin \tau} \in \mathbb{R}^{n-r}$ *along* Y *at* $p \in Y$ *if the series*

$$\rho_{\mathbf{s}}^\tau(f) := \sum_{\alpha \in \mathbb{N}^{n-r}} \frac{f_\alpha(x_\tau)}{\alpha!^{\mathbf{s}-\mathbf{1}}} x_{\overline{\tau}}^\alpha$$

is convergent at p. *Here we denote* $\alpha!^{\mathbf{s}-\mathbf{1}} = \prod_{i \notin \tau}(\alpha_i!)^{s_i-1}$.

Definition 2.2. *A formal series*

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\overline{\tau}}^\alpha \in \mathbb{C}\{x_\tau - p_\tau\}[[x_{\overline{\tau}}]]$$

is said to be Gevrey of order $s \in \mathbb{R}$ *along* Y *at* $p \in Y$ *if the series*

$$\rho_s^\tau(f) := \sum_{\alpha \in \mathbb{N}^{n-r}} \frac{f_\alpha(x_\tau)}{(\alpha!)^{s-1}} x_{\overline{\tau}}^\alpha$$

is convergent at p.

Moreover, if $\rho_{s'}^\tau(f)$ *is not convergent at* p *for any* $s' < s$ *then* s *is said to be the Gevrey index of* f *along* Y *at* p. *It is clear that such a series* f *belongs to* $\mathcal{O}_{\widehat{X|Y},p}$ *and we denote by* $\mathcal{O}_{X|Y}(s)$ *the subsheaf of* $\mathcal{O}_{\widehat{X|Y}}$ *whose germs are Gevrey series of order* s *along* Y.

Remark 2.3. *Notice that any Gevrey series of multi-order* $\mathbf{s} = (s_i)_{i \notin \tau}$ *along* Y *at* $p \in Y$ *is also a Gevrey series of order* $s = \max\{s_i : i \notin \tau\}$ *along* Y *at* p.

For $s = 1$ we have that $\mathcal{O}_{X|Y}(1) = \mathcal{O}_{X|Y}$ is the restriction of $\mathcal{O}_X$ to Y and by convention $\mathcal{O}_{X|Y}(+\infty) = \mathcal{O}_{\widehat{X|Y}}$.

We denote by $\mathcal{Q}_Y$ the quotient sheaf $\mathcal{O}_{\widehat{X|Y}}/\mathcal{O}_{X|Y}$ and by $\mathcal{Q}_Y(s)$ its subsheaf $\mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y}$ for $1 \leq s \leq \infty$.

Definition 2.4. [19, Definition 6.3.1] *For each $1 \leq s \leq \infty$, the irregularity complex of order s of $\mathcal{M}$ along Y is*

$$\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s)).$$

The irregularity complex of $\mathcal{M}$ along Y is $\mathrm{Irr}_Y(\mathcal{M}) := \mathrm{Irr}_Y^{(\infty)}(\mathcal{M})$.

Z. Mebkhout proved in [19, Th. 6.3.3] that for any holonomic $\mathcal{D}_X$–module $\mathcal{M}$ and any smooth hypersurface $Y \subset X$ the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a perverse sheaf on Y for $1 \leq s \leq \infty$. Furthermore, the sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$, $s \geq 1$, determine an increasing filtration of $\mathrm{Irr}_Y(\mathcal{M})$. This filtration is called the Gevrey filtration of $\mathrm{Irr}_Y(\mathcal{M})$ (see [19, Sec. 6]).

Assume for the remainder of this section that Y is a smooth hypersurface.

Definition 2.5. [16, Sec. 2.4] *A number $s > 1$ is said to be an analytic slope of $\mathcal{M}$ along Y at a point $p \in Y$ if p belongs to the analytic closure of the set:*

$$\{q \in Y : \mathrm{Irr}_Y^{(s')}(\mathcal{M})_q \neq \mathrm{Irr}_Y^{(s)}(\mathcal{M})_q, \ \forall s' < s\}.$$

Let us denote $\mathcal{O}_{X|Y}(< s) := \cup_{s'<s}\mathcal{O}_{X|Y}(s')$ for $s \in \mathbb{R}$. For the sake of completeness we include a proof of the following result.

Lemma 2.6. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module such that there exists a series $f \in \mathcal{O}_{X|Y}(s)_p$ with Gevrey index $s > 1$ whose class in*

$$(\mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y}(< s))_p$$

is a solution of $\mathcal{M}$, for all p in a relatively open set $U \subseteq Y$. Then s is a slope of $\mathcal{M}$ along Y at any point in the closure of U.

Proof. Any holonomic $\mathcal{D}$-module is cyclic (see [4, Proposition 3.1.5]). Thus, we can assume without loss of generality that $\mathcal{M} = \mathcal{D}/\mathcal{I}$ with $\mathcal{I}$ a sheaf of ideals generated by some differential operators $P_1, \ldots, P_m \in \mathcal{D}(U)$. Then, by the assumption, there exists $s_i < s$ such that $P_i(f_p) \in \mathcal{O}_{X|Y}(s_i)_p$, $i = 1, \ldots, m$. For $s' = \max\{s_i\} < s$ we have that $(\overline{P_i(f)})_{i=1}^m \in (\mathcal{Q}_Y(s'))^m$ verifies all the left $\mathcal{D}$-relations verified by $(P_i)_{i=1}^m$. Thus, we can consider its class in $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))$.

Since Y is a smooth hypersurface, $\mathrm{Irr}_Y^{(s')}(M)$ is a perverse sheaf on Y [19]. In particular, the support S of the sheaf $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))$ has at most dimension equal to $\dim Y - 1$, so the relatively open set $U' = U \setminus \overline{S} \subseteq Y$ verifies that $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))_{|U'} = 0$ and its closure is equal to the one of U.

In particular the class of $(\overline{P_i(f)})_{i=1}^m \in (\mathcal{Q}_Y(s'))^m$ in $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))_{|U'}$ is zero. This implies the existence of $\overline{h} \in \mathcal{Q}_Y(s')$ such that $(\overline{P_i(h)})_{i=1}^m = (\overline{P_i(f)})_{i=1}^m$ in $(\mathcal{Q}_Y(s')_p)^m$. Equivalently, $P_i(f - h)$ is convergent at any point of U' for all

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$i = 1, \ldots, m$, and we also have that $f - h$ has Gevrey index s because f has Gevrey index s and h has Gevrey index $s' < s$.

The last assertion means that

$$\overline{f - h} \in \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{Q}_Y(s))_{|U'} \setminus \bigcup_{s' < s} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{Q}_Y(s'))_{|U'}$$

and therefore s is a slope of $\mathcal{M}$ along Y at any point in the closure of U. $\qquad\square$

Remark 2.7. *For all $s \geq 1$, $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a constructible sheaf (see [19]). Then there exists a Whitney stratification $\{Y_\alpha(s)\}_\alpha$ of Y such that $\mathcal{H}^i(\mathrm{Irr}_Y^{(s)}(\mathcal{M}))_{|Y_\alpha(s)}$ are locally constant sheaves. If Y is an irreducible algebraic hypersurface and $Y_\alpha(s)$ are algebraic subvarieties then the set $Y_\gamma(s) = Y \setminus \cup_{\dim Y_\alpha(s) < n-1} Y_\alpha(s)$ is a connected stratum (see [10, Théorème 2.1.]). Since $U \cap Y_\gamma(s)$ is a relatively open set in $Y_\gamma(s)$ and s is a slope of $\mathcal{M}$ along Y at any point of U, we have that s is a slope of $\mathcal{M}$ along Y at any point of $Y_\gamma(s)$. This implies that s is a slope of $\mathcal{M}$ along Y at any point of Y by Definition 2.5 because Y is the analytic closure of $Y_\gamma(s)$.*

3   Gevrey solutions of $\mathcal{M}_A(\beta)$ associated with a simplex

Let $A = (a_1 \cdots a_n)$ be a full rank matrix with columns $a_j \in \mathbb{Z}^d$ and $\beta \in \mathbb{C}^d$.

For any set $\tau \subseteq \{1, \ldots, n\}$ let $\mathrm{conv}(\tau)$ be the convex hull of $\{a_i : i \in \tau\} \subseteq \mathbb{R}^d$ and let Δ_τ be the convex hull of $\{a_i : i \in \tau\} \cup \{\mathbf{0}\} \subseteq \mathbb{R}^d$. We shall identify τ with the set $\{a_i : i \in \tau\}$ and with $\mathrm{conv}(\tau)$. We also denote by A_τ the matrix given by the columns of A indexed by τ.

We fix a set $\sigma \subseteq \{1, \ldots, n\}$ with cardinal d and $\det(A_\sigma) \neq 0$ throughout this section. Then Δ_σ is a d-simplex and σ is a $(d-1)$-simplex. The normalized volume of Δ_σ with respect to $\mathbb{Z}A$ is

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = \frac{d! \, \mathrm{vol}(\Delta_\sigma)}{[\mathbb{Z}^d : \mathbb{Z}A]} = \frac{|\det(A_\sigma)|}{[\mathbb{Z}^d : \mathbb{Z}A]}$$

where $\mathrm{vol}(\Delta_\sigma)$ denotes the Euclidean volume of Δ_σ. The aims of this section are: 1) to explicitly construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal solutions of $\mathcal{M}_A(\beta)$ along the subspace $Y_\sigma = \{x_i = 0 : i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0 : j \in \sigma\}$ and 2) to prove that these series are Gevrey series along Y_σ of multi-order $(s_i)_{i \notin \sigma}$ with $s_i = |A_\sigma^{-1} a_i|$.

We reorder the variables in order to have $\sigma = \{1, \ldots, d\}$ for simplicity. Then a basis of $\ker(A) = \{u \in \mathbb{Q}^n : Au = 0\}$ is given by the columns of the matrix:

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1}A_{\overline{\sigma}} \\ I_{n-d} \end{pmatrix} = \begin{pmatrix} -A_\sigma^{-1}a_{d+1} & -A_\sigma^{-1}a_{d+2} & \cdots & -A_\sigma^{-1}a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

For $v \in \mathbb{C}^n$ with $Av = \beta$ the Γ–series defined in [9]:

$$\varphi_v := \sum_{u \in L_A} \frac{1}{\Gamma(v+u+1)} x^{v+u}$$

is formally annihilated by the differential operators (1) and (2). Here Γ is the Euler Gamma function and $L_A := \ker(A) \cap \mathbb{Z}^n$. Notice that φ_v is zero if and only if $(v+L_A) \cap (\mathbb{C} \setminus \mathbb{Z}_{<0})^n = \emptyset$. In contrast, the series φ_v does not define a formal power series at any point if $v \in \mathbb{C}^n$ is very generic.

Observe that $v := (A_\sigma^{-1}\beta, \mathbf{0}) \in \mathbb{C}^n$ satisfies $Av = \beta$ and so the vectors

$$v^{\mathbf{k}} = (A_\sigma^{-1}(\beta - \sum_{i \notin \sigma} k_i a_i), \mathbf{k})$$

where $\mathbf{k} = (k_i)_{i \notin \sigma} \in \mathbb{N}^{n-d}$. Hence, according to Lemma 1 in Section 1.1 of [9], we have that the formal series along $Y_\sigma := \{x_i = 0 : i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0 : j \in \sigma\}$:

$$\varphi_{v^{\mathbf{k}}} = x_\sigma^{A_\sigma^{-1}\beta} \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} \frac{x_\sigma^{-A_\sigma^{-1}(\sum_{i \notin \sigma}(k_i+m_i)a_i)} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}}{\Gamma(A_\sigma^{-1}(\beta - \sum_{i \notin \sigma}(k_i+m_i)a_i)+1)(\mathbf{k}+\mathbf{m})!}$$

where
$$\Lambda_{\mathbf{k}} := \{\mathbf{k}+\mathbf{m} = (k_i+m_i)_{i \in \overline{\sigma}} \in \mathbb{N}^{n-d} : \sum_{i \in \overline{\sigma}} a_i m_i \in \mathbb{Z}A_\sigma\}$$

is annihilated by the operators (1) and (2). Notice that $\varphi_{v^{\mathbf{k}}}$ is zero if and only if for all $\mathbf{m} \in \Lambda_{\mathbf{k}}$, $A_\sigma^{-1}(\beta - \sum_{i \notin \sigma}(k_i+m_i)a_i)$ has at least one negative integer coordinate.

Let us consider the lattice $\mathbb{Z}\sigma = \mathbb{Z}A_\sigma = \sum_{i \in \sigma} \mathbb{Z}a_i$ contained in $\mathbb{Z}A$.

Lemma 3.1. *The following statements are equivalent for all* $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^{n-d}$:

1) $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$

2) $[A_{\overline{\sigma}}\mathbf{k}] = [A_{\overline{\sigma}}\mathbf{k}']$ *in* $\mathbb{Z}A/\mathbb{Z}\sigma$.

3) $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}'}$.

Lemma 3.2. *We have the equality:*

$$\{\Lambda_{\mathbf{k}} : \ \mathbf{k} \in \mathbb{Z}^{n-d}\} = \{\Lambda_{\mathbf{k}} : \ \mathbf{k} \in \mathbb{N}^{n-d}\}$$

and the cardinal of this set is $[\mathbb{Z}A : \mathbb{Z}\sigma] = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$.

Proof. The equality is clear because $A_{\overline{\sigma}}\mathbf{c} \in \mathbb{Z}\sigma$ for $\mathbf{c} = |\det(A_\sigma)| \cdot (1,\ldots,1) \in (\mathbb{N}^*)^{n-d}$ and then for any $\mathbf{k} \in \mathbb{Z}^{n-d}$ there exists $\alpha \in \mathbb{N}$ such that $\mathbf{k} + \alpha\mathbf{c} \in \mathbb{N}^{n-d}$ and $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}+\alpha\mathbf{c}}$.

$\forall \overline{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$ there exists $\mathbf{k} \in \mathbb{Z}^{n-d}$ with $\overline{A_{\overline{\sigma}}\mathbf{k}} = \overline{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$. Then by the equivalence of 2) and 3) in Lemma 3.1 we have that $\{\Lambda_{\mathbf{k}} : \ \mathbf{k} \in \mathbb{Z}^{n-d}\}$ has the same cardinal as the finite group $\mathbb{Z}A/\mathbb{Z}\sigma$. $\qquad\square$

Remark 3.3. *Recall that the support of a series $\sum_v c_v x^v$ is the set*

$$\{v \in \mathbb{C}^n : \ c_v \neq 0\}.$$

Then, for all $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^{n-d}$ such that $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$ we have that $\varphi_{v^{\mathbf{k}}} = \varphi_{v^{\mathbf{k}'}}$ and in other case we have that $\varphi_{v^{\mathbf{k}}}$, $\varphi_{v^{\mathbf{k}'}}$ have disjoint supports.

Remark 3.4. *One may consider $\mathbf{k}(1),\ldots,\mathbf{k}(r) \in \mathbb{N}^{n-d}$ such that*

$$\mathbb{Z}A/\mathbb{Z}\sigma = \{[A_{\overline{\sigma}}\mathbf{k}(i)] : \ i = 1,\ldots,r\}$$

with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$. Then the set in Lemma 3.2 is equal to $\{ \Lambda_{\mathbf{k}(i)} : \ i = 1,\ldots,r\}$ and it determines a partition of $\mathbb{N}^{n-d}$, i.e.,

1) $\Lambda_{\mathbf{k}(i)} \cap \Lambda_{\mathbf{k}(j)} = \emptyset$ *if $i \neq j$.*

2) $\cup_{i=1}^r \Lambda_{\mathbf{k}(i)} = \mathbb{N}^{n-d}$.

We have described $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ formal solutions of $\mathcal{M}_A(\beta)$ along Y_σ associated with a simplex σ having pairwise disjoint supports. Thus, they are linearly independent if none of them is zero.

Definition 3.5. *$\beta \in \mathbb{C}^d$ is said to be generic if it runs in a Zariski open set. β is said to be very generic if it runs in a countable intersection of Zariski open sets.*

Remark 3.6. *For generic β we have that β is non-rank-jumping. For very generic β we have that $v^{\mathbf{k}}$ does not have any negative integer coordinate for all $\mathbf{k} \in \mathbb{N}^{n-d}$ and that β is generic. In particular, if β is very generic we have that $\varphi_{v^{\mathbf{k}}} \neq 0$, $\forall \mathbf{k}$. More precisely, non generic parameter vectors β (resp. non very generic) lie in the complement of a hyperplane arrangement (resp. a countable union of hyperplane arrangements) that depends on A.*

These Γ–series are handled in [24] in such a way that they are not zero for any $\beta \in \mathbb{C}^d$:

$$\phi_v := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}$$

where $v \in \mathbb{C}^n$ verifies $Av = \beta$ and $N_v = \{u \in L_A : \text{nsupp}(v+u) = \text{nsupp}(v)\}$. Here $\text{nsupp}(w) := \{i \in \{1, \ldots, n\} : w_i \in \mathbb{Z}_{<0}\}$ for $w \in \mathbb{C}^n$, $[v]_u = \prod_i [v_i]_{u_i}$ and $[v_i]_{u_i} = \prod_{j=1}^{u_i}(v_i - j + 1)$ is the Pochhammer symbol for $v_i \in \mathbb{C}$, $u_i \in \mathbb{N}$. When $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ we have:

$$\phi_v = \Gamma(v+1)\varphi_v.$$

Since $Av = \beta$ the series ϕ_v is annihilated by the operators (2). It is annihilated by the toric ideal I_A if and only if the negative support of v is minimal, i.e., $\nexists u \in L_A := \ker(A) \cap \mathbb{Z}^n$ with $\text{nsupp}(v+u) \subsetneq \text{nsupp}(v)$ (see [24], Section 3.4.).

Observe that any $u \in L_A$ has the form $(-\sum_{j \notin \sigma} r_j A_\sigma^{-1} a_j, \mathbf{r})$ with $\mathbf{r} = (r_j)_{j \notin \sigma} \in \mathbb{Z}^{n-d}$ such that $A_{\overline{\sigma}} \mathbf{r} = \sum_{j \notin \sigma} r_j a_j \in \mathbb{Z}\sigma$. Then we can choose $\mathbf{k} \in \mathbb{N}^{n-d}$ such that $v^{\mathbf{k}}$ has minimal negative support because we do not change the class of $\sum_{j \notin \sigma} k_j a_j$ modulo $\mathbb{Z} A_\sigma$ when replacing $\mathbf{k}$ by $\mathbf{k} + \mathbf{r} \in \mathbb{N}^{n-d}$. Then the new series $\phi_\sigma^{\mathbf{k}} := \phi_{v^{\mathbf{k}}} \neq 0$ has the form:

$$\phi_\sigma^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m} \in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} x^{v^{\mathbf{k}}+u(\mathbf{m})}$$

where

$$S_{\mathbf{k}} := \{\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k}} : \text{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \text{nsupp}(v^{\mathbf{k}})\} \subseteq \Lambda_{\mathbf{k}}$$

and $u(\mathbf{m}) = (-\sum_{i \notin \sigma} m_i A_\sigma^{-1} a_i, \mathbf{m})$ for $\mathbf{m} = (m_i)_{i \notin \sigma} \in \mathbb{Z}^{n-d}$. It is clear that $\mathbf{k} + \mathbf{m} \in S_{\mathbf{k}}$ if and only if $v^{\mathbf{k}+\mathbf{m}} \in v^{\mathbf{k}} + N_{v^{\mathbf{k}}}$.

Remark 3.7. *Using that $S_{\mathbf{k}} \subseteq \Lambda_{\mathbf{k}}$, $\forall \mathbf{k} \in \mathbb{N}^{n-d}$, and Remark 3.4 we have that two series in $\{\phi_\sigma^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ are either equal up to multiplication by a nonzero scalar or they have disjoint supports. Thus, the set $\{\phi_\sigma^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ has $\text{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal series solutions of $\mathcal{M}_A(\beta)$ along Y_σ at any point of $Y_\sigma \cap \{x_j \neq 0 : j \in \sigma\}$ for all $\beta \in \mathbb{C}^d$.*

Example 3.8. *Let $A = (a_1 \; a_2 \; a_3) \in \mathbb{Z}^{2 \times 3}$ be the matrix with columns:*

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad a_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The kernel of A is generated by $u = (6, 1, -2)$ and so $L_A = \mathbb{Z}u$. Then the hypergeometric system associated with A and $\beta \in \mathbb{C}^2$ is generated by the differential operators:

$$\Box_u = \partial_1^6 \partial_2 - \partial_3^2, \; E_1 - \beta_1 = x_1 \partial_1 + 3x_3 \partial_3 - \beta_1, \; E_2 - \beta_2 = 2x_2 \partial_2 + x_3 \partial_3 - \beta_2.$$

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In this example $\mathbb{Z}A = \mathbb{Z}^2$, A is pointed and $\sigma = \{1, 2\}$ is a simplex with normalized volume $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = |\det(A_\sigma)| = 2$ (see Figure 1).
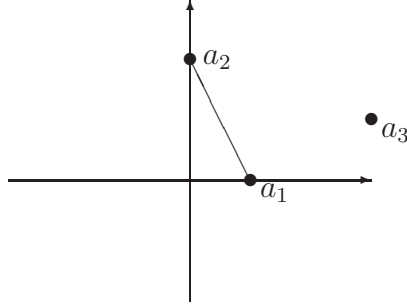

Figure 1

Two convenient vectors associated with σ are

$$v^0 = (\beta_1, \beta_2/2, 0) \text{ and } v^1 = (\beta_1 - 3, (\beta_2 - 1)/2, 1).$$

The associated series:

$$\phi_{v^0} = \sum_{m \geq 0} \frac{[\beta_1]_{6m}[\beta_2/2]_m}{(2m)!} x_1^{\beta_1 - 6m} x_2^{\beta_2/2 - m} x_3^{2m}$$

and

$$\phi_{v^1} = \sum_{m \geq 0} \frac{[\beta_1 - 3]_{6m}[(\beta_2 - 1)/2]_m}{(2m+1)!} x_1^{\beta_1 - 3 - 6m} x_2^{(\beta_2 - 1)/2 - m} x_3^{1 + 2m}$$

are formal series along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$ that are annihilated by the Euler operators $E_1 - \beta_1$, $E_2 - \beta_2$ because $Av^k = \beta$ and by the toric operator $\Box_u$ since v^k has minimal negative support for all $\beta \in \mathbb{C}^2$ for $k = 0, 1$.

The following Lemma is very related with Lemma 1 in [21] (see also [9, Proposition 1, Section 1.1.] and [22, Proposition 5]).

Lemma 3.9. *Assume that $\{b_i\}_{i=d+1}^n$ is a set of vectors in $\mathbb{Q}^d \times \mathbb{N}^{n-d}$, $\mathbf{k} \in \mathbb{Z}^{n-d}$. Let us denote $u(\mathbf{m}) = \sum_{i=d+1}^n m_i b_i$ and consider a set $D_\mathbf{k} \subseteq \{\mathbf{k} + \mathbf{m} \in \mathbb{N}^{n-d} : u(\mathbf{m}) \in \mathbb{Z}^n\}$ and a vector $v \in \mathbb{C}^n$ such that $\mathrm{nsupp}(v + u(\mathbf{m})) = \mathrm{nsupp}(v)$ for any $\mathbf{m} \in D_\mathbf{k} - \mathbf{k}$. Then for all $\mathbf{s} \in \mathbb{R}^{n-d}$ the following statements are equivalent:*

1) $\displaystyle\sum_{\mathbf{k}+\mathbf{m} \in D_\mathbf{k}} \frac{[v]_{u(\mathbf{m})_-}}{[v + u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.*

2) $\displaystyle\sum_{\mathbf{k}+\mathbf{m} \in D_\mathbf{k}} \frac{u(\mathbf{m})_-!}{u(\mathbf{m})_+!} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.*

3) $\sum_{\mathbf{k}+\mathbf{m}\in D_{\mathbf{k}}} \prod_{j=d+1}^{n} (k_j + m_j)!^{-|b_j|} y^{\mathbf{k}+\mathbf{m}}$ is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

In particular, for $\mathbf{s} = (s_{d+1}, \ldots, s_n)$ with $s_i = 1 - |b_i|$, $i = d+1 \ldots, n$, 1),2) and 3) are satisfied. Moreover, 1), 2) and 3) are also equivalent if we write order s instead of multi-order $\mathbf{s}$ and all these series are Gevrey of order $s = \max_i \{1 - |b_i|\}$.

Proof. $\forall \alpha \in \mathbb{C}$, $\forall m \in \mathbb{N}$ with $[\alpha]_m \neq 0$ there exists $C, D > 0$ such that:

$$C^m |[\alpha]_m| \leq m! \leq |[\alpha]_m| D^m \tag{3}$$

For the proof of (3) it is enough to consider $c_m := [\alpha]_m / m!$ and see that $\lim_{m\to\infty} |c_{m+1}/c_m| = 1$. The proof for (3) with $\alpha + m$ instead of α is analogous. It follows that 1) and 2) are equivalent.

We can use Stirling's formula $m! \sim \sqrt{2\pi m}(m/e)^m$ in order to prove that $\forall m \in \mathbb{N}$, $\forall q \in \mathbb{Q}_+$ with $qm \in \mathbb{N}$, there exist $C', D' > 0$ verifying:

$$(C')^m m!^q \leq (qm)! \leq (D')^m m!^q \tag{4}$$

Take $\lambda \in \mathbb{N}^*$ such that $\lambda b_i \in \mathbb{Z}^n$ for all $i = d + 1, \ldots, n$. Then by (4) we have 2) if and only if the series $\sum_{\mathbf{k}+\mathbf{m}\in D_{\mathbf{k}}} \frac{((\lambda u(\mathbf{m})_-)!)^{1/\lambda}}{((\lambda u(\mathbf{m})_+)!)^{1/\lambda}} y^{\mathbf{k}+\mathbf{m}}$ is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

For the rest of the proof we assume for simplicity that $D_{\mathbf{k}} \subseteq (\mathbf{k}+\mathbb{N}^{n-d})\cap\mathbb{N}^{n-d}$. The equivalence of 2) and 3) can be proven without this assumption but it is necessary to distinguish more cases.

Observe that $\lambda u(m)_+ - \lambda u(m)_- = \sum_{i=d+1}^{n} \lambda(b_i)_+ m_i - \sum_{i=d+1}^{n} \lambda(b_i)_- m_i$ and $u(m)_+, u(m)_-, \sum_{i=d+1}^{n} \lambda(b_i)_+ m_i, \sum_{i=d+1}^{n} \lambda(b_i)_- m_i \in \mathbb{N}^n$. However, $u(m)_+, u(m)_-$ have disjoint supports while, in general, $\sum_{i=d+1}^{n} \lambda(b_i)_+ m_i$ and $\sum_{i=d+1}^{n} \lambda(b_i)_- m_i$ do not.

On the other hand, for all $m, n \in \mathbb{N}$ with $n \leq m$ we have that:

$$(m - n)! \leq \frac{m!}{n!} \leq 2^m (m - n)! \tag{5}$$

Then by (5) we have 2) if and only if

$$\sum_{\mathbf{k}+\mathbf{m}\in S} \left(\frac{(\sum_{i=d+1}^{n} \lambda(b_i)_- m_i)!}{(\sum_{i=d+1}^{n} \lambda(b_i)_+ m_i)!} \right)^{1/\lambda} y^{\mathbf{k}+\mathbf{m}} \tag{6}$$

is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

If we replace m by $m + n$ in (5) and multiply by $n!$ we obtain a formula that can be generalized by induction. We obtain that $\forall m_{d+1}, \ldots, m_n \in \mathbb{N}$ there exist $C'', D'' > 0$ such that:

$$(C'')^{\Sigma_j m_j} \prod_i m_i! \leq (\sum m_i)! \leq (D'')^{\Sigma_j m_j} \prod_i m_i! \tag{7}$$

A combination of (7) and (4) proves that 3) holds if and only if (6) is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

Finally, it is clear that 3) is true for $s_i = 1 - |b_i|$, $i = d + 1, \ldots, n$. $\qquad\square$

Remark 3.10. *From the proof of the equivalence of 1) and 2) in Lemma 3.9 it can be deduced that after applying $\rho_{\mathbf{s}}$ to the series in 1) there exists an open set W such that this series converges in W and $0 \in W$ does not depend on v but in $D_k - k$.*

Consider $\mathbf{s} = (s_j)_{j \notin \sigma}$ with

$$s_j := |A_\sigma^{-1} a_j|, \ j \notin \sigma$$

and $s = \max_i\{s_i\}$ throughout this section.

Lemma 3.11. *For all $\mathbf{k} \in \mathbb{N}^{n-d}$ the series*

$$\psi_\sigma^{\mathbf{k}} := \sum_{\mathbf{k}+\mathbf{m} \in S_k} \frac{[v^k]_{u(m)_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$$

is Gevrey of multi-order $\mathbf{s}$ along $y = \mathbf{0} \in \mathbb{C}^{n-d}$. Moreover, if β is very generic then it has Gevrey index s along $y = \mathbf{0} \in \mathbb{C}^{n-d}$.

Proof. It follows from Lemma 3.9 (if we take b_{d+i} equal to the i-th column of B_σ, $D_{\mathbf{k}} = S_{\mathbf{k}}$ and $v = v^{\mathbf{k}}$) that $\psi_\sigma^{\mathbf{k}}$ is Gevrey of multi-order $\mathbf{s}$ along $y = \mathbf{0}$.

If β is very generic we have that $S_{\mathbf{k}} = \Lambda_{\mathbf{k}}$ and it is obvious that the series in 3) of Lemma 3.9 has Gevrey index s in this case. $\qquad\square$

Corollary 3.12. *The series $\phi_\sigma^{\mathbf{k}}$ is Gevrey of multi-order $\mathbf{s} = (s_j)_{j \notin \sigma}$ along Y_σ at any point of $Y_\sigma \cap \{x_i \neq 0 : i \in \sigma\}$. If β is very generic then it is Gevrey with index s along Y_σ.*

Proof. If we take $y = (y_j)_{j \notin \sigma}$ with $y_j := x_\sigma^{-A_\sigma^{-1} a_j} x_j$, $j \notin \sigma$, then $\phi_\sigma^{\mathbf{k}}(x) = x_\sigma^{A_\sigma^{-1}\beta} \psi_\sigma^{\mathbf{k}}(y)$ and the result follows from Lemma 3.11. $\qquad\square$

Example 3.13. *(Continuation of Example 3.8)* We have that

$$\rho_s(\phi_{v^0}) = x_1^{\beta_1} x_2^{\beta_2/2} \sum_{m \geq 0} \frac{[\beta_1]_{6m}[\beta_2/2]_m}{(2m)!^s} \left(\frac{x_3^2}{x_1^6 x_2}\right)^m$$

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It is easy to see that $\rho_s(\phi_{v^0})$ has a nonempty domain of convergence if and only if $s \geq 7/2$ when $\beta_1, \beta_2/2 \notin \mathbb{N}$ (use D'Alembert criterion for the series in one variable $y = x_3^2/(x_1^6 x_2)$). Then ϕ_{v^0} is a Gevrey series solution of $\mathcal{M}_A(\beta)$ with index $s = 7/2$ along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. Nevertheless, ϕ_{v^0} is a finite sum if either $\beta_1 \in \mathbb{N}$ or $\beta_2/2 \in \mathbb{N}$ and so it has the same convergence domain as the (multi-valued) function $x_1^{\beta_1} x_2^{\beta_2/2}$. If both $\beta_1, \beta_2/2 \in \mathbb{N}$, then ϕ_{v^0} is a polynomial.

Analogously, ϕ_{v^1} is a Gevrey series solution of order $s = 7/2$ along Y_σ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. It has Gevrey index $s = 7/2$ if $\beta_1 - 3, (\beta_2 - 1)/2 \notin \mathbb{N}$ and it is convergent in other case.

Notice that $s = 7/2$ is the unique algebraic slope of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_3 = 0\}$ at $\mathbf{0} \in \mathbb{C}^3$ (see [25] or [11]).

The convergence domain of $\rho_{\mathbf{s}}^{\emptyset}(\psi_\sigma^{\mathbf{k}})$ contains $\{y \in \mathbb{C}^{n-d} : |y_j| < R, \ j \notin \sigma\}$ for certain $R > 0$. In particular, $\rho_{\mathbf{s}}^\sigma(\phi_{v^{\mathbf{k}}})$ converges in

$$\{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_\sigma^{A_\sigma^{-1} a_j}|, \ \forall j \notin \sigma\}.$$

The unique hyperplane that contains σ is

$$H_\sigma = \{\mathbf{y} \in \mathbb{R}^d : |A_\sigma^{-1}\mathbf{y}| = 1\}$$

and we denote by $H_\sigma^- := \{\mathbf{y} \in \mathbb{R}^d : |A_\sigma^{-1}\mathbf{y}| < 1\}$ (resp. by $H_\sigma^+ := \{\mathbf{y} \in \mathbb{R}^d : |A_\sigma^{-1}\mathbf{y}| > 1\}$) the open affine half-space that contains (resp. does not contain) the origin $\mathbf{0} \in \mathbb{R}^d$.

Recall that $\mathbf{s} = (s_i)_{i \notin \sigma}$ where $s_i = |A_\sigma^{-1} a_i|$ is the unique rational number such that $a_i/s_i \in H_\sigma$. Moreover, $s_i > 1$ (resp. $s_i < 1$) if and only if $a_i \in H_\sigma^+$ (resp. $a_i \in H_\sigma^-$). Taking the set

$$\tau = \{i : a_i \notin H_\sigma^+\}$$

and $\mathbf{s}' = (s_i)_{i \notin \tau}$ we have that $\rho_{\mathbf{s}'}^\tau(\phi_{v^{\mathbf{k}}})$ converges in the open set

$$U_\sigma' := \{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_\sigma^{A_\sigma^{-1} a_j}|, \ \forall a_j \in (H_\sigma \setminus \sigma) \cup H_\sigma^+\}.$$

This implies that $\phi_{v^{\mathbf{k}}}$ is Gevrey of multi-order $\mathbf{s}'$ along Y_τ at any point of $U_\sigma' \cap Y_\tau$. Then, if we consider

$$U_\sigma := \{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_\sigma^{A_\sigma^{-1} a_j}|, \ \forall a_j \in H_\sigma \setminus \sigma\} \tag{8}$$

the following result is obtained.

Theorem 3.14. *For any set ς with $\sigma \subseteq \varsigma \subseteq \tau$ the series*

$$\phi_\sigma^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m}\in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} x^{v^{\mathbf{k}}+u(\mathbf{m})}$$

is a Gevrey series solution of $\mathcal{M}_A(\beta)$ of order $s = \max\{s_i = |A_\sigma^{-1}a_i| : i \notin \sigma\}$ along Y_ς at any point of $Y_\varsigma \cap U_\sigma$. If β is very generic then s is its Gevrey index.

Remark 3.15. *If $H_\sigma \cap \{a_i : i = 1, \ldots, n\} = \sigma$ then $U_\sigma = \{\prod_{i \in \sigma} x_i \neq 0\}$.*

Remark 3.16. *Recall that in Theorem 3.14 the vector $v^{\mathbf{k}} = (A_\sigma^{-1}(\beta - A_{\bar\sigma}\mathbf{k}), \mathbf{k})$ has minimal negative support because we have chosen $\mathbf{k} \in \Lambda_{\mathbf{k}}$ this way. This guarantees that $\phi_{v^{\mathbf{k}}}$ is a solution of $\mathcal{M}_A(\beta)$. However, this series is also Gevrey of order s when $\mathbf{k}$ does not satisfy this condition.*

4   Slopes of $\mathcal{M}_A(\beta)$ associated with a simplex

In the context of Section 3 we fix a simplex $\sigma \subseteq A$ with $\det(A_\sigma) \neq 0$ and consider $\mathbf{s} = (s_i)_{i \notin \sigma}$ where $s_i = |A_\sigma^{-1}a_i|$. We consider $\tau = \{j : a_j \notin H_\sigma^+\} \supseteq \sigma$ and the coordinate subspace $Y_\tau = \{x_j = 0 : j \notin \tau\}$ in this section.

Our purpose here is to construct one nonzero Gevrey series solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s))_p$ for $p \in Y_\tau \cap U_\sigma$ with support contained in the set $\Lambda_{\mathbf{k}} \subseteq \mathbb{N}^{n-d}$ in the partition of $\mathbb{N}^{n-d}$ (see Remark 3.4) for all $\beta \in \mathbb{C}^d$. In particular we will prove the following result:

Proposition 4.1. *For $s = \max\{s_i = |A_\sigma^{-1}a_i| : i \notin \sigma\}$, for all $p \in Y_\tau \cap U_\sigma$ and for all $\beta \in \mathbb{C}^d$:*

$$\dim(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s)))_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma).$$

As a consequence of Proposition 4.1 and Lemma 2.6, we obtain the following result that justifies the name of this section:

Corollary 4.2. *If Y_τ is a coordinate hyperplane (equivalently, the cardinal of τ is $n-1$) and $s = |A_\sigma^{-1}a_{\bar\tau}| > 1$ then s is an analytic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point in the closure of $Y_\tau \cap U_\sigma$.*

Remark 4.3. *By Theorem 3.14 we only need Lemma 2.6 for the proof of Corollary 4.2 if β is not very generic.*

Remark 4.4. *Observe that $\mathbf{0}$ is in the closure of $Y_\tau \cap U_\sigma$. However, by Remark 2.7 we have that s is a slope along Y_τ at any point of Y_τ.*

Remark 4.5. *Corollary 4.2 uses that Y_τ has codimension one because the analytic slopes along Y_τ are not defined if the codimension is greater. However, a good definition for the analytic slopes of a holonomic $\mathcal{D}$-module along a variety Z of codimension greater than 1 should include the indices of the Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Z. One reason is that $\{\mathcal{H}^0(\mathrm{Irr}_Z^{(s)}(\mathcal{M}))\}_{s\geq 1}$ determines a Gevrey filtration of $\mathcal{H}^0(\mathrm{Irr}_Z(\mathcal{M}))$, although $\{\mathrm{Irr}_Z^{(s)}(\mathcal{M})\}_{s\geq 1}$ does not determine in general a filtration of $\mathrm{Irr}_Z(\mathcal{M})$.*

Let us proceed with the construction of the announced series and the proof of Proposition 4.1.

We identify $\mathbf{k} + \mathbf{m} \in \mathbb{N}^{n-d}$ with $v^{\mathbf{k}+\mathbf{m}} = (A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k} + \mathbf{m})), \mathbf{k} + \mathbf{m}) \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ and establish a partition of $\Lambda_{\mathbf{k}}$ in terms of the negative support of the vector $v^{\mathbf{k}+\mathbf{m}} \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ as follows. For any subset $\eta \subseteq \sigma$ set:

$$\Lambda_{\mathbf{k},\eta} := \{\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k}} : \ \mathrm{nsupp}(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k} + \mathbf{m}))) = \eta\}.$$

Consider the set

$$\Omega_{\mathbf{k}} := \{\eta \subseteq \sigma : \ \Lambda_{\mathbf{k},\eta} \neq \emptyset\}.$$

Then it is clear that $\{\Lambda_{\mathbf{k},\eta} : \eta \in \Omega_{\mathbf{k}}\}$ is a partition of $\Lambda_{\mathbf{k}}$. Moreover $\Lambda_{\mathbf{k},\eta}$ is the intersection of a polytope with $\Lambda_{\mathbf{k}}$ because the conditions

$$\mathrm{nsupp}(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k} + \mathbf{m}))) = \eta$$

are equivalent to inequalities of type:

$$(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k} + \mathbf{m})))_i < 0$$

for $i \in \eta$ and

$$(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k} + \mathbf{m})))_j \geq 0$$

for $j \notin \eta$ such that $(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k}))_j \in \mathbb{Z}$.

For any $\eta \in \Omega_{\mathbf{k}}$ the series $\phi_{v^{\mathbf{k}+\mathbf{m}}}$ for $\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ depends on $\Lambda_{\mathbf{k},\eta}$ but not on $\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ up to multiplication by nonzero scalars. Let us fix any $\widetilde{\mathbf{k}} \in \Lambda_{\mathbf{k},\eta}$ and define:

$$\phi_{\mathbf{k},\eta} := \phi_{v^{\widetilde{\mathbf{k}}}}.$$

Observe that the support of the series $\phi_{\mathbf{k},\eta}$ is:

$$\mathrm{supp}(\phi_{\mathbf{k},\eta}) = \{v^{\mathbf{k}+\mathbf{m}} : \ \mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k},\eta}\}.$$

If the set $\Omega_{\mathbf{k}}$ has only one element η then $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k},\eta}$ and the series $\phi_\sigma^{\mathbf{k}} = \phi_{\mathbf{k},\eta}$ is a nonzero Gevrey series solution of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{X|Y_\tau}(s) \setminus \mathcal{O}_{X|Y_\tau}(< s)$ at any point of $Y_\tau \cap U_\sigma$ (see Theorem 3.14).

All the series in the finite set $\{\phi_{\mathbf{k},\eta} : \mathbf{k} \in \mathbb{N}^{n-d}, \eta \in \Omega_{\mathbf{k}}\}$ are Gevrey series along $Y_\sigma = \{x_i = 0 : i \notin \sigma\}$ with multi-order $\mathbf{s}$ at points of $Y_\sigma \cap \{x_j \neq 0 : j \in \sigma\}$. In fact, these series are Gevrey of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ and they are all annihilated by the Euler operators.

For all $\eta \in \Omega_{\mathbf{k}}$, the support of the series $\phi_{\mathbf{k},\eta}$ is $\mathrm{supp}(\phi_{\mathbf{k},\eta}) = \{v^{\mathbf{k}+\mathbf{m}} : \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}\}$ and $\cup_{\eta \in \Omega_{\mathbf{k}}} \Lambda_{\mathbf{k},\eta} = \Lambda_{\mathbf{k}}$. Then there exists $\eta \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\eta} \in \mathcal{O}_{X|Y_\tau}(s)$ has Gevrey index s. But a series ϕ_v is annihilated by I_A if and only if v has minimal negative support (see [24], Section 3.4.) so if we take $\eta' \in \Omega_{\mathbf{k}}$ with minimal cardinal then $\phi_{\mathbf{k},\eta'} \in \mathcal{O}_{X|Y_\tau}(s)$ is a solution of $\mathcal{M}_A(\beta)$. In general, we cannot take $\eta = \eta'$.

The following Lemma is the key of the proof of Proposition 4.1.

Lemma 4.6. *Consider an element η of the set*

$$\{\eta' \in \Omega_{\mathbf{k}} : \phi_{\mathbf{k},\eta'} \text{ has Gevrey index } s\}$$

with minimal cardinal. Then $\square_u(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(< s)$ for all $u \in L_A$.

Proof. Consider $\Lambda_{\mathbf{k},\eta}$ with η as above and $u \in L_A$. Then there exists $\widetilde{\mathbf{m}} \in \mathbb{Z}^{n-d}$ such that $u = (-A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}}, \widetilde{\mathbf{m}})$ and then

$$\square_u = \partial_\sigma^{(A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}})_-} \partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_+} - \partial_\sigma^{(A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}})_+} \partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_-}.$$

On the other hand, the series $\phi_{\mathbf{k},\eta}$ has the form:

$$\phi_{\mathbf{k},\eta} = \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}} c_{\mathbf{k}+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}$$

where $c_{\mathbf{k}+\mathbf{m}} \in \mathbb{C}$ verifies that $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$ is a rational function on $\mathbf{m}$ (recall that there exists $\widetilde{\mathbf{k}} \in \Lambda_{k,\eta}$ such that $c_{\mathbf{k}+\mathbf{m}} = \frac{[v^{\widetilde{\mathbf{k}}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_-}}{[v^{\mathbf{k}+\mathbf{m}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_+}}$ by definition of $\phi_{\mathbf{k},\eta}$).

A monomial $x^{v^{\mathbf{k}+\mathbf{m}}-u_-} = x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+}$ appearing in $\square_u(\phi_{\mathbf{k},\eta})$ comes from the monomials $x^{v^{\mathbf{k}+\mathbf{m}}}$ and $x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}}$ after one applies ∂^{u_-} and ∂^{u_+} respectively.

If $\mathbf{k}+\mathbf{m}, \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ then the monomial $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ appears in $\partial^{u_-}(\phi_{\mathbf{k},\eta})$ and $\partial^{u_+}(\phi_{\mathbf{k},\eta})$ with the same coefficients so it doesn't appear in the difference.

If $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \notin \Lambda_{\mathbf{k},\eta}$ (the case $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ is analogous), we can distinguish two cases:

1) There exists i such that $v_i^{\mathbf{k}+\mathbf{m}} \in \mathbb{N}$ but $v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} < 0$ so $u_i = v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - v_i^{\mathbf{k}+\mathbf{m}} < 0$. Then $\partial^{u_-}(x^{v^{\mathbf{k}+\mathbf{m}}}) = 0$ and $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ does not appear in $\square_u(\phi_{\mathbf{k},\eta})$.

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2) We have $\mathrm{nsupp}(v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}) = \varsigma \subsetneq \mathrm{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \eta$. Then $[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$ and the coefficient of $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ in $\square_u(\phi_{\mathbf{k},\eta})$ is $c_{\mathbf{k}+\mathbf{m}}[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$. Furthermore, $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}$ with $\varsigma \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\varsigma}$ is Gevrey of index $s' < s$ because we chose η that way.

By 1), 2) and the analogous cases when $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$, we have:

$$\square_u(\phi_{\mathbf{k},\eta}) = \sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}]_{u_+} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+} -$$

$$- \sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}}[v^{\mathbf{k}+\mathbf{m}}]_{u_-} x^{v^{\mathbf{k}+\mathbf{m}}-u_-} \tag{9}$$

Here, $\varsigma, \varsigma' \subseteq \eta$ varies in a subset of the finite set $\Omega_{\mathbf{k}}$ whose elements ς'' verify that the series $\phi_{\mathbf{k},\varsigma''}$ has Gevrey index $s'' < s$. Let us denote by $\widetilde{s} < s$ the maximum of these s''.

Since $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$, $[v^{\mathbf{k}+\mathbf{m}}]_{u_-}$ and $[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}]_{u_+}$ are rational functions on $\mathbf{m}$ the series $\square_u(\phi_{\mathbf{k},\eta})$ has Gevrey index at most the maximum of the Gevrey index of the series

$$\sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+}$$

$$\sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$$

which is at most $\widetilde{s} < s$.

It follows that $I_A(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(< s)$ while $\phi_{\mathbf{k},\eta}$ has Gevrey index s. $\square$

Moreover the classes of the series $\{\phi_{\mathbf{k},\eta_{\mathbf{k}}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ (with $\eta_{\mathbf{k}} \in \Omega_{\mathbf{k}}$ chosen as η in Lemma 4.6) in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s))_p$, $p \in Y_\tau \cap U_\sigma$, are linearly independent since the support of $\phi_{\mathbf{k},\eta}$ restricted to the variables x_i with $i \notin \sigma$ is $\Lambda_{\mathbf{k},\eta_{\mathbf{k}}} \subseteq \Lambda_{\mathbf{k}}$ and $\{\Lambda_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ is a partition of $\mathbb{N}^{n-d}$. This finishes the proof of Proposition 4.1.

5 Slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes

In this section we will describe all the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes. First, we recall here the definition of (A, L)-umbrella [25], but we will slightly modify the notation in [25] for technical reasons. Consider any full rank matrix $A = (a_1 \ \cdots \ a_n) \in \mathbb{Z}^{d\times n}$ and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n_{>0}$.

Definition 5.1. *Set $a_j^{\mathbf{s}} := a_j/s_j$, $j = 1, \ldots, n$, and let*

$$\Delta_A^{\mathbf{s}} := \mathrm{conv}(\{a_i^{\mathbf{s}} : i = 1, \ldots, n\} \cup \{\mathbf{0}\})$$

be the so-called $(A, \mathbf{s})$-polyhedron.

The $(A, \mathbf{s})$-umbrella is the set $\Phi_A^{\mathbf{s}}$ of faces of $\Delta_A^{\mathbf{s}}$ which do not contain the origin. $\Phi_A^{\mathbf{s},q} \subseteq \Phi_A^{\mathbf{s}}$ denotes the subset of faces of dimension q for $q = 0, \ldots, d-1$.

The following statement is [25, Lemma 2.13]. The difference here is that we do not assume that A is pointed but we just consider $\mathbf{s} \in \mathbb{R}^n$ such that $s_i > 0$ for all $i = 1, \ldots, n$. For this reason we slightly modify a part of the proof of [25, Lemma 2.13].

Lemma 5.2. *Let $\widetilde{I}_A^{\mathbf{s}}$ be the ideal of $\mathbb{C}[\xi_1, \ldots, \xi_n]$ generated by the following elements:*

  i) *$\xi_{i_1} \cdots \xi_{i_r}$ where $a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r}$ do not lie in a common facet of $\Phi_A^{\mathbf{s}}$.*

  ii) *$\xi^{u_+} - \xi^{u_-}$ where $u \in \ker_{\mathbb{Z}} A$ and $\mathrm{supp}(u)$ is contained in a facet of $\Phi_A^{\mathbf{s}}$.*

Then $\widetilde{I}_A^{\mathbf{s}} = \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$.

Proof. The proofs of [25, Lemma 2.12] and [25, Lemma 2.13] use [25, Lemma 2.8] and [25, Lemma 2.10], but they do no use that A is pointed elsewhere. We rewrite here the proofs of [25, Lemma 2.8] and [25, Lemma 2.10] without the pointed assumption on A but for $\mathbf{s} \in \mathbb{R}^n_{>0}$. Let us prove in particular that $\mathrm{in}_{\mathbf{s}}(I_A) \subseteq \widetilde{I}_A^{\mathbf{s}} \subseteq \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$.

For the proof of the inclusion $\mathrm{in}_{\mathbf{s}}(I_A) \subseteq \widetilde{I}_A^{\mathbf{s}}$ we only need to prove that $\forall u \in \mathbb{Z}^n$ with $Au = 0$ then $\mathrm{in}_{\mathbf{s}}(\square_u) \in \widetilde{I}_A^{\mathbf{s}}$.

If $\mathrm{supp}(u) \subseteq \tau \in \Phi_A^{\mathbf{s}}$ then $\exists h_\tau \in \mathbb{Q}^d$ such that $\langle h_\tau, a_i/s_i \rangle = 1$, $\forall i \in \tau$, i.e., $\langle h_\tau, a_i \rangle = s_i$, $\forall i \in \tau$. Since $Au = 0$ and $\mathrm{supp}(u) \subseteq \tau$ we have

$$0 = \langle h_\tau, Au \rangle = \langle h_\tau A, u \rangle = \sum_{i \in \tau} s_i u_i = \sum_{i=1}^n s_i u_i$$

so $\mathrm{in}_{\mathbf{s}}(\square_u) = \xi^{u_+} - \xi^{u_-}$ which lies in $\widetilde{I}_A^{\mathbf{s}}$ by definition.

Assume there exists $\tau \in \Phi_A^{\mathbf{s}}$ such that $\mathrm{supp}(u_+) \subseteq \tau$ and $\mathrm{supp}(u_-) \subsetneq \tau'$ for any $\tau' \in \Phi_A^{\mathbf{s}}$. $h_\tau(a_i) = s_i \ \forall i \in \tau$ but $h_\tau(a_j) < s_i \ \forall i \notin \tau$. Since $Au = 0$ then $Au_+ = Au_-$ and by the assumption

$$\sum_{i=1}^n s_i(u_+)_i = \langle h_\tau A, u_+ \rangle = \langle h_\tau, Au_+ \rangle = \langle h_\tau, Au_- \rangle < \sum_{i=1}^n s_i(u_-)_i$$

so $\operatorname{in}_\mathbf{s}(\square_u) = \xi^{u_-}$ is a multiple of $\displaystyle\prod_{j\in\operatorname{supp}(u_-)} \xi_j$ which is an element of the type of

$i)$ by assumption.

The case $\operatorname{supp}(u_-) \subseteq \tau$ and $\operatorname{supp}(u_-) \subsetneq \tau'$ for any $\tau' \in \Phi^\mathbf{s}_A$ is analogous to the previous case. If there is no face containing $\operatorname{supp}(u_+)$ nor $\operatorname{supp}(u_-)$ it is trivial that $\xi^{u_+}, \xi^{u_+} \in \widetilde{I}^\mathbf{s}_A$ and so $\xi^{u_+} - \xi^{u_+} \in \widetilde{I}^\mathbf{s}_A$. Finally, since $Au_+ = Au_-$ it is not possible that $\varsigma = \operatorname{supp}(u_+) \subseteq \tau$ and $\varsigma' = \operatorname{supp}(u_-) \subseteq \tau'$ for any $\tau, \tau' \in \Phi^{\mathbf{s},d-1}_A$ such that $\tau \neq \tau'$ (because this implies that $\operatorname{pos}(\varsigma) \cap \operatorname{pos}(\varsigma') = \{\mathbf{0}\}$).

Let us prove the inclusion $\widetilde{I}^\mathbf{s}_A \subseteq \sqrt{\operatorname{in}_\mathbf{s}(I_A)}$. It is clear that the elements of type $ii)$ lies in $\operatorname{in}_\mathbf{s}(I_A) \subseteq \sqrt{\operatorname{in}_\mathbf{s}(I_A)}$ so we only need to proof that the elements of type $i)$ belong to $\sqrt{\operatorname{in}_\mathbf{s}(I_A)}$:

If $a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r}$ do not lie in a common facet of $\Phi^\mathbf{s}_A$ then $\exists \mathbf{a}$ such that:

(1) $\mathbf{a} \in \operatorname{conv}(a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r})$.

(2) $\mathbf{a}$ lies in the interior of $\Delta^\mathbf{s}_A$.

By (1) we can write:

$$\mathbf{a} = \sum_{j=1}^{r} \epsilon_j a_{i_j}/s_{i_j}$$

with $\sum_j \epsilon_j = 1$ and $\epsilon_j \geq 0, \forall j$.

And by (2) there exists $t > 1$ such that $t\mathbf{a}$ still belongs to $\Delta^\mathbf{s}_A$ and we can write:

$$t\mathbf{a} = \sum_{i=1}^{n} \eta_i a_i/s_i$$

with $\sum_j \eta_j = 1$ and $\eta_j \geq 0, \forall j$.

Finally we put together both equalities and get:

$$\sum_{i=1}^{r}(t\epsilon_j)a_{i_j}/s_{i_j} = \sum_{i=1}^{n} \eta_i a_i/s_i.$$

Then there exists $\lambda \in \mathbb{N}^*$ such that

$$P = \prod_{j=1}^{r} \partial_{i_j}^{\lambda t \epsilon_j / s_j} - \prod_{j=1}^{n} \partial_j^{\lambda \eta_j / s_j} \in I_A$$

and the $\mathbf{s}$-degree of the first monomial is λt while the $\mathbf{s}$-degree of the second monomial is λ so $\operatorname{in}_\mathbf{s}(P) = \prod_{j=1}^{r} \partial_{i_j}^{\lambda t \epsilon_j / s_j} \in \operatorname{in}_\mathbf{s}(I_A)$. This implies that $\xi_{i_1} \cdots \xi_{i_r} \in \sqrt{\operatorname{in}_\mathbf{s}(I_A)}$.

$\square$

Let $\tau \subseteq \{1, \ldots, n\}$ be a set with cardinal $l \geq 0$ and consider the coordinate subspace $Y_\tau = \{x_i = 0 : i \notin \tau\}$ with dimension l.

The special filtration
$$L_s := F + (s-1)V_\tau$$
with $s \geq 1$ is an intermediate filtration between the filtration F by the order of the differential operators and the Malgrange-Kashiwara filtration with respect to Y_τ that we denote by V_τ. Recall that V_τ is associated with the weights -1 for the variables $x_{\overline{\tau}}$, 1 for $\partial_{\overline{\tau}}$ and 0 for the rest of the variables.

We shall identify $s \in \mathbb{R}_{>0}$ with $(s_1, \ldots, s_n)$ throughout this section, where $s_i = 1$ if $i \in \tau$ and $s_i = s$ if $i \notin \tau$. Then $(L_s)_{n+j} = s_j$ for all $j = 1, \ldots, n$.

Lemma 5.3. *Assume $s > 1$ is such that $\Phi_A^s = \Phi_A^{s+\epsilon} = \Phi_A^{s-\epsilon}$ for sufficiently small $\epsilon > 0$. Then the ideal $\widetilde{I}_A^s$ is homogeneous with respect to F and V_τ. In particular $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\underline{\xi} \rangle)$ is a bi-homogeneous variety in $\mathbb{C}^{2n}$.*

Proof. We only need to prove that the elements in Lemma 5.2, ii), are bi-homogeneous. From the proof of Lemma 5.2 we deduce that they are L_s-homogeneous. By assumption we have that they are also $(L_s \pm \epsilon V_\tau)$-homogeneous for all $\epsilon > 0$ small enough. Since $L_s \pm \epsilon V_\tau = F + (s \pm \epsilon - 1)V_\tau$ we obtain that they are F-homogeneous and V_τ-homogeneous. $\square$

Lemma 5.4. $\dim_{\mathbb{C}}(\mathcal{V}(\mathrm{in}_{L_s}(I_A)) \cap \mathcal{V}(A\mathbf{x}\xi)) \leq n$.

Proof. Let $\omega \in \mathbb{R}_{>0}^n$ be a generic weight vector such that $\mathrm{in}_\omega(\mathrm{in}_{L_s}(I_A))$ is a monomial ideal. For $\epsilon > 0$ small enough $\mathrm{in}_\omega(\mathrm{in}_{L_s}(I_A)) = \mathrm{in}_{\widetilde{\omega}}(I_A)$ for $\widetilde{\omega} = s + \epsilon\omega \in \mathbb{R}_{>0}^n$.

Choose any monomial order $<$ in $\mathbb{C}[x, \xi]$ that refines the partial order given by $(u, v) := (1 - \epsilon\omega_1, \ldots, 1 - \epsilon\omega_n; \epsilon\omega_1, \ldots, \epsilon\omega_n) \in \mathbb{R}_{>0}^{2n}$. It is clear that $\mathrm{in}_{(u,v)}(Ax\xi)_i = (Ax\xi)_i$ for all $i = 1, \ldots, d$ and that $\mathrm{in}_{(u,v)}(\mathrm{in}_{L_s}(I_A)) = \mathrm{in}_{\widetilde{\omega}}(I_A)$. Then
$$\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle Ax\xi \rangle \subseteq \mathrm{in}_{(u,v)}(\mathrm{in}_{L_s}(I_A) + \langle Ax\xi \rangle)$$
and so we have that:

$$E_<(\mathrm{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi \rangle) = E_<(\mathrm{in}_{(u,v)}(\mathrm{in}_{L_s}(I_A) + \langle Ax\xi \rangle)) \supset E_<(\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle) \tag{10}$$

where $E_<(I) := \{(\alpha, \gamma) \in \mathbb{N}^{2n} : \mathrm{in}_<(P) = c_{\alpha,\gamma}x^\alpha\xi^\gamma, \ P \in I \setminus \{0\}\}$ for any ideal $I \subseteq \mathbb{C}[x, \xi]$. The inclusion (10) implies that the Krull dimension of the residue ring $\mathbb{C}[x, \xi]/(\mathrm{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi \rangle)$ is at most the one of $\mathbb{C}[x, \xi]/(\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle)$.

Then it is enough to prove that $\mathbb{C}[x, \xi]/(\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle)$ has Krull dimension n. Since $M = \mathrm{in}_{\widetilde{\omega}}(I_A)$ is a monomial ideal then:

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$$\mathrm{in}_{\widetilde{\omega}}(I_A) = \cap_{(\partial^b,\sigma)\in\mathcal{S}(M)}\langle \xi_j^{b_j+1} \ : \ j \notin \sigma\rangle$$

where $S(M)$ denotes the set of standard pairs of M (see [24, Section 3.2]). This implies that

$$\mathcal{V}(\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi\rangle) = \cup_{(\partial^b,\sigma)\in\mathcal{S}(M)}\mathcal{V}(\langle \xi_j \ : \ j \notin \sigma\rangle + \langle Ax\xi\rangle).$$

By [24, Corollary 3.2.9.], the columns of A indexed by σ are linearly independent when $(\partial^b,\sigma) \in \mathcal{S}(M)$, so the dimension of each component

$$\mathcal{V}(\langle \xi_j \ : \ j \notin \sigma\rangle + \langle Ax\xi\rangle) = \mathcal{V}(\langle \xi_j \ : \ j \notin \sigma\rangle + \langle x_j\xi_j \ : \ j \in \sigma\rangle)$$

is n. $\qquad\square$

Lemma 5.5. *Under the assumptions of lemma 5.3 we have that s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ.*

Proof. We know that:

$$\mathrm{Ch}^s(\mathcal{M}_A(\beta)) = \mathcal{V}(\sqrt{\mathrm{in}_{L_s}(H_A(\beta))}) \subseteq \mathcal{V}(\sqrt{\mathrm{in}_{L_s}(I_A)}) \cap \mathcal{V}(A\mathbf{x}\xi) = \mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle).$$

Hence the s-characteristic variety of $\mathcal{M}_A(\beta)$ is contained in a bi-homogeneous variety of dimension at most n when the assumptions in Lemma 5.3 are satisfied. Since $\mathrm{Ch}^s(\mathcal{M}_A(\beta))$ is known to be purely n-dimensional, each irreducible component is an irreducible component of $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle)$ and so it is also bi-homogeneous. Moreover, this is true not only at the origin $x = 0 \in \mathbb{R}^n$ but also at any point of Y_τ because $(L_s)_i = 0$ for $i \in \tau$ and $Y_\tau = \{x_i = 0 : i \notin \tau\}$. Then s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ. $\qquad\square$

Remark 5.6. *Observe that after the proof of Lemma 5.5 we have the equality in Lemma 5.4.*

Remark 5.7. *A consequence of Lemma 5.5 is that $\mathcal{M}_A(\beta)$ has no algebraic slopes along $\mathbf{0} \in \mathbb{R}^n$ at $\mathbf{0}$.*

Example 5.8. *Let $A = (a_1 \ a_2 \ a_3 \ a_4)$ be the non-pointed matrix with columns*

$$a_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ a_3 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \ a_4 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and consider the associated hypergeometric system:

$$H_A(\beta) = I_A + \langle x_1\partial_1 - 3x_3\partial_3 + 2x_4\partial_4 - \beta_1, -x_1\partial_1 + x_2\partial_2 - 2x_3\partial_3 + 2x_4\partial_4 - \beta_2\rangle$$

where $I_A = \langle \partial_1\partial_2\partial_3\partial_4 - 1, \partial_1\partial_2^3 - \partial_3\partial_4^2, \partial_3^2\partial_4^3 - \partial_2^2\rangle$ and $\beta_1, \beta_2 \in \mathbb{C}$.
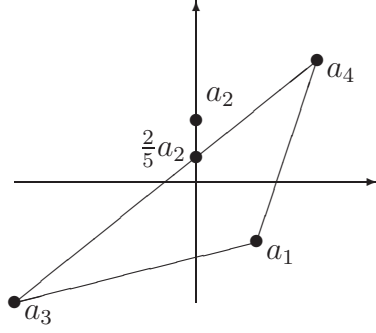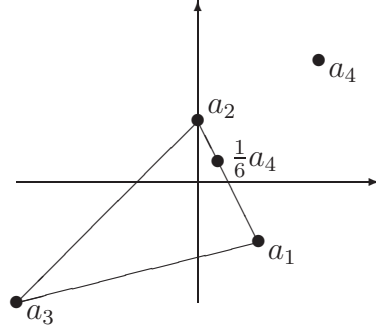
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Figure 2   Figure 3

From Lemma 5.5 we deduce that there is not any algebraic slope along a co-ordinate subspace different from $Y = \{x_2 = 0\}$ and $Z = \{x_4 = 0\}$. By Corollary 4.2 and using again Lemma 5.5 we know that the unique slope of $\mathcal{M}_A(\beta)$ along Y is $|A_\sigma^{-1}a_2| = 5/2$ with $\sigma = \{3,4\}$ and that the unique slope of $\mathcal{M}_A(\beta)$ along Z is $|A_{\overline{\sigma}}^{-1}a_4| = 6$ with $\overline{\sigma} = \{1,2\}$. Notice that $2a_2/5$ lies in the affine line pass-ing through a_3 and a_4 (see Figure 1) and that $a_4/6$ lies in the affine line passing through a_1 and a_2 (see Figure 2). We also can construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = 2$ Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y (it is analogous for Z) as follows.

The matrix B_σ is

$$B_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ 5/2 & -3/2 \end{pmatrix}$$

and we consider the vectors $v^1 = (0,0,A_\sigma^{-1}\beta) = (0,0,-\beta_1+\beta_2,-\beta_1+\frac{3}{2}\beta_2)$ and $v^2 = (0,1,A_\sigma^{-1}(\beta-a_2)) = (0,1,-\beta_1+\beta_2-1,-\beta_1+\frac{3}{2}(\beta_2-1))$.

If none of $\beta_1 - \beta_2, -\beta_1 + \frac{3}{2}\beta_2$ and $-\beta_1 + \frac{3}{2}(\beta_2 - 1)$ are integers then the series ϕ_{v^1} and ϕ_{v^2} are Gevrey series solutions along Y of $\mathcal{M}_A(\beta)$ with index $5/2$ at any point of $Y \cap \{x_1 x_2 \neq 0\}$. In other case, we can replace the vectors v^i by $v^{i,k} := v^i + k(0,1,-1,-3/2)$ with $k \in \mathbb{N} \setminus 2\mathbb{N}$ big enough in order to obtain Gevrey solutions $\phi_{v^{i,k}}$ of $\mathcal{M}_A(\beta)$ modulo convergent series at any point of $Y \cap \{x_1 x_2 \neq 0\}$ with index $5/2$.

Denote for $s > 1$:

$$\Omega_{Y_\tau}^{(s)} = \{\sigma \subseteq \tau : \ \det(A_\sigma) \neq 0, \max\{|A_\sigma^{-1}a_i| : \ i \notin \tau\} = s, |A_\sigma^{-1}a_j| \leq 1, \forall j \in \tau\}.$$

Then we have the following result.

Lemma 5.9. *If $\sigma \in \Omega_{Y_\tau}^{(s_0)} \neq \emptyset$ then for all $p \in Y_\tau \cap U_\sigma$:*

1) *s_0 is the Gevrey index of a solution of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}$ for very generic parameters $\beta \in \mathbb{C}^d$.*

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2) s_0 is the Gevrey index of a solution of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}/\mathcal{O}_{X|Y_\tau(<s_0),p}$ for all $\beta \in \mathbb{C}^d$.

3) If Y_τ is a hyperplane, then s_0 is the Gevrey index of a solution of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}/\mathcal{O}_{X|Y_\tau,p}$ for all $\beta \in \mathbb{C}^d$.

Proof. We consider any $\sigma \in \Omega_{Y_\tau}^{(s_0)}$. If β is very generic the Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ associated with σ, $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ (see Section 3), have Gevrey index $s_0 = \max\{|A_\sigma^{-1}a_i| : i \in \tau\}$ along Y_τ at $p \in Y_\tau \cap U_\sigma$. If β is not very generic we can proceed as in Section 4 in order to construct a Gevrey series associated with σ with index s_0 which is a solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y}(s_0)/\mathcal{O}_{X|Y}(<s_0))_p$ for all $p \in Y \cap U_\sigma$. By a similar argument to the one in the proof of Lemma 2.6 the result is obtained. $\qquad\square$

Assume that Y is a coordinate hyperplane for the remainder of this section. We can reorder the variables so that $Y = \{x_n = 0\}$.

In the following result the equivalence of 3) and 4) is a particular case of the Comparison Theorem of the slopes [16]. However, we just need to use this theorem for the implication 3) $\Longrightarrow$ 4).

Theorem 5.10. *For all $p \in Y$ the following statements are equivalent:*

1) Φ_A^s *jumps at* $s = s_0$.

2) $\Omega_Y^{(s_0)} \neq \emptyset$.

3) s_0 *is an analytic slope of* $\mathcal{M}_A(\beta)$ *along* Y *at* p.

4) s_0 *is an algebraic slope of* $\mathcal{M}_A(\beta)$ *along* Y *at* p.

Proof. We will prove first the equivalence of 1) and 2). Assume there exists $\sigma \in \Omega_Y^{(s_0)} \neq \emptyset$, then $H_\sigma = \{y \in \mathbb{R}^d : |A_\sigma^{-1}y| = 1\}$ is the only hyperplane containing a_i for all $i \in \sigma$ and $|A_\sigma^{-1}(a_n/(s_0+\epsilon))| = s_0/(s_0+\epsilon) < 1, \forall \epsilon > 0$. Hence $a_n/s_0 \in H_\sigma$ but $a_n/(s_0 + \epsilon) \notin H_\sigma, \forall \epsilon > 0$.

Consider $\eta = \{i : a_i \in H_\sigma\}$, then $\eta \in \Phi_A^{s_0+\epsilon,d-1}, \forall \epsilon > 0$ and $n \notin \eta$ while $\eta \cup \{n\} \in \Phi_A^{s_0,d-1}$, so Φ_A^s jumps at $s = s_0$.

Conversely if $\Omega_Y^{(s_0)} = \emptyset$ then $\forall \sigma \subseteq \{1, 2, \ldots, n-1\}$ such that $|A_\sigma^{-1}a_i| \leq 1$ for all $i = 1, \ldots, n-1$ we have $|A_\sigma^{-1}a_n| < s_0$ or $|A_\sigma^{-1}a_n| > s_0$.

Consider $\epsilon > 0$ small enough such that $|A_\sigma^{-1}a_n| < s_0 \pm \epsilon$ if $|A_\sigma^{-1}a_n| < s_0$ and $|A_\sigma^{-1}a_n| > s_0 \pm \epsilon$ if $|A_\sigma^{-1}a_n| > s_0$ for all simplices σ such that $|A_\sigma^{-1}a_i| \leq 1$ for all $i = 1, \ldots, n-1$.

Let us prove that $\Phi_A^{s_0,d-1} = \Phi_A^{s_0 \pm \epsilon, d-1}$.

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Assume first that $n \notin \eta \subseteq \{1, \ldots, n\}$. Then:

$\eta \in \Phi_A^{s_0, d-1} \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \pm \epsilon \iff \eta \in \Phi_A^{s_0 \pm \epsilon, d-1}$.

If $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\mathrm{conv}(\eta \setminus \{n\})) = d-1$ then there exists a simplex $\sigma \subseteq \eta \setminus \{n\}$ such that $\det(A_\sigma) \neq 0$. Then $\eta \notin \Phi_A^{s_0, d-1}$ because in such a case $|A_\sigma^{-1} a_i| \leq 1$ for all $i \neq n$, $|A_\sigma^{-1} a_n| = s_0$ and so $\sigma \in \Omega_Y^{(s_0)}$, a contradiction. Moreover $\eta \notin \Phi_A^{s_0 \pm \epsilon, d-1}$ for $\epsilon > 0$ small enough because $|A_\sigma^{-1} a_n|$ is a fixed value while $s_0 \pm \epsilon$ varies with ϵ.

Finally, if $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\mathrm{conv}(\eta \setminus \{n\})) < d-1$ then there exists a hyperplane $H' = \{\mathbf{y} \in \mathbb{R}^d : h'(\mathbf{y}) = 0\}$ that contains $\mathbf{0} \in \mathbb{R}^d$ and a_i for all $i \in \eta \setminus \{n\}$. We also can choose the linear function h' in the definition of H' such that $h'(a_n) = 1$. In this case:

$\eta \in \Phi_A^{s_0, d-1} \iff \eta \setminus \{n\} \in \Phi_A^{s_0, d-2}$ and $\exists H'' = \{\mathbf{y} \in \mathbb{R}^d : h''(\mathbf{y}) = 1\}$ such that $h''(a_i) = 1$ for $i \in \eta \setminus \{n\}$, $h''(a_n) = s_0$ and $h''(a_j) < 1$ for $j \notin \eta$. This imply for $h := h'' \pm \epsilon h'$ that $h(a_i) = 1$ for all $i \in \eta \setminus \{n\}$, $h(a_n) = s_0 \pm \epsilon$ and $h(a_j) = h''(a_j) \pm \epsilon h'(a_j) < 1$ for $j \notin \eta$ and $\epsilon > 0$ small enough because $h''(a_j) < 1$ for $j \notin \eta$. Hence $\eta \in \Phi_A^{s_0 \pm \epsilon, d-1}$.

We have proved that $\Phi_A^{s_0, d-1} \subseteq \Phi_A^{s_0 \pm \epsilon, d-1}$. This implies equality since they are (A, s)-umbrellas of the same matrix A and $s > 0$ for $s = s_0$ and $s = s_0 \pm \epsilon$ (in particular $\cup_{\eta \in \Phi_A^{s, d-1}} \mathrm{pos}(\eta) = \mathrm{pos}(A)$ do not depends on $s > 0$). Moreover, the (A, s)-umbrellas are determined by their facets, so $\Phi_A^{s_0} = \Phi_A^{s_0 \pm \epsilon}$.

The implication 2) $\implies$ 3) is a direct consequence of Lemma 5.9 if p belongs to the closure of $Y \cap U_\sigma$ for some $\sigma \in \Omega_Y^{(s_0)}$ (for example, if $p = 0$). Nevertheless, since the analytic slopes are found in relatively open subsets of the hyperplane Y we can use the constructibility of the slopes in order to prove the result at any point of Y (see Remark 2.7). For the implication 3) $\implies$ 4) we use the Comparison Theorem of the slopes [16]. Finally, the implication 4) $\implies$ 1) is nothing but Lemma 5.5. $\qquad\square$

Remark 5.11. *Notice that if Y is a coordinate hyperplane then every algebraic slope s_0 of $\mathcal{M}_A(\beta)$ along Y is the Gevrey index of certain Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y modulo convergent series. Example 5.12 shows that this is not true for coordinate subspaces of codimension greater than one.*

Example 5.12. *Let $\mathcal{M}_A(\beta)$ be the hypergeometric $\mathcal{D}$-module associated with the matrix*

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

and the parameter vector $\beta \in \mathbb{C}^2$. In this case $n = 3 = d + 1$ and so the toric ideal is principal $I_A = \langle \partial_1^3 - \partial_2 \partial_3 \rangle$.
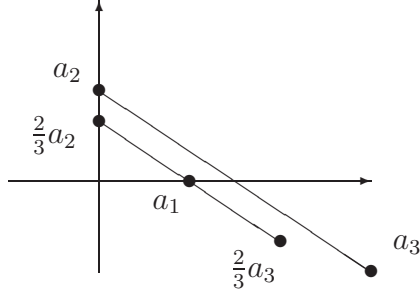

Figure 4

If we take $Y = \{x_2 = x_3 = 0\}$ then the only algebraic slope of $\mathcal{M}_A(\beta)$ along Y at $p \in Y$ is $s_0 = 3/2$ (observe Figure 4 and see [25] since A is pointed). Nevertheless, we will prove that if $\beta_2 \notin \mathbb{Z}$ then for all $s \geq 1$, $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)) = 0$:

For any formal series $f = \sum_{m \in \mathbb{N}^2} f_m(x_1) x_2^{m_2} x_3^{m_3}$ along Y at $p = (p_1, 0, 0) \in Y$ then

$$(E_2 - \beta_2)(f) = \sum_{m \in \mathbb{N}^2} (m_2 - m_3 - \beta_2) f_m(x_1) x_2^{m_2} x_3^{m_3}$$

and hence $(E_2 - \beta_2)(f) \in \mathcal{O}_{X,p}$ (resp. $(E_2 - \beta_2)(f) = 0$) if and only if $f \in \mathcal{O}_{X,p}$ (resp. $f = 0$) because $(m_2 - m_3 - \beta_2) \neq 0$, $\forall m_2, m_3 \in \mathbb{N}$.

On the other hand, if $\beta_2 \in \mathbb{Z}$ we can take $k \in \mathbb{N}$ the minimum natural number such that $v = (\beta_1 - 3k, \beta_2 + k, k) \in \mathbb{C} \times \mathbb{N}^2$ has minimal negative support. Since $Av = \beta$ then

$$\phi_v = \sum_{m \geq 0} \frac{k! [\beta_1 - 3k]_{3m}}{(k+m)! [\beta_2 + k + m]_m} x_1^{\beta_1 - 3(k+m)} x_2^{\beta_2 + k + m} x_3^{k+m}$$

is a formal solution of $\mathcal{M}_A(\beta)$ along Y at any point $p \in Y$ with $p_1 \neq 0$. In fact ϕ_v has Gevrey index $s_0 = 3/2$ if $\beta_1 - 3k \notin \mathbb{N}$ and it is a polynomial when $\beta_1 - 3k \in \mathbb{N}$. In this last case, if we consider $v' = v + k'u$ with $u = (-3, 1, 1) \in L_A$ and $k' \in \mathbb{N}$ such that $v_1' < 0$ then $\phi_{v'}$ is a Gevrey series of index s_0 and $P(\phi_{v'})$ is convergent along Y at any point $p \in Y \setminus \{0\}$.

So the algebraic slope appears as the index of a Gevrey series solution along Y of $\mathcal{M}_A(\beta)$ if and only if $\beta_2 \in \mathbb{Z}$. Observe that "the special parameters" are not contained in a Zariski closed set but in a countable union of them. Note also

that I_A is Cohen-Macaulay and then it is known that the set of rank-jumping parameters is empty.

6   Regular Triangulations and $(A, \mathbf{s})$-umbrellas

The aims of this section are to compare the notion of $(A, \mathbf{s})$-umbrella in [25] with the one of regular triangulation of the matrix A (see for example [26]), to show that the common domain of definition of the constructed Gevrey series solutions $\phi_\sigma^{\mathbf{k}}$ is nonempty when σ varies in a regular triangulation and to prove the existence of convenient regular triangulations.

For any subset $\sigma \subseteq \{1, \ldots, n\}$ we will write $\mathrm{pos}(\sigma) = \sum_{i \in \sigma} \mathbb{R}_{\geq 0} a_i \subseteq \mathbb{R}^d$. Recall that we identify σ with $\{a_i : i \in \sigma\}$.

Definition 6.1. *A triangulation of A is a set* T *whose elements are subsets of columns of A verifying:*

1) $\{\mathrm{pos}(\sigma) : \sigma \in \mathrm{T}\}$ is a simplicial fan.

2) $\mathrm{pos}(A) = \cup_{\sigma \in \mathrm{T}} \mathrm{pos}(\sigma)$.

A vector $\omega \in \mathbb{R}^n$ defines a collection T_ω of subsets of columns of A as follows:

$\sigma \subseteq \{a_1, \ldots, a_n\}$ belongs to T_ω if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that

$$\langle \mathbf{c}, a_j \rangle = \omega_j \text{ for all } j \in \sigma$$

and

$$\langle \mathbf{c}, a_j \rangle < \omega_j \text{ for all } j \notin \sigma.$$

Remark 6.2. *We will say that $\omega \in \mathbb{R}^n$ is generic when the collection T_ω is a simplicial complex and a triangulation of A.*

Definition 6.3. *A triangulation* T *is said to be regular if there exists a generic $\omega \in \mathbb{R}^n$ such that* $\mathrm{T} = \mathrm{T}_\omega$.

Observe that the collection $\{\mathrm{pos}(\sigma) : \sigma \in \Phi_A^{\mathbf{s}}\}$ is a polyhedral fan. When $\mathbf{s} \in \mathbb{R}_{>0}^n$ is generic it is a simplicial fan and so $\Phi_A^{\mathbf{s}}$ is a triangulation of A. In fact it is a regular triangulation because for any $\mathbf{s} \in \mathbb{R}_{>0}^n$ we have that $\Phi_A^{\mathbf{s}} = \mathrm{T}_{\mathbf{s}}$:

$\sigma \in \Phi_A^{\mathbf{s}} \iff \exists \mathbf{c} \in \mathbb{R}^d | \langle \mathbf{c}, a_i/s_i \rangle = 1, \ \forall i \in \sigma, \text{ and } \langle \mathbf{c}, a_i/s_i \rangle < 1, \ \forall i \notin \sigma \iff \exists \mathbf{c} | \langle \mathbf{c}, a_i \rangle = s_i, \ \forall i \in \sigma \text{ and } \langle \mathbf{c}, a_i \rangle < s_i, \ \forall i \notin \sigma \iff \sigma \in \mathrm{T}_{\mathbf{s}}$.

Given a $(d-1)$-simplex $\sigma \in \mathrm{T}_\omega$ there exists $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbf{c}A_\sigma = \omega_\sigma$ and $\mathbf{c}A_{\bar\sigma} < \omega_{\bar\sigma}$. This is equivalent to:

$$\mathbf{c} = \omega_\sigma A_\sigma^{-1}, \ \omega_\sigma A_\sigma^{-1} A_{\bar\sigma} < \omega_{\bar\sigma}.$$

But this happens if and only if $\omega \in C(\sigma) := \{\omega \in \mathbb{R}^n : \omega B_\sigma > 0\}$ which is an open convex polyhedral rational cone of dimension n. Then we can write

$$C(\sigma) = \{\omega \in \mathbb{R}^n : \sigma \in T_\omega\}$$

and for any regular triangulation $T = T_{\omega_0}$ we have

$$\omega_0 \in C(T) := \bigcap_{\sigma \in T} C(\sigma).$$

Hence $C(T) = \{\omega \in \mathbb{R}^n : T_\omega = T\}$ is a nonempty open rational convex polyhedral cone. It is clear that $\cup_T \overline{C(T)} = \mathbb{R}^n$ where T runs over all regular triangulations of A and $\overline{C(T)}$ denotes the Euclidean closure of $C(T)$. More precisely, there exists a polyhedral fan with support $\mathbb{R}^n$ such that T_ω is constant for $\omega \in \mathbb{R}^n$ running in any relatively open cone of this polyhedral fan. We also can restrict this fan to $\mathbb{R}^n_{>0}$ and obtain that the $(A, \mathbf{s})$-umbrella is constant for $\mathbf{s} \in \mathbb{R}^n_{>0}$ running in any relatively open cone.

Recall from (8) that

$$U_\sigma = \{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0,\ (-\log|x_1|, \ldots, -\log|x_n|)B_{\sigma,j} > -\log R,\ \forall a_j \in H_\sigma \backslash \sigma\}$$

where $B_{\sigma,j}$ is the j-th column of B_σ, i.e. the vector with σ-coordinates $-A_\sigma^{-1}a_j$ and $\overline{\sigma}$-coordinates equal to the j-th column of the identity matrix of order $n - d$. Then U_σ contains those points $x \in \mathbb{C}^n \cap \{\prod_{i \in \sigma} x_i \neq 0\}$ for which

$$(-\log|x_1|, \ldots, -\log|x_n|)$$

lies in a sufficiently far translation of the cone $C(\sigma)$ inside itself. Then for any regular triangulation T of A we have that $\cap_{\sigma \in T} U_\sigma$ is a nonempty open set since it contains those points $x \in \mathbb{C}^n \cap \{\prod_{i \in \sigma} x_i \neq 0 : \sigma \in T\}$ for which $(-\log|x_1|, \ldots, -\log|x_n|) \in \mathbb{R}^n$ lies in a sufficiently far translation of the nonempty open cone $C(T)$ inside itself.

Lemma 6.4. *Given a full rank matrix $A \in \mathbb{Z}^{d \times n}$ with $d \leq n$ and a lattice Λ with $A \subseteq \Lambda \subseteq \mathbb{Z}^d$ there exists a regular triangulation T of A such that*

$$\mathrm{vol}_\Lambda(\Delta_A) = \sum_{\sigma \in T, \dim \sigma = d-1} \mathrm{vol}_\Lambda(\Delta_\sigma) \tag{11}$$

Proof. The volume function with respect to a lattice Λ is nothing but the Euclidean volume function normalized so that the unit simplex in Λ has volume one. Hence, it is enough to proof the result for the Euclidean volume.

If A is such that the facets of Δ_A contain exactly d columns of A, then they are simplicial facets and so the regular triangulation T_ω with $\omega = (\omega_1, \ldots, \omega_n)$ and $\omega_i = 1$, $\forall i$, verifies the desired condition.

Assume now that there exists $\tau \in T_\omega$ with at least cardinal $d + 1$. Then we can take $i \in \tau$ such that the columns of A in $\tau \setminus \{i\}$ determines a hyperplane H_τ. Consider all the hyperplanes $H_{\tau'} \neq H_\tau$ determined by facets τ' of Δ_A not containing neither the origin nor a_i. Then a_i lies in the interior of the polytope $\cap_{\tau'}(H_{\tau'} \cup H_{\tau'}^-)$ and so $\frac{a_i}{1-\epsilon}$ does too for $\epsilon > 0$ small enough. This means that we do not modify the facets that not contain a_i via replacing it by $\frac{a_i}{1-\epsilon}$. The facets τ' containing a_i such that $\tau' \setminus \{a_i\}$ does not determine $H_{\tau'}$ are not modified neither. We just modify the facets τ' of Δ_A that contain a_i and $\tau' \setminus \{a_i\}$ determine $H_{\tau'}$. Such a kind of facet is replaced by more than one facet with vertices contained in τ' and hence each of the new facets contain less columns of A than the original one. This process finishes in a finite number of steps and yields to a polytope with simplicial facets. We have replaced each a_i by a_i/ω_i with $\omega_i \in \mathbb{R}_{>0}$ and this is equivalent to consider T_ω with $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_{>0}^n$. Moreover, we do not modify the set of vertices and thus T_ω satisfies (11). $\qquad \square$

7 Gevrey solutions of $\mathcal{M}_A(\beta)$ along coordinate subspaces

7.1 Lower bound for the dimension

In this section we provide an optimal lower bound in terms of volumes of polytopes of the dimension of $\mathcal{H}om_\mathcal{D}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p$, $s \in \mathbb{R}$, for generic points $p \in Y_\tau = \{x_i = 0 : i \notin \tau\}$ and for all $\beta \in \mathbb{C}^d$.

Consider the submatrix $A_\tau = (a_i)_{i \in \tau}$ of A. If the rank of A_τ is d then there exists a regular triangulation $\mathrm{T}(\tau)$ of A_τ such that

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \sum_{\sigma \in \mathrm{T}(\tau), \dim \sigma = d-1} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{12}$$

because of Lemma 6.4. If the rank of A_τ is lower than d then this equality holds for any regular triangulation of the matrix A_τ since all the volumes in (12) are zero.

In this section we shall identify $s \in \mathbb{R}_{>0}$ with $(s_1, \ldots, s_n)$ such that $s_i = 1$ for $i \in \tau$ and $s_i = s$ for $i \notin \tau$. Since A_τ is a submatrix of A there exists a regular triangulation T of A such that $\mathrm{T}(\tau) \subseteq \mathrm{T}$.

For all $s \in \mathbb{R}$ we consider the following subset of $\mathrm{T}(\tau)$:

$$\mathrm{T}(\tau, s) := \{\sigma \in \mathrm{T}(\tau) : \dim(\sigma) = d - 1, \ a_j/s \notin H_\sigma^+ \ \forall j \notin \tau\}.$$

The following theorem is the main result in this section.

Theorem 7.1. *For all $\tau \subseteq \{1, \ldots, n\}$,*

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) \tag{13}$$

for p in the nonempty relatively open set $W_{\mathrm{T}(\tau)} := Y_\tau \cap (\bigcap_{\sigma \in \mathrm{T}(\tau)} U_\sigma)$. More precisely,

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p \geq \sum_{\sigma \in \mathrm{T}(\tau,s)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{14}$$

for all $s \in \mathbb{R}$ and p in the nonempty relatively open set $W_{\mathrm{T}(\tau,s)} := Y_\tau \cap (\bigcap_{\sigma \in \mathrm{T}(\tau,s)} U_\sigma)$.

Proof. $W_{\mathrm{T}(\tau)} \subseteq W_{\mathrm{T}(\tau,s)}$ are nonempty relatively open subsets of Y_τ because $\mathrm{T}(\tau)$ is a regular triangulation of A_τ (see Section 6).

For each fixed $(d-1)$-simplex $\sigma \in \mathrm{T}(\tau, s)$, we have that $|A_\sigma^{-1} a_j| \leq 1$ for all $j \in \tau$ and $|A_\sigma^{-1} a_j| \leq s$ for all $j \notin \tau$ and we can construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ Gevrey solutions of $\mathcal{M}_A(\beta)$ of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ (see Section 3). These $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ series $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ are linearly independent because they have pairwise disjoint supports. The linear independency of the set of all $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ when σ varies in $\mathrm{T}(\tau)$ is also clear if we assume that β is very generic (because this implies that they have pairwise disjoint supports).

If β is not very generic some of the series could be equal up to multiplication by a nonzero scalar. In such a case one can proceed similarly to the proof of Theorem 3.5.1. in [24]:

We introduce a perturbation $\beta \mapsto \beta + \epsilon\beta'$ with $\beta' \in \mathbb{C}^d$ such that $\beta + \epsilon\beta'$ is very generic for $\epsilon \in \mathbb{C}$ with $|\epsilon| > 0$ small enough (it is enough to consider $\beta' \in \mathbb{C}^d$ such that $(A_\sigma^{-1}\beta')_i \neq 0$ for $i = 1, \ldots, d$ and $\sigma \in \mathrm{T}(\tau)$).

Consider the set $\{\phi_\sigma^{\mathbf{k}} : \sigma \in \mathrm{T}(\tau), \mathbf{k} \in \mathbb{N}^{n-d}\}$ with $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta + \epsilon\beta')$ with disjoint supports. We will denote these series by $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')$ in this proof. It is clear that $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = \phi_{v_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')}$ for

$$v_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta').$$

Here $v_\sigma^{\mathbf{k}}(\beta)$ has σ-coordinates $A_\sigma^{-1}(\beta - A_{\bar{\sigma}}\mathbf{k})$ and $\bar{\sigma}$-coordinates $\mathbf{k}$. Similarly, $v_\sigma^{\mathbf{0}}(\beta')$ has σ-coordinates $A_\sigma^{-1}\beta'$ and $\bar{\sigma}$-coordinates $\mathbf{0}$. Let T be a regular triangulation of A such that $\mathrm{T}(\tau) \subseteq \mathrm{T}$. For any $\phi_\sigma^{\mathbf{k}}(\beta)$ we can assume without loss of generality that $v_\sigma^{\mathbf{k}}(\beta)$ has minimal negative support, $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}(\beta)}$ and $\mathrm{in}_\omega(\phi_\sigma^{\mathbf{k}}(\beta)) = x^{v_\sigma^{\mathbf{k}}(\beta)}$ for a fixed generic $\omega \in \mathcal{C}(\mathrm{T})$. Then for two simplices $\sigma, \sigma' \in \mathrm{T}(\tau)$ we have that $\phi_{v_\sigma^{\mathbf{k}}(\beta)} = \phi_{v_{\sigma'}^{\mathbf{k}'}(\beta)}$ if and only if $v_\sigma^{\mathbf{k}}(\beta) = v_{\sigma'}^{\mathbf{k}'}(\beta)$.

Let us denote $\nu = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ and consider the ν linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ of the form

$$\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = \sum_{\mathbf{k+m}\in\Lambda_{\mathbf{k}}} q_{k+m}(\epsilon)x^{v_\sigma^{\mathbf{k}}(\beta+\epsilon\beta')+u(\mathbf{m})}$$

where

$$q_{k+m}(\epsilon) = \frac{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta')]_{u(\mathbf{m})_-}}{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta') + u(m)]_{u(\mathbf{m})_-}}$$

for $\sigma \in \mathrm{T}(\tau)$ and $\mathbf{k} \in \mathbb{N}^{n-d}$ verifying that $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}}(\beta)$. Observe that for all $\mathbf{k+m} \in \Lambda_{\mathbf{k}}$ we can write

$$x^{v_\sigma^{\mathbf{k}}(\beta)+\epsilon v_\sigma^{\mathbf{0}}(\beta')+u(m)} = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} x^{v_\sigma^{\mathbf{k}}(\beta)}.$$

Then we have:

$$\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} \sum_{\mathbf{k+m}\in\Lambda_{\mathbf{k}}} q_{k+m}(\epsilon)x^{v_\sigma^{\mathbf{k}}(\beta)+u(\mathbf{m})}.$$

It is clear that $q_{\mathbf{k+m}}(\epsilon)$ is a rational function on ϵ and it has a pole of order $\mu_{\mathbf{k+m}}$ with $0 \le \mu_{\mathbf{k+m}} \le d$. On the other hand $e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} = \sum_{l\ge 0} \frac{(\log(x_\sigma^{A_\sigma^{-1}(\beta')}))^l}{l!}\epsilon^l$ so we can expand the series $\epsilon^\mu \phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')$ (with $\mu = \max\{\mu_{\mathbf{k+m}}\} \le d$) and write it in the form $\sum_{j\ge 0} \phi_j(x)\epsilon^j$ where $\phi_0(x) \ne 0$ and $\phi_j(x)$ are Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y_τ that converge in a common relatively open subset of Y_τ for all j.

After a reiterative process making convenient linear combinations of the series and dividing by convenient powers of ϵ, one obtain ν Gevrey solutions of $\mathcal{M}_A(\beta + \epsilon\beta')$ of the form $\sum_{j\ge 0} \psi_{i,j}(x)\epsilon^j$ where $\psi_{i,0}(x) \ne 0$, $i = 1, \ldots, \nu$, are linearly independent. Then we can substitute $\epsilon = 0$ and obtain the desired ν linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$. The logarithms $\log(x_i)$ just appear for $i \in \sigma$ with σ varying in $\mathrm{T}(\tau)$ at any step of the process. Thus the $\nu = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ final series just have logarithms $\log(x_i)$ with $i \in \tau$. Hence these series are Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ at points of $W_{\mathrm{T}(\tau)}$. Moreover, it is clear that the Gevrey index cannot increase with this process. $\square$

Remark 7.2. *The proof of Proposition 5.2. in [23] guarantees that all the series solutions obtained after the process that we mention in the proof of Theorem 7.1 have the form*

$$\sum_v g_v(log(x_i): \ i \in \tau)x^v$$

with $g_v(y_\tau)$ a polynomial in $\mathbb{C}[y_\tau^u: \ u \in L_{A_\tau}]$.

Remark 7.3. *Theorem 7.1 generalizes [24, Theorem 3.5.1] and [27, Corollary 1] (taking $\tau = \{1, \ldots, n\}$ and $s = 1$ in (14)), that establish that the holonomic rank of a hypergeometric system (i.e. the dimension of the space of holomorphic solutions at nonsingular points) is greater than or equal to $\mathrm{vol}_{\mathbb{Z}A}(\Delta_A)$. A more*

precise statement than [27, Corollary 1] is given in [17]: the holonomic rank is upper semi-continuous in β for holonomic families, including hypergeometric systems $\mathcal{M}_A(\beta)$ with A a pointed matrix.

Remark 7.4. *Different regular triangulations $\mathrm{T}(\tau)$ of A_τ verifying the condition (12) will produce different sets with $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ linearly independent solutions of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}$ for p in pairwise disjoint open subsets $W_{\mathbf{T}(\tau)}$ of Y_τ. It is natural to ask whether $\cup_{\mathbf{T}(\tau)}\overline{W_{\mathbf{T}(\tau)}} = Y_\tau$ for $\mathbf{T}(\tau)$ running over all possible regular triangulations $\mathbf{T}(\tau)$ of A_τ verifying (11). We have that $\cup_{\mathbf{T}(\tau)}\overline{C(\mathbf{T}(\tau))} = \mathbb{R}^l$ and that there exists $w, w' \in C(\mathbf{T}(\tau))$ verifying two-sided Abel lemma:*

$$w + C(\mathbf{T}(\tau)) \subseteq -\mathrm{Log}\, W_{\mathrm{T}(\tau)} \subseteq w' + C(\mathbf{T}(\tau))$$

where $\mathrm{Log} : \mathbb{C}^l \longrightarrow \mathbb{R}^l$ is the map $\mathrm{Log}(x_1,\ldots,x_l) = (\log|x_1|,\ldots,\log|x_l|)$. This should be contrasted with [22, Lemma 11].

Remark 7.5. *If there are no more than d columns of A_τ in the same facet of Δ_τ then by Theorem 3.14 all the series above are Gevrey of the corresponding order along Y_τ at any point of $Y_\tau \cap (\cap_{\sigma \in \mathrm{T}(\tau)}\{\prod_{i \in \sigma} x_i \neq 0\})$.*

Remark 7.6. *An anonymous referee of the paper [7] made us the following question. Is there some understanding how Gevrey solutions of $\mathcal{M}_A(\beta)$ relate to solutions of $\mathcal{M}_{A^h}(\beta^h)$ with A^h the matrix obtained from A by adding a row of 1's and then a column equal to the first unit vector? The idea is to consider a regular triangulation T for the matrix A^h containing a regular triangulation $\mathrm{T}(\tau)$ of A_τ^h. For any simplex $\sigma \in T(\tau)$, the dehomogenization (in the sense of [21, Definition 2]) of the holomorphic solutions $\phi_\sigma^{\mathbf{k}}$ of $M_{A^h}(\beta^h)$ are Gevrey solutions of $M_A(\beta)$ with respect to Y_τ. We will give more details about this subject in a forthcoming paper.*

7.2 Dimension for very generic parameters

Let $\tau \subseteq \{1,\ldots,n\}$ be a subset with cardinal l, $1 \leq l \leq n - 1$, and recall that we denote $Y_\tau = \{x_i = 0 : i \notin \tau\}$.

The aim of this section is to prove the following result:

Theorem 7.7. *For generic $p \in Y_\tau$ and very generic β,*

$$\dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau).$$

Remark 7.8. *Theorem 7.7 also implies that equality holds in (14) for very generic parameters $\beta \in \mathbb{C}^d$ because the $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in \mathrm{T}(\tau)$ have pairwise disjoint supports and their index along Y_τ is $\max\{|A_\sigma^{-1}a_j| : j \notin \tau\}$.*

Corollary 7.9. *If $\beta \in \mathbb{C}^d$ is very generic then*

$$\dim_{\mathbb{C}} \mathcal{H}^0(\mathrm{Irr}_{Y_\tau}^{(s)}(\mathcal{M}_A(\beta)))_p \geq \sum_{\sigma \in \mathrm{T}(\tau,s) \setminus \mathrm{T}(\tau,1)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{15}$$

for generic $p \in Y_\tau$.

In Section 7.1 we proved the lower bound (13) by explicitly constructing $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ in certain relatively open subsets of Y_τ. Now we are going to prove that equality holds if β is very generic.

Lemma 7.10. *If $\mathrm{rank}(A_\tau) = d$ then*

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau)[\mathbb{Z}A : \mathbb{Z}\tau].$$

Proof. We have that $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \frac{d! \, \mathrm{vol}(\Delta_\tau)}{[\mathbb{Z}^d : \mathbb{Z}A]}$ and $\mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau) = \frac{d! \, \mathrm{vol}(\Delta_\tau)}{[\mathbb{Z}^d : \mathbb{Z}\tau]}$. Since $\mathbb{Z}\tau \subseteq \mathbb{Z}A \subseteq \mathbb{Z}^d$ then $[\mathbb{Z}^d : \mathbb{Z}\tau] = [\mathbb{Z}^d : \mathbb{Z}A][\mathbb{Z}A : \mathbb{Z}\tau]$ and the result is obtained. $\square$

Lemma 7.11. *If $f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau},p}$ is a formal solution of $\mathcal{M}_A(\beta)$, then $f_m(x_\tau) \in \mathcal{O}_{Y_\tau,p}$ is a holomorphic solution of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m)$ for all $m \in \mathbb{N}^{n-l}$.*

Proof. It is clear that $I_A \cap \mathbb{C}[\partial_\tau] = I_{A_\tau}$. Then for any differential operators $P \in I_{A_\tau} \subseteq \mathbb{C}[\partial_\tau]$ we have that

$$0 = P(f) = \sum_{m \in \mathbb{N}^{n-l}} P(f_m(x_\tau)) x_{\overline{\tau}}^m$$

and this implies that $P(f_m(x_\tau)) = 0$ for all $m \in \mathbb{N}^{n-l}$.

Let Θ denote the vector with coordinates $\Theta_i = x_i \partial_i$ for $i = 1, \ldots, n$. Then $A\Theta - \beta = A_\tau \Theta_\tau + A_{\overline{\tau}} \Theta_{\overline{\tau}} - \beta$ and

$$\mathbf{0} = (A\Theta - \beta)(f) = \sum_{m \in \mathbb{N}^{n-l}} (A_\tau \Theta_\tau + A_{\overline{\tau}}m - \beta)(f_m(x_\tau)) x_{\overline{\tau}}^m$$

so $f_m(x_\tau)$ must be annihilated by the Euler operators $A_\tau \Theta_\tau - (\beta - A_{\overline{\tau}}m)$. $\square$

Corollary 7.12. *If $\mathrm{rank}(A_\tau) < d$ and $\beta \in \mathbb{C}^d$ is very generic then*

$$dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}}) = 0.$$

Proof. If $\mathrm{rank}(A_\tau) < d$, then there exists a nonzero vector $\gamma \in \mathbb{Q}^d$ such that the vector γA_τ is zero. If β is very generic $(\gamma A_\tau \Theta_\tau - \gamma(\beta - A_{\overline{\tau}}m) = -\gamma(\beta - A_{\overline{\tau}}m) \neq 0$ is a nonzero constant that is a linear combination of the Euler operators in the definition of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m)$ and so $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m) = 0$. By Lemma 7.11, the coefficients in $\mathcal{O}_{Y_\tau,p}$ of any formal solution f of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}$ must be solutions of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m) = 0$. This implies that the coefficients of f are zero and so $f = 0$. $\square$

Remark 7.13. *By Corollary 7.12 we have the equality in Theorem 7.7 holds when* $\operatorname{rank}(A_\tau) < d$. *For the remainder of this section we shall assume that* $\operatorname{rank}(A_\tau) = d$ *and then* $l \geq d$.

The following Lemma is a direct consequence of results from [1] and [9].

Lemma 7.14. *If* β *is very generic and* $p \in Y_\tau$, *then for all* $m \in \mathbb{N}^{n-l}$:

$$\dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m), \mathcal{O}_{Y_\tau})_p \leq \operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau).$$

Equality holds if p *does not lie in the singular locus of* $\mathcal{M}_{A_\tau}(\beta)$ *(which does not depend on* β*).*

Let us consider $\mathrm{T}(\tau)$ a regular triangulation of A_τ verifying (12).

Lemma 7.15. *Any formal solution* $f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau},p}$ *of* $\mathcal{M}_A(\beta)$, $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$, *can be written as follows:*

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{\mathbf{m} \in \mathbb{N}^{n-d}} c_{\sigma,\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{m})} x_{\overline{\sigma}}^{\mathbf{m}}.$$

Proof. By Lemma 7.14 a basis of $\mathcal{H}om(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}\mathbf{m}_{\overline{\tau}}, \mathcal{O}_{Y_\tau,p})$ for $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is given by the $\operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ with σ running in the $(d-1)$-simplices of $\mathrm{T}(\tau)$ and $\Lambda_{\mathbf{k}}$ running in the partition of $\mathbb{N}^{l-d}$ (see Remark 3.4 and apply it to the matrix A_τ with l columns and $\sigma \subseteq \tau$). In particular we obtain that:

$$f_{m_{\overline{\tau}}}(x_\tau) = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{m_{\overline{\sigma} \cap \tau} \in \mathbb{N}^{l-d}} c_{\sigma,\mathbf{m}_{\overline{\sigma}}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\tau}}\mathbf{m}_{\overline{\tau}} - A_{\overline{\sigma} \cap \tau}\mathbf{m}_{\overline{\sigma} \cap \tau})} x_{\overline{\sigma} \cap \tau}^{\mathbf{m}_{\overline{\sigma} \cap \tau}}$$

and this implies the result. $\qquad\square$

Using the partition $\{\Lambda_{\mathbf{k}(i)} : i = 1, \ldots, r\}$ of $\mathbb{N}^{n-d}$ (see Remark 3.4) with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$ we can write the formal solution in Lemma 7.15 as:

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{i=1}^r \sum_{\mathbf{k}(i)+\mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma,\mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}.$$

Let us denote by $v_{\sigma,\mathbf{k}(i)+\mathbf{m}}$ the exponent of the monomial $x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$.

Since Euler operators $E_i - \beta_i$ annihilate every monomial $x^{v_{\sigma,\mathbf{k}(i)+\mathbf{m}}}$ appearing in f we just need to use toric operators $\square_u = \partial^{u_+} - \partial^{u_-}$ with $u \in L_A = \ker(A) \cap \mathbb{Z}^n$ in order prove that f is annihilated by $H_A(\beta)$ if and only if the formal series

$$\sum_{\mathbf{k}(i)+\mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma,\mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$$

is annihilated by $H_A(\beta)$ for all $\sigma \in \mathrm{T}(\tau)$ and $i = 1, \ldots, r$.

This is clear because $v_{\sigma,\mathbf{k}(i)+\mathbf{m}} - v_{\sigma',\mathbf{k}(j)+\mathbf{m}} \in \mathbb{Z}^n$ if and only if $\sigma = \sigma'$ and $i = j$ (because β is very generic and for fixed σ we have Lemma 3.1). Recall here that for $u \in L_A$ any pair of monomials x^v, $x^{v'}$ verify that $\partial^{u_-}(x^v) = [v]_{u_-} x^{v-u_-}$ and $\partial^{u_+}(x^{v'}) = [v']_{u_+} x^{v'-u_+}$ and $x^{v-u_-} = x^{v'-u_+}$ if and only if $v - v' = u$.

Moreover, a series $\sum_{\mathbf{k}(i)+\mathbf{m}\in\Lambda_{\mathbf{k}(i)}} c_{\sigma,\mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$ is annihilated by I_A if and only if it is $c\phi_\sigma^{\mathbf{k}(i)}$ for certain $c \in \mathbb{C}$.

Thus we obtain that any formal solution of $\mathcal{M}_A(\beta)$ along Y_τ at $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is a linear combination of the linearly independent formal solutions $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in T(\tau)$ and $\{\Lambda_{k(i)} : 1 \leq i \leq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = [\mathbb{Z}A : \mathbb{Z}\sigma]\}$ the partition of $\mathbb{N}^{n-d}$ associated with σ (see Remark 3.4). That is we have a basis with cardinal:

$$\sum_{\sigma\in T(\tau)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = \sum_{\sigma\in T(\tau)} \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\sigma)[\mathbb{Z}A : \mathbb{Z}\tau] = \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau)[\mathbb{Z}A : \mathbb{Z}\tau] = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau).$$

This finishes the proof of Theorem 7.7.

8 Irregularity of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes under some conditions on (A, β)

Assume throughout this section that A is a pointed matrix such that $\mathbb{Z}A = \mathbb{Z}^d$ and that Y is a coordinate hyperplane. Then we have that the irregularity complex of order s, $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$, is a perverse sheaf on Y (see [20]). This implies in particular the existence of an analytic subvariety $S \subseteq Y$ with codimension $q > 0$ in Y such that for all $p \in Y \setminus S$:

$$\chi(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))))_p = \dim(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) \tag{16}$$

Here $\chi(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \dim(\mathcal{H}^i(\mathcal{F}))$ denotes the Euler-Poincaré characteristic of a bounded constructible complex of sheaves $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$. The characteristic cycle of $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$ is the unique lagrangian cycle

$$\mathrm{CCh}(\mathcal{F}) = m_Y T_Y^* Y + \sum_{\alpha:\, \dim Y_\alpha < \dim Y} m_\alpha T_{Y_\alpha}^* Y \subseteq T^* Y$$

that satisfies the index formula:

$$\chi(\mathcal{F}) = \mathrm{Eu}(m_Y Y + \sum_{\alpha:\, \dim Y_\alpha < \dim Y} (-1)^{\mathrm{codim}_Y (Y_\alpha)} m_\alpha \overline{Y_\alpha})$$

where Eu denotes the Euler's morphism between the group of cycles on Y and the group of constructible functions on Y with integer values. Thus by (16) we have that for all $p \in Y \setminus S$:

$$\dim(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))))_p = \mathrm{Eu}(\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))))_p = m_Y \qquad (17)$$

where m_Y is the multiplicity of T_Y^*Y in $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$.

Y. Laurent and Z. Mebkhout provided a formula in [16] to obtain the cycle $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ in terms of the $(1+\epsilon)$-characteristic cycle and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ for $\epsilon > 0$ small enough. Because of the correspondence established in [16] in order to compute the multiplicity m_Y of T_Y^*Y in $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ we only need to know the multiplicity of T_X^*X and T_Y^*X in the $(1+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ with respect to Y for $\epsilon > 0$ small enough.

We are going to use the multiplicities formula for the s-characteristic cycle of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [25] in the case when A is pointed and β is non-rank-jumping. First of all we need to recall some definitions given in [25].

Let us consider $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$ and the natural projection

$$\pi_{\tau,\tau'} : \mathbb{Z}\tau' \to \mathbb{Z}\tau'/(\mathbb{Z}\tau' \cap \mathbb{Q}\tau).$$

Definition 8.1. *In a lattice Λ, the volume function vol_Λ is normalized so that the unit simplex of Λ has volume 1. We abbreviate $\mathrm{vol}_{\tau,\tau'} := \mathrm{vol}_{\pi_{\tau,\tau'}(\mathbb{Z}\tau')}$.*

Definition 8.2. *For $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$, define the polyhedra*

$$P_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \cup \{0\})), \quad Q_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \setminus \tau))$$

where conv *means to take the convex hull.*

The following theorem was proven by M. Schulze and U. Walther (see [25, Th. 4.21] and [25, Cor. 4.12]).

Theorem 8.3. *For generic $\beta \in \mathbb{C}^d$ (more precisely, non-rank-jumping) and $\tau \in \Phi_A^s$, the multiplicity of $\overline{C}_A^\tau$ in the s-characteristic cycle of $\mathcal{M}_A(\beta)$ is:*

$$\mu_A^{s,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_A^s} [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \mathrm{vol}_{\tau,\tau'}(P_{\tau,\tau'} \setminus Q_{\tau,\tau'}).$$

Here $\overline{C}_A^\tau$ is the closure in T^*X of the conormal space to the orbit $O_A^\tau \subseteq T_0^*X$, where O_A^τ is the orbit of $1_\tau \in \{0,1\}^n$ ($(1_\tau)_i = 1$ if $a_i \in \tau$, $(1_\tau)_i = 0$ if $a_i \notin \tau$) by the d-torus action:

$$\begin{aligned}(\mathbb{C}^*)^d \times T_0^*X &\longrightarrow T_0^*X \\ (t,\xi) &\mapsto t \cdot \xi := (t^{a_1}\xi_1, \ldots, t^{a_n}\xi_n)\end{aligned}$$

Assume that $Y = \{x_n = 0\}$ by reordering the variables. We are interested in the multiplicities of $\overline{C}_A^\emptyset = T_X^* X$ and $\overline{C}_A^{\{n\}} = T_Y^* X$ in the r-characteristic cycles of $\mathcal{M}_A(\beta)$ for $r = s + \epsilon$ and $r = 1 + \epsilon$ with $\epsilon > 0$ small enough. In particular, we need to compute $\mu_A^{s+\epsilon,\emptyset}, \mu_A^{s+\epsilon,\{n\}}, \mu_A^{1+\epsilon,\emptyset}$ and $\mu_A^{1+\epsilon,\{n\}}$.

It is a classical result that $\mu_A^{1,\emptyset} = \operatorname{rank}(\mathcal{M}_A(\beta)) = \operatorname{vol}_{\mathbb{Z}^d}(\Delta_A)$ for generic β (see [9], [1]).

From [25, Corollay 4.22] if $\tau = \emptyset$ then

$$\mu_A^{s,\emptyset} = \operatorname{vol}_{\mathbb{Z}^d}(\cup_{\tau' \in \Phi_A^{s,d-1}}(\Delta_{\tau'}^1 \setminus \operatorname{conv}(\tau'))).$$

Since $\Phi_A^{s+\epsilon}$ is constant for $\epsilon > 0$ small enough we have that all its faces τ are F-homogeneous and then $\operatorname{vol}_{\mathbb{Z}^d}(\operatorname{conv}(\tau)) = 0$. As a consequence,

$$\mu_A^{s+\epsilon,\emptyset} = \operatorname{vol}_{\mathbb{Z}^d}(\cup_{\tau' \in \Phi_A^{s+\epsilon,d-1}}(\Delta_{\tau'}^1)) = \operatorname{vol}_{\mathbb{Z}^d}(\cup_{\tau' \in \Phi_A^{s,d-1}}(\Delta_{\tau'}^1)).$$

Let us compute $\mu_A^{r,\{n\}}$ for $r = s + \epsilon$ and $r = 1 + \epsilon$.

Consider any $\tau \in \Phi_A^{s+\epsilon,d-1}$ such that $n \in \tau$. Since $\epsilon > 0$ is generic ($\Phi_A^{t,d-1}$ is locally constant at $t = s + \epsilon$) we have that $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ and hence there exists certain $(d-1)$-simplices $\sigma_1, \ldots, \sigma_r$ such that $n \in \sigma_i \subseteq \tau$, $\tau = \cup_i \sigma_i$, $\sigma_i \cap \sigma_j$ is an k-simplex with $k \leq d - 2$ ($\sigma_1, \ldots, \sigma_r$ is a triangulation of τ). Then $\operatorname{vol}_{\mathbb{Z}^d}(\Delta_\tau) = \sum_{i=1}^r \operatorname{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i})$ and we want to prove that

$$\operatorname{vol}_{\mathbb{Z}^d}(\Delta_\tau) = [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \operatorname{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}) \qquad (18)$$

Since $\mathbb{Z}\sigma_i \subseteq \mathbb{Z}\tau \subseteq \mathbb{Z}^d$ then $\operatorname{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i}) = [\mathbb{Z}^d : \mathbb{Z}\sigma_i] = [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [\mathbb{Z}\tau : \mathbb{Z}\sigma_i]$ so we only need to prove:

$$\sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i] = [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \operatorname{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}).$$

But $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ implies that $[\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] = 1$ and τ is F-homogeneous so we have to prove that:

$$\operatorname{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We observe that $\pi_{\{n\},\tau}(\tau \cup \{0\}) = (\tau \setminus \{n\}) \cup \{0\}$ in $\mathbb{Z}\tau/(\mathbb{Z}\tau \cap \mathbb{Q}a_n) = \mathbb{Z}(\tau \setminus \{n\})$. Consider a $(d-2)$-simplex $\widetilde{\sigma}$ such that $\mathbb{Z}\widetilde{\sigma} = \mathbb{Z}(\tau \setminus \{n\})$. Since $a_n \notin \sum_{i \in \tau \setminus \{n\}} \mathbb{Q}a_i$ there exists a hyperplane H such that $a_i \in H$ for all $i \in \tau \setminus \{n\}\}$, $0 \in H$ and $\widetilde{\sigma} \subseteq H$. Recall that the Euclidean volume of the convex hull of a bounded polytope Δ contained in a hyperplane $H \subseteq \mathbb{R}^d$ and a point $c \notin H$ is the product

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of the relative volume of the polytope $\mathrm{vol}_{rel}(\Delta)$ and the distance from c to H, $\mathrm{d}(c, H)$, divided by $d!$. Hence, we have the following equalities:

$$\mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \frac{\mathrm{vol}_{rel}(\Delta_{\tau\setminus\{n\}})}{\mathrm{vol}_{rel}(\Delta_{\widetilde{\sigma}})} = \frac{\mathrm{vol}(\Delta_\tau)}{\mathrm{vol}(\Delta_{\widetilde{\sigma}\cup\{n\}})} = \sum_{i=1}^{r} \frac{\mathrm{vol}(\Delta_{\sigma_i})}{\mathrm{vol}(\Delta_{\widetilde{\sigma}\cup\{n\}})} =$$

$$= \sum_{i=1}^{r} \frac{[\mathbb{Z}^d : \mathbb{Z}\sigma_i]}{[\mathbb{Z}^d : \mathbb{Z}\tau]} = \sum_{i=1}^{r} [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We have proved (18) and as a consequence the following Lemma.

Lemma 8.4. *Consider $s \geq 1$ and β non-rank-jumping. Then for all $\epsilon > 0$ small enough:*

$$\mu_A^{s+\epsilon,\{n\}} = \sum_{n\in\tau\in\Phi_A^{s+\epsilon}} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau).$$

We close this section with the following result about the irregularity along any coordinate hyperplane Y of the hypergeometric system $\mathcal{M}_A(\beta)$ associated with a full rank pointed matrix A with $\mathbb{Z}A = \mathbb{Z}^d$. It is a consequence of Lemma 8.4 and the results in [16].

Theorem 8.5. *If $\beta \in \mathbb{C}^d$ is generic (more precisely, non-rank-jumping) then*

$$\dim_{\mathbb{C}}(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) = \sum_{n\notin\tau\in\Phi_A^s} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) - \sum_{n\notin\tau\in\Phi_A^1} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau)$$

for all $p \in Y \setminus S$, where S is a subvariety of Y with $\dim S < \dim Y$. Then, for very generic β the nonzero classes in $\mathcal{Q}_Y(s)$ of the constructed series $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in \mathrm{T}'$ form a basis in their common domain of definition $U \subseteq Y$.

Remark 8.6. *Notice that Theorem 8.5 implies that under the assumptions of this section equality holds in (15).*

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