KATZ-RADON TRANSFORM OF ℓ-ADIC REPRESENTATIONS

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Abstract. We prove a simple explicit formula for the local Katz-Radon transform of an ℓ-adic representation of the Galois group of the fraction field of a strictly henselian discrete valuation ring with positive residual characteristic, which can be defined as the local additive convolution with a fixed tame character. The formula is similar to one proved by D. Arinkin in the $\mathcal{D}$-module setting, and answers a question posed by N. Katz.

1. Introduction

In [10, 3.4.1], N. Katz defines some functors on the category of continuous ℓ-adic representations of the inertia groups I_0 and I_∞ of the projective line over $\bar{k}$ at 0 and infinity, where $\bar{k}$ is the algebraic closure of a finite field of characteristic p and ℓ is a prime different from p. These functors arise during his study of middle convolution of sheaves on the affine line and, roughly speaking, correspond to locally convolving a representation with a fixed tame character $\mathcal{L}_\chi$ of I_0 or I_∞. They are defined using G. Laumon's local Fourier transform functors, and in fact correspond to taking the tensor product with the conjugate tame character $\mathcal{L}_{\bar{\chi}}$ on the other side of the equivalence of categories given by these functors. Katz asks [10, 3.4.1] whether there is a simple expression for the functors defined in this way.

Recently, D. Arinkin [1] has studied the analog of Katz's functor in $\mathcal{D}$-module theory: if K is a field of characteristic 0, $K((x))$ is the field of Laurent series over K and $\mathcal{D}_x$ the ring of differential operators with coefficients in $K((x))$, the local Katz-Radon transform for a given $\lambda \in K - \mathbb{Z}$ is an equivalence of categories $\rho_\lambda : \mathcal{D}_x\text{-mod} \to \mathcal{D}_x\text{-mod}$, originally defined in [3]. Arinkin proves the simple formula [1, Theorem C]

$$\rho_\lambda(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{K}^{\lambda(a+1)}$$

for any $\mathcal{F} \in \mathcal{D}_x$-mod with a single slope a, where $\mathcal{K}^\mu$ is the Kummer $\mathcal{D}_x$-module of rank 1 generated by $\mathbf{e}$, on which the derivative acts by

$$\frac{d}{dx}\mathbf{e} = \frac{\mu}{x}\mathbf{e}.$$

In this article we will prove a similar formula in the ℓ-adic case. More precisely, for a fixed tame ℓ-adic character $\mathcal{L}_\chi$ and an ℓ-adic representation $\mathcal{F}$ of I_0, let

$$\rho_\chi(\mathcal{F}) := \mathrm{FT}^{\psi,-1}_{(0,\infty)}(\mathcal{L}_{\bar{\chi}} \otimes \mathrm{FT}^{\psi}_{(0,\infty)}\mathcal{F})$$

where $\mathrm{FT}^{\psi}_{(0,\infty)}$ denotes Laumon's local Fourier transform functor. If $\mathcal{F}$ has a single slope $a = c/d$ (with c, d relatively prime positive integers), we will prove that there is an isomorphism of I_0-representations

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi^{\otimes(a+1)}$$

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where $\mathcal{L}_\chi^{\otimes(a+1)}$ is any d-th root of the character $\mathcal{L}_\chi^{\otimes(c+d)}$.

For a large class of representations $\mathcal{F}$ of I_0 (in particular for many of those who appear in applications), the isomorphism can be proven via the explicit formulas for the local Fourier transforms given by L. Fu [5] and A. Abbes and T. Saito [2]. In this article we take a different approach that works for any $\mathcal{F}$, and is independent of any explicit expression for the local Fourier transforms.

2. The Katz-Radon transform

Fix a finite field k of characteristic $p > 0$ and an algebraic closure $\bar{k}$. Let $\mathbb{P}^1_{\bar{k}}$ be the projective line over $\bar{k}$ and, for every $t \in \mathbb{P}^1(\bar{k}) = \bar{k} \cup \{\infty\}$, denote by I_t its inertia group at t: for $t \neq \infty$, if $x - t$ denotes a local coordinate at t, it is the Galois group of the fraction field of the henselization of the local ring $\bar{k}[x]_{(x-t)}$. We have an exact sequence [8, 1.0]

$$0 \to P_t \to I_t \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 0$$

for every $t \in \mathbb{P}^1(\bar{k})$, where P_t is the only p-Sylow subgroup of I_t. Moreover, there is a canonical filtration of I_t by the higher ramification groups

$$I_t^{(r)} \supseteq I_t^{(s)} \text{ for } 0 \leq r < s \in \mathbb{R}$$

which are normal in I_t.

Fix a prime $\ell \neq p$, and denote by $\mathcal{R}_t$ the abelian category of continuous ℓ-adic representations of I_t (i.e. continuous representations $\mathcal{F} : I_t \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, whose image is in $\mathrm{GL}_n(E_\lambda)$ for some finite extension E_λ of $\mathbb{Q}_\ell$). For every irreducible $\mathcal{F} \in \mathcal{R}_t$, the *slope* of $\mathcal{F}$ is $\inf\{r \geq 0 | \mathcal{F}_{|I_t^{(r)}} \text{ is trivial}\}$. It is a non-negative rational number. In general, the slopes of $\mathcal{F}$ are the slopes of the irreducible components of $\mathcal{F}$. For every $\mathcal{F}$ there is a canonical direct sum decomposition [8, Lemma 1.8]

$$\mathcal{F} \cong \bigoplus_{r \geq 0} \mathcal{F}^r \tag{1}$$

with $\mathcal{F}^r$ having a single slope r. The slope 0 (tame) part will be denoted by $\mathcal{F}^t$. $\mathcal{F}$ is said to be *tame* (respectively *totally wild*) if $\mathcal{F} = \mathcal{F}^t$ (resp. $\mathcal{F}^t = 0$).

For every $r \geq 0$ let $\mathcal{R}_t^r$ denote the full subcategory of $\mathcal{R}_t$ consisting of representations with a single slope r. We have a decomposition

$$\mathcal{R}_t = \bigoplus_{r \geq 0} \mathcal{R}_t^r$$

in the sense that every $\mathcal{F} \in \mathcal{R}_t$ has a decomposition (1) and $\mathrm{Hom}_{\mathcal{R}_t}(\mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{F} \in \mathcal{R}_t^r$, $\mathcal{G} \in \mathcal{R}_t^s$ and $r \neq s$ [8, Proposition 1.1].

Let $k' \subseteq \bar{k}$ be a finite extension of k, and $\chi : k'^* \to \bar{\mathbb{Q}}_\ell^*$ a multiplicative character. By [4, 1.4-1.8] there is an associated smooth Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_{m,\bar{k}}$, which is a tame character of I_0 (and of I_∞) of the same order as χ. If $k' \subseteq k''$ is another extension, the sheaves defined by χ and $\chi \circ \mathrm{Nm}_{k''/k'} : k''^* \to \bar{\mathbb{Q}}_\ell^*$ are isomorphic. Moreover, every tame character of I_0 (and of I_∞) can be obtained in this way. Whenever we speak about a tame character of I_0, we will implicitly assume that we have made a choice of such a finite extension of k and of a character.

Fix a non-trivial additive character $\psi : k \to \bar{\mathbb{Q}}_\ell^*$. The local Fourier transform functors, defined by G. Laumon in [11], give equivalences of categories

$$\mathrm{FT}^{\psi}_{(0,\infty)} : \mathcal{R}_0 \to \mathcal{R}_\infty^{<1},$$

$$\mathrm{FT}^{\psi}_{(\infty,\infty)} : \mathcal{R}_\infty^{>1} \to \mathcal{R}_\infty^{>1}$$

and

$$\mathrm{FT}^{\psi}_{(\infty,0)} : \mathcal{R}_\infty^{<1} \to \mathcal{R}_0$$

(where $\mathcal{R}_\infty^{<1} = \bigoplus_{r<1} \mathcal{R}_\infty^r$ and $\mathcal{R}_\infty^{>1} = \bigoplus_{r>1} \mathcal{R}_\infty^r$) that describe the relationship between the local monodromies of an ℓ-adic sheaf on $\mathbb{A}^1_{\bar{k}}$ and its Fourier transform with respect to ψ. The Katz-Radon transform is defined in terms of them.

Definition 2.1. *Fix a tame character $\mathcal{L}_\chi$ of I_0. The (local)* Katz-Radon transform *(with respect to $\mathcal{L}_\chi$) is the functor $\rho_\chi : \mathcal{R}_0 \to \mathcal{R}_0$ given by*

$$\rho_\chi(\mathcal{F}) = \mathrm{FT}^{\psi,-1}_{(0,\infty)}(\mathrm{FT}^{\psi}_{(0,\infty)}\mathcal{L}_\chi \otimes \mathrm{FT}^{\psi}_{(0,\infty)}\mathcal{F}) = \mathrm{FT}^{\psi,-1}_{(0,\infty)}(\mathcal{L}_{\bar{\chi}} \otimes \mathrm{FT}^{\psi}_{(0,\infty)}\mathcal{F}).$$

The Katz-Radon transform is an auto-equivalence of the category $\mathcal{R}_0$ (since it is a composition of three equivalences of categories). It preserves dimensions and slopes, and for tame $\mathcal{F}$ it is given by $\rho_\chi(\mathcal{F}) = \mathcal{F} \otimes \mathcal{L}_\chi$ [10, 3.4.1]. For totally wild $\mathcal{F}$, it can be interpreted as the "local additive convolution" of $\mathcal{F}$ and $\mathcal{L}_\chi$ [10, 3.4.3]: if we extend $\mathcal{F}$ to a smooth sheaf on $\mathbb{G}_{m,\bar{k}}$, tamely ramified at infinity, then $\rho_\chi(\mathcal{F})$ is the wild part of the local monodromy at 0 of $\mathcal{F} * \mathcal{L}_\chi$, where

$$\mathcal{F} * \mathcal{L}_\chi = \mathrm{R}^1\sigma_!(\mathcal{F} \boxtimes \mathcal{L}_\chi)$$

and $\sigma : \mathbb{A}^2_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ denotes the addition map (in [10], the "middle convolution" is used instead, but that one differs from the one used here only by Artin-Shreier components, which are smooth at 0 and therefore do not affect the local monodromy). Notice that, in particular, ρ_χ is independent of the choice of the additive character ψ.

More intrinsically, it can be described in terms of vanishing cycles functors [11, 2.7.2]: If $X = \mathbb{A}^2_{(0,0)}$ (respectively $S = \mathbb{A}^1_{(0)}$) denotes the henselization of $\mathbb{A}^2_{\bar{k}}$ at $(0,0)$ (resp. the henselization of $\mathbb{A}^1_{\bar{k}}$ at 0) then $\rho_\chi(\mathcal{F}) \cong \mathrm{R}^1\Phi(\sigma, \mathcal{F} \boxtimes \mathcal{L}_\chi)_{(0,0)}$, where $\mathrm{R}\Phi(\sigma, \mathcal{F} \boxtimes \mathcal{L}_\chi)$ is the vanishing cycles complex for the addition map $\sigma : X \to S$ with respect to the sheaf $\mathcal{F} \boxtimes \mathcal{L}_\chi$ on X.

Similarly, it also has an interpretation as a "local multiplicative convolution" [12, Corollary 5.6]: If $X = \mathbb{G}^2_{m,(1,1)}$ (respectively $S = \mathbb{G}_{m,(1)}$) denotes the henselization of $\mathbb{G}_{m,\bar{k}}$ at $(1,1)$ (resp. the henselization of $\mathbb{G}_{m,\bar{k}}$ at 1) then $\rho_\chi(\mathcal{F}) \cong \mathrm{R}^1\Phi(\mu, \mathcal{F} \boxtimes \mathcal{L}_\chi)_{(1,1)}$, where $\mathrm{R}\Phi(\mu, \mathcal{F} \boxtimes \mathcal{L}_\chi)$ is the vanishing cycles complex for the multiplication map $\mu : X \to S$ with respect to the sheaf $\mathcal{F} \boxtimes \mathcal{L}_\chi$ on X, and $\mathcal{F}$ and $\mathcal{L}_\chi$ are viewed as representations of I_1 via the isomorphism $I_0 \cong I_1$ that maps the uniformizer x at 0 to the uniformizer $x-1$ at 1.

The main result of this article is the following simple expression for ρ_χ:

Theorem 2.2. *Let $\mathcal{F} \in \mathcal{R}_0$ be totally wild with a single slope $a > 0$. Write $a = c/d$, where c and d are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of I_0 such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes(c+d)}$. Then*

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

In other words, we have the formula

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi^{\otimes(a+1)} \tag{2}$$

where $\mathcal{L}_\chi^{\otimes(a+1)}$ stands for "any character that can reasonably be called $\mathcal{L}_\chi^{\otimes(a+1)}$".

By the decomposition $\mathcal{R}_0 = \bigoplus_{r\geq 0} \mathcal{R}_0^r$, this determines $\rho_\chi(\mathcal{F})$ for any $\mathcal{F} \in \mathcal{R}_0$, thus answering the question posed by N. Katz in [10, 3.4.1].

A question that remains open is the following: in the article we prove that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$, independently for any $\mathcal{F}$ with slope a. So the functors $\mathcal{R}_0^a \to \mathcal{R}_0^a$ given by ρ_χ and $(-) \otimes \mathcal{L}_\eta$ map any $\mathcal{F}$ to isomorphic objects. Is there an actual isomorphism of functors between them? In the affirmative case, is there a simple way to construct it?

3. Proof of the main theorem

In this section we will prove theorem 2.2. We will start with the case where $\mathcal{F} \in \mathcal{R}_0$ is irreducible.

Lemma 3.1. *Let $\mathcal{F} \in \mathcal{R}_0$. Then $\mathcal{F}^t \neq 0$ if and only if there exists $\epsilon > 0$ such that for every $\mathcal{G} \in \mathcal{R}_0$ with a single slope $b \in (0, \epsilon)$ we have*

$$\mathrm{Swan}(\mathcal{F} \otimes \mathcal{G}) > \mathrm{Swan}(\mathcal{F}) \dim(\mathcal{G}).$$

Proof. Suppose that $\mathcal{F}^t \neq 0$, and let $a_0 = 0 < a_1 < \cdots < a_r$ be the slopes of $\mathcal{F}$, with multiplicities $n_0, n_1, \ldots, n_r$. Then $\mathrm{Swan}(\mathcal{F}) = \sum n_i a_i$. Let $\epsilon = a_1$. Then for every $\mathcal{G} \in \mathcal{R}_0$ with a single slope $b \in (0, \epsilon)$ the tensor product $\mathcal{F} \otimes \mathcal{G}$ has slopes $b < a_1 < \cdots < a_r$ with multiplicities $n_0 m, n_1 m, \ldots, n_r m$ where $m = \dim(\mathcal{G})$ by [8, Lemma 1.3]. Therefore

$$\mathrm{Swan}(\mathcal{F} \otimes \mathcal{G}) = n_0 m b + \sum_{i=1}^r n_i m a_i > \sum_{i=1}^r n_i m a_i = \mathrm{Swan}(\mathcal{F}) \dim(\mathcal{G}).$$

Conversely, suppose that $\mathcal{F}^t = 0$, and let $a_1 < \cdots < a_r$ be the slopes of $\mathcal{F}$. Then for every $\mathcal{G} \in \mathcal{R}_0$ with a single slope $b \in (0, a_1)$ the tensor product $\mathcal{F} \otimes \mathcal{G}$ has the same slopes as $\mathcal{F}$ by [8, Lemma 1.3], and in particular $\mathrm{Swan}(\mathcal{F} \otimes \mathcal{G}) = \mathrm{Swan}(\mathcal{F}) \dim(\mathcal{G})$. This proves the lemma, since for every $\epsilon > 0$ there exist representations in $\mathcal{R}_0$ with slope $b \in (0, \epsilon)$ (for instance, one may take $[n]_* \mathcal{H}$, where $\mathcal{H} \in \mathcal{R}_0$ has slope $a > 0$ and n is a prime to p integer greater than a/ϵ [8, 1.13.2]). $\square$

For any two objects $K, L \in \mathcal{D}_c^b(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$, we will denote by $K * L \in \mathcal{D}_c^b(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ their additive convolution:

$$K * L = \mathrm{R}\sigma_!(K \boxtimes L)$$

where $\sigma : \mathbb{A}^2_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ is the addition map.

Lemma 3.2. *Let $K, L, M \in \mathcal{D}_c^b(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$. Then*

$$\mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, (K * L) \otimes M) \cong \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, K \otimes ((\tau_{-1}^* L) * M))$$

where $\tau_{-1} : \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ is the additive inversion.

Proof. We have

$$\mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, (K * L) \otimes M) = \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}\sigma_!(K \boxtimes L) \otimes M) =$$
$$= \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}\sigma_!((K \boxtimes L) \otimes \sigma^* M)) = \mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, (K \boxtimes L) \otimes \sigma^* M)$$

by the projection formula. If $\pi_1, \pi_2 : \mathbb{A}^2_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ are the projections then

$$\mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, (K \boxtimes L) \otimes \sigma^* M) = \mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M).$$

Consider the automorphism $\phi : \mathbb{A}^2_{\bar{k}} \to \mathbb{A}^2_{\bar{k}}$ given by $(x, y) \mapsto (x + y, -y)$. Then $\sigma = \pi_1 \circ \phi$, $\pi_1 = \sigma \circ \phi$ and $\tau_{-1} \circ \pi_2 = \pi_2 \circ \phi$. It follows that

$$\begin{aligned}
&\mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M) \cong \mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, \phi^*\pi_1^* K \otimes \phi^*\pi_2^* L \otimes \phi^*\sigma^* M) = \\
&= \mathrm{R}\Gamma_c(\mathbb{A}^2_{\bar{k}}, \sigma^* K \otimes \pi_2^*\tau_{-1}^* L \otimes \pi_1^* M) = \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}\sigma_!(\sigma^* K \otimes \pi_2^*\tau_{-1}^* L \otimes \pi_1^* M)) \cong \\
&\cong \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, K \otimes \mathrm{R}\sigma_!((\tau_{-1}^* L) \boxtimes M)) = \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, K \otimes ((\tau_{-1}^* L) * M)).
\end{aligned}$$

□

If $\mathcal{F}$ is a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_{m,\bar{k}}$ which is totally wild at 0, then for every $t \in \bar{k}$ the sheaf $\mathcal{F} \otimes \mathcal{L}_{\chi(t-x)}$ (extended by zero to $\mathbb{A}^1_{\bar{k}}$) is totally wild at 0 and has no punctual sections (where $\mathcal{L}_{\chi(t-x)}$ is the pull-back of the Kummer sheaf $\mathcal{L}_\chi$ under the map $x \mapsto t-x$), so its only non-zero cohomology group with compact support is H^1_c. We conclude that the only non-zero cohomology sheaf of $\mathcal{F}[0] * \mathcal{L}_\chi[0] \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ is $\mathcal{H}^1 = \mathrm{R}^1\sigma_!(\mathcal{F} \otimes \mathcal{L}_\chi)$. We will denote this sheaf by $\mathcal{F} * \mathcal{L}_\chi$.

Lemma 3.3. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{R}_0$. Then*

$$\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) = \mathrm{Swan}(\mathcal{F} \otimes \mathcal{G}).$$

Proof. By additivity of the Swan conductor, we may assume that $\mathcal{F}$ is irreducible, and in particular that it has a single slope $a \geq 0$. If $a = 0$ then $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi$, so the equality is clear. Suppose that $a > 0$. By [7, Theorem 1.5.6], $\mathcal{F}$ and $\mathcal{G}$ can be extended to smooth sheaves on $\mathbb{G}_{m,\bar{k}}$, tamely ramified at infinity, which we will also denote by $\mathcal{F}$ and $\mathcal{G}$. Let $\mathcal{F}$ and $\mathcal{G}$ be also their extensions by zero to $\mathbb{A}^1_{\bar{k}}$.

Using the compatibility between Fourier transform with respect to ψ and convolution [11, Proposition 1.2.2.7], we have

$$\mathcal{F} * \mathcal{L}_\chi = \mathrm{FT}^{\bar{\psi}}(\mathrm{FT}^\psi \mathcal{F} \otimes \mathrm{FT}^\psi \mathcal{L}_\chi) = \mathrm{FT}^{\bar{\psi}}(\mathrm{FT}^\psi \mathcal{F} \otimes \mathcal{L}_{\bar{\chi}}),$$

where $\mathrm{FT}^\psi \mathcal{F}$ denotes the "naive" Fourier transform in the sense of [8, 8.2], that is, the (-1)-th cohomology sheaf of the Fourier transform of $\mathcal{F}[1] \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ (which is its only non-zero cohomology sheaf, since $\mathcal{F}$ is totally wild at zero and therefore it is Fourier [8, Lemma 8.3.1]).

Let n be the rank of $\mathcal{F}$, and denote by $\mathcal{F}_{(\infty)} \in \mathcal{R}_\infty$ its local monodromy at infinity, which is a tame representation of I_∞. By Ogg-Shafarevic [6, Exposé X, Corollaire 7.12], $\mathrm{FT}^\psi \mathcal{F}$ is smooth on $\mathbb{G}_{m,\bar{k}}$ of rank $na + n = n(a+1)$. By Laumon's local Fourier transform theory [9, Theorem 13], $\mathrm{FT}^\psi \mathcal{F}$ has a single slope $\frac{a}{a+1}$ at infinity, with multiplicity $n(a+1)$, and its monodromy at 0 has a trivial part of dimension na and its quotient is the dual $\widehat{\mathcal{F}_{(\infty)}}$ of $\mathcal{F}_{(\infty)}$. Then $\mathrm{FT}^\psi \mathcal{F} \otimes \mathcal{L}_{\bar{\chi}}$ also has a single slope $\frac{a}{a+1}$ at infinity with multiplicity $n(a+1)$, and its monodromy $\mathcal{M}$ at 0 sits in an exact sequence

$$0 \to \mathcal{L}_{\bar{\chi}}^{\oplus na} \to \mathcal{M} \to \widehat{\mathcal{F}_{(\infty)}} \otimes \mathcal{L}_{\bar{\chi}} \to 0. \tag{3}$$

Its inverse Fourier transform, by Ogg-Shafarevic, is smooth of rank $n(a+1)$ on $\mathbb{G}_{m,\bar{k}}$, and by local Fourier transform its wild part at 0 has slope a with multiplicity n.

In fact, this wild part is simply $\rho_\chi(\mathcal{F})$ by the additive convolution interpretation of ρ_χ. Its monodromy at infinity sits in an exact sequence

$$0 \to \mathcal{L}_\chi^{\oplus na} \to (\mathcal{F} * \mathcal{L}_\chi)_{(\infty)} \to \mathcal{F}_{(\infty)} \otimes \mathcal{L}_\chi \to 0 \tag{4}$$

obtained from (3) by local Fourier transform.

So $\mathcal{F} * \mathcal{L}_\chi$ has rank $n(a+1)$ on $\mathbb{G}_{m,\bar{k}}$, and its monodromy at 0 is the direct sum of $\rho_\chi(\mathcal{F})$ and a constant part of dimension $na = \mathrm{Swan}(\mathcal{F})$. So

$$\mathrm{Swan}_0((\mathcal{F} * \mathcal{L}_\chi) \otimes \mathcal{G}) = \mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) + \mathrm{Swan}(\mathcal{F})\mathrm{Swan}(\mathcal{G}).$$

In particular, by Ogg-Shafarevic, the Euler characteristic of the sheaf $(\mathcal{F} * \mathcal{L}_\chi) \otimes \mathcal{G}$ (extended by zero to $\mathbb{A}^1_{\bar{k}}$) is $-\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) - \mathrm{Swan}(\mathcal{F})\mathrm{Swan}(\mathcal{G})$. Using lemma 3.2, proper base change, and the fact that $\chi(\mathbb{G}_{m,\bar{k}}, K \otimes \mathcal{L}_\chi) = \chi(\mathbb{G}_{m,\bar{k}}, K)$ for any object $K \in \mathcal{D}^b_c(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell)$, we get

$$\begin{aligned}
&\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) + \mathrm{Swan}(\mathcal{F})\mathrm{Swan}(\mathcal{G}) = \chi(\mathbb{A}^1_{\bar{k}}, (\mathcal{F}[1] * \mathcal{L}_\chi[1]) \otimes \mathcal{G}) = \\
&= \chi(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_\chi \otimes (\tau^*_{-1}\mathcal{F}[1] * \mathcal{G}[1])) = \chi(\mathbb{G}_{m,\bar{k}}, \tau^*_{-1}\mathcal{F}[1] * \mathcal{G}[1]) = \\
&= \chi(\mathbb{A}^1_{\bar{k}}, \tau^*_{-1}\mathcal{F}[1] * \mathcal{G}[1]) - \mathrm{rank}_0(\tau^*_{-1}\mathcal{F}[1] * \mathcal{G}[1]) = \\
&= \chi(\mathbb{A}^1_{\bar{k}}, \mathcal{F}[1])\chi(\mathbb{A}^1_{\bar{k}}, \mathcal{G}[1]) - \chi(\mathbb{A}^1_{\bar{k}}, \mathcal{F}[1] \otimes \mathcal{G}[1]) = \\
&= \mathrm{Swan}(\mathcal{F})\mathrm{Swan}(\mathcal{G}) + \mathrm{Swan}(\mathcal{F} \otimes \mathcal{G})
\end{aligned}$$

where rank_0 of a derived category object denotes the alternating sum of the ranks at 0 of its cohomology sheaves, so $\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) = \mathrm{Swan}(\mathcal{F} \otimes \mathcal{G})$. □

Proposition 3.4. *Let $\mathcal{F} \in \mathcal{R}_0$ be totally wild and irreducible. Then there exists a tame character $\mathcal{L}_\eta$ of I_0 such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$.*

Proof. Let $\widehat{\mathcal{F}}$ be the dual representation. We claim that the tame part of $\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}$ is non-zero. By lemma 3.1, it suffices to show that there is an $\epsilon > 0$ such that, for any $\mathcal{G} \in \mathcal{R}_0$ with slope $b \in (0, \epsilon)$, $\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) > \mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}) \dim(\mathcal{G})$. But by lemma 3.3, we have

$$\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) = \mathrm{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}} \otimes \mathcal{G})$$

and

$$\mathrm{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}) = \mathrm{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}})$$

and, since $\widehat{\mathcal{F}}$ is the dual of $\mathcal{F}$, the tensor product $\mathcal{F} \otimes \widehat{\mathcal{F}}$ has a trivial quotient and, in particular, has non-trivial tame part. By lemma 3.1, there exists $\epsilon > 0$ such that, for any $\mathcal{G} \in \mathcal{R}_0$ with slope $b \in (0, \epsilon)$, $\mathrm{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) > \mathrm{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}}) \dim(\mathcal{G})$.

Since the tame part of $\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}$ is non-zero and it is a direct summand, it contains a tame character $\mathcal{L}_\eta$ of I_0 as a subrepresentation. Then

$$\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{L}_{\bar{\eta}} = \rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F} \otimes \mathcal{L}_\eta} = \mathrm{Hom}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F}))$$

contains a trivial subrepresentation, so $\mathrm{Hom}_{I_0}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F})) \neq 0$. Since both $\rho_\chi(\mathcal{F})$ and $\mathcal{F} \otimes \mathcal{L}_\eta$ are irreducible, any non-zero I_0-equivariant map $\mathcal{F} \otimes \mathcal{L}_\eta \to \rho_\chi(\mathcal{F})$ must be an isomorphism. □

Proposition 3.5. *Let $\mathcal{F} \in \mathcal{R}_0$ be totally wild and irreducible of dimension n and slope a, and let $\mathcal{L}_\eta$ be a tame character of I_0 such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$. Then $\mathcal{L}_\eta^{\otimes n} \cong \mathcal{L}_\chi^{\otimes n(a+1)}$.*

Proof. Extend $\mathcal{F}$ to a smooth ℓ-adic sheaf on $\mathbb{G}_{m,\bar{k}}$, tamely ramified at infinity, also denoted by $\mathcal{F}$. Let $\mathcal{F}$ also denote its extension by zero to $\mathbb{A}^1_{\bar{k}}$. By the proof of lemma 3.3, the sheaf $\mathcal{F} * \mathcal{L}_\chi$ is smooth on $\mathbb{G}_{m,\bar{k}}$, its monodromy at 0 is the direct sum of $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$ and a trivial part of dimension na, and its monodromy at infinity sits in the exact sequence (4). Its determinant is then a smooth sheaf of rank 1 on $\mathbb{G}_{m,\bar{k}}$, whose monodromy at 0 is $\det(\mathcal{F}) \otimes \mathcal{L}_\eta^{\otimes n}$, and whose monodromy at ∞ is $\det(\mathcal{F}_{(\infty)}) \otimes \mathcal{L}_\chi^{\otimes n(a+1)}$.

Then $\widehat{\det(\mathcal{F})} \otimes \mathcal{L}_{\bar{\eta}}^{\otimes n} \otimes \det(\mathcal{F} * \mathcal{L}_\chi)$ is a rank 1 smooth sheaf on $\mathbb{G}_{m,\bar{k}}$, with trivial monodromy at 0 and tamely ramified at infinity. Since the tame fundamental group of $\mathbb{A}^1_{\bar{k}}$ is trivial, we conclude that

$$\det(\mathcal{F} * \mathcal{L}_\chi) \cong \det(\mathcal{F}) \otimes \mathcal{L}_\eta^{\otimes n}$$

as sheaves on $\mathbb{G}_{m,\bar{k}}$. Comparing their monodromies at infinity gives the desired isomorphism. □

It remains to show that any such $\mathcal{L}_\eta$ works.

Lemma 3.6. *Let $\mathcal{F} \in \mathcal{R}_0$ be irreducible of dimension n, and let $\mathcal{L}_\eta$ be a tame character of I_0 such that $\mathcal{L}_\eta^{\otimes n}$ is trivial. Then $\mathcal{F} \otimes \mathcal{L}_\eta \cong \mathcal{F}$.*

Proof. Write $n = n_0 p^\alpha$, where $\alpha \geq 0$ and n_0 is prime to p. Since the p-th power operation permutes the tame characters of I_0 preserving their order, $\mathcal{L}_\eta^{\otimes n_0}$ must be the trivial character. Now by [8, 1.14.2], $\mathcal{F}$ is induced from a p^α-dimensional representation $\mathcal{G}$ of $I_0(n_0)$, the unique open subgroup of I_0 of index n_0. Then

$$\mathcal{F} \otimes \mathcal{L}_\eta = (\mathrm{Ind}_{I_0(n_0)}^{I_0} \mathcal{G}) \otimes \mathcal{L}_\eta \cong \mathrm{Ind}_{I_0(n_0)}^{I_0}(\mathcal{G} \otimes \mathrm{Res}_{I_0(n_0)}^{I_0} \mathcal{L}_\eta) = \mathrm{Ind}_{I_0(n_0)}^{I_0}(\mathcal{G}) = \mathcal{F}$$

since the restriction of $\mathcal{L}_\eta$ to $I_0(n_0)$ is trivial. □

We can now finish the proof of theorem 2.2 for irreducible representations

Proposition 3.7. *Let $\mathcal{F} \in \mathcal{R}_0$ be irreducible of slope $a > 0$. Write $a = c/d$, where c and d are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of I_0 such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes (c+d)}$. Then*

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

Proof. Let n be the dimension of $\mathcal{F}$. By propositions 3.4 and 3.5, there exists a tame character $\mathcal{L}_{\eta'}$ of I_0 such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_{\eta'}$, and $\mathcal{L}_{\eta'}^{\otimes n} \cong \mathcal{L}_\chi^{\otimes n(a+1)}$. Since the Swan conductor $na = nc/d$ of $\mathcal{F}$ is an integer, n must be divisible by d. Then

$$\begin{aligned}(\mathcal{L}_{\bar{\eta}'} \otimes \mathcal{L}_\eta)^{\otimes n} = \mathcal{L}_{\bar{\eta}'}^{\otimes n} \otimes \mathcal{L}_\eta^{\otimes d(n/d)} = \\ \mathcal{L}_{\bar{\chi}}^{\otimes n(a+1)} \otimes \mathcal{L}_\chi^{\otimes (c+d)n/d} = \mathcal{L}_{\bar{\chi}}^{\otimes n(a+1)} \otimes \mathcal{L}_\chi^{\otimes n(a+1)} = \mathbf{1}\end{aligned}$$

so, by lemma 3.6,

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_{\eta'} \cong (\mathcal{F} \otimes \mathcal{L}_{\eta'}) \otimes (\mathcal{L}_{\bar{\eta}'} \otimes \mathcal{L}_\eta) = \mathcal{F} \otimes \mathcal{L}_\eta.$$

□

Proof of theorem 2.2. The functors $\mathcal{R}_0^a \to \mathcal{R}_0^a$ given by $\mathcal{F} \mapsto \rho_\chi(\mathcal{F})$ and $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}_\eta$ are equivalences of categories, so they preserve direct sums. It is enough then to prove the isomorphism for indecomposable representations.

So let $\mathcal{F} \in \mathcal{R}_0^a$ be indecomposable of length m. Then by [10, Lemma 3.1.6, Lemma 3.1.7(3)] there exist an irreducible $\mathcal{F}_0 \in \mathcal{R}_0^a$ and a (necessarily tame) indecomposable unipotent $\mathcal{U}_m \in \mathcal{R}_0$ of dimension m such that $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{U}_m$. Since $\mathcal{F}$ is a succesive extension of m copies of $\mathcal{F}_0$, by exactness $\rho_\chi(\mathcal{F})$ is a succesive extension of m copies of $\rho_\chi(\mathcal{F}_0) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta$, which is irreducible. By [10, Lemma 3.1.7(2)], there is a unipotent $\mathcal{U} \in \mathcal{R}_0$ of dimension m such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta \otimes \mathcal{U}$.

Since ρ_χ is an equivalence of categories, $\rho_\chi(\mathcal{F})$ must be indecomposable, so $\mathcal{U}$ itself must be indecomposable. Therefore $\mathcal{U} \cong \mathcal{U}_m$ and

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta \otimes \mathcal{U}_m \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

□

4. Some variants

We will consider now representations of the inertia group I_∞ at infinity. For any $\mathcal{F} \in \mathcal{R}_\infty$ of slope > 1, we can take its local Fourier transform $\mathrm{FT}^\psi_{(\infty,\infty)}\mathcal{F}$, which is again in the same category. In [10, 3.4.4], N. Katz asks about a simple formula for

$$\rho'_\chi(\mathcal{F}) := \mathrm{FT}^{\psi,-1}_{(\infty,\infty)}(\mathcal{L}_{\bar\chi} \otimes \mathrm{FT}^\psi_{(\infty,\infty)}\mathcal{F}),$$

which is an auto-equivalence of the category of continuous ℓ-adic representations of $\mathcal{R}_\infty$ with slopes > 1. It can be interpreted as the wild part of the monodromy at infinity of the (additive) convolution $\mathcal{F} * \mathcal{L}_\chi$ [10, 3.4.6], where $\mathcal{F}$ is any extension of the representation $\mathcal{F}$ to a smooth sheaf on $\mathbb{G}_{m,\bar k}$ tamely ramified at 0. In this section we will prove

Theorem 4.1. *Let $\mathcal{F} \in \mathcal{R}_\infty$ be totally wild with a single slope $a > 1$. Write $a = c/d$, where c and d are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of I_∞ such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_{\bar\chi}^{\otimes(c-d)}$. Then*

$$\rho'_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

In other words, we have the formula

$$\rho'_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_{\bar\chi}^{\otimes(a-1)} \tag{5}$$

where $\mathcal{L}_{\bar\chi}^{\otimes(a-1)}$ stands for "any character that can reasonably be called $\mathcal{L}_{\bar\chi}^{\otimes(a-1)}$".

The proof is very similar to the one for ρ_χ. Since every representation in $\mathcal{R}_\infty$ is a direct sum of representations with single slopes, we can assume that $\mathcal{F}$ has a single slope a.

Lemma 4.2. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{R}_\infty$ be totally wild, with $\mathcal{F}$ having all slopes > 1. Then*

$$\mathrm{Swan}(\rho'_\chi(\mathcal{F}) \otimes \mathcal{G}) = \mathrm{Swan}(\mathcal{F} \otimes \mathcal{G}).$$

Proof. We can assume that $\mathcal{F}$ has a single slope $a > 1$. Extend $\mathcal{F}$ and $\mathcal{G}$ to smooth sheaves on $\mathbb{G}_{m,\bar k}$, tamely ramified at 0, which we will also denote by $\mathcal{F}$ and $\mathcal{G}$ (as well as their extensions by zero to $\mathbb{A}^1_{\bar k}$).

Let n be the rank of $\mathcal{F}$, and denote by $\mathcal{F}_{(0)}$ its local monodromy at 0, which is a tame representation of I_0. Since all slopes of $\mathcal{F}$ at infinity are > 1, it is a Fourier sheaf [8, Lemma 8.3.1], so its Fourier transform is a single sheaf that we will denote by $\mathrm{FT}^\psi\mathcal{F}$. By Ogg-Shafarevic, $\mathrm{FT}^\psi\mathcal{F}$ is smooth on $\mathbb{G}_{m,\bar k}$ of rank na. By Laumon's local Fourier transform theory [9, Remark 9], it has a single positive slope $\frac{a}{a-1}$ at infinity with multiplicity $n(a-1)$ and tame part isomorphic to $\widehat{\mathcal{F}_{(0)}}$, and it is unramified at 0. Then $\mathrm{FT}^{\psi}\mathcal{F}\otimes\mathcal{L}_{\bar\chi}$ also has a single slope $\frac{a}{a-1}$ at infinity with multiplicity $n(a-1)$, tame part isomorphic to $\mathcal{L}_{\bar\chi}\otimes\widehat{\mathcal{F}_{(0)}}$, and its monodromy at 0 is a direct sum of na copies of $\mathcal{L}_{\bar\chi}$.

Its inverse Fourier transform, by Ogg-Shafarevic, is smooth of rank $n(a-1)\frac{a}{a-1}+n=n(a+1)$ on $\mathbb{G}_{m,\bar k}$, and by local Fourier transform its monodromy at infinity is the direct sum of $\rho'_\chi(\mathcal{F})$ and $na=\mathrm{Swan}(\mathcal{F})$ copies of $\mathcal{L}_\chi$. At 0 is has trivial part of rank na, whith quotient isomorphic to $\mathcal{L}_\chi\otimes\mathcal{F}_{(0)}$. So

$$\mathrm{Swan}_\infty((\mathcal{F}*\mathcal{L}_\chi)\otimes\mathcal{G})=\mathrm{Swan}(\rho'_\chi(\mathcal{F})\otimes\mathcal{G})+\mathrm{Swan}(\mathcal{F})\mathrm{Swan}(\mathcal{G}).$$

We conclude exactly as in lemma 3.3. □

Using lemma 3.1 as in proposition 3.4 we deduce

Proposition 4.3. *Let $\mathcal{F}\in\mathcal{R}_\infty$ be irreducible with slope >1. Then there exists a tame character $\mathcal{L}_\eta$ of I_∞ such that $\rho'_\chi(\mathcal{F})\cong\mathcal{F}\otimes\mathcal{L}_\eta$.*

Proposition 4.4. *Let $\mathcal{F}\in\mathcal{R}_\infty$ be irreducible of dimension n and slope $a>1$, and let $\mathcal{L}_\eta$ be a tame character of I_∞ such that $\rho'_\chi(\mathcal{F})\cong\mathcal{F}\otimes\mathcal{L}_\eta$. Then $\mathcal{L}_\eta^{\otimes n}\cong\mathcal{L}_{\bar\chi}^{\otimes n(a-1)}$.*

Proof. Extend $\mathcal{F}$ to a smooth ℓ-adic sheaf on $\mathbb{G}_{m,\bar k}$, tamely ramified at 0, also denoted by $\mathcal{F}$, and let $\mathcal{F}$ also denote its extension by zero to $\mathbb{A}^1_{\bar k}$. By the proof of lemma 4.2, the sheaf $\mathcal{F}*\mathcal{L}_\chi$ is smooth on $\mathbb{G}_{m,\bar k}$, its monodromy at infinity is the direct sum of $\rho'_\chi(\mathcal{F})\cong\mathcal{F}\otimes\mathcal{L}_\eta$ and na copies of $\mathcal{L}_\chi$, and its monodromy at 0 has trivial part of dimension na with quotient isomorphic to $\mathcal{L}_\chi\otimes\mathcal{F}_{(0)}$. Its determinant is then a smooth sheaf of rank 1 on $\mathbb{G}_{m,\bar k}$, whose monodromy at ∞ is $\det(\mathcal{F})\otimes\mathcal{L}_\eta^{\otimes n}\otimes\mathcal{L}_\chi^{\otimes na}$, and whose monodromy at 0 is $\det(\mathcal{F}_{(0)})\otimes\mathcal{L}_\chi^{\otimes n}$.

We conclude, as in proposition 3.5, that

$$\det(\mathcal{F}*\mathcal{L}_\chi)\cong\det(\mathcal{F})\otimes\mathcal{L}_\eta^{\otimes n}\otimes\mathcal{L}_\chi^{\otimes na}$$

as sheaves on $\mathbb{G}_{m,\bar k}$. Comparing their monodromies at 0 gives the desired isomorphism. □

The remainder of the proof of theorem 4.1 is identical to the one for ρ_χ.

We have a third variant, for representations $\mathcal{F}\in\mathcal{R}_\infty$ with slopes <1:

$$\rho''_\chi(\mathcal{F}):=\mathrm{FT}^{\psi,-1}_{(\infty,0)}(\mathcal{L}_{\bar\chi}\otimes\mathrm{FT}^{\psi}_{(\infty,0)}(\mathcal{F})),$$

which is again an auto-equivalence of the category of continuous ℓ-adic representations of $\mathcal{R}_\infty$ with slopes <1. As in the ρ_χ case we have $\rho''_\chi(\mathcal{F})\cong\mathcal{F}\otimes\mathcal{L}_\chi$ for $\mathcal{F}$ tame. The corresponding formula for wild $\mathcal{F}$ is

Theorem 4.5. *Let $\mathcal{F}\in\mathcal{R}_\infty$ be totally wild with a single slope $a<1$. Write $a=c/d$, where c and d are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of I_∞ such that $\mathcal{L}_\eta^{\otimes d}=\mathcal{L}_\chi^{\otimes(d-c)}$. Then*

$$\rho''_\chi(\mathcal{F})\cong\mathcal{F}\otimes\mathcal{L}_\eta.$$

Proof. Let $\mathcal{G}:=\mathrm{FT}^{\psi}_{(\infty,0)}(\mathcal{F})\in\mathcal{R}_0$, which has slope $\frac{a}{1-a}=\frac{c}{d-c}$ [9, Theorem 13]. The statement is then equivalent to

$$\mathrm{FT}^{\psi,-1}_{(\infty,0)}(\mathcal{L}_{\bar\chi}\otimes\mathcal{G})\cong\mathcal{L}_\eta\otimes\mathrm{FT}^{\psi,-1}_{(\infty,0)}(\mathcal{G})$$

or

$$\mathrm{FT}^{\psi}_{(\infty,0)}(\mathcal{L}_\eta\otimes\mathrm{FT}^{\psi,-1}_{(\infty,0)}(\mathcal{G}))\cong\mathcal{G}\otimes\mathcal{L}_{\bar\chi}.$$

But the left hand side is just $\rho_{\bar{\eta}}(\mathcal{G})$, since the inverse of $\mathrm{FT}^{\psi}_{(\infty,0)}$ is $\mathrm{FT}^{\bar{\psi}}_{(0,\infty)}$ with respect to the conjugate additive character, and ρ_{χ} does not depend on the choice of the non-trivial additive character ψ. So the isomorphism follows from theorem 2.2. □

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