Estimates for Singular Multiplicative Character Sums

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Abstract

We give some estimates for multiplicative character sums on quasi-projective varieties over finite fields depending on the severity of the singularities of the variety at infinity. We also remove the hypothesis of non-divisibility by the characteristic of the base field in the known estimates for the non-singular case.

1 Introduction

In [Ka4], Katz proved the following estimate for multiplicative character sums. Let k be a finite field of characteristic p and cardinality q, and X/k a projective smooth scheme of dimension n endowed with a k-embedding in $\mathbb{P}^N_k$. Let Z (resp. H) be a hyperplane (resp. a hypersurface of degree d) in $\mathbb{P}^N_k$, and suppose that $X \cap H$, $X \cap Z$ and $X \cap H \cap Z$ are all smooth of the right codimension. Then, if $V = X - X \cap Z$ and $f : V \to \mathbb{A}^1_k$ denotes the map $f(x) = H(x)/Z(x)^d$,

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for any non-trivial multiplicative character $\chi : k^\star \to \mathbb{C}^\star$ we have the estimate

$$\left| \sum_{x \in V(k)} \chi(f(x)) \right| \leq C \cdot q^{n/2}$$

where C depends only on d, n and the total Chern class of X.

This article extends this result to the singular case, in the same way that [Ka2] extended the results in [Ka1] for additive character sums. Let X be a scheme which is projective over k and purely of dimension $n \geq 2$, embedded in $\mathbb{P}^N_k$ as the closed subscheme defined by r homogeneous forms $F_1, \ldots, F_r$ of degrees $a_1, \ldots, a_r$. Let H and Z be homogeneous forms in $k[X_0, \ldots, X_N]$ of degrees d and e. We will also denote by H and Z the hypersurfaces they define in $\mathbb{P}^N_k$. Assume that (H, Z) is a regular sequence in the graded coordinate ring $\oplus_{i \geq 0} \Gamma(X, \mathcal{O}_X(i))$ of X (If X is Cohen-Macaulay, this just means that $X \cap H \cap Z$ has pure codimension 2 in X). For simplicity we will also assume that d and e are coprime. See the remarks at the end of section 3 for the case where they are not.

Following [Ka2], we define δ to be the dimension of the singular locus of $X \cap H \cap Z$, and ε to be that of the singular locus of $X \cap Z$. We also define ε' as the dimension of the singular locus of $X \cap H$. We have the a priori inequalities (cf. [Ka2], Lemma 3)

$$\varepsilon \leq \delta + 1, \ \varepsilon' \leq \delta + 1.$$

since the singular locus of $X \cap H \cap Z$ contains the intersection of Z and the singular locus of $X \cap H$ and the intersection of H and the singular locus of $X \cap Z$.

Fix a non-trivial multiplicative character $\chi : k^\star \to \mathbb{C}^\star$. Let $V = X - (H \cup Z)$ and $f : V \to \mathbb{G}_{m,k}$ be the map defined by $f(x) = H(x)^e / Z(x)^d$. Our main result is:

Theorem 1. *Denote by S the sum $\sum_{x \in V(k)} \chi(f(x))$.*

a) Suppose that e is prime to p and χ^e is non-trivial (for instance $e = 1$). Let $C = 3(3 + \sup(a_1, \ldots, a_r, e) + d)^{N+r+2}$. We have the estimate

$$|S| \leq C \cdot q^{(n+\delta+2)/2}.$$

Furthermore, if $\varepsilon' \leq \delta$, we have the sharper estimate

$$|S| \leq C \cdot q^{(n+\delta+1)/2}.$$

b) Suppose that d *is prime to* p *and* χ^d *is non-trivial. Let* $C = 3(3 + \sup(a_1, \ldots, a_r, d) + e)^{N+r+2}$*. We have the estimate*

$$|S| \leq C \cdot q^{(n+\delta+2)/2}.$$

Furthermore, if $\varepsilon \leq \delta$*, we have the sharper estimate*

$$|S| \leq C \cdot q^{(n+\delta+1)/2}.$$

c) Suppose that $\gcd(d,p) = \gcd(e,p) = 1$*. Let*

$$C = 3(3 + \sup(a_1, \ldots, a_r, d, e) + \sup(d,e))^{N+r+2}.$$

We have the estimate

$$|S| \leq C \cdot q^{(n+\delta+2)/2}.$$

Furthermore, if $\varepsilon \leq \delta$ *and* $\varepsilon' \leq \delta$*, we have the sharper estimate*

$$|S| \leq C \cdot q^{(n+\delta+1)/2}.$$

Part (b) of the theorem is deduced from part (a) by just switching the roles of H and Z and replacing χ by $\bar{\chi}$, since

$$\bar{\chi}(Z(x)^d/H(x)^e) = \chi(H(x)^e/Z(x)^d).$$

Part (c) follows immediately from (a) and (b). It remains to prove (a).

2 Cohomological interpretation of the sums

Fix a prime $\ell \neq p$, we will work with ℓ-adic cohomology. We will pick an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ so that we can freely speak of absolute values of element of $\bar{\mathbb{Q}}_\ell$ and weights. This also gives a way to look at a $\mathbb{C}^\star$-valued character as a $\bar{\mathbb{Q}}_\ell^\star$-valued character and viceversa. Given a non-trivial multiplicative character $\chi : k^\star \to \mathbb{C}^\star$, there is an associated Kummer $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_{m,k}$ (cf. [De2], 1.7) such that for every finite extension k'/k and every $t \in \mathbb{G}_m(k') = k'^\star$, the trace of the geometric Frobenius element in $\mathrm{Gal}(\bar{k}/k')$ acting on the stalk of $\mathcal{L}_\chi$ at a geometric point $\bar{t}$ over t is $\chi(\mathrm{N}_{k'/k}(t))$. In particular, $\mathcal{L}_\chi$ is pure of weight zero.

If we denote by $\mathcal{L}_{\chi(f)}$ the pull-back of $\mathcal{L}_\chi$ to V by f, it follows from Grothendieck trace formula that

$$\sum_{x\in V(k)} \chi(f(x)) = \sum_{i=0}^{2n} (-1)^i \operatorname{Trace}(F|\mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_{\chi(f)}))$$

where $F \in \mathrm{Gal}(\bar{k}/k)$ is the geometric Frobenius element. Furthermore, Deligne's theorem (cf. [De1], Corollaire 3.3.4) implies that all eigenvalues of F acting on $\mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_{\chi(f)})$ have absolute value at most $q^{i/2}$. Therefore, Theorem 1 will be a consequence of the following two cohomological results:

Theorem 2. *With the previous notation, suppose that e is prime to p and χ^e is non-trivial. Then the cohomology group $\mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_{\chi(f)})$ vanishes for $i > n+\delta+2$. Furthermore, if $\varepsilon' \leq \delta$, it also vanishes for $i = n+\delta+2$.*

Theorem 3. *Suppose that e is prime to p and χ^e is non-trivial. Then we have the bound*

$$\sum_i \dim \mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_{\chi(f)}) \leq 3(3+\sup(a_1,\ldots,a_r,e)+d)^{N+r+2}.$$

Consider the finite étale covering $\pi : W \to V$ given by

$$W := \{(x,s)\in V\times\mathbb{G}_{m,k} : s^e = f(x)\}$$

mapping to V via the first projection, and let $g : W \to \mathbb{G}_{m,k}$ be the restriction of the second projection. We have a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{\pi} & V \\ \downarrow g & & \downarrow f \\ \mathbb{G}_{m,k} & \xrightarrow{[e]} & \mathbb{G}_{m,k} \end{array}$$

where $[e]$ is the e-th power map $\lambda \mapsto \lambda^e$. Since π is finite, we have $\mathrm{R}\pi_\star = \pi_\star = \pi_!$. Combining that with proper base change and the projection formula we get

$$\mathrm{H}^i_c(W\otimes\bar{k},\mathcal{L}_{\chi^e(g)}) = \mathrm{H}^i_c(W\otimes\bar{k},g^\star[e]^\star\mathcal{L}_\chi) = \mathrm{H}^i_c(W\otimes\bar{k},\pi^\star f^\star\mathcal{L}_\chi) =$$
$$= \mathrm{H}^i_c(V\otimes\bar{k},\pi_\star\pi^\star\mathcal{L}_{\chi(f)}) = \mathrm{H}^i_c(V\otimes\bar{k},(\pi_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi(f)}) =$$
$$= \mathrm{H}^i_c(V\otimes\bar{k},(\pi_\star g^\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi(f)}) = \mathrm{H}^i_c(V\otimes\bar{k},(f^\star[e]_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi(f)}) =$$
$$= \mathrm{H}^i_c(V\otimes\bar{k},(f^\star \textstyle\bigoplus_{\rho^e=\mathbf{1}}\mathcal{L}_\rho)\otimes\mathcal{L}_{\chi(f)}) =$$
$$= \mathrm{H}^i_c(V\otimes\bar{k},\textstyle\bigoplus_{\rho^e=\mathbf{1}}\mathcal{L}_{\rho(f)}\otimes\mathcal{L}_{\chi(f)}) = \textstyle\bigoplus_{\rho^e=\mathbf{1}}\mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_{\rho\chi(f)})$$

where the direct sum is taken over the set of characters of $k^\star$ whose e-th power is trivial. In particular, $\mathrm{H}^i_c(V\otimes\bar{k},\mathcal{L}_\chi(f))$ is a direct summand of $\mathrm{H}^i_c(W\otimes\bar{k},\mathcal{L}_{\chi^e(g)})$, so in order to prove Theorems 2 and 3 it suffices to show

Theorem 4. *With the previous notation, suppose that* e *is prime to* p *and* χ^e *is non-trivial. Then the cohomology group* $\mathrm{H}^i_c(W\otimes\bar{k},\mathcal{L}_{\chi^e(g)})$ *vanishes for* $i>n+\delta+2$*. Furthermore, if* $\varepsilon'\leq\delta$*, it also vanishes for* $i=n+\delta+2$*.*

Theorem 5. *Suppose that* e *is prime to* p *and* χ^e *is non-trivial. Then we have the bound*

$$\sum_i \dim \mathrm{H}^i_c(W\otimes\bar{k},\mathcal{L}_{\chi^e(g)})\leq 3(3+\sup(a_1,\ldots,a_r,e)+d)^{N+r+2}.$$

We will now construct a new scheme Y as the closed subscheme of $\mathbb{P}^{N+2}_k$ (with coordinates $X_0,\ldots,X_N,T,U$) defined by the homogeneous forms $F_1,\ldots,F_r,T^e-Z$ and $H-UT^{d-1}$. Roughly speaking, we are adding two more variables to X, one representing the e-th root of Z and the other one the e-th root of the value of f. Then, define the incidence variety $\tilde{Y}$ as the divisor in $Y\times\mathbb{P}^1_k$ (with coordinates λ_0,λ_1 for the second factor) given by the vanishing of $\lambda_0U-\lambda_1T$, thus

$$\tilde{Y}(\bar{k})=\{(x_0,\ldots,x_N,t,u),(\lambda_0,\lambda_1))\in Y(\bar{k})\times\mathbb{P}^1(\bar{k}):\lambda_0u=\lambda_1t\}.$$

Let $\tilde{g}:\tilde{Y}\to\mathbb{P}^1_k$ be the restriction of the projection $Y\times\mathbb{P}^1_k\to\mathbb{P}^1_k$, it is a proper map. We can embed W as a dense open subset of $\tilde{Y}$ in the following way: first pick $\alpha,\beta\in\mathbb{Z}$ such that $\alpha d+\beta e=1$. We map the point $(x,s)\in W$ (where $x=(x_0,\ldots,x_N)$) to the pair $\tau(x,s):=((x_0,\ldots,x_N,t,u),s)\in\tilde{Y}$, where

$$t=\frac{H(x)^\alpha Z(x)^\beta}{s^\alpha},\ u=\frac{H(x)^\alpha Z(x)^\beta}{s^{\alpha-1}}.$$

Notice that $H^\alpha Z^\beta$ is a rational function of total degree 1 defined at every point of V, therefore the map is well defined. This gives an isomorphism between W and the dense open subset of $\tilde{Y}$ where $T\neq 0$ and $U\neq 0$, the inverse map being given by $((x_0,\ldots,x_N,t,u),s)\mapsto(x,s)$. We have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ \tau\ } & \tilde{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbb{G}_{m,k} & \longrightarrow & \mathbb{P}^1_k \end{array}$$

where the horizontal arrows are open embeddings. We extend by zero the sheaf $\mathcal{L}_{\chi^e}$ to all of $\mathbb{P}^1_k$, and take its pull-back to $\tilde{Y}$ by $\tilde{g}$, which we will denote by $\mathcal{L}_{\chi^e(g)}$. Notice that its restriction to W is just the previously defined $\mathcal{L}_{\chi^e(g)}$.

Lemma 6. *There is a quasi-isomorphism*

$$\mathrm{R}\Gamma_c(W \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{Y} \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}).$$

Proof. By excision, it suffices to show that $\mathrm{R}\Gamma_c((\tilde{Y} - W) \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}) = 0$. We have a decomposition of $\tilde{Y} - W$ as the disjoint union of W_0, W_1 and W_2, where (identifying T and U with the divisors they define)

$$\begin{aligned} W_0 &= (Y \cap T \cap U) \times \mathbb{P}^1_k \\ W_1 &= (\tilde{Y} \cap T) - U \\ W_2 &= (\tilde{Y} \cap U) - T. \end{aligned}$$

Since W_1 maps to infinity under $\tilde{g}$, the sheaf $\mathcal{L}_{\chi^e(g)}$ vanishes on W_1. Similarly, W_2 maps to zero under $\tilde{g}$, so $\mathcal{L}_{\chi^e(g)}$ vanishes on W_2 too. Again by excision we deduce

$$\mathrm{R}\Gamma_c((\tilde{Y} - W) \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c((Y \cap T \cap U) \times \mathbb{P}^1_{\bar{k}}, \mathcal{L}_{\chi^e(g)}).$$

Now on $(Y \cap T \cap U) \times \mathbb{P}^1$ the sheaf $\mathcal{L}_{\chi^e(g)}$ is the external tensor product $\bar{\mathbb{Q}}_\ell \boxtimes \mathcal{L}_{\chi^e}$, so by Künneth we conclude that

$$\mathrm{R}\Gamma_c((Y \cap T \cap U) \times \mathbb{P}^1_{\bar{k}}, \mathcal{L}_{\chi^e(g)}) =$$

$$= \mathrm{R}\Gamma_c((Y \cap T \cap U) \otimes \bar{k}, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_{\chi^e}) = 0$$

since $\mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_{\chi^e}) = \mathrm{R}\Gamma_c(\mathbb{G}_{m,\bar{k}}, \mathcal{L}_{\chi^e}) = 0$ when χ^e is non-trivial (cf. [De2], Théorème 2.7*)

□

By the projection formula, we have

$$\mathrm{R}\Gamma_c(\tilde{Y} \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}) = \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}\tilde{g}_\star \tilde{g}^\star \mathcal{L}_{\chi^e}) = \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, (\mathrm{R}\tilde{g}_\star \bar{\mathbb{Q}}_\ell) \otimes \mathcal{L}_{\chi^e}).$$

Furthermore, there is a spectral sequence

$$\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, (\mathrm{R}^b \tilde{g}_\star \bar{\mathbb{Q}}_\ell) \otimes \mathcal{L}_{\chi^e}) \Rightarrow \mathrm{H}^{a+b}_c(\mathbb{P}^1_{\bar{k}}, (\mathrm{R}\tilde{g}_\star \bar{\mathbb{Q}}_\ell) \otimes \mathcal{L}_{\chi^e}).$$

In particular, in order to prove Theorem 4 it suffices to show that $\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, (\mathrm{R}^b \tilde{g}_\star \bar{\mathbb{Q}}_\ell) \otimes \mathcal{L}_{\chi^e})$ vanishes when $a + b > n + \delta + 2$, and when $a + b = n + \delta + 2$ if $\varepsilon' \leq \delta$.

Lemma 7. *The map $\tilde{g} : \tilde{Y} \to \mathbb{P}^1_k$ is flat.*

Proof. Following ([Ka2], Lemma 9) we will show that all geometric fibers of $\tilde{g}$ have the same Hilbert polynomial. Let $C(X) \subset \mathbb{P}^{N+1}_k$ be the cone over X, i.e. the subscheme defined in $\mathbb{P}^{N+1}_k$ (with coordinates $X_0, \ldots, X_N, T$) by the same ideal that defines X in $\mathbb{P}^N_k$. First of all, notice that T is not a zero divisor in (the homogeneous coordinate ring of) $C(X)$, and the section of $C(X)$ it defines is isomorphic to X. Since (H, Z) is a regular sequence for X by hypothesis, we conclude that (T, H, Z) in a regular sequence for $C(X)$. Recall that the property of a sequence of *homogeneous* elements in a graded ring being a regular sequence is invariant under permutation of the elements of the sequence (cf. [BT], Lemma 23.5).

The fiber of $\tilde{g}$ over a finite point $\lambda \in \bar{k}$ is defined in $\mathbb{P}^{N+2}_{\bar{k}}$ (with coordinates $X_0, \ldots, X_N, T, U$) by the vanishing of $F_1, \ldots, F_r, Z - T^e, H - \lambda T^d$ and $U - \lambda T$. So in $\mathbb{P}^{N+1}_{\bar{k}}$ (which we identify with the hyperplane $U - \lambda T = 0$ in $\mathbb{P}^{N+2}_{\bar{k}}$) it is obtained from $C(X)$ by taking the hypersurface sections $Z - T^e = 0$ and $H - \lambda T^d = 0$. But $(Z - T^e, H - \lambda T^d)$ is a regular sequence in $C(X)$ for every λ (because it is if we add T), so the Hilbert polynomial of any such fiber is given by

$$P(m) = Q(m) - Q(m-d) - Q(m-e) + Q(m-d-e),$$

where Q is the Hilbert polynomial of $C(X)$.

Similarly, the fiber over infinity is defined in $\mathbb{P}^{N+2}_{\bar{k}}$ by the vanishing of $F_1, \ldots, F_r, Z, H$ and T, so in $\mathbb{P}^{N+1}_{\bar{k}}$ (identified with the hyperplane $T = 0$ in $\mathbb{P}^{N+2}_{\bar{k}}$) it is obtained from $C(X)$ by taking the hypersurface sections $Z = 0$ and $H = 0$. Again (Z, H) is a regular sequence in $C(X)$, so the Hilbert polynomial of this fiber is also given by

$$P(m) = Q(m) - Q(m-d) - Q(m-e) + Q(m-d-e).$$

□

The proof of the previous lemma shows that the intersection of the fiber of $\tilde{g}$ over a finite point $\lambda \in \bar{k}$ with the hyperplane $T = 0$ is just $X \cap H \cap Z$, which has singular locus of dimension δ. Therefore, the fiber itself has singular locus of dimension at most $\delta + 1$. Similarly, the fiber over infinity is the cone over $X \cap H \cap Z$, so it has singular locus of dimension $\delta + 1$. From ([SGA7I], Exposé I, Cor. 4.3) we deduce that for every $\lambda \in \mathbb{P}^1(\bar{k})$ the I_λ-invariant specialization map

$(\mathrm{R}^b\tilde{g}_\star\bar{\mathbb{Q}}_\ell)_\lambda \to (\mathrm{R}^b\tilde{g}_\star\bar{\mathbb{Q}}_\ell)_{\bar\eta}$ (where $\bar\eta$ is a geometric generic point of $\mathbb{P}^1_{\bar k}$ and I_λ the inertia group at λ) is an isomorphism for $b > n+\delta+1$ and surjective for $b = n+\delta+1$. This implies that $\mathrm{R}^b\tilde{g}_\star\bar{\mathbb{Q}}_\ell$ is lisse on $\mathbb{P}^1_{\bar k}$ for $b > n+\delta+1$, and that we have an exact sequence

$$0 \to \mathcal{G} \to \mathrm{R}^{n+\delta+1}\tilde{g}_\star\bar{\mathbb{Q}}_\ell \to \mathcal{H} \to 0$$

where $\mathcal{H}$ is lisse on $\mathbb{P}^1_{\bar k}$ and $\mathcal{G}$ is punctual (cf. [Ka2], Theorem 13).

Since $\mathbb{P}^1_{\bar k}$ is simply connected, every lisse sheaf on it is constant. In particular, for $b > n+\delta+1$ and any a we get

$$\mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, (\mathrm{R}^b\tilde{g}_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi^e}) = (\mathrm{R}^b\tilde{g}_\star\bar{\mathbb{Q}}_\ell)_{\bar\eta}\otimes\mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, \mathcal{L}_{\chi^e}) = 0$$

since χ^e is non-trivial (cf. [De2], Théorème 2.7*). Similarly $\mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, \mathcal{H}\otimes\mathcal{L}_{\chi^e}) = 0$, so from the exact sequence above we get isomorphisms

$$\mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, \mathcal{G}\otimes\mathcal{L}_{\chi^e}) \cong \mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, (\mathrm{R}^{n+\delta+1}\tilde{g}_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi^e}).$$

Now $\mathcal{G}$ is punctual, so we conclude that $\mathrm{H}^a_c(\mathbb{P}^1_{\bar k}, (\mathrm{R}^{n+\delta+1}\tilde{g}_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi^e}) = 0$ for $a > 0$. Since H^a_c of any constructible sheaf on $\mathbb{P}^1_{\bar k}$ vanishes for all $a > 2$, this covers all possible cases where $a+b > n+\delta+2$. The only case with $a+b = n+\delta+2$ that has not yet been considered is $a = 2$, $b = n+\delta$. So it remains to show that $\mathrm{H}^2_c(\mathbb{P}^1_{\bar k}, (\mathrm{R}^{n+\delta}\tilde{g}_\star\bar{\mathbb{Q}}_\ell)\otimes\mathcal{L}_{\chi^e}) = 0$ when $\varepsilon' \le \delta$.

Lemma 8. *The sheaf* $\mathcal{F} := \mathrm{R}^{n+\delta}\tilde{g}_\star\bar{\mathbb{Q}}_\ell$ *is lisse at* $0 \in \mathbb{P}^1_{\bar k}$.

Proof. Let $I = I_0 \subset \mathrm{Gal}(\bar k(t)^{sep}/\bar k(t))$ be the inertia group at zero. If $\bar\eta$ is a geometric generic point of $\mathbb{P}^1_{\bar k}$, the lemma states that I acts trivially on $\mathcal{F}_{\bar\eta}$. Therefore, it suffices to show that the I-invariant specialization map $\mathcal{F}_0 \to \mathcal{F}_{\bar\eta}$ is surjective. By ([SGA7I], Exposé I, Cor. 4.3), this will happen if the fiber of $\tilde g$ at zero has singular locus of dimension at most δ.

Such fiber $\tilde Y_0$ is given in $\mathbb{P}^{N+2}_{\bar k}$ (with coordinates $X_0,\dots,X_N,T,U$) by the vanishing of $F_1,\dots,F_r,T^e-Z,H$ and U. We have an obvious finite projection map $\pi : \tilde Y_0 \to X\cap H$, which is étale outside $\tilde Y_0 - T$. In particular, the singularities of $\tilde Y_0 - T$ map to singularities of $X\cap H$. But the singular locus of $X\cap H$ has dimension $\varepsilon' \le \delta$ and π is finite, so the singular locus of $\tilde Y_0$ outside T has dimension at most δ.

On the other hand, a singular point of $\tilde Y_0$ in $\tilde Y_0\cap T$ must also be a singular point of $\tilde Y_0\cap T$ (cf. [Ka2], Lemma 3). But $\tilde Y_0\cap T$ is isomorphic to $X\cap H\cap Z$, so its singular locus has dimension δ. Therefore, the singular locus of $\tilde Y_0$ in T also has dimension at most δ. This proves the lemma. ☐

Let $U \subset \mathbb{G}_{m,\bar{k}}$ be a dense open subset on which $\mathcal{F}$ is lisse. By the birational invariance of H^2_c it suffices to show that $\mathrm{H}^2_c(U, \mathcal{F} \otimes \mathcal{L}_{\chi^e}) = 0$. By Lemma 8 $\mathcal{F}$ is lisse at zero. Therefore if $I = I_0 \subset \mathrm{Gal}(\bar{k}(t)^{sep}/\bar{k}(t))$ is the inertia group at zero, I acts trivially on the stalk $\mathcal{F}_{\bar{\eta}}$ of $\mathcal{F}$ at a geometric generic point $\bar{\eta}$ of $\mathbb{P}^1_{\bar{k}}$. On the other hand, since χ^e is non-trivial, $\mathcal{L}_{\chi^e}$ is totally ramified at zero, so $(\mathcal{L}_{\chi^e})^I_{\bar{\eta}} = 0$. Hence

$$(\mathcal{F} \otimes \mathcal{L}_{\chi^e})^I_{\bar{\eta}} = (\mathcal{F}_{\bar{\eta}} \otimes (\mathcal{L}_{\chi^e})_{\bar{\eta}})^I = \mathcal{F}_{\bar{\eta}} \otimes (\mathcal{L}_{\chi^e})^I_{\bar{\eta}} = 0.$$

In particular the coinvariants $((\mathcal{F} \otimes \mathcal{L}_{\chi^e})_{\bar{\eta}})_I$ also vanish and a fortiori

$$\mathrm{H}^2_c(U, \mathcal{F} \otimes \mathcal{L}_{\chi^e}) = ((\mathcal{F} \otimes \mathcal{L}_{\chi^e})_{\bar{\eta}})_{\pi_1(U,\bar{\eta})}(-1) = 0.$$

This completes the proof of Theorem 4.

3 An upper bound for the sum of the Betti numbers

In this section we will prove Theorem 5. The main tool will be the following bound of Katz:

Theorem 9. ([Ka3], Theorem 12) *Let $V \subset \mathbb{A}^N_k$ be a closed subscheme, defined by the vanishing of r polynomials $f_1, \ldots, f_r$ of degrees $a_1, \ldots, a_r$. Let $h, h_1, \ldots, h_s \in k[x_1, \ldots, x_N]$, $s \geq 0$, be polynomials of degrees $e, e_1, \ldots, e_s$. Fix a non-trivial additive character $\psi : k \to \mathbb{C}^\star$ and s non-trivial multiplicative characters $\chi_1, \ldots, \chi_s : k^\star \to \mathbb{C}^\star$. Let $\mathcal{L}_\psi$ and $\mathcal{L}_{\chi_j}$ be the corresponding Artin-Schreier and (extension by zero of) Kummer $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathbb{A}^1_k$, and denote by $\mathcal{L}_{\psi(h)}$ and $\mathcal{L}_{\chi_j(h_j)}$ their pull-backs to V by h and h_j respectively. Then we have the upper bound*

$$\sum_i \dim \mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\psi(h)} \otimes (\bigotimes_{j=1}^s \mathcal{L}_{\chi_j(h_j)})) \leq$$

$$\leq 3(s + 1 + \sup_i(e, 1 + a_i) + \sum_j e_j)^{N+r}.$$

In order to optimize the bound, we will not embed W in Y, but in a new projective scheme Y' defined in $\mathbb{P}^{N+1}_k$ (with coordinates $X_0, \ldots, X_N, T$) by the homogeneous forms $F_1, \ldots, F_r$ and $T^e - Z$. We now embed W as a dense open subscheme of Y' by mapping (x, s), where $x = (x_0, \ldots, x_N)$, to $(x_0, \ldots, x_N, t) \in Y'$, with $t = s^{-\alpha} H(x)^\alpha Z(x)^\beta$ (Recall that α and β are integers such that $\alpha d + \beta e =$

1). This gives an isomorphism between W and the open subset of Y' where $T \neq 0$ and $H \neq 0$, the inverse map being

$$(x_0, \ldots, x_N, t) \mapsto ((x_0, \ldots, x_N), t^{-d}H(x)).$$

Take the ambient space $\mathbb{A}_k^{N+1}$ to be the projective space $\mathbb{P}_k^{N+1}$ minus the hyperplane $T = 0$. So we have coordinates $x_0 = X_0/T, \ldots, x_N = X_N/T$. With this notation, the closure $\overline{W}$ of W is defined by the vanishing of $F_i(x_0, \ldots, x_N)$ for $i = 1, \ldots, r$ and $Z(x_0, \ldots, x_N) - 1$, and g is given by the polynomial $H(x_0, \ldots, x_N)$ on W. If we apply Theorem 9 to this data, with $s = 1$, $h = 0$ and $h_1 = g$, we get the desired bound, since $\mathrm{H}_c^i(W \otimes \bar{k}, \mathcal{L}_{\chi^e(g)}) = \mathrm{H}_c^i(\overline{W} \otimes \bar{k}, \mathcal{L}_{\chi^e(g)})$ (where we extended g by zero to $\overline{W}$).

Remarks. 1) The following example, multiplicative analogue of the one given in [Ka2], will show that the exponent of q is optimal in these estimates and that the sharper estimate does not hold without some extra hypothesis. Let $N = n + 1$, and let X be the hypersurface in $\mathbb{P}_k^{n+1}$ (with coordinates $X_0, \ldots, X_{n+1}$) defined by the equation $X_2^{q-1} - X_1 X_0^{q-2} = 0$. Let Z be the hyperplane defined by $X_0 = 0$, H the one defined by $X_1 = 0$. Hence $d = e = 1$, $X \cap Z$ (resp. $X \cap H$) is the everywhere singular $(n-1)$-dimensional linear subspace $X_0 = X_2^{q-1} = 0$ (resp. $X_1 = X_2^{q-1} = 0$) of $\mathbb{P}_k^{n+1}$, and $X \cap H \cap Z$ is the everywhere singular $(n-2)$-dimensional linear subspace $X_0 = X_1 = X_2^{q-1} = 0$. So $\varepsilon = \varepsilon' = n - 1$ and $\delta = n - 2$.

Then V is defined in $\mathbb{A}_k^{n+1}$ (with coordinates $x_i = X_i/X_0$, $i = 1, \ldots, n+1$) by $x_1 = x_2^{q-1}$, $x_1 \neq 0$; and $f : V \to \mathbb{G}_{m,k}$ is the map given by $f(x_1, \ldots, x_{n+1}) = x_1$. So in this case, for every finite extension k_m/k of degree m the sum is

$$\sum_{(x_1, \ldots, x_{n+1}) \in V(k_m)} \chi(\mathrm{N}_{k_m/k}(x_1)) = \sum_{\substack{x_2 \in k_m^\star \\ x_3, \ldots, x_{n+1} \in k_m}} \chi(\mathrm{N}_{k_m/k}(x_2^{q-1})) =$$

$$= \sum_{\substack{x_2 \in k_m^\star \\ x_3, \ldots, x_{n+1} \in k_m}} \chi(\mathrm{N}_{k_m/k}(x_2))^{q-1} = q^{m(n-1)}(q^m - 1) \neq O(q^{m\alpha/2})$$

for any $\alpha < 2n = n + \delta + 2$.

2) What if d and e are not coprime? In that case, let a be their greatest common divisor, $d' = d/a$ and $e' = e/a$. Let f be the map

defined on V by $H(x)^{e'}/Z(x)^{d'}$. Consider the a-uple embedding $\iota_a : \mathbb{P}^N_k \hookrightarrow \mathbb{P}^{N'}_k$, where $N' = \binom{N+a}{a} - 1$. Denote by X' the image of X under this embedding. Let H' and Z' be forms of degrees d' and e' in $k[Y_0, \ldots, Y_{N'}]$ such that $\iota_a^\star H' = H$ and $\iota_a^\star Z' = Z$. Then $V \cong V' := X' - (H' \cup Z')$, $X \cap H \cong X' \cap H'$, $X \cap Z \cong X' \cap Z'$ and $X \cap H \cap Z \cong X' \cap H' \cap Z'$. Let $f' : V' \to \mathbb{G}_{m,k}$ be the map defined by $f'(x) = H'(x)^{e'}/Z'(x)^{d'}$. Clearly $f = f' \circ \iota_{a|V}$. Since $\iota_{a|V} : V \to V'$ is an isomorphism we deduce

$$\mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) \cong \mathrm{H}^i_c(V' \otimes \bar{k}, \mathcal{L}_{\chi(f')}).$$

Hence Theorems 2 and 3 still hold in this case for the map f defined above, after replacing d by d' and e by e'.

4 The smooth case

Here we take $e = 1$, so Z is now a linear form. Suppose that X , $X \cap H$ and $X \cap H \cap Z$ are all smooth. Then Theorem 2 implies

Theorem 10. *Under these hypotheses,* $\mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)})$ *vanishes for* $i \neq n$.

Proof. For $i > n$, $\mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = 0$ by Theorem 2, since $\varepsilon' = \delta = -1$ here. For $i < n$, $\mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = 0$ by Poincaré duality, since V is smooth and affine and $\mathcal{L}_{\chi(f)}$ is lisse on V. □

In this case

$$\textstyle\sum_i \dim \mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) =$$

$$= \dim \mathrm{H}^n_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = (-1)^n \chi_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)})$$

where $\chi_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = \sum_i (-1)^i \dim \mathrm{H}^i_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)})$ is the compact Euler characteristic. Theorem 10 is proved in [Ka4] under the additional hypotheses that $X \cap Z$ is also smooth and either d is prime to p or χ^d is trivial, and an exact formula for the Euler characteristic is given (Theorem 5.1), namely

$$\chi_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = \int_X \frac{c(X)}{(1+L)(1+dL)} \tag{1}$$

where $c(X)$ is the total Chern class of X and L is the class of a hyperplane. But in fact the formula is still valid when p divides d:

Lemma 11. *The compact Euler characteristic of $\mathcal{L}_{\chi(f)}$ on $V \otimes \bar{k}$ is given by (1) for any d.*

Proof. Since $\mathcal{L}_\chi$ is lisse of rank 1 on $\mathbb{G}_{m,\bar{k}}$ and tame at both 0 and ∞, by the Grothendieck-Neron-Ogg-Shafarevic formula (cf. [SGA5], Exposé X, Théorème 7.1) we have

$$\chi_c(\mathbb{G}_{m,\bar{k}}, \mathrm{R}^j f_! \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\chi) = \chi_c(\mathbb{G}_{m,\bar{k}}, \mathrm{R}^j f_! \bar{\mathbb{Q}}_\ell)$$

for every $j \geq 0$. In particular, using the spectral sequences

$$\mathrm{H}^i_c(\mathbb{G}_{m,\bar{k}}, \mathrm{R}^j f_! \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\chi) \Rightarrow \mathrm{H}^{i+j}_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)})$$

and

$$\mathrm{H}^i_c(\mathbb{G}_{m,\bar{k}}, \mathrm{R}^j f_! \bar{\mathbb{Q}}_\ell) \Rightarrow \mathrm{H}^{i+j}_c(V \otimes \bar{k}, \bar{\mathbb{Q}}_\ell)$$

we deduce that

$$\chi_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)}) = \chi_c(V \otimes \bar{k}) := \sum_i (-1)^i \dim \mathrm{H}^i_c(V \otimes \bar{k}, \bar{\mathbb{Q}}_\ell).$$

Furthermore, by excision we have

$$\chi_c(V \otimes \bar{k}) = \chi_c(X \otimes \bar{k}) - \chi_c((X \cap H) \otimes \bar{k}) - \chi_c((X \cap L) \otimes \bar{k}) + \chi_c((X \cap H \cap L) \otimes \bar{k})$$

and we conclude by using the formulas (cf. [SGA7II], Exposé XVII)

$$\chi_c(X \otimes \bar{k}) = \int_X c(X)$$

$$\chi_c((X \cap H) \otimes \bar{k}) = \int_X c(X) \frac{dL}{1 + dL}$$

$$\chi_c((X \cap L) \otimes \bar{k}) = \int_X c(X) \frac{L}{1 + L}$$

$$\chi_c((X \cap H \cap L) \otimes \bar{k}) = \int_X c(X) \frac{dL^2}{(1 + L)(1 + dL)}.$$

□

In particular we get

Corollary 12. *Let $C = (-1)^n \int_X c(X)/((1 + L)(1 + dL))$. Then we have the estimate*

$$\left| \sum_{x \in V(k)} \chi(f(x)) \right| \leq C \cdot q^{n/2}.$$

When X is a complete intersection of $r = N - n$ hypersurfaces of degrees $a_1, \ldots, a_r$ we can compute $\chi_c(V \otimes \bar{k}, \mathcal{L}_{\chi(f)})$ explicitly:

$$\int_X \frac{c(X)}{(1+L)(1+dL)} = \int_{\mathbb{P}^N_{\bar{k}}} \frac{a_1 \cdots a_r L^r c(\mathbb{P}^N_{\bar{k}})}{(1+a_1L)\cdots(1+a_rL)(1+L)(1+dL)} =$$

$$= \int_{\mathbb{P}^N_{\bar{k}}} \frac{a_1 \cdots a_r L^r (1+L)^N}{(1+a_1L)\cdots(1+a_rL)(1+dL)} =$$

$$= \text{coeff. of } L^N \text{ in } \frac{a_1 \cdots a_r L^r (1+L)^N}{(1+a_1L)\cdots(1+a_rL)(1+dL)} =$$

$$= \text{coeff. of } L^n \text{ in } \frac{a_1 \cdots a_r (1+L)^N}{(1+a_1L)\cdots(1+a_rL)(1+dL)} =$$

$$= \text{coeff. of } L^n \text{ in}$$

$$a_1 \cdots a_r (\sum_m \binom{N}{m} L^m)(\sum_{b_1} (-a_1)^{b_1} L^{b_1}) \cdots (\sum_{b_r} (-a_r)^{b_r} L^{b_r})(\sum_c (-d)^c L^c) =$$

$$= a_1 \cdots a_r \sum_{m+b_1+\cdots+b_r+c=n} \binom{N}{m} (-a_1)^{b_1} \cdots (-a_r)^{b_r} (-d)^c$$

$$= (-1)^n a_1 \cdots a_r \sum_{m+b_1+\cdots+b_r+c=n} \binom{N}{m} (-1)^m a_1^{b_1} \cdots a_r^{b_r} d^c.$$

For instance, if $r = 0$ (i.e. $X = \mathbb{P}^n_k$) this is $(-1)^n(d-1)^n$, and we have the following generalization (to the case where p divides d) of ([Ka4], Theorems 2.1 and 2.2):

Theorem 13. *Let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree d, and f_d its degree d homogeneous component. Suppose that*

a) The equation $f = 0$ defines a smooth hypersurface in $\mathbb{A}^n_k$.

b) The equation $f_d = 0$ defines a smooth hypersurface in $\mathbb{P}^{n-1}_k$.

Then we have the estimate

$$\left| \sum_{x \in k^n} \chi(f(x)) \right| \leq (d-1)^n \cdot q^{n/2}.$$

In general we will not be able to compute the Euler characteristic explicitly, but we can always use the bound given by Theorem 9. Thus we always have

Corollary 14. *Suppose that X, $X \cap H$ and $X \cap H \cap Z$ are all smooth, and let $C = 3(3 + d + \sup_i a_i)^{N+r}$. Then we have the estimate*

$$\left| \sum_{x \in V(k)} \chi(f(x)) \right| \leq C \cdot q^{n/2}.$$

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