Free divisors and duality for $\mathcal{D}$-modules *

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March 14, 2001

Abstract

The relationship between $\mathcal{D}$-modules and free divisors has been studied in a general setting by L. Narváez and F.J. Calderón. Using the ideas of these works we prove in this article a duality formula between two $\mathcal{D}$-modules associated to a class of free divisors on $\mathbf{C}^n$ and we give some applications.

Keywords: $\mathcal{D}$-MODULES, DIFFERENTIAL OPERATORS, GRÖBNER BASES, LOGARITHMIC COMPARISON THEOREM.
Math. Classification: 32C38, 13N10, 14F40, 13P10.

1 Free divisors

Here we summarize some results of K. Saito [14].

Let us denote $X = \mathbf{C}^n$. Denote by $\mathcal{O} = \mathcal{O}_X$ the sheaf of holomorphic functions on X. Let $D \subset X$ be a divisor and $x \in D$. Denote by $Der(\mathcal{O}_x)$ the $\mathcal{O}_x$-module of $\mathbf{C}$-derivations of $\mathcal{O}_x$ (the elements in $Der(\mathcal{O}_x)$ are called *vector fields*).

A vector field $\delta \in Der(\mathcal{O}_x)$ is said to be *logarithmic* w.r.t. D if $\delta(f) = af$ for some $a \in \mathcal{O}_x$, where f is a local (reduced) equation of the germ $(D, x) \subset (\mathbf{C}^n, x)$. The $\mathcal{O}_x$-module of logarithmic vector fields (or logarithmic derivations) is denoted by $Der(\log D)_x$. This yields a $\mathcal{O}$-module sheaf denoted by $Der(\log D)$.

The divisor D is said to be *free at the point* $x \in D$ if the $\mathcal{O}_x$-module $Der(\log D)_x$ is free (and, in this case, of rank n). The divisor D is called *free* if it is free at each point $x \in D$.

Smooth divisors are free. A normal crossing divisor $D \equiv (x_1 \cdots x_t = 0) \subset \mathbf{C}^n$ is free because we have $Der(\log D) = \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbf{C}^n} x_i \partial_i \oplus \bigoplus_{j=t+1}^{n} \mathcal{O}_{\mathbf{C}^n} \partial_j$. By [14] any reduced germ of plane curve $D \subset \mathbf{C}^2$ is a free divisor.

Saito's criterium to test the freedom of a divisor D at a point p is:

Lemma 1.1.1. *([14, (1.9)]) Let $\delta_i = \sum_{j=1}^{n} a_{ij}\partial_j$, $i = 1, \ldots, n$ be a system of holomorphic vector fields at $p \in \mathbf{C}^n$, such that:*
i) $[\delta_i, \delta_j] \in \sum_{k=1}^{n} \mathcal{O}_p \delta_k$, for $i, j = 1, \ldots, n$.
ii) $det(a_{ij}) = h$ defines a reduced hypersurface D.
Then, for $D \equiv (h = 0)$, $\delta_1, \ldots, \delta_n$ belong to $Der(\log D)_p$ and hence $\{\delta_1, \ldots, \delta_n\}$ is a free basis of $Der(\log D)_p$.

*Partially supported by DGESIC-97-0723 and HF-1998-0105. Second author partially supported by MSRI (Berkeley)

2 The logarithmic comparison theorem

Let X be a complex manifold and $D \subset X$ a divisor. We have a canonical inclusion

$$i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$$

where $\Omega^\bullet(\star D)$ is the meromorphic de Rham complex and $\Omega^\bullet(\log D)$ is the de Rham logarithmic complex, both w.r.t D. A meromorphic form $\omega \in \Omega^p(\star D)$ is said to be *logarithmic* if $fw \in \Omega^p$ and $df \wedge \omega \in \Omega^{p+1}$ for each local equation f of D.

A classical natural problem is to find the class of divisors $D \subset X$ for which $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$ is a quasi-isomorphism (i.e. i_D induces an isomorphism on cohomology).

By Grothendieck's comparison theorem we know that the complexes $\Omega^\bullet(\star D)$ and $\mathbf{R}j_*(\mathbf{C})$ are naturally quasi-isomorphic, where $j : U = X \setminus D \to X$ is the natural inclusion. So, if i_D is a quasi-isomorphism we say that the logarithmic comparison theorem holds for D (or simply LCT holds for D).

Definition 2.1.2. ([8]) *A divisor $D \subset X$ is locally quasi-homogeneous if for all $q \in D$ there exist local coordinates $(V; x_1, \ldots, x_n)$ centered at q such that $D \cap V$ has a weighted homogeneous defining equation w.r.t. $(x_1, \ldots, x_n)$.*

Smooth divisors and normal crossing divisors are locally quasi-homogeneous. A weighted homogeneous polynomial $f \in \mathbf{C}[x,y]$ defines a locally quasi-homogeneous divisor $D \equiv (f = 0) \subset \mathbf{C}^2$.

Suppose $D \subset X$ is a locally quasi-homogeneous free divisor. The main result of [8] is that LCT holds for D, i.e.

$$i_D : \Omega^\bullet(\log D) \to \mathbf{R}j_*(\mathbf{C})$$

is a quasi-isomorphism.

3 Logarithmic $\mathcal{D}$-modules

Let us denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf (of rings) of linear differential operators with holomorphic coefficients on X.

A local section P of $\mathcal{D}$ is a finite sum

$$P = \sum_\alpha a_\alpha \partial^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, a_α is a local section of $\mathcal{O}$ on some chart $(U; x_1, \ldots, x_n)$ and $\partial = (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

The sheaf $\mathcal{D}$ is naturally filtered by the order of the differential operators. The associated graded ring $\mathrm{gr}(\mathcal{D})$ is commutative. In fact, we can identify $\mathrm{gr}(\mathcal{D})$ with the sheaf $\mathcal{O}[\xi_1, \ldots, \xi_n]$ of polynomials in the variables $\xi = (\xi_1, \ldots, \xi_n)$ and with coefficients in $\mathcal{O}$.

Assume the operator $P = \sum a_\alpha \partial^\alpha$ has order d (i.e. $d = \max\{|\alpha| = \alpha_1 + \cdots + \alpha_n \mid a_\alpha \neq 0\}$) then the *principal symbol* of P is

$$\sigma(P) = \sum_{|\alpha|=d} a_\alpha \xi^\alpha \in \mathcal{O}[\xi].$$

For each left ideal I in $\mathcal{D}$ the graded ideal associated to I is the ideal of $\mathrm{gr}(\mathcal{D})$ generated by the set of principal symbol $\sigma(P)$ for $P \in I$. This ideal is denoted by $\mathrm{gr}(I)$.

The *characteristic variety* of the $\mathcal{D}$-module $M = \frac{\mathcal{D}}{I}$ is, by definition, the analytic sub-variety of the cotangent bundle T^*X defined by $\mathcal{O}_{T^*X}\mathrm{gr}(I)$. This characteristic variety if denoted by $Ch(M)$. The cycle defined in T^*X by the ideal $\mathcal{O}_{T^*X}\mathrm{gr}(I)$ is denoted by $CCh(M)$ and it is called the *characteristic cycle* of the $\mathcal{D}$-module M.

For any divisor $D \subset \mathbf{C}^n$ the sheaf $\mathcal{O}[\star D]$ of meromorphic functions with poles along D is naturally a left coherent $\mathcal{D}$-module (that follows from the results of Bernstein-Björk on the existence of the b-function for each local equation f of D, [1], [2]). Even more, Kashiwara proved that the dimension of $Ch(\mathcal{O}[\star D])$ is equal to n (i.e. $\mathcal{O}[\star D]$ is holonomic, [11]).

In [3] and [4] the author considers the (left) ideal $I^{\log D} \subset \mathcal{D}$ generated by the logarithmic vector fields $Der(\log D)$ (see 1). We will denote simply $I^{\log} = I^{\log D}$ and $M^{\log}$ the quotient $\mathcal{D}/I^{\log}$ if no confusion is possible.

3.1 Koszul free divisors

Let us give the main result of F.J. Calderón, [3] (see also [4]). Let $D \subset X$ be a divisor and $x \in D$.

Definition 3.1.1. ([4, Def. 4.1.1]) *The divisor D is said to be Koszul free at the point $x \in D$ if it is free at x and there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $Der(\log D)_x$ such that the sequence $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ of principal symbols is a regular sequence in the ring $\mathrm{gr}^F(\mathcal{D})$. The divisor D is Koszul free if it is Koszul free at any point of D.*

By [14] and [4, 4.2.2.] any plane curve $D \subset \mathbf{C}^2$ is a Koszul free divisor. By [4, Prop. 4.1.2] if D is a Koszul free divisor then $M^{\log}$ is holonomic and

Theorem 3.1.2. ([4, Th. 4.2.1]) *If D is a Koszul free divisor then $\Omega^\bullet(\log D)$ and $\mathbf{R}\mathcal{H}om_{\mathcal{D}}(M^{\log}, \mathcal{O})$ are naturally quasi-isomorphic.*

3.2 $\widetilde{M}^{\log}$

In [17] (see also [9]) L. Narváez suggested the study of the $\mathcal{D}$-module $\widetilde{M}^{\log}$ defined as follows: Let us denote by $\widetilde{I}^{\log}$ the left ideal of $\mathcal{D}$ generated by the set $\{\delta + a \,|\, \delta \in I^{\log} \text{ and } \delta(f) = af\}$. Let us write $\widetilde{M}^{\log} = \mathcal{D}/\widetilde{I}^{\log}$. There exists a natural morphism $\phi_D : \widetilde{M}^{\log} \to \mathcal{O}[\star D]$ defined by $\phi_D(\overline{P}) = P(1/f)$ where $\overline{P}$ denotes the class of the operator $P \in \mathcal{D}$ modulo $\widetilde{I}^{\log}$. The image of ϕ_D is $\mathcal{D}\frac{1}{f}$. As a natural question we ask for the class of D such that the morphism ϕ_D is an isomorphism (see 5.2).

4 The duality theorem

Suppose here that the divisor $D \subset X$ is free, and let $f \in \mathcal{O}$ be a local equation of D and let $\{\delta_1, ..., \delta_n\}$ be a basis of the logarithmic derivations. We will use the following notation:

- $\delta_i(f) = m_i f$ for some $m_i \in \mathcal{O}$.
- $\delta_i = \sum_{k=1}^n a_{ik}\partial_k$ for some $a_{ik} \in \mathcal{O}$.
- $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$
- $[\delta_i, \delta_j] = \sum_{k=1}^n \alpha_k^{ij}\delta_k$ for some $\alpha_k^{ij} \in \mathcal{O}$.

Lemma 4.1.1. *For any $i = 1, ..., n$ we have*

$$\delta_i(|A|) = \sum_{k=1}^n (\delta_i(a_{k1}), ..., \delta_i(a_{kn})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}$$

where A_{kj} is the adjoint matrix of a_{kj}.

Proof. From the very definition of the determinant developed from the k-th row. □

The lemma above is true in fact for any derivation, not only for elements in the basis.

Lemma 4.1.2. *We have*

$$f(\alpha_1^{ij},\dots,\alpha_n^{ij}) = (\delta_i(a_{j1})-\delta_j(a_{i1}),...,\delta_i(a_{jn})-\delta_j(a_{in}))Adj(A)^t.$$

Proof. It is only necessary to consider that

$$[\delta_i,\delta_j] = (\alpha_1^{ij},...,\alpha_n^{ij})A\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix} =$$

$$= (\delta_i(a_{j1})-\delta_j(a_{i1}),...,\delta_i(a_{jn})-\delta_j(a_{in}))\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}.$$

□

We consider the augmented Spencer logarithmic complex as in [4, page 712]. We have

$$\mathcal{D}\otimes_{\mathcal{O}}\wedge^\bullet Der(\log D)\to M^{\log}\to 0.$$

We say that a free divisor D is *of Spencer type* if this complex is a (locally) free resolution of $M^{\log}$ and this last $\mathcal{D}$-module is holonomic. By [4, Prop. 4.1.3] if D is Koszul free (in particular if D is a plane curve) then it is of Spencer type but the converse is not true, see [4, Remark 4.2.4] and section 5.3.

The following proposition is a consequence of [4, Th. 4.2.1].

Proposition 4.1.3. *If D is of Spencer type then $Sol(M^{\log})$ is naturally quasi-isomorphic to $\Omega^\bullet(\log D)$.*

Theorem 4.1.4. *Suppose D is of Spencer type. Then $(M^{log})^* \simeq \widetilde{M}^{log}$.*

Proof. Using the Spencer logarithmic free resolution of the holonomic $\mathcal{D}$-module $M^{\log}$, we first compute a presentation of the right $\mathcal{D}$-module $\mathcal{E} := Ext^n_{\mathcal{D}}(M^{\log},\mathcal{D})$ and then we prove that left $\mathcal{D}$-module associated to $\mathcal{E}$ is $\widetilde{M}^{\log}$.

The matrix of the n-th morphism in the resolution of M^{log} (see [4, page 712]) has components of the form

$$(-1)^{i-1}\delta_i + (-1)^i\sum_{l\neq i}\alpha_l^{il},$$

so it is enough to prove that

$$(-\delta_i+\sum_{k\neq i}\alpha_k^{ik})^* = \delta_i + \sum_{k=1}^n \partial_k(a_{ik}) + \sum_{k\neq i}\alpha_k^{ik} = \delta_i + m_i.$$

In order to prove the last equality, we will show that

$$m_i f = \delta_i(f) = \delta_1(|A|) =$$

$$= \sum_{k=1}^n f\partial_k(a_{ik}) + \sum_{k\neq i} f\alpha_k^{ik}.$$

Using 4.1.2, we obtain

$$\sum_{k\neq i} f\alpha_k^{ik} = \sum_{k\neq i} (\delta_i(a_{k1}) - \delta_k(a_{i1}), ..., \delta_i(a_{kn}) - \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} =$$

$$= \sum_{k=1}^{n} (\delta_i(a_{k1}), ..., \delta_i(a_{kn})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} - \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}$$

So we have collected in the first sum precisely (see 4.1.1) $\delta_i(|A|)$. It remains to check that

$$\sum_{k=1}^{n} f\partial_k(a_{ik}) = \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}.$$

As $f = (a_{k1}, ..., a_{kn})(A_{k1}, ..., A_{kn})^t$, we have

$$\sum_{k=1}^{n} f\partial_k(a_{ik}) = \sum_{k=1}^{n} \partial_k(a_{ik})(a_{k1}, ..., a_{kn}) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} =$$

$$= \sum_{k=1}^{n} (\sum_{j=1}^{n} a_{kj}\partial_j(a_{i1}), ..., \sum_{j=1}^{n} a_{kj}\partial_j(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} =$$

$$= \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}.$$

□

5 Some applications

5.1 LCT in dimension 2. Regularity of $M^{\log}$ and $\widetilde{M}^{\log}$

Let $D \subset \mathbf{C}^2$ be a plane curve.

Theorem 5.1.1. ([5, Theorem 3.9]) *The morphism $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$ is a quasi-isomorphism if and only if D is locally quasi-homogeneous.*

Proof. We show here how to read the original (topological) proof of [5] to give a differential proof of "only if" part. Part "if" is a consequence of [8] because any plane curve is a free divisor [14].

The problem is local. Suppose the local equation f of D is defined in a small open neighbourhood such that the only singular point of $f = 0$ is the origin. Denote $\mathcal{O}[1/f] = \mathcal{O}[\star D]$.

Let us consider (see 3.2) the natural surjective morphism

$$\phi_D : \widetilde{M}^{\log} \to \mathcal{D}\frac{1}{f} \simeq \mathcal{O}[\star D]$$

where the last isomorphism follows by a result of Varchenko (i.e. the local b-function $b_f(s)$ of f verifies $b_f(-k) \neq 0$ for any integer $k \geq 2$, [18]). The kernel K of ϕ_D is supported by the origin (because f is smooth outside $(0,0)$) and $CCh(\widetilde{M}^{\log}) = CCh(K) + CCh(\mathcal{O}[\star D])$. In particular $\widetilde{M}^{\log}$ and $M^{\log} = (\widetilde{M}^{\log})^*$ are regular holonomic (cf. [12]) because as we said before D satisfies the hypothesis of theorem 4.1.4.

Let us denote $Sol(M^{\log}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}}(M^{\log}, \mathcal{O})$ the solution complex of $M^{\log}$.

Assume LCT holds for D. Then we have

$$DR(\mathcal{O}[\star D]) \simeq \Omega^\bullet(\star D) \simeq \Omega^\bullet(\log D) \simeq Sol(M^{\log}) \simeq DR((M^{\log})^*) \simeq DR(\widetilde{M}^{\log}).$$

Then both $\mathcal{D}$-modules $\mathcal{O}[\star D]$ and $\widetilde{M}^{\log}$ have the same de Rham complex and then the same characteristic cycle. In this case $K = 0$ and $\widetilde{M}^{\log} \simeq \mathcal{O}[\star D]$. Finally, by [16, page 88] (or by [17, 2.2.6], see also [9]) f is weighted homogeneous in suitable coordinates. That proves the "only if" part of the theorem. □

5.2 On the comparison of $\widetilde{M}^{\log}$ and $\mathcal{O}[\star D]$

In the previous section we proved (in dimension 2) that if $\widetilde{M}^{\log} \simeq \mathcal{O}[\star D]$ then f is weighted homogeneous and the converse is also true (cf. [9]). So, $\widetilde{M}^{\log} \simeq \mathcal{O}[\star D]$ if and only if f is weighted homogeneous if and only if LCT holds for D.

Now we return to dimension n.

Theorem 5.2.1. *If $D \subset \mathbf{C}^n$ is a free, locally quasi-homogeneous (l.q-h.) divisor then $M^{\log}$ and $\widetilde{M}^{\log}$ are regular holonomic. Moreover $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are naturally isomorphic.*

Proof. By [6] D is Koszul free and then $M^{\log}$ is holonomic and the dual of $M^{\log}$ is $\widetilde{M}^{\log}$ (see 4.1.4). So, $\widetilde{M}^{\log}$ is also holonomic. It is enough to prove that $\widetilde{M}^{\log}$ is regular.

To avoid confusion we will denote $\widetilde{M}^{\log D}$ to emphasize the divisor D. In fact we will prove, by induction on n, that the natural morphism

$$\phi_D : \widetilde{M}^{\log D} \to \mathcal{O}[\star D]$$

is an isomorphism. We follows here the argument of [7, 4.3.]. There is nothing to prove in the case $n = 1$. We note that in dimension 2 the result is proved in [17] (see 5.1). Suppose the result is true for any free, l.q-h. divisor in dimension $\leq n-1$. Let $D \subset \mathbf{C}^n$ be a free, l.q-h. For any $x \in D$ there exists an open neighbourhood U of x such that for any $y \in U \cap D \setminus \{x\}$ the germ $(\mathbf{C}^n, D, y)$ is isomorphic to $(\mathbf{C}^{n-1} \times \mathbf{C}, D' \times \mathbf{C}, (0,0))$, where D' is a free, l.q-h. divisor in $\mathbf{C}^{n-1}$ (see [8, prop. 2.4, lemma 2.2]). So, by induction hypothesis the morphism $\phi_{D'} : \widetilde{M}^{\log D'} \to \mathcal{O}[\star D']$ is an isomorphism. Then, by applying the functor ϕ^* (where $\pi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ is the projection), we have that for any $y \in U \cap D$, $y \neq x$, the morphism $\phi_{D,y}$ is an isomorphism between $\widetilde{M}_y^{\log D}$ and $\mathcal{O}[\star D]_y$. We owe this argument to L. Narváez. So, the kernel of $\phi_D : \widetilde{M}^{\log D} \to N^D$ is concentrated on a discrete set and it is regular holonomic (here N^D is the $\mathcal{D}$-module $\mathcal{D}\frac{1}{f}$, where f is a local equation of D). As $N^D \subset \mathcal{O}[\star D]$ is regular holonomic we deduce the regularity of $\widetilde{M}^{\log D}$. On the other hand, by [8] the logarithmic comparison theorem holds for D. So, by using duality 4.1.4 and the natural quasi-isomorphism $Sol(M^{\log D}) \to \Omega^\bullet(\log D)$ (3.1.2), we deduce (as in 5.1) that $DR(\widetilde{M}^{\log D})$ and $DR(\mathcal{O}[\star D])$ are naturally quasi-isomorphic and therefore, by Riemann-Hilbert correspondence, $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are naturally isomorphic, i.e. ϕ_D is an isomorphism. Thus we have concluded the induction. □

5.3 An example in dimension 3

In [5] the authors give an example of a non Koszul free divisor –in dimension 3– for which LCT holds. We will treat here, following the same lines as in [9], the case of the surface $D \subset \mathbf{C}^3$ defined by $f = y(x^2+y)(x^2z+y) = 0$.

The surface is free because computing the syzygies among f, f_x, f_y, f_z we obtain

$$\begin{gathered}(-3, \tfrac{1}{2}x, y, 0)\\ (-x^2, 0, 0, x^2z+y)\\ (-xz-x, \tfrac{1}{2}x^2+\tfrac{1}{2}y, 0, xz^2-xz),\end{gathered}$$

which produce the logarithmic vector fields

$$\begin{aligned}\delta_1 &= \tfrac{1}{2}x\partial_x + y\partial_y\\ \delta_2 &= (x^2z+y)\partial_z\\ \delta_3 &= (\tfrac{1}{2}x^2+\tfrac{1}{2}y)\partial_x + (xz^2-xz)\partial_z,\end{aligned}$$

whose coefficients have a determinant equal to $1/2f$.

This surface it is not Koszul-free because the set of the symbols (with respect to the total order) $\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)$ do not form a regular sequence. If we write $\sigma(\partial_x) = \xi$, $\sigma(\partial_y) = \eta$, $\sigma(\partial_3) = \zeta$. We have $yz\eta^2\zeta + \frac{1}{4}\xi^2\zeta \notin (\sigma(\delta_1), \sigma(\delta_2))$ but $yz\eta^2\zeta + \frac{1}{4}\xi^2\zeta\sigma(\delta_3) \in (\sigma(\delta_1), \sigma(\delta_2))$.

We compute a free resolution of $M^{log} = \mathcal{D}/\mathcal{D}(\delta_1, \delta_2, \delta_3)$ using Gröbner basis. We obtain that the module of syzygies $Syz(\delta_1, \delta_2, \delta_3)$ is generated by $\mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23}$ deduced from the commutators $[\delta_i, \delta_j]$ where:

- $[\delta_1, \delta_2] = \delta_2$
- $[\delta_1, \delta_3] = \frac{1}{2}\delta_3$
- $[\delta_1, \delta_2] = (xz - x)\delta_2$.

The second module of syzygies (among the $\mathbf{s}_{ij}$) it is generated by only one element $\mathbf{t}$, so we have finished the resolution. This element is

$$\mathbf{t} = (t_1, t_2, t_3) = (-x^2z\partial_z + xz\partial_z - \frac{1}{2}x^2\partial_x - \frac{1}{2}y\partial_x - xz + x,\ x^2z\partial_z + y\partial_z,\ -y\partial_y - \frac{1}{2}x\partial_x + \frac{3}{2}),$$

precisely the one required in the Spencer logarithmic complex for $M^{\log}$ in dimension 3, which is in fact a free resolution of $M^{\log}$. We can check that in this case $M^{\log}$ is holonomic (use for example [10] or [15]), so D is of Spencer type and we apply:
a) The theorem 4.1.4 to obtain that $(M^{\log})^* \simeq \widetilde{M}^{\log}$.
b) Proposition 4.1.3 to obtain $Sol(M^{\log}) \simeq \Omega^\bullet(\log D)$.

Besides, the global b-function of f is

$$(6s+5)(3s+2)(2s+1)(3s+4)(6s+7)(s+1)^3,$$

so we can assure that $\mathcal{O}[\frac{1}{f}] \simeq \widetilde{M}^{\log}$, because $\widetilde{I}^{\log} = Ann_{\mathcal{D}}(1/f)$. We have used [10] and [15] again to compute the b-function and the annihilating ideal of $1/f$, that is to say, the algorithms of [13].

Finally, we have the following chain of quasi-isomorphisms

$$DR(\mathcal{O}[\star D]) \simeq \Omega^\bullet(\star D) \simeq \Omega^\bullet(\log D) \simeq Sol(M^{\log}) \simeq DR((M^{\log})^*) \simeq DR(\widetilde{M}^{\log}).$$

and the LCT holds for D.

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