



Programa de Doctorado “Matemáticas”

PHD DISSERTATION

**Stability in the combinatorics
of representation theory**

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SAM: This is it.

FRODO: This is what?

SAM: If I take one more step, it'll be the farthest away from home I've ever been.

FRODO: Come on, Sam. Remember what Bilbo used to say:

It's a dangerous business, Frodo, going out your door. You step onto the road, and if you don't keep your feet, there's no knowing where you might be swept off to.

The Lord of the Rings: The Fellowship of the Ring.

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Introduction

In this thesis we study two families of coefficients: plethysm coefficients and Kronecker coefficients. These two families arise from representation theory and theory of symmetric functions. For representation theory, they are the coefficients appearing in the decomposition into irreducible of different constructions from irreducible representations both of the symmetric group and of the general linear group. For theory of symmetric functions, they corresponds to the coefficients that emerge when we decompose in the Schur basis several operations of Schur functions.

In the framework of representation theory of general linear groups, the plethysm coefficients arise from decomposition into irreducible representations of the composition of Schur functors. This construction induces an operation $(f, g) \mapsto f[g]$ on the ring of symmetric functions, also called *plethysm*. In this setting, the plethysm coefficients are the coefficients of the decomposition of the plethysm of Schur functions in the Schur basis.

In 1950, Foulkes observed some stability properties in sequences of plethysm coefficients, [Fou54]. These properties were proved in the 1990's by Carré and Thibon [CT92], using vertex operators and other arguments from the combinatorics of symmetric functions, and by Brion [Bri93] for algebraic groups in general (rather than just general linear groups), using tools from geometric representation theory. In this thesis, we reproduce a detailed proof of the results proved by Carré and Thibon in order to obtain the bounds for which the coefficients are constant.

We also present a combinatorial interpretation of other plethysm coefficients, the *h-plethysm coefficients*, defined by the complete homogeneous basis. These coefficients are directly related with the plethysm coefficients by the *Jacobi-Trudi formula*. The combinatorial interpretation of the *h-plethysm coefficients* describes them as the number of integral points in a polytope depending on the partitions indexing the coefficients. In fact, they count the non-negative solutions of systems of linear Diophantine equations whose constant terms depend linearly on the parts of the partitions. This new interpretation provides a combinatorial proof for the stability properties already proved by Brion, and Carré and Thibon.

This complete work has been presented at the conference *Jornadas de Matemática Discreta y Algorítmica* (JMDA'14), [BC14], and at the conference *27th International conference on formal power series and algebraic combinatorics* (FPSAC'15), [Col15a], and it has been submitted, [Col15b].

In the framework of symmetric functions, the Kronecker coefficients are the coefficients appearing when we decompose into the Schur basis $\{s_\lambda[X] \cdot s_\mu[Y]\}$, the Schur function $s_\nu[XY]$, for any alphabets X and Y . In general, there is no combinatorial interpretation of the Kronecker coefficients.

In 1938, Murnaghan observed a stability phenomenon in the Kronecker coefficients: the sequence of Kronecker coefficients whose indexing partitions have an increasing first part is eventually constant. The *reduced Kronecker coefficients* can be defined as the stable value of these sequence of Kronecker coefficients. The reduced Kronecker coefficients are interesting objects of their own right. For instance, Littlewood observed that they coincide with the Littlewood–Richardson coefficients when $|\alpha| + |\beta| = |\gamma|$. In fact, it has been shown by E. Briand, R. Orellana and M. Rosas in [BOR11] that they contain enough information to compute from them the Kronecker coefficients.

One open problem related to both families, Kronecker coefficients and reduced Kronecker coefficients, is the understanding of the rate of growth experienced by the Kronecker coefficients as we increase the sizes of the rows of their indexing partitions. Recently, in [BRR16], Briand, Rattan and Rosas describe what happens when we increase the first column of the partitions indexing the reduced Kronecker coefficients (stability) and when we increase their first row (linear growth). Given three partitions, the Kronecker coefficients indexed by them stabilize when we increase these partitions with n new boxes in their first row and n new boxes in their first column. They call this phenomena *hook stability*. They also show that the resulting sequence obtained by increasing the sizes of the second rows (keeping the first one very long in comparison) of the partitions indexing the Kronecker coefficients are described by a linear quasipolynomial of period 2.

In this thesis we investigate what happens when we add boxes to other rows and columns of the partitions indexing reduced Kronecker coefficients. We present a study related to the following four families:

- ▷ **Family 1** $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$
- ▷ **Family 2** $\bar{g}_{(k^a), (k^b)}^{(k)}$
- ▷ **Family 3** $\bar{g}_{((k+i)^a), (k^b)}^{(k)}$

▷ **Family 4** $\bar{g}_{(k^b), (k+i, k^a)}^{(k)}$

In Family 1, for $i = 0$, due to Vergne and Baldoni, [BV15], we know that they are given by a quasipolynomial. In general, using results of Meinrenken and Sjamaar, [MS99], we already know that they are given by a piecewise quasipolynomial. We asked when they stop being defined by a quasipolynomial to be defined by a piecewise quasipolynomial. In this thesis we present a study that shows that even in case that $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 1$, the coefficients $\left\{ \bar{g}_{k\mu, k\nu}^{k\lambda+(i)} \right\}$ are given by a piecewise quasipolynomial for $i \geq 1$. We include explicit formulas for this case. The results obtained show the existence of a stabilization phenomenon, reminiscent of the one described for Murnaghan for the reduced Kronecker coefficients.

For the other three families, we present a complete study, which includes the generating function for the reduced Kronecker coefficients and two combinatorial descriptions: one in terms of plane partitions fitting in a rectangle and the other as quasipolynomials, specifying the degree and the period. The generating functions provide an efficient way to compute the reduced Kronecker coefficient, and as a consequence, the Kronecker coefficients when the first part of the indexed partitions is large enough. In terms of Kronecker coefficients, we obtain three families of Kronecker coefficients, indexed by partitions whose first part is large enough, which are described by quasipolynomials. In fact, we determine the degree and the period of the quasipolynomials. We note that the degree of these quasipolynomials can be as large as we want, involving partitions with longer length.

The study is based on a paper of C. Ballantine and R. Orellana, [BO07], in which they give a combinatorial interpretation of the coefficients appearing in the Kronecker product $s_\lambda * s_{(n-p, p)}$ in terms of special tableaux, the *Kronecker tableaux*. Actually, the generating functions are obtained defining a bijection between the corresponding set of Kronecker tableaux and the set of coloured partitions that describes the generating function. Recently, C. Ballantine and B. Hallahan, in [BH12] also study the stability of the Kronecker product of a Schur function indexed by a hook partition and another Schur function indexed by a rectangle partition. They use the Blasiak's combinatorial rule, which describes the Kronecker coefficients in terms of *Yamanouchi coloured tableaux*, introduced in [Bla12], as we do with the Kronecker tableaux. They are able to give bounds for the size of the partition with which the Kronecker coefficients are stable and that once the bound is reached, no new Schur functions appear in the decomposition of the Kronecker product.

Another interesting approach for the reduced Kronecker coefficients is to use the *vertex operators*. In this thesis we include a proof of Murnaghan's Theorem using vertex operators. This proof provides a description of the reduced Kronecker

coefficients obtained by Brion, [Bri93]. Vertex operators are also used to give a description of the reduced Kronecker coefficients with one partition equals to (k) in terms of the Littlewood–Richardson coefficients.

The study of Family 1 was presented with M. Rosas at the conference *Décimoquarto Encuentro de Álgebra Computacional y Aplicaciones* (EACA 2014), [CR14]. The study of Family 2 is published in [CR15].

The structure of the thesis is as follows: In Chapter 1 we move from the framework of representation theory to symmetric functions describing how appear the main fundamental coefficients in both settings. We also give a brief introduction to the structure of *Sym*. In Chapter 2 we present the results related to the h -plethysm coefficients and the stability of the plethysm coefficients. In Chapter 3 we present the four families of reduced Kronecker coefficients and their study. The Appendices A, B and C include details of representation theory and symmetric functions, as well as the relation between them.

Chapter 1.

Framework: representation theory and symmetric functions

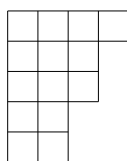
This chapter is dedicated to describe briefly the background in which this thesis is established.

1.1. Preliminaries

A *partition* λ of a non-negative integer n is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\sum_i \lambda_i = n$. Any $\lambda_i = 0$ is considered irrelevant and we can identify λ with the infinite sequence $(\lambda_1, \dots, \lambda_k, 0, 0, \dots)$. The length of a partition λ , $\ell(\lambda)$, is the number of non-zero λ_i , also called the *parts* of the partition. If we write m_i for the number of parts of λ that equal i , we can also denote the partition λ by $\lambda = (1^{m_1} 2^{m_2} \dots)$. We denote by $\lambda \vdash n$ or $|\lambda| = n$ that λ is a partition of n . We set $\text{Par}(n)$ for the set of partitions of n , with $\text{Par}(0)$ consisting of the empty partition. We also set

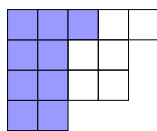
$$\text{Par} := \bigcup_{n \geq 0} \text{Par}(n)$$

Partitions can be graphically visualized using Young diagrams. A *Young diagram* is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing. The Young diagram is said to be of shape λ if the i^{th} row of the Young diagram has exactly λ_i boxes. For instance, the Young diagram of $\lambda = (4, 3, 3, 2, 2)$ is



Listing the number of boxes of a Young diagram in each column gives another partition, called *conjugate partition*. Its Young diagram can be obtained by reflecting the original diagram along its main diagonal. If we consider a partition λ , its conjugate is denoted by λ' . For example, for $\lambda = (4, 3, 3, 2, 2)$, its conjugate is $\lambda' = (5, 5, 3, 1)$.

Partitions and Young diagrams carry the same information. Moreover, containment of one Young diagram in another, defines a partial ordering on the set Par . We say that $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ , i.e. $\mu_i \leq \lambda_i$, for all i . We define a *skew shape* λ/μ as a pair of partitions (λ, μ) such that $\mu \subseteq \lambda$. Then, a Young diagram of skew shape λ/μ or a *skew Young diagram of shape* λ/μ is obtained by considering the set of boxes that belong to the Young diagram of λ but not to the Young diagram of μ . For example, for $\lambda = (5, 4, 4, 2)$ and $\mu = (3, 2, 2, 2)$, the skew diagram of shape λ/μ is



where the *blue boxes* are the ones that are not in the skew diagram.

We can fill the boxes of a Young diagram with numbers. In general, this is called *tableau*. But we will consider a special type of filling. A *semi-standard Young tableau of shape* λ is obtained by filling the boxes of the Young diagram of λ with positive (non-zero) integers, with the entries weakly increasing along each row and strictly increasing down each column. Recording the number of times each number appears in a semi-standard Young tableau gives a sequence known as the *type* of the semi-standard Young tableau. Instead of a partition λ , we can consider a skew shape λ/μ and extend the concept to *semi-standard Young tableaux of shape* λ/μ , with the same conditions over the entries in each row and each column. For a semi-standard Young tableau T , its shape is denoted by $sh(T)$, and its type is denoted by $ty(T)$. The *reading word of* T is the sequence of entries of T obtained by concatenating the rows of T bottom to top. The *reverse reading word* is simply its reading word read backwards.

For instance, for $\lambda = (5, 4, 4, 2)$ and $\mu = (3, 2, 2, 2)$, we have the following examples of a semi-standard Young tableau of shape λ and a semi-standard Young tableau of shape λ/μ

1	1	2	2	3
2	3	3	3	
3	4	5	5	
5	5			

T_1

			1	1
		2	2	
		3	5	

T_2

There is a special kind of semi-standard Young tableaux, the *Littlewood–Richardson tableaux*. They are defined as the skew semi-standard Young tableaux such that their reverse reading word is a *lattice permutation*, i.e. it is a sequence of positive integers such that in every initial part of it, the number of occurrences of i is bigger than or equal to the number of occurrences of $i + 1$.

1.2. Moving from representation theory to symmetric functions

Representation theory is directly related to the theory of symmetric functions, which is the setting of this thesis. In Appendix A and Appendix B we include a brief introduction to both frameworks. Both theories are related by the *Frobenius characteristic map*, described in Appendix C.

The main objects of this thesis are two fundamental coefficients arising from both representation theory and symmetric functions: the Kronecker coefficients and the plethysm coefficients. The Littlewood–Richardson coefficients also appear in both theories. They constitute a golden standard of what we would like to obtain for the Kronecker coefficients and the plethysm coefficients.

The following list summarizes where these three families of coefficients appear in representation theory for the symmetric group, representation theory for the general linear group, and symmetric functions.

Representation theory of symmetric groups

Let us denote by V_λ the irreducible representations of the symmetric group. Fix λ a partition of n , and μ a partition of m .

LITTLEWOOD-RICHARDSON COEFFICIENTS $\boxed{c'_{\lambda\mu}}$:

$$\text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{m+n}} V_\lambda \otimes V_\mu \cong \bigoplus c'_{\lambda\mu} V_\nu,$$

where the sum carries over all the partitions ν of $m + n$.

KRONECKER COEFFICIENTS $\boxed{g'_{\lambda\mu}}$:

For $m = n$,

$$V_\lambda \otimes V_\mu \cong \bigoplus g'_{\lambda\mu} V_\nu,$$

where the sum carries over all partitions of m .

PLETHYSM COEFFICIENTS $\boxed{a'_{\lambda\mu}}$:

$$\text{Ind}_{\mathfrak{S}_m[\mathfrak{S}_n]}^{\mathfrak{S}_{mn}} (V_\lambda [V_\mu]) \cong \bigoplus a'_{\lambda\mu} V_\nu,$$

where the sum carries over all partitions of $m \cdot n$.

Representation theory of general linear groups

Let us denote by S_λ the irreducible representation of $GL_d(\mathbb{C})$ associated to the partition λ of n . Fix partitions λ and μ of m and n , respectively.

LITTLEWOOD-RICHARDSON COEFFICIENTS $\boxed{c'_{\lambda\mu}}$:

$$S_\lambda \otimes S_\mu \cong \bigoplus c'_{\lambda\mu} S_\nu,$$

where the sum carries over all partitions of $m + n$.

KRONECKER COEFFICIENTS $\boxed{g'_{\lambda\mu}}$:

For $m = n$,

$$\text{Ind}_{GL_{d_1}(\mathbb{C}) \otimes GL_{d_2}(\mathbb{C})}^{GL_{d_1 d_2}(\mathbb{C})} S_\lambda \otimes S_\mu \cong \bigoplus g'_{\lambda\mu} S_\nu,$$

where the sum carries over all partitions of m .

PLETHYSM COEFFICIENTS $\boxed{a'_{\lambda\mu}}$:

In this case, we consider the irreducible representations of the general linear group obtained by applying to V the Schur functor \mathbb{S}_λ , for V any finite dimensional vector space and λ a partition. These irreducible

representations are denoted by $\mathbb{S}_\lambda(V)$. Then, the plethysm coefficients appear once we consider the composition of Schur functors.

$$\mathbb{S}_\lambda[\mathbb{S}_\mu(V)] \cong \bigoplus a_{\lambda\mu}^\nu \mathbb{S}_\nu(V),$$

where the sum carries over all partitions of $m \cdot n$.

Symmetric functions

LITTLEWOOD-RICHARDSON COEFFICIENTS $\boxed{c_{\lambda\mu}^\nu}$:

This family appears twice in symmetric functions theory. For the usual product on *Sym*: fix partitions λ and μ , then

$$s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu,$$

where we sum over all partitions ν such that $|\lambda| + |\mu| = |\nu|$.

They also appear in the coproduct defined using the sum of alphabets. For any alphabets X and Y , fix a partition ν . Then,

$$s_\nu[X + Y] = \sum c_{\lambda\mu}^\nu s_\lambda[X] \cdot s_\mu[Y], \tag{1.1}$$

where we sum over all partitions λ and μ such that $|\lambda| + |\mu| = |\nu|$.

KRONECKER COEFFICIENTS $\boxed{g_{\lambda\mu}^\nu}$:

This family appears when we consider the product of alphabets. For any alphabets X and Y , consider a partition ν . Then,

$$s_\nu[XY] = \sum g_{\lambda\mu}^\nu s_\lambda[X] \cdot s_\mu[Y],$$

where the sum carries over all partitions λ and μ such that $|\lambda| = |\mu| = |\nu|$. The Kronecker coefficients also defines a new product over *Sym*. For any partitions of the same integer λ and μ , the Kronecker product of Schur functions is defined as

$$s_\lambda * s_\mu = \sum g_{\lambda\mu}^\nu s_\nu,$$

where the sum carries over all ν such that $|\nu| = |\lambda| = |\mu|$.

PLETHYSM COEFFICIENTS $\boxed{a_{\lambda\mu}^\nu}$:

For any partitions λ and μ , the plethysm of two Schur functions decomposes as

$$s_\lambda[s_\mu] = \sum a_{\lambda\mu}^\nu s_\nu,$$

where the sum carries over all partitions ν such that $|\nu| = |\lambda| \cdot |\mu|$.

1.3. The structure of *Sym*

In this section we include a brief introduction to the structure *Sym*. For this purpose, we describe the Hopf algebra structure defined with the usual product of functions and the coproduct associated to the sum of alphabets, and another bialgebra structure, defined by the Kronecker product and the coproduct associated to the product of alphabets. Both structures will bring to scene two families of coefficients: the Littlewood–Richardson coefficients and the Kronecker coefficients. We introduce the λ -ring notation, which is very useful in the framework of symmetric functions. Finally, we include several results related to symmetric functions that we will use among the thesis.

1.3.1. λ -ring notation

The ring of symmetric functions is setting in terms of a set of countable many independent variables, $X = \{x_1, x_2, \dots\}$, called the *underlying alphabet*. One can think in another infinite alphabet A and try to figure out how is *Sym* in this new alphabet. Any infinite alphabet A gives rise to a copy $Sym(A)$ of *Sym*. In this copy, the element corresponding to $f \in Sym$ is denoted by $f[A]$. The map $f \mapsto f[A]$ is a *specialization*, [Sta99, Section 7.8]. In particular, it is a morphism of algebras from *Sym* to $Sym(A)$. It is convenient to write it as $f[A]$, rather than $A(f)$.

In fact, for some commutative algebra \mathcal{A} , we can consider any morphism of algebras from *Sym* to \mathcal{A} . Then, we consider it as a *specialization at a virtual alphabet* A . Once again, it is convenient to write $f \mapsto f[A]$. Since the power sums symmetric functions, p_k with $k \geq 1$, generate *Sym* and are algebraically independent, the map

$$A \longmapsto (p_1[A], p_2[A], \dots) \tag{1.2}$$

is a bijection from the set of all morphisms of algebras from Sym to \mathcal{A} , to the set of infinite sequences of elements from \mathcal{A} . This set of sequences is endowed with this operations of component-wise sum, product, and multiplication by a scalar. The bijection (1.2) is used to lift these operations to the set of morphism from Sym to \mathcal{A} .

By definition, for any $k \geq 1$, any virtual alphabets A and B , and any scalar c :

$$\begin{aligned} p_k[A + B] &= p_k[A] + p_k[B], \\ p_k[AB] &= p_k[A] \cdot p_k[B], \\ p_k[cA] &= c \cdot p_k[A]. \end{aligned}$$

For instance,

$$\begin{aligned} p_k[x_1 + x_2 + \cdots + x_n] &= x_1^k + x_2^k + \cdots + x_n^k, \\ p_k[n] &= p_k[\underbrace{1 + \cdots + 1}_n] = \underbrace{p_k[1] + \cdots + p_k[1]}_n = n. \end{aligned}$$

These expressions are extended by linearity to describe $f[A + B]$, $f[AB]$, and $f[cA]$, for any symmetric function f . But also to describe $f[P(A, B, \dots)]$, where $P(A, B, \dots)$ is a polynomial in several virtual alphabets A, B, \dots with coefficients in the basis field. The morphism $f \mapsto f[X]$ is just the identity of Sym . Among this thesis, we abuse notation and we denote by X both the general virtual alphabet and the underlying alphabet.

The ring Sym , together with the operations over the alphabets and the morphisms involved, has the structure of what is known as λ -ring. This is a natural concept, involving many rings considered in K -theory. In the case of symmetric functions, the λ -ring notation provides a powerful formalism to study Sym , recovering and extending many classical results stated by A. Lascoux in [Las03].

1.3.2. Sym , more than a ring

The product of two symmetric functions is also a symmetric function. Thus, Sym is an algebra. This basic structure brings into the situation one of the most important and interesting families of coefficients.

Definition 1.3.1. The *Littlewood–Richardson coefficients* are the coefficients, $c_{\lambda\mu}^\nu$, appearing in the decomposition of the product of $s_\lambda s_\mu$ in terms of the Schur basis,

$$s_\lambda s_\mu = \sum_{\nu \in \text{Par}} c_{\lambda,\mu}^\nu s_\nu.$$

Since the Schur basis is an orthonormal basis, we can describe the Littlewood–Richardson coefficients in terms of the scalar product:

$$c_{\lambda,\mu}^\nu = \langle s_\lambda s_\mu, s_\nu \rangle.$$

Thus, the Littlewood–Richardson coefficients are non–negative integers. They also appear in the decomposition of $s_{\nu/\lambda}$ in terms of the Schur basis:

$$s_{\nu/\lambda} = \sum_{\mu} s_{\lambda,\mu}^\nu s_\mu,$$

where $|\lambda| + |\mu| = |\nu|$. The Littlewood–Richardson coefficients have the following combinatorial description.

Theorem 1.3.2 (Littlewood–Richardson rule, [LR34]). *Let λ , μ and ν be three partitions. Then, $c_{\lambda,\mu}^\nu$ is equal to the number of tableaux T of shape ν/λ and type μ , such that the reverse reading word is a lattice permutation. These tableaux are called Littlewood–Richardson tableaux.*

Example 1. For $\nu = (4, 3, 2)$, $\lambda = (2, 1)$, and $\mu = (3, 2, 1)$, we have $c_{\lambda,\mu}^\nu = 2$. Let us see the two corresponding Littlewood–Richardson tableaux:

		1	1
	1	2	
2	3		

		1	1
	2	2	
1	3		

From this combinatorial description, we observe the following.

Proposition 1.3.3. *If $|\lambda| + |\mu| \neq |\nu|$ or $\lambda, \mu \notin \nu$, then $c_{\lambda,\mu}^\nu = 0$.*

The Littlewood–Richardson coefficients joint with their combinatorial interpretation stated in the Littlewood–Richardson rule are a golden standard of what we would like to obtain for other families of coefficients appearing in the theory of symmetric functions and in the representation theory. In the bibliography, the list of properties is long drawn out to state here. We simply mention the interpretation in terms of *honeycombs*, by A. Knutson and T. Tao in [KT99], in terms of *hives*, by Berenstein and Zelevinsky in [BZ92], and the interpretation in terms of *BZ patterns*, in [Sta99].

We consider the following operations of alphabets.

Definition 1.3.4. Let $X = \{x_1, x_2, x_3, \dots\}$ and $Y = \{y_1, y_2, y_3, \dots\}$ be two alphabets. Then, we define the *sum of alphabets* and the *product of alphabets* as

$$\begin{aligned} X + Y &= x_1 + x_2 + x_3 + \dots + y_1 + y_2 + y_3 + \dots \\ XY &= x_1y_1 + x_1y_2 + \dots + x_2y_1 + x_2y_2 + \dots + x_iy_j + \dots \end{aligned}$$

The sum of alphabets is the tool we use to define the coproduct on Sym . We identify $Sym \otimes Sym$ with the functions in X and Y which are symmetric for each alphabet. Then, we define the product

$$\begin{aligned} Sym \otimes Sym &\longrightarrow Sym \\ f \otimes g &\longmapsto f[X] \cdot g[Y] \end{aligned}$$

The coproduct is defined using the sum of alphabets

$$\begin{aligned} \Delta : Sym &\longrightarrow Sym \otimes Sym \\ f &\longmapsto f[X + Y] \end{aligned} \tag{1.3}$$

with the counit defined by the specialization $\epsilon(f) = f(0, 0, 0, \dots)$.

Theorem 1.3.5. *(Sym, \cdot, Δ) is a Hopf algebra, where the antipode corresponds to a variation of the involution ω , defined using the complete homogeneous basis in terms of the elementary basis*

$$\begin{aligned} \bar{\omega} : Sym &\longrightarrow Sym \\ h_i &\longmapsto (-1)^i e_i \end{aligned}$$

Proposition 1.3.6. *The Hopf algebra Sym has the following properties:*

- For the Schur basis,

$$\Delta(s_\nu) = \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu s_\lambda \otimes s_\mu.$$

- We can define the following scalar product over $Sym \otimes Sym$

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle \cdot \langle f_2, g_2 \rangle,$$

for any $f_1, f_2, g_1, g_2 \in Sym$.

- The antipode $\bar{\omega}$ in an involution.
- For any $f \in Sym$, $\langle f, 1 \rangle = \epsilon(f)$.
- For any $f, g, h \in Sym$, $\langle \Delta(f), g \otimes h \rangle = \langle f, gh \rangle$.

By the λ -ring notation introduced in Subsection 1.3.1, the power sum symmetric functions of the product of two alphabet is

$$p_\lambda[XY] = p_\lambda[X] \cdot p_\lambda[Y].$$

For the sum of alphabets and the Schur basis, we have the following result.

Proposition 1.3.7. *For any partition ν ,*

$$s_\nu[X + Y] = \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu s_\lambda[X] s_\mu[Y],$$

where the coefficients $c_{\lambda, \mu}^\nu$ are the Littlewood–Richardson coefficients.

The understanding of what happens when we consider the product of alphabets instead of the sum is still an open problem. We define the *Cauchy kernel* as $\sigma_1 = \sum_{n \in \mathbb{Z}} h_n$. Then, we have the following result.

Proposition 1.3.8. *Two bases $\{u_\lambda\}$ and $\{v_\lambda\}$ of Sym are dual bases if and only if*

$$\sigma_1[XY] = \sum_{\lambda \vdash n} u_\lambda[X] \cdot v_\lambda[Y].$$

In particular,

$$\sum_{n \geq 0} s_n[XY] = \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda[Y].$$

We can define another bialgebra structure over Sym using the product of alphabets.

$$\begin{aligned} \Delta^* : Sym &\longrightarrow Sym \otimes Sym \\ f &\longmapsto f[XY] \end{aligned} \tag{1.4}$$

with the counit defined by the specialization $\epsilon^*(f) = f(1, 0, 0, \dots)$.

For instance, by Proposition 1.3.8, we know that

$$\Delta^*(h_n) = \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda.$$

It can be also proved the following properties:

Proposition 1.3.9.

$$\begin{aligned} \Delta^*(e_n) &= \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda, & \epsilon^*(h_n) &= 1, \\ \Delta^*(p_n) &= p_n \otimes p_n, & \epsilon^*(e_n) &= \delta_{1,n} + \delta_{0,n}, \\ & & \epsilon^*(p_n) &= 1. \end{aligned}$$

This operation induces the following definition.

Definition 1.3.10. The decomposition of $\Delta^*(s_\nu)$ in the Schur basis can be described as

$$\Delta^*(s_\nu) = \sum_{\lambda, \mu} g_{\lambda, \mu}^\nu s_\lambda[X] \cdot s_\mu[Y].$$

where the coefficients $g_{\lambda, \mu}^\nu$ form a new family of coefficients for the structure of Sym , the *Kronecker coefficients*.

These coefficients satisfy a similar property than the one presented in Proposition 1.3.3 for the Littlewood–Richardson coefficients.

Proposition 1.3.11. *The Kronecker coefficients $g_{\lambda, \mu}^\nu$ are zero if the indexing partitions do not satisfy $|\lambda| = |\mu| = |\nu|$.*

The Kronecker coefficients can be used to define a new product over $Sym \otimes Sym$, the *Kronecker product*. We define it on the Schur basis, and extend it by linearity:

$$s_\lambda \star s_\mu = \sum_{\nu} g_{\lambda, \mu}^\nu s_\nu.$$

This is the second product defined on Sym .

Proposition 1.3.12. *We have the following list of properties.*

1. *The elements $s_\lambda[X] \cdot s_\mu[Y]$, with $\lambda, \mu \in Par$, form an orthonormal basis of $Sym \otimes Sym$.*
2. *For each $f, g, h \in Sym$, $\langle \Delta^*(f), g \otimes h \rangle = \langle f, g \star h \rangle$. This means that Δ^* is the adjoin of the Kronecker product.*
3. *For the power sums basis, we have that*

$$\begin{aligned} p_\lambda \star p_\mu &= \delta_{\lambda, \mu} \cdot z_\lambda \cdot p_\lambda, \\ p_\lambda \star p_\mu &= \langle p_\lambda, p_\mu \rangle \cdot p_\lambda. \end{aligned}$$

4. *The Kronecker product is commutative.*
5. *For all $f \in Sym^n$, $h_k \star f = f \star h_k = f$. Then, the unit with respect to the Kronecker product is given by $\sum_{k \geq 0} h_k$, which is not an element of Sym . It is an element of the completion of Sym .*

Trying to understand the Kronecker coefficients is one of the most important problems in representation theory. It has captured the attention of mathematicians for more than a century, but that has remained unsolved. The problem of finding a combinatorial interpretation for the Kronecker coefficients can be restated as whether computing the coefficients is in $\#P$. It remains a major open problem,

one of the oldest unsolved problems in algebraic combinatorics. More recently, the interest in computing Kronecker coefficients has come back to the forefront because of their connections with geometric complexity theory, pioneered recently by Mulmuley as an approach to the P vs. NP problem.

On one hand, in 2006, Narayanan showed that the computation of the Littlewood–Richardson coefficients is a #P–complete problem, [Nar06]. In 2008, Bürgisser and Ikenmeyer showed that the computation of the Kronecker coefficients is #P–hard, [BI08]. In [PP15b], Pak and Panova also give explicit bounds for the complexity of computing Kronecker coefficients in terms of the number of parts in the partitions, their largest part size and the smallest second part of the three partitions.

On the other hand, deciding the positivity of the Littlewood–Richardson coefficients can be done in polynomial time, as a consequence of the *saturation hypothesis* established by Knutson and Tao in 1999, [KT99]: if $c_{\mu\nu}^\lambda > 0$, then $c_{N\mu, N\nu}^{N\lambda}$, for any $N \geq 0$, where $N\lambda$ is the partition $(N\lambda_1, N\lambda_2, \dots)$.

The scarce information available about Kronecker coefficients makes difficult even the experimental checking of these conjectures. In 2009, E. Briand, R. Orellana and M. Rosas showed that the Kronecker coefficients do not satisfy the saturation conjecture, [BOR09].

1.3.3. Results related to symmetric functions

In this subsection we include results we will use among the thesis.

Theorem 1.3.13 (Jacobi–Trudi formula). *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition, and $\mu = (\mu_1, \dots, \mu_n)$ other partition such that $\mu \subseteq \lambda$ and $\mu_i = 0$, for $i > \ell(\mu)$. Then,*

$$s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{i,j=1}^n.$$

Example 2. Consider $\lambda = (3, 2)$ and $\mu = (2, 1)$, then

$$s_{32/21} = \det \begin{pmatrix} h_1 & h_2 \\ h_{-1} & h_1 \end{pmatrix} = h_1 \cdot h_1 - h_2 \cdot h_{-1} = h_{11}.$$

The next two results have an important role in the Chapter 2. The first result is a shifted version of the Jacobi–Trudi formula.

Proposition 1.3.14. *For a sequence of positive integers $\mu = (\mu_1, \dots, \mu_k)$, for any $1 \leq i \leq k$,*

$$s_\mu = -s_{(\mu_1, \dots, \mu_{i+1}-1, \mu_i+1, \dots, \mu_k)}.$$

In fact, $s_\mu = 0$, or there exists a partition $\tilde{\mu}$ such that $s_\mu = \pm s_{\tilde{\mu}}$.

Proof. Consider the determinant that defines s_μ :

$$s_\mu = \det \begin{pmatrix} h_{\mu_1} & \cdots & h_{\mu_i-i+1} & h_{\mu_{i+1}-i} & \cdots & h_{\mu_k-k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\mu_1+i-1} & \cdots & h_{\mu_i} & h_{\mu_{i+1}-1} & \cdots & h_{\mu_k-k+i} \\ h_{\mu_1+i} & \cdots & h_{\mu_i+1} & h_{\mu_{i+1}} & \cdots & h_{\mu_k-k+i+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\mu_1+k-1} & \cdots & h_{\mu_i+k-i} & h_{\mu_{i+1}+k-i-1} & \cdots & h_{\mu_k} \end{pmatrix}.$$

If we exchange the i^{th} column with the $(i+1)^{\text{th}}$ column, we obtain

$$s_\mu = -\det \begin{pmatrix} h_{\mu_1} & \cdots & h_{\mu_{i+1}-i} & h_{\mu_i-i+1} & \cdots & h_{\mu_k-k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\mu_1+i-1} & \cdots & h_{\mu_{i+1}-1} & h_{\mu_i} & \cdots & h_{\mu_k-k+i} \\ h_{\mu_1+i} & \cdots & h_{\mu_{i+1}} & h_{\mu_i+1} & \cdots & h_{\mu_k-k+i+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\mu_1+k-1} & \cdots & h_{\mu_{i+1}+k-i-1} & h_{\mu_i+k-i} & \cdots & h_{\mu_k} \end{pmatrix}.$$

Then,

$$s_\mu = -s_{(\mu_1, \dots, \mu_{i+1}-1, \mu_i+1, \dots, \mu_k)}.$$

When μ is not a partition, we apply successively the first formula, exchanging columns in each step, until we get that the integers indexing the diagonal elements of the determinant form a partition, $\tilde{\mu}$. \square

The second result is a property of the skew Schur functions indexed by a disconnected skew diagram.

Proposition 1.3.15. *Let λ/μ be a skew diagram such that λ/μ is disconnected, i.e. $\lambda/\mu = \alpha \oplus \beta$. Then,*

$$s_{\lambda/\mu} = s_\alpha \cdot s_\beta.$$

Proof. This proposition follows immediately from the combinatorial definition of the skew Schur functions. \square

Chapter 2.

Stability of plethysm coefficients

The *plethysm* was introduced by D. E. Littlewood [Lit44]. The operation of plethysm arises naturally in both the representation of the general linear group $GL_n(\mathbb{C})$ and the symmetric group \mathfrak{S}_n .

Given any finite dimensional complex vector space V , the irreducible finite dimensional (polynomial) representations of $GL(V)$ are obtained by applying to V the Schur functor \mathbb{S}_λ , for a partition λ . These constructions generalize the symmetric powers, Sym^n , and the exterior powers, \wedge^n . The Schur functors can be composed, giving rise up to representations $\mathbb{S}_\lambda[\mathbb{S}_\mu(V)]$. Such a representation decomposes into irreducible as

$$\mathbb{S}_\lambda[\mathbb{S}_\mu(V)] \simeq \bigoplus a_{\lambda\mu}^\nu \mathbb{S}_\nu(V), \quad (2.1)$$

and is thus described by the multiplicities $a_{\lambda\mu}^\nu$. These multiplicities are the *plethysm coefficients*. The character of the irreducible representation $\mathbb{S}_\mu(V)$ is the Schur function $s_\mu(x_1, \dots, x_k)$, and the character of the representation $\mathbb{S}_\lambda[\mathbb{S}_\mu(V)]$ is the plethysm of Schur functions, $s_\lambda[s_\mu](x_1, \dots, x_k)$. The multiplicities $a_{\lambda\mu}^\nu$ are thus recovered by decomposing the plethysm of Schur functions, $s_\lambda[s_\mu]$, in the basis of Schur functions. The plethysm is sometimes called *outer plethysm* to distinguish it from the operation of *inner plethysm*, a term used for Kronecker product of representations.

This chapter is structured as follows: In Section 2.1, we define the plethysm in the framework of the symmetric functions. In Section 2.2, we introduce the *h-plethysm coefficients* and their combinatorial interpretation in terms of counting integer points in polytopes. The rest of the sections are dedicated to describe and study the stability phenomena presented in different sequences of plethysm coefficients. In Section 2.4 we present an adaptation of the proof given by Carré and Thibon in [CT92]. In Section 2.5 we present alternative combinatorial proofs

of these results, and of another two results proved by Brion in [Bri93] in a more general setting. Finally, Section 2.6 is dedicated to shorten all the bounds.

2.1. Definition and properties

The plethysm $f[g]$ is essentially the symmetric function obtained by substituting the monomials of g for the variables of f . Let us introduce this binary operation, using the power sums basis, as N. A. Loher and J. B. Remmel in [LR11].

Definition 2.1.1. The *plethysm* is the unique binary operation on Sym such that:

1. For all $m, n \geq 1$, $p_m[p_n] = p_{mn}$.
2. For all $m \geq 1$, the map

$$\begin{aligned} Sym &\longrightarrow Sym \\ g &\longmapsto p_m[g], \end{aligned}$$

is a \mathbb{Q} -algebra homomorphism, i.e. for all $m \geq 1$, $g_1, g_2 \in Sym$ and $c \in \mathbb{Q}$

- $p_m[g_1 + g_2] = p_m[g_1] + p_m[g_2]$,
- $p_m[g_1 \cdot g_2] = p_m[g_1] \cdot p_m[g_2]$,
- $p_m[c] = c$.

3. For all $g \in Sym$, the map

$$\begin{aligned} Sym &\longrightarrow Sym \\ f &\longmapsto f[g], \end{aligned}$$

is a \mathbb{Q} -algebra homomorphism, i.e. for all $g, f_1, f_2 \in Sym$ and $c \in \mathbb{Q}$

- $(f_1 + f_2)[g] = f_1[g] + f_2[g]$,
- $(f_1 \cdot f_2)[g] = f_1[g] \cdot f_2[g]$,
- $c[g] = c$.

By this definition, it follows next result.

Proposition 2.1.2. Consider partitions λ and μ . We set $\lambda \circ \mu$ for the partition obtained reordering the sequence $(\lambda_i \cdot \mu_j)_{i,j}$. Then, $p_\lambda[p_\mu] = p_{\lambda \circ \mu}$.

To compute $f[g]$ in practice, we first express f in terms of the power sums basis, $f = \sum_{\nu} c_{\nu} p_{\nu}$. By (3) in 2.1.1,

$$f[g] = \sum_{\nu} c_{\nu} \prod_i p_{\nu_i}[g].$$

Next, we expand g in the power sums basis, $g = \sum_{\mu} d_{\mu} p_{\mu}$. Then, using (1) and (2) in 2.1.1, we get

$$p_{\nu_i}[g] = \sum_{\mu} d_{\mu} \prod_j p_{\nu_i}[p_{\mu_j}] = \sum_{\mu} d_{\mu} \prod_j p_{\nu_i \mu_j}.$$

Then,

$$f[g] = \sum_{\nu} c_{\nu} \cdot \left(\sum_{\mu} d_{\mu} p_{\nu \circ \mu} \right).$$

Example 3. Consider $g = \sum_{i \geq 1} 2x_i$ and $f = m_{31}$. We express g as:

$$g = \sum_{i \geq 1} 2x_i = \sum_{i \geq 1} x_i + \sum_{i \geq 1} x_i = p_1 + p_1.$$

We express also $f = m_{31}$ in the power sums basis: $m_{31} = p_{31} - p_4$. Then,

$$\begin{aligned} m_{31}[g] &= (p_{31} - p_4)[g] = p_{31}[p_1 + p_1] - p_4[p_1 + p_1] = \\ &= p_3[p_1 + p_1] \cdot p_1[p_1 + p_1] - p_4[p_1 + p_1] = 4p_{31} - 2p_4. \end{aligned}$$

We can express the result in terms of the monomial basis as $m_{31}[g] = 4m_{31} + 2m_4$.

Another way to compute $f[g]$ is the evaluation of f in the alphabet defined by the monomials of g : $f[g] = f(x^{u_1}, x^{u_2}, \dots)$, for $g = \sum x^{u_i}$.

2.2. Plethysm coefficients: a combinatorial interpretation

We have introduced the plethysm coefficients $a_{\lambda\mu}^{\nu}$ in (2.1) in the setting of representation theory. Moving to the framework of symmetric functions, the plethysm coefficient $a_{\lambda\mu}^{\nu}$ is the coefficient of s_{ν} in the expansion in the Schur basis of the plethysm of Schur functions $s_{\lambda}[s_{\mu}]$. Alternatively, by the orthonormality of the

Schur basis, this coefficient is extracted by means of a scalar product:

$$a'_{\lambda\mu} = \langle s_\lambda[s_\mu], s_\nu \rangle. \quad (2.2)$$

They have the following property.

Proposition 2.2.1. *If $|\lambda| \cdot |\mu| \neq |\nu|$, then $a'_{\lambda\mu} = 0$.*

Proof. Using properties (2) and (3) of Definition 2.1.1, we check that $p_n[p_m] = p_m[p_n] = p_{mn}$. By linearity, the plethysm of homogeneous symmetric functions f and g of degree m and n , respectively, is an homogeneous symmetric function of degree mn . \square

Let us see how we can decompose the plethysm coefficients as a sum of other coefficients easier to calculate. The Jacobi–Trudi formula, recalled in Theorem 1.3.13, gives us a description of s_λ as a determinant depending on the complete homogeneous basis $\{h_\gamma\}$. If we expand explicitly the determinant in this expression, we decompose the Schur function as a sum over the permutations σ in the symmetric group \mathfrak{S}_N , with $N \geq \ell(\lambda)$:

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) h_{\lambda+\omega(\sigma)},$$

where $\omega(\sigma)_j = \sigma(j) - j$, for all j between 1 and N , and $\varepsilon(\sigma)$ is the sign of the permutation.

We now perform this Jacobi–Trudi expansion for s_λ and s_ν in (2.2). We get the following alternating decomposition for the plethysm coefficients.

Lemma 2.2.2. *Let N and N' be positive integers. Let λ , μ and ν be partitions, such that λ has length at most N and ν has length at most N' . Then*

$$a'_{\lambda\mu} = \sum_{\sigma, \tau} \varepsilon(\sigma) \varepsilon(\tau) \langle h_{\lambda+\omega(\sigma)}[s_\mu], h_{\nu+\omega(\tau)} \rangle,$$

where the sum is carried over all permutations $\sigma \in \mathfrak{S}_N$ and $\tau \in \mathfrak{S}_{N'}$.

Consider these new coefficients appearing in Lemma 2.2.2.

Definition 2.2.3. For any partition μ and any finite sequences λ and ν of integers, the h -plethysm coefficients are defined as

$$b'_{\lambda\mu} = \langle h_\lambda[s_\mu], h_\nu \rangle.$$

These coefficients are interesting by their own. We will show that they count the non-negative solutions of systems of linear Diophantine equations whose constant terms depend linearly on the parts of λ and ν .

Before specifying the description, we introduce some notation. For any partition μ and any positive integer N , let $t(\mu, N)$ be the set of semi-standard Young tableaux of shape μ with entries between 1 and N . Each element of $t(\mu, N)$, T , has associated a weight, $\rho(T)$, which is a vector of length N . Let $\mathcal{P}_{\mu, N} = (\rho_j(T))_{T, j}$ be the matrix whose rows are the weight of the elements of $t(\mu, N)$. This means that the rows of $\mathcal{P}_{\mu, N}$ are indexed by the semi-standard Young tableaux $T \in t(\mu, N)$, and that $\rho_j(T)$ is the number of occurrences of j in T .

Proposition 2.2.4. *Let λ and ν be finite sequences of positive integers and let μ be a partition. Consider an integer $N \geq \ell(\lambda), \ell(\nu)$. The coefficient $b_{\lambda, \mu}^\nu$ is the cardinal of the set $Q_{\lambda, \mu}^\nu(N)$ of matrices $\mathcal{M} = (m_{i, T})$ with non-negative integer entries whose rows are indexed by the integers i between 1 and N , and whose columns are indexed by the semi-standard Young tableaux $T \in t(\mu, N)$ such that:*

- ROW SUM CONDITION FOR \mathcal{M} : *the sum of the entries in the i^{th} row of \mathcal{M} is λ_i .*
- COLUMN SUM CONDITION FOR $\mathcal{M} \cdot \mathcal{P}_{\mu, N}$: *the sum of the entries in the j^{th} column of $\mathcal{M} \cdot \mathcal{P}_{\mu, N}$ is ν_j .*

We include an example of how Proposition 2.2.4 works after its proof.

Proof. Let x_1, x_2, \dots be the underlying variables of the symmetric functions and, for any finite sequence $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, let us denote x^μ for $x_1^{\mu_1} x_2^{\mu_2} \dots x_k^{\mu_k}$. The scalar product of any symmetric function with h_ν extracts the coefficient of m_ν in its expansion in the basis of monomial functions. Therefore, $b_{\lambda, \mu}^\nu$ can be interpreted as the coefficient of the monomial x^ν in $h_\lambda[s_\mu]$.

Instead of working with symmetric functions (with infinitely many variables) we can work with symmetric polynomials in N variables, provided N is at least the length of λ and ν . By the combinatorial definition of the Schur polynomial $s_\mu(x_1, x_2, \dots, x_N)$:

$$s_\mu(x_1, x_2, \dots, x_N) = \sum_{T \in t(\mu, N)} x^{\rho(T)} = \sum_{T \in t(\mu, N)} x_1^{\rho_1(T)} x_2^{\rho_2(T)} \dots x_N^{\rho_N(T)}.$$

We use the notation $x^{\rho(T)}$ instead of simply x^T , in order to keep in mind that the exponents correspond to the weight of the semi-standard Young tableaux.

The plethysm $f[g]$ can be seen as the evaluation $f(x^{u_1}, x^{u_2}, \dots)$ once we have written g as a sum of monomials, $g = \sum_i x^{u_i}$, and the complete homogeneous symmetric polynomial h_n can be defined as the sum of all monomials of degree n . We set k as the number of semi-standard Young tableaux of $t(\mu, N)$. Then, we get that

$$h_n[s_\mu] = h_n \left[\sum_{T \in t(\mu, N)} x_1^{\rho_1(T)} x_2^{\rho_2(T)} \dots x_N^{\rho_N(T)} \right] = \sum_{m_1 + \dots + m_k = n} \prod_{i=1}^k x^{m_i \cdot \rho(T_i)},$$

where $x^{m_i \cdot \rho(T_i)}$ means $(x^{\rho(T_i)})^{m_i} = x_1^{\rho_1(T_i) \cdot m_i} \dots x_N^{\rho_N(T_i) \cdot m_i}$. Since $N \geq \ell(\lambda)$, we can consider λ as a partition of length N by adding as many zeros as we need to the sequence. Therefore, we have the following decomposition for the complete sum h_λ

$$h_\lambda[s_\mu] = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}[s_\mu] = \prod_{i=1}^{\ell(\lambda)} \left(\sum_{m_{i1} + \dots + m_{ik} = \lambda_i} \prod_{j=1}^k x^{\rho(T_j) \cdot m_{ij}} \right) = \sum_{\mathcal{M}} \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^k x^{m_{ij} \cdot \rho(T_j)} = \sum_{\mathcal{M}} x^{\sum_{i,j} m_{ij} \rho(T_j)}, \quad (2.3)$$

where the sum is carried over all matrices $\mathcal{M} = (m_{ij})$ such that

$$\sum_{j=1}^k m_{ij} = \lambda_i \quad \text{and} \quad \sum_{i,j} m_{ij} \cdot \rho_n(T_j) = \nu_n. \quad (2.4)$$

This proves the proposition. □

Example 4. Consider the partition $\mu = (2)$ and the finite sequences $\lambda = (\lambda_1, \lambda_2)$ and $\nu = (\nu_1, \nu_2, \nu_3)$, and take $N = 3$. First, we compute the set $t(\nu, N)$:

$$\begin{array}{cccccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \end{array}$$

Then, the corresponding matrix $\mathcal{P}_{\mu N}$ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The matrices \mathcal{M} described in Proposition 2.2.4 are the matrices

$$\begin{pmatrix} m_{1T_1} & m_{1T_2} & m_{1T_3} & m_{1T_4} & m_{1T_5} & m_{1T_6} \\ m_{2T_1} & m_{2T_2} & m_{2T_3} & m_{2T_4} & m_{2T_5} & m_{2T_6} \\ m_{3T_1} & m_{3T_2} & m_{3T_3} & m_{3T_4} & m_{3T_5} & m_{3T_6} \end{pmatrix},$$

that satisfy the following conditions

$$\begin{cases} \sum_j m_{1T_j} = \lambda_1 \\ \sum_j m_{2T_j} = \lambda_2 \\ \sum_j m_{3T_j} = 0 \end{cases} \quad \begin{cases} \sum_i (2m_{iT_1} + m_{iT_2} + m_{iT_3}) = \nu_1 \\ \sum_i (m_{iT_2} + 2m_{iT_4} + m_{iT_5}) = \nu_2 \\ \sum_i (m_{iT_3} + m_{iT_5} + 2m_{iT_6}) = \nu_3. \end{cases}$$

As a consequence of Proposition 2.2.4, we give the following result.

Proposition 2.2.5. *Fix a partition μ and finite sequences of positive integers λ and ν , such that $|\lambda| \cdot |\mu| = |\nu|$. Let N be an integer bigger than or equal to $\ell(\lambda)$, $\ell(\mu)$ and $\ell(\nu)$. Then, the coefficient $b_{\lambda\mu}^\nu$ will be zero unless ν satisfies the following conditions:*

- For any $j \leq N - \ell(\mu)$, $\nu_j \leq |\lambda| \cdot \mu_1$.
- For any $j > N - \ell(\mu)$, $\nu_j \leq |\lambda| \cdot \mu_{j-(N-\ell(\mu))}$.

Proof. Both inequalities come from the fact that we can bound the number of times that j appears in any semi-standard Young tableau T of $t(\mu, N)$ and use these estimates in the j^{th} column sum condition for $\mathcal{M} \cdot \mathcal{P}_{\mu N}$. \square

2.3. Stability Properties

We understand by stability of a sequence of plethysm coefficients the fact that the sequence is eventually constant.

Example 5. Let us see different sequences that stabilize.

- For the partitions $\lambda = (4, 1)$, $\mu = (3)$ and $\nu = (6, 6, 3)$, the sequence of plethysm coefficients $a_{\lambda+(n), \mu}^{\nu+(n \cdot |\mu|)}$ is constant for $n \geq 2$. The first terms are

$$a_{(4,1)+(n),(3)}^{(6,6,3)+(3n)} = 1, 6, 10, \mathbf{12, 12, 12}, \dots$$

- For the partitions $\lambda = (3, 1)$, $\mu = (2, 1)$ and $\nu = (3, 2, 2, 2, 1, 1, 1)$, the sequence of plethysm coefficients $a_{\lambda+(n),\mu}^{\nu+n\cdot\mu}$ is constant for $n \geq 4$. The first terms are

$$a_{(3,1)+(n),(2,1)}^{(3,2,2,2,1,1,1)+n\cdot(2,1)} = 1, 23, 75, 104, \mathbf{109, 109}, \dots$$

- For the partitions $\lambda = (3, 2, 1)$, $\mu = (2)$ and $\nu = (6, 4, 2)$, the sequence of plethysm coefficients $a_{\lambda,\mu+(n)}^{\nu+(n\cdot|\lambda|)}$ is constant for $n \geq 2$. The first terms are

$$a_{(3,2,1),(2)+(n)}^{(6,4,2)+(6n)} = 3, 8, \mathbf{9, 9, 9, 9}, \dots$$

In 1950, Foulkes observed the stability phenomena in some sequences of plethysm coefficients. In [Fou54], he presented specific formulas for some of the coefficients in $s_m[s_\mu]$, with $\mu \vdash 4$. For instance, he presented the following properties, stated in our language:

- For any m , $\langle s_{(\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_4+1)}, s_{(m+1)}[s_\mu] \rangle = \langle s_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}, s_{(m)}[s_{\mu'}] \rangle$.
- If $\lambda_2 \leq m$, then μ , $\langle s_{(\lambda_1+4, \lambda_2, \lambda_3, \lambda_4)}, s_{(m+1)}[s_\mu] \rangle = \langle s_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}, s_{(m)}[s_{\mu'}] \rangle$.

He also gave a method for the explicit determination of the coefficient of s_λ in $s_m[s_\mu]$, for any partitions λ of $4m$ and μ of 4. This method did not involve usual product of Schur functions nor recursive relations.

The stability properties observed by Foulkes were proved in the 90's by Brion in [Bri93], and by Carré and Thibon in [CT92]. Brion stated two stability properties in a more general setting: geometric representation theory of algebraic groups. Carré and Thibon stated another two properties in the framework of symmetric functions, and they used *vertex operators* to prove them. We summarize these four stability properties in the language of symmetric functions here.

(P1) In [CT92, Theorem 4.2], it is proved that the sequence with general term

$$a_{\lambda+(n),\mu}^{\nu+(|\mu|\cdot n)} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(|\mu|\cdot n)} \rangle$$

stabilizes. It has limit zero when $\ell(\mu) > 1$.

(Q1) In [Bri93, Corollary 1, Section 2.6], it is proved that the sequence of general term

$$a_{\lambda+(n),\mu}^{\nu+n\cdot\mu} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+n\cdot\mu} \rangle$$

is increasing and stabilizes.

(R1) In [CT92, Theorem 4.1], it is proved that the sequence with general term

$$a_{\lambda, \mu+(n)}^{\nu+(|\lambda| \cdot n)} = \langle s_{\lambda}[s_{\mu+(n)}], s_{\nu+(|\lambda| \cdot n)} \rangle$$

stabilizes.

(R2) In [Bri93, Corollary 1, Section 2.6], it is proved that the sequence of general term

$$a_{\lambda, \mu+n \cdot \pi}^{\nu+n \cdot |\lambda| \cdot \pi} = \langle s_{\lambda}[s_{\mu+n \cdot \pi}], s_{\nu+n \cdot |\lambda| \cdot \pi} \rangle$$

is increasing and stabilizes.

We refer to the sequences of these stability properties as (P1), (Q1), (R1) and (R2). The sequence appearing in (R1) is a particular case of the one in (R2), considering $\pi = (1)$. The sequences appearing in (P1) and (Q1) are not related one to each other.

The idea of testing the stability of some coefficients through other coefficients can be also found in other references. For instance, in [KM14], Kahle and Michałek study stability properties for the plethysm coefficients $\langle p_{\alpha}[s_{(k)}], h_{\beta} \rangle$. Also in the thesis of R. Abebe, [Abe], we find a study of the plethysm coefficient $\langle s_{\lambda}[s_{\mu}], h_{\beta} \rangle$.

In this thesis we present a combinatorial proof for these stability properties stated for the h -plethysm coefficients. We state this as a theorem.

Theorem 2.3.1. *For any partition μ , and any integer sequences λ and ν , such that $|\lambda| \cdot |\mu| = |\nu|$, the sequences of general terms*

$$b_{\lambda, \mu+(n)}^{\nu+(n \cdot |\lambda|)}, \quad b_{\lambda, \mu+n \pi}^{\nu+n \cdot |\lambda| \cdot \pi}, \quad b_{\lambda+(n), \mu}^{\nu+n \cdot \mu}, \quad \text{and} \quad b_{\lambda+(n), \mu}^{\nu+(n \cdot |\mu|)}$$

stabilize.

The stability properties of (P1), (Q1), (R1), and (R2) are a consequence of Theorem 2.3.1.

Corollary 2.3.2. *For any partitions λ , μ and ν , such that $|\lambda| \cdot |\mu| = |\nu|$, the sequences of general terms*

$$a_{\lambda, \mu+(n)}^{\nu+(n \cdot |\lambda|)}, \quad a_{\lambda, \mu+n \pi}^{\nu+n \cdot |\lambda| \cdot \pi}, \quad a_{\lambda+(n), \mu}^{\nu+n \cdot \mu}, \quad \text{and} \quad a_{\lambda+(n), \mu}^{\nu+(n \cdot |\mu|)}$$

stabilize.

2.4. Stability using vertex operators

The stability properties (P1) and (R1) were proved by Carré and Thibon using vertex operators. In this section, we present a reproduction of the original proofs published by Carré and Thibon in [CT92] in order to obtain explicit bounds for which the coefficients are constant. This adaptation also includes a change of notation for the partitions involved. Let λ be a partition and n an integer. Carré and Thibon denote λn for the sequence $(n, \lambda_1, \lambda_2, \dots)$. Note that this sequence is a partition when $n \geq \lambda_1$. With this notation, Carré and Thibon studied the generating functions of the coefficients $a_{\lambda, \mu}^{\nu m}$ and $a_{\lambda n, \mu}^{\nu m}$. Instead of adding n as the first part of the partition λ , we consider the partition $\lambda + (n) = (\lambda_1 + n, \lambda_2, \dots)$. With our notation, we will study the generating functions of the coefficients $a_{\lambda, \mu + (n)}^{\nu + (m)}$ and $a_{\lambda + (n), \mu}^{\nu + (m)}$.

2.4.1. Symmetric functions as operators

First, we introduce the vertex operators and some useful properties.

Let X be an alphabet and z an extra indeterminate. The usual scalar product defined on $Sym(X)$ can be extended by $\mathbb{C}[z]$ -linearity to the ring of symmetric functions in X with coefficients in $\mathbb{C}[z]$, $\mathbb{C}[z] \otimes Sym(X)$. For $f \in \mathbb{C}[z] \otimes Sym(X)$, we denote by D_f the adjoint of the multiplication operator, $g \mapsto f \cdot g$, with respect to the scalar product, i.e. for any $h, g \in \mathbb{C}[z] \otimes Sym(X)$, $\langle D_f(h), g \rangle = \langle h, f \cdot g \rangle$. For an infinite series, $f = \sum_n f_n z^n$, we set $D_f = \sum_n z^n D_{f_n}$.

Consider the complete homogeneous symmetric functions, $h_n(X)$, and the elementary symmetric functions, $e_n(X)$, with $n \geq 0$. Their generating functions are

$$\sigma_z(X) = \sum_{n \geq 0} h_n(X) z^n = \prod_{x \in X} \frac{1}{1 - zx}, \quad (2.5)$$

$$\lambda_z(X) = \sum_{n \geq 0} e_n(X) z^n = \prod_{x \in X} (1 + zx). \quad (2.6)$$

Then, D_{σ_z} and D_{λ_z} are the adjoint operators to the multiplication by σ_z and λ_z , respectively.

Proposition 2.4.1. *For any pair of dual basis $\{u_\lambda\}$ and $\{v_\lambda\}$, and any symmetric function f , we have the following fundamental formula*

$$f[X + Y] = \sum_{\lambda \in Par} D_{u_\lambda} f[X] \cdot v_\lambda[Y].$$

In particular, for Schur functions, we get

$$s_\mu[X + Y] = \sum_{\lambda \in \text{Par}} D_{s_\lambda} s_\mu[X] \cdot s_\lambda[Y].$$

By Proposition 1.1, this implies that $D_{s_\lambda} s_\mu = s_{\mu/\lambda}$.

Moreover, the action of the operators associated to σ_z and λ_z is given by the following result.

Proposition 2.4.2 ([CT92]). *The action of the operators D_{σ_z} and $D_{\lambda_{-z}}$ on an element F of $\mathbb{C}[z] \otimes \text{Sym}$ is given by*

$$\begin{aligned} D_{\sigma_z} f[X] &= f[X + z]. \\ D_{\lambda_{-z}} f[X] &= f[X - z]. \end{aligned}$$

Finally, we have the following result for the generating function of $s_{\lambda+(n)}$.

Proposition 2.4.3. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition. Then,*

$$\sum_{n \in \mathbb{Z}} z^n \cdot s_{\lambda+(n)} = \frac{1}{z^{\lambda_1}} \cdot \sigma_z(X) \cdot D_{\lambda_{-\frac{1}{z}}} s_{\bar{\lambda}},$$

where $\bar{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_k)$.

Proof. Consider the description of $s_{\lambda+(n)}$ as a determinant given by the Jacobi–Trudi formula, Theorem 1.3.13

$$s_{\lambda+(n)} = \det \begin{pmatrix} h_{\lambda_1+n} & h_{\lambda_2-1} & \dots & h_{\lambda_k-k+1} \\ h_{\lambda_1+n+1} & h_{\lambda_2} & \dots & h_{\lambda_k-k+2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+n+k-1} & h_{\lambda_2+k-2} & \dots & h_{\lambda_k} \end{pmatrix}.$$

Expand it along its first column, and note that $s_{\bar{\lambda}/(1^r)} = D_{e_r} s_{\bar{\lambda}}$. We get that

$$s_{\lambda+(n)} = \sum_{r \geq 0} (-1)^r \cdot s_{\lambda_1+n+r} \cdot D_{e_r} s_{\bar{\lambda}},$$

where $D_{e_r} s_{\bar{\lambda}}$ is zero, for $r > \ell(\bar{\lambda})$. Therefore,

$$z^n \cdot s_{\lambda+(n)} = \sum_{r \geq 0} \left(\frac{-1}{z} \right)^r z^{n+r} \cdot s_{\lambda_1+n+r} \cdot D_{e_r} s_{\bar{\lambda}}.$$

Then, summing over all $n \in \mathbb{Z}$,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} z_n \cdot s_{\lambda+(n)} &= \sum_{n \in \mathbb{Z}} \left[\sum_{r \geq 0} \left(\frac{-1}{z} \right)^r z^{n+r} \cdot s_{\lambda_1+n+r} \cdot D_{e_r} s_{\bar{\lambda}} \right] = \\ &= \sum_{r \geq 0} \left(\frac{-1}{z} \right)^r \cdot D_{e_r} s_{\bar{\lambda}} \cdot \frac{1}{z^{\lambda_1}} \underbrace{\left(\sum_{n \in \mathbb{Z}} z^{\lambda_1+n+r} s_{\lambda_1+n+r} \right)}_{\sigma_z(X)} = \sum_{r \geq 0} \frac{1}{z^{\lambda_1}} \cdot \sigma_z(X) \cdot \left(\frac{-1}{z} \right)^r \cdot D_{e_r} s_{\bar{\lambda}} = \\ &= \frac{1}{z^{\lambda_1}} \cdot \sigma_z(X) \cdot \sum_{r \geq 0} \left(\frac{-1}{z} \right)^r \cdot D_{e_r} s_{\bar{\lambda}} = \frac{1}{z^{\lambda_1}} \cdot \sigma_z(X) \cdot D_{\lambda_{-\frac{1}{z}}} s_{\bar{\lambda}}. \end{aligned}$$

□

The operator $\sigma_z D_{\lambda_{-\frac{1}{z}}}$ is known as the basic *vertex operator*.

As mentioned by B. Fauser, P. D. Jarvis, and R. C. King in [FJK10], since their introduction in string theory, vertex operators have played a fruitful role in mathematical constructions of group representations as well as combinatorial objects. Among all known applications, we cite for example applications to affine Lie algebras [FK81], quantum affine algebras [FJ88] or applications to variations on the symmetric functions, like Hall–Littlewood functions [Jin91] or Macdonald polynomials [Mac95, EK95].

2.4.2. The stability property (R1)

Set the coefficients $u_{n,m} = \langle s_{\lambda}[s_{\mu+(n)}], s_{\nu+(m)} \rangle$. We consider their generating function

$$f(z) = \sum_{n,m \in \mathbb{Z}} u_{n,m} z^m.$$

The coefficients $u_{n,m}$ with $m = n \cdot |\lambda|$ are exactly the plethysm coefficients $a_{\lambda, \mu+(n)}^{\nu+(n \cdot |\mu|)}$. Moreover, by Proposition 2.2.1, if $m \neq n \cdot |\lambda|$, then $u_{n,m} = 0$.

In [CT92], Carré and Thibon proved that $f(z) = \frac{P(z)}{1-z^{|\lambda|}}$, with $P(z)$ a Laurent polynomial. The stability of the sequence $\left\{ a_{\lambda, \mu+(n)}^{\nu+(n \cdot |\mu|)} \right\}_n$ is a consequence of this result. Moreover, we get that the stable value is $P(1)$ and that $a_{\lambda, \mu+(n)}^{\nu+(n \cdot |\mu|)} = P(1)$ for $n \geq \frac{\deg(P(z))}{|\lambda|}$. The bounds will be studied in Section 2.6.

Theorem 2.4.4. *Suppose that λ , μ and ν are partitions. Then,*

$$\sum_{n,m \in \mathbb{Z}} \langle s_\lambda [s_{\mu+(n)}], s_{\nu+(m)} \rangle z^m = \frac{P(z)}{1 - z^{|\lambda|}},$$

where $P(z)$ is a Laurent polynomial.

Proof. We consider the generating function of the theorem

$$f(z) = \sum_{n,m \in \mathbb{Z}} \langle s_\lambda [s_{\mu+(n)}], s_{\nu+(m)} \rangle z^m.$$

Then,

$$f(z) = \sum_{n \in \mathbb{Z}} \left\langle s_\lambda [s_{\mu+(n)}], \sum_{m \in \mathbb{Z}} s_{\nu+(m)} z^m \right\rangle.$$

We apply Proposition 2.4.3 to the right-hand side of the scalar product, and we get that

$$f(z) = \sum_{n \in \mathbb{Z}} \left\langle s_\lambda [s_{\mu+(n)}], \frac{1}{z^{\nu_1}} \cdot \sigma_z(X) \cdot D_{\lambda_{-\frac{1}{z}}} s_{\bar{\nu}} \right\rangle.$$

We can move $\sigma_z(X)$ from the right-hand side to the left-hand side of the scalar product as the adjoint operator:

$$f(z) = \sum_{n \in \mathbb{Z}} \left\langle D_{\sigma_z} s_\lambda [s_{\mu+(n)}], \frac{1}{z^{\nu_1}} \cdot D_{\lambda_{-\frac{1}{z}}} s_{\bar{\nu}} \right\rangle.$$

Then, applying Proposition 2.4.2 to both sides of the scalar product, We obtain that

$$f(z) = \sum_{n \in \mathbb{Z}} \left\langle s_\lambda [s_{\mu+(n)}[X+z]], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We separate the sum into two parts: when $\mu + (n)$ is a partition and when it is not. The sequence $\mu + (n)$ is a partition if $\mu_1 + n \geq \mu_2$. We set $f(z) = Q(z) + R(z)$, with

$$\begin{aligned} Q(z) &= \sum_{n \leq \mu_2 - \mu_1 - 1} \left\langle s_\lambda [s_{\mu+(n)}[X+z]], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle, \\ R(z) &= \sum_{n \geq \mu_2 - \mu_1} \left\langle s_\lambda [s_{\mu+(n)}[X+z]], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle. \end{aligned}$$

First, we consider $Q(z)$. We look at $s_{\mu+(n)}[X+z]$. Let $k = \ell(\mu+(n))$. Then, by the Jacobi–Trudi formula,

$$s_{\mu+(n)} = \det \begin{pmatrix} h_{\mu_1+n} & h_{\mu_2-1} & \cdots & h_{\mu_k-k+1} \\ h_{\mu_1+n+1} & h_{\mu_2} & \cdots & h_{\mu_k-k+2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mu_1+n+k-1} & h_{\mu_2+k-2} & \cdots & h_{\mu_k} \end{pmatrix}.$$

Note that, for $n = \mu_2 - \mu_1 - 1$, the first two columns are equal. Then, $s_{\mu+(n)} = 0$. Since this factor does not contribute to the sum over n , we have the upper bound $n \leq \mu_2 - \mu_1 - 2$. Moreover, for $\mu_1 + n + k - 1 \leq 0$, the first column is zero and we get the lower bound: $n \geq 1 - \mu_1 - k$. Then,

$$Q(z) = \sum_{n=1-\mu_1-k}^{\mu_2-\mu_1-2} \left\langle s_\lambda [s_{\mu+(n)}[X+z]], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

By Proposition 1.3.14, there exists a partition γ such that $s_{\mu+(n)} = \pm s_\gamma$, or it is zero. Since $\mu_1 + n < \mu_2$, we apply Proposition 1.3.14 at least once for exchanging the first two columns. As a consequence, $\gamma_1 = \mu_2 - 1$ is the largest part of the partition γ .

Since $s_\beta[z] = z^i$ for $\beta = (i)$, and zero if β has more than one part, we get that

$$s_\gamma[X+z] = \sum_{\beta \subseteq \gamma} s_{\gamma/\beta}[X] \cdot s_\beta[z] = \sum_{i=0}^{\gamma_1} s_{\gamma/(i)}[X] \cdot z^i.$$

Then, $Q(z)$ can be written as

$$Q(z) = \sum_{n=1-k-\mu_1}^{\mu_2-\mu_1-2} \left\langle s_\lambda \left[\pm \sum_{i=0}^{\gamma_1} s_{\gamma/(i)}[X] \cdot z^i \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle. \quad (2.7)$$

The sum over n is finite, and $Q(z)$ is a Laurent polynomial. Using (2.7), we estimate its degree $\deg(Q(z)) \leq |\lambda| \cdot \gamma_1 - \nu_1$. Since $\gamma_1 = \mu_2 - 1$, we have that

$$\deg(Q(z)) \leq |\lambda| \cdot (\mu_2 - 1) - \nu_1. \quad (2.8)$$

Let us see what happens with $R(z)$. Recall that

$$R(z) = \sum_{n \geq \mu_2 - \mu_1} \left\langle s_\lambda [s_{\mu+(n)}[X+z]], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

As before, we can write

$$s_{\mu+(n)}[X+z] = \sum_{i=0}^{\mu_1+n} s_{\mu+(n)/(i)} \cdot z^i.$$

Therefore,

$$R(z) = \sum_{n \geq \mu_2 - \mu_1} \left\langle s_\lambda \left[\sum_{i=0}^{\mu_1+n} s_{\mu+(n)/(i)} \cdot z^i \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We look at the degree in X on both sides of the scalar product. The right-hand side has degree $|\bar{\nu}|$ in X . The left-hand side involves a plethysm. If we expand the plethysm, we have a term of degree $|\mu| + n - i$ and $|\lambda| - 1$ terms of degree bigger than or equal to $|\bar{\mu}|$. Then, the left-hand side has degree bigger than or equal to $(|\lambda| - 1) \cdot |\bar{\mu}| + (|\mu| + n - i)$, with $0 \leq i \leq \mu_1 + n$. Only the terms with

$$(|\lambda| - 1) \cdot |\bar{\mu}| + (|\mu| + n - i) \leq |\bar{\nu}| \tag{2.9}$$

contribute to the scalar product. Define $r = |\lambda| \cdot \mu_1 - \mu_1 - \nu_1$. Together with the fact that $|\lambda| \cdot |\mu| = |\nu|$, we have that (2.9) is equivalent to $n - i \leq r$. Since, $i \geq 0$ and we do not know if $n - r \geq 0$, we should consider $i \geq \max\{0, n - r\}$. For simplifying, we define $s_{\mu+(n)/(i)}[X] = 0$, if $n - r < 0$. Then, we get that

$$\sum_{i=0}^{\mu_1+n} s_{\mu+(n)/(i)}[X] \cdot z^i = \sum_{i=n-r}^{\mu_1+n} s_{\mu+(n)/(i)}[X] \cdot z^i.$$

Now, we consider the following index change: $j = i - n + r$. Then,

$$\sum_{i=n-r}^{\mu_1+n} s_{\mu+(n)/(i)}[X] \cdot z^i = \sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^{j+n-r}.$$

Thus,

$$R(z) = \sum_{n \geq \mu_2 - \mu_1} \left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^{j+n-r} \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We split $R(z)$ into two parts: $R(z) = R_1(z) + R_2(z)$, where

$$\begin{aligned} R_1(z) &= \sum_{n=\mu_2-\mu_1}^{\mu_2+r-1} \left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^{j+n-r} \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle, \\ R_2(z) &= \sum_{n \geq \mu_2+r-1} \left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^{j+n-r} \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle. \end{aligned}$$

On one hand, we consider $R_1(z)$. Since the sum over n is finite, $R_1(z)$ is a Laurent polynomial. Its degree fulfils

$$\deg(R_1(z)) \leq |\lambda| \cdot (\mu_1 + \mu_2 + r - 1) - \nu_1. \quad (2.10)$$

On the other hand, we can write $R_2(z)$ as

$$R_2(z) = \sum_{n \geq \mu_2 + r - 1} z^{|\lambda| \cdot (n-r)} \cdot \left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^j \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

Set $H(z)$ for the scalar product in $R_2(z)$:

$$\left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\mu+(n)/(j+n-r)}[X] \cdot z^j \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

Note that the skew diagram $\mu + (n)/(j+n-r)$, with $0 \leq j \leq \mu_1 + r$ and $n \geq \mu_2 + r$, is always disconnected. Therefore, by Proposition 1.3.15,

$$s_{\mu+(n)/(j+n-r)}[X] = s_{\bar{\mu}}[X] \cdot s_{(\mu_1-j+r)}[X]. \quad (2.11)$$

Thus,

$$H(z) = \left\langle s_\lambda \left[\sum_{j=0}^{\mu_1+r} s_{\bar{\mu}}[X] \cdot s_{(\mu_1-j+r)}[X] \right], \frac{1}{z^{\nu_1}} s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle$$

is a Laurent polynomial in z , and does not depend on n . Moreover, its degree fulfils $\deg(H(z)) \leq |\lambda| \cdot (\mu_1 + r) - \nu_1$. As a consequence, we can write $R_2(z)$ as

$$R_2(z) = \sum_{n \geq \mu_2 + r} z^{|\lambda| \cdot (n-r)} H(z) = H(z) \cdot \left(\sum_{n \geq \mu_2 + r} z^{|\lambda| \cdot (n-r)} \right) = \frac{z^{|\lambda| \cdot \mu_2}}{1 - z^{|\lambda|}} H(z)$$

In short, we have proved that

$$f(z) = Q(z) + R_1(z) + R_2(z) = \frac{P(z)}{1 - z^{|\lambda|}},$$

where $P(z) = [Q(z) + R_1(z)] \cdot (1 - z^{|\lambda|}) + z^{|\lambda| \cdot \mu_2} H(z)$.

Finally, putting together all the bounds about the degrees, and the value of r , we get that

$$\deg(P(z)) \leq \max \{ |\lambda| \cdot \mu_2 - \nu_1, |\lambda| \cdot (|\lambda| \cdot \mu_1 + \mu_2 - \nu_1) - \nu_1 \}.$$

By Proposition 2.2.5, $\nu_1 \leq |\lambda| \cdot \mu_1$. Then, we obtain that

$$\deg(P(z)) \leq |\lambda| \cdot (\mu_2 + |\lambda| \cdot \mu_1 - \nu_1) - \nu_1. \quad (2.12)$$

□

2.4.3. The stability property (P1)

The stability property (P1) is proved in a similar way. We set the coefficients $v_{n,m} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(m)} \rangle$. We consider their generating function, and we observe that for $m \neq n \cdot |\mu|$, $v_{n,m} = 0$. This means that they do not contribute to the sum of their generating function. We proceed in a similar way than in Theorem 2.4.4.

Theorem 2.4.5. *Suppose that λ , μ , and ν are partitions. Let*

$$g(z) = \sum_{n,m \in \mathbb{Z}} \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(m)} \rangle z^m.$$

Then,

1. If $\ell(\mu) > 1$, $g(z)$ is a Laurent polynomial.
2. If μ has at most one part, $\mu = (p)$, then $g(z) = \frac{P(z)}{1-z^p}$, where $P(z)$ is a Laurent polynomial.

Proof. Consider $g(z)$ as in the statement of the theorem. First, we have that

$$g(z) = \sum_{n \in \mathbb{Z}} \left\langle s_{\lambda+(n)}[s_\mu], \sum_{m \in \mathbb{Z}} s_{\nu+(m)} \cdot z^m \right\rangle.$$

Applying Proposition 2.4.3 to the sum over m , we get that

$$g(z) = \sum_{n \in \mathbb{Z}} \left\langle s_{\lambda+(n)}[s_\mu], \frac{1}{z^{\nu_1}} \cdot \sigma_z(X) \cdot D_{\lambda_{\frac{-1}{z}}} s_{\bar{\nu}} \right\rangle = \sum_{n \in \mathbb{Z}} \left\langle D_{\sigma_z} s_{\lambda+(n)}[s_\mu], \frac{1}{z^{\nu_1}} \cdot D_{\lambda_{\frac{-1}{z}}} s_{\bar{\nu}} \right\rangle.$$

We apply Proposition 2.4.2 to both sides of the scalar product. We get that

$$g(z) = \sum_{n \in \mathbb{Z}} \left\langle s_{\lambda+(n)}[s_\mu[X+z]], \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We proceed to consider separately each case of the theorem.

Case $\ell(\mu) > 1$ We have that

$$s_\mu[X + z] = \sum_{i=0}^{\mu_1} s_{\mu/(i)}[X] \cdot z^i.$$

In $s_{\lambda+(n)}[s_\mu[X + z]]$, all terms have degree at least $(|\lambda| + n) \cdot |\bar{\mu}|$ in X . They contribute to the scalar product only if their degree is less than or equal to the degree in X of the right-hand side factor of the scalar product: $(|\lambda| + n) \cdot |\bar{\mu}| \leq |\bar{\nu}|$. Recall that $|\lambda| \cdot |\mu| = |\nu|$ and that $|\nu| = |\bar{\nu}| + \nu_1$. Then, we have the following upper bound for n : $n \leq \frac{|\lambda| \cdot \mu_1 - \nu_1}{|\bar{\mu}|}$. Using the Jacobi–Trudi formula and setting $k = \ell(\lambda + (n))$, we know that $s_{\lambda+(n)}$ is zero if $\lambda_1 + n + k - 1 \leq 0$. This yields the lower bound for n : $n \geq 1 - \lambda_1 - k$. Then,

$$g(z) = \sum_{n=1-\lambda_1-k}^{\lfloor \frac{|\lambda| \cdot \mu_1 - \nu_1}{|\bar{\mu}|} \rfloor} \left\langle s_{\lambda+(n)}[s_\mu[X + z]], \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

As a consequence, $g(z)$ is a Laurent polynomial. We can give an upper bound for its degree: $\deg(g(z)) \leq (|\lambda| + n) \cdot |\mu| - \nu_1$, for any $n \leq \frac{|\lambda| \cdot \mu_1 - \nu_1}{|\bar{\mu}|}$. Using that $|\lambda| \cdot |\mu| = |\nu|$, we get

$$\deg(g(z)) \leq \frac{|\bar{\nu}| \cdot |\mu|}{|\bar{\mu}|} - \nu_1. \tag{2.13}$$

Case $\ell(\mu) \leq 1$ Let $\mu = (p)$. We consider

$$g(z) = \sum_{n \in \mathbb{Z}} \left\langle s_{\lambda+(n)}[s_p[X + z]], \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We can write

$$s_p[X + z] = \sum_{i=0}^p s_{(p)/(i)} z^i = \underbrace{\left[\sum_{i=0}^{p-1} s_{(p)/(i)} z^i \right]}_Y + z^p.$$

By Proposition 2.4.1,

$$s_{\lambda+(n)}[s_p[X + z]] = s_{\lambda+(n)}[Y + z^p] = \sum_{j \geq 0} D_{s_j} s_{\lambda+(n)}[Y] \cdot z^{jp}.$$

Replacing it on $g(z)$, we get

$$g(z) = \sum_{n \in \mathbb{Z}} \left\langle \sum_{j \geq 0} D_{s_j} s_{\lambda+(n)}[Y] \cdot z^{jp}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We split $g(z)$ into two parts: $g(z) = R_1(z) + R_2(z)$, where

$$\begin{aligned} R_1(z) &= \sum_{n \in \mathbb{Z}} \left\langle \sum_{j \geq \lambda_2} D_{s_j} s_{\lambda+(n)}[Y] \cdot z^{jp}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle, \\ R_2(z) &= \sum_{n \in \mathbb{Z}} \left\langle \sum_{j < \lambda_2} D_{s_j} s_{\lambda+(n)}[Y] \cdot z^{jp}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle. \end{aligned}$$

First we consider $R_1(z)$. Suppose that $\lambda + (n)$ is not a partition. Then, by Proposition 1.3.14, there exists a partition γ , with $\gamma_1 = \lambda_2 - 1$, such that $s_{\lambda+(n)} = \pm s_\gamma$, or $s_{\lambda+(n)} = 0$. Since $j \geq \lambda_2$, by Proposition 2.4.1, $D_{s_j} s_{\lambda+(n)} = \pm D_{s_j} s_\gamma = \pm s_{\gamma/(j)} = 0$.

We assume that $\lambda + (n)$ is a partition. Then, $D_{s_j} s_{\lambda+(n)} = s_{\lambda+(n)/(j)}$. Since $j \geq \lambda_2$, the skew diagram $\lambda + (n)/(j)$ is disconnected and, by Proposition 1.3.15, we have that $s_{\lambda+(n)/(j)} = s_{\bar{\lambda}} \cdot s_{(\lambda_1+n-j)}$. Thus,

$$R_1(z) = \sum_{n \in \mathbb{Z}} \left\langle \sum_{j \geq \lambda_2} s_{\bar{\lambda}}[Y] \cdot s_{(\lambda_1+n-j)}[Y] \cdot z^{jp}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

By linearity,

$$R_1(z) = \left\langle \sum_{j \geq \lambda_2} s_{\bar{\lambda}}[Y] \cdot \underbrace{\left(\sum_{n \in \mathbb{Z}} s_{(\lambda_1+n-j)}[Y] \right)}_{\sigma_1[Y]}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

Then, we get that

$$R_1(z) = \left\langle s_{\bar{\lambda}}[Y] \cdot \sigma_1[Y] \cdot \left(\sum_{j \geq \lambda_2} z^{jp} \right), \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

Since the sum over j is a geometric series, we obtain that

$$R_1(z) = \frac{z^{\lambda_2 p - \nu_1}}{1 - z^p} \cdot \left\langle s_{\bar{\lambda}}[Y] \cdot \sigma_1[Y], s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We set $H(z) = \langle s_{\bar{\lambda}}[Y] \cdot \sigma_1[Y], s_{\bar{\nu}}[X - \frac{1}{z}] \rangle$. Then,

$$H(z) = \left\langle s_{\bar{\lambda}} \left[\sum_{i=0}^{p-1} s_{(p)/(i)} z^i \right] \cdot \left(\sum_{k \geq 0} s_k \left[\sum_{i=0}^{p-1} s_{(p)/(i)} z^i \right] \right), s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

Furthermore, for each k , the term $s_k[Y]$ contributes to the scalar product only if $k + |\bar{\lambda}| \leq |\bar{\nu}|$. Therefore, the degree in z is

$$\deg_z(H(z)) \leq (p-1) \cdot (k + |\bar{\lambda}|) \leq (p-1) \cdot |\bar{\nu}|. \quad (2.14)$$

We consider now $R_2(z)$. Set $r_{n,j} = D_{s_j} s_{\lambda+(n)}[Y] z^{jp}$. Then,

$$R_2(z) = \sum_{n \in \mathbb{Z}} \left\langle \sum_{j < \lambda_2} r_{n,j}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

For each n and j , $r_{n,j}$ contributes to the scalar product only if $|\lambda| + n - j \leq |\bar{\nu}|$. In this case, $\deg_z(r_{n,j}) \leq j \cdot p + (|\lambda| + n - j)(p-1) \leq j \cdot p + |\bar{\nu}| \cdot (p-1)$. Since $j < \lambda_2$ in $R_2(z)$, for any n , $\deg_z(r_{n,j}) \leq (\lambda_2 - 1)p + (p-1)|\bar{\nu}|$. In fact, $R_2(z)$ is a Laurent polynomial with

$$\deg(R_2(z)) \leq (\lambda_2 - 1)p + (p-1) \cdot |\bar{\nu}| - \nu_1. \quad (2.15)$$

In short, $g(z) = \frac{P(z)}{1-z^p}$, with $P(z) = R_1(z) \cdot (1 - z^p) + z^{\lambda_2 p - \nu_1} H(z)$. As a consequence of (2.14) and (2.15),

$$\deg(P(z)) \leq \lambda_2 \cdot p + (p-1) \cdot |\bar{\nu}| - p \cdot \nu_1. \quad (2.16)$$

In fact, we analyse the Laurent polynomial $P(z)$ in order to figure out its leading term. We denote by $LT(f)$ the leading term of f , as a Laurent polynomial in z . First, we look at $H(z) = \langle s_{\bar{\lambda}}[Y] \cdot \sigma_1[Y], s_{\bar{\nu}}[X - \frac{1}{z}] \rangle$, where $Y = \sum_{i=0}^{p-1} s_{(p-i)}[X] \cdot z^i$. Then, the leading term of Y , as a polynomial in z , is $s_1[X] \cdot z^{p-1}$. We can also write σ_1 as $\sum_k s_k$ and then, for each k , the factor contributes to the scalar product if $|\bar{\lambda}| + k \leq |\bar{\nu}|$. Therefore, the leading term of $H(z)$ is

$$LT(H(z)) = z^{(p-1) \cdot |\bar{\nu}|} \langle s_{\bar{\lambda}}[X] s_{(|\bar{\nu}| - |\bar{\lambda}|)}[X], s_{\bar{\nu}}[X] \rangle. \quad (2.17)$$

In $R_2(z)$, we also use that the leading term of Y as polynomial in z is $s_1[X] \cdot z^{p-1}$. Then, we get that

$$\begin{aligned} LT(R_2(z)) &= LT \left[\sum_{n \in \mathbb{Z}} \left\langle \sum_{j < \lambda_2} D_{s_j} s_{\lambda+(n)}[X] z^{(p-1)(|\bar{\lambda}|+n)+jp}, \frac{1}{z^{\nu_1}} \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle \right] = \\ &= LT \left[\sum_{n \in \mathbb{Z}} \left\langle \sum_{j < \lambda_2} s_{\lambda+(n)}[X] z^{(p-1)(|\bar{\lambda}|+n)+jp-\nu_1}, s_{(j)}[X] \cdot s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle \right]. \end{aligned}$$

Since for each j , the scalar product contributes to the sum if $|\lambda| + n \leq j + \nu_1$, and since we sum over all $j < \lambda_2$, we get that

$$LT(R_2(z)) = z^{|\bar{\nu}| \cdot (p-1) + (\lambda_2-1) \cdot p - \nu_1} \cdot \left\langle s_{\bar{\lambda} + (|\bar{\nu}| + \lambda_2 - 1 - |\lambda|)}[X], s_{(\lambda_2-1)}[X] s_{\bar{\nu}}[X] \right\rangle. \quad (2.18)$$

Setting together the leading terms of $H(z)$ and $R_2(z)$, computed in (2.17) and (2.18), we get the leading term of $P(z)$

$$LT(P(z)) = LT(H(z)) \cdot z^{(\lambda_2-1) \cdot p - \nu_1} - z^p \cdot LT(R_2(z)).$$

Note that $\deg(LT(P(z))) = \lambda_2 \cdot p + (p-1) \cdot |\nu| - p \cdot \nu_1$. Moreover, the coefficient is

$$\left\langle s_{\bar{\lambda}}[X] \cdot s_{(|\bar{\nu}| - |\bar{\lambda}|)}[X], s_{\bar{\nu}}[X] \right\rangle - \left\langle s_{\bar{\lambda} + (|\bar{\nu}| + \lambda_2 - 1 - |\lambda|)}[X], s_{(\lambda_2-1)}[X] \cdot s_{\bar{\nu}}[X] \right\rangle.$$

□

Theorem 2.4.5 gives us a description of the plethysm coefficients in terms of their stable value. We set $\bar{a}_{\lambda,p}^{\bar{\nu}}$ for the stable value of the plethysm coefficients $a_{\lambda+(n),p}^{\nu+(n,p)}$. We also set $\alpha^{\dagger k}$ for the following partition obtained from $\alpha = (\alpha_1, \dots, \alpha_l)$:

$$\alpha^{\dagger k} = (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_{k-1} + 1, \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_l). \quad (2.19)$$

Then, we have the following result.

Corollary 2.4.6. *Let n and p be non-negative integers and λ and α partitions. Then*

$$a_{(n,\bar{\lambda}),p}^{\alpha} = \sum_{k=0}^{\infty} (-1)^{k+1} \bar{a}_{\lambda,p}^{\alpha^{\dagger k}}.$$

Proof. By Theorem 2.4.5, the stable value $\bar{a}_{\lambda,p}^{\bar{\nu}}$ is $H(1)$, where

$$H(z) = \left\langle (s_{\bar{\lambda}} \cdot \sigma) [s_p[X+z] - z^p], s_{\bar{\nu}} \left[X - \frac{1}{z} \right] \right\rangle.$$

We simplify $H(z)$ as

$$\begin{aligned} H(z) &= \left\langle (s_{\bar{\lambda}} \cdot \sigma) [s_p[X+z] - z^p] \cdot \sigma \left[-\frac{X}{z} \right], s_{\bar{\nu}} [X] \right\rangle = \\ &= \left\langle s_{\bar{\lambda}} [s_p[X+z] - z^p] \cdot \sigma \left[-\frac{X}{z} + s_p[X+z] - z^p \right], s_{\bar{\nu}} [X] \right\rangle. \end{aligned}$$

We consider the following decomposition

$$s_{\bar{\lambda}} [s_p[X+z] - z^p] \cdot \sigma \left[-\frac{X}{z} + s_p[X+z] - z^p \right] = \sum_{\bar{\nu}} \bar{a}_{\lambda,p}^{\bar{\nu}}(z) s_{\bar{\nu}} [X].$$

We apply the transformation $X+z \mapsto X$, and we get

$$s_{\bar{\lambda}} [s_p[X] - z^p] \cdot \sigma \left[-\frac{X}{z} + 1 + s_p[X] - z^p \right] = \sum_{\bar{\nu}} \bar{a}_{\lambda,p}^{\bar{\nu}}(z) s_{\bar{\nu}} [X-z].$$

Specializing at $z=1$, we obtain that

$$s_{\bar{\lambda}} [s_p[X] - 1] \cdot \sigma [s_p[X] - X] = \sum_{\bar{\nu}} \bar{a}_{\lambda,p}^{\bar{\nu}} s_{\bar{\nu}} [X-1],$$

where the coefficients on the right-hand side of the equation are exactly the stable value of the plethysm coefficients. Multiplying both sides by $\sigma[X]$, we get that

$$\sum_n s_{(n,\bar{\lambda})} [s_p[X]] = \sum_{m,\bar{\nu}} \bar{a}_{\lambda,p}^{\bar{\nu}} s_{(m,\bar{\nu})} [X].$$

Performing the scalar product with s_{α} in the preceding equation yields:

$$a_{(n,\bar{\lambda}),p}^{\alpha} = \sum_{m,\bar{\nu}} \bar{a}_{\lambda,p}^{\bar{\nu}} \langle s_{(m,\bar{\nu})}, s_{\alpha} \rangle. \tag{2.20}$$

Consider $\bar{\nu}$ such that $\langle s_{(m,\bar{\nu})}, s_{\alpha} \rangle \neq 0$. Then, $s_{(m,\bar{\nu})}$ and s_{α} have the same columns in their Jacobi–Trudi determinants, (1.3.13), up to order. That is $v = (m, \bar{\nu}) + (k+1, k, k-1, \dots, 1)$ is a permutation of $u = \alpha + (k+1, k, k-1, \dots, 1)$, for some $k \geq 0$, with $\alpha_j = 0$ for $j > \ell(\alpha)$.

By construction, we have that v is decreasing starting at v_2 . Therefore, there exists an index i such that $u_j = v_j + 1$, for all $j < i$, and $u_j = v_j$, for all $j > i$. This means that $\bar{v} = \alpha^{\dagger^i}$ for some $i \leq k + 1$.

Finally, $\langle s_{(m, \bar{v})}, s_\alpha \rangle$ is the sign of the permutation that transforms v into u . This permutation is $(i, i - 1, \dots, 2, 1)$, which has sign $(-1)^{i+1}$. This shows that only the partitions $\bar{v} = \alpha^{\dagger^i}$, for $i \geq 0$, contribute to the sum in the right-hand side of (2.20), and that the contribution is $(-1)^{i+1} \bar{a}_{\lambda, p}^{\bar{v}}$. \square

2.5. Combinatorial Proof of Theorem 2.3.1

In Section 2.2, we establish a relation between the plethysm coefficients, $a_{\lambda, \mu}^\nu$, and the h -plethysm coefficients, $b_{\lambda, \mu}^\nu$. We also give a combinatorial interpretation of the coefficients $b_{\lambda, \mu}^\nu$. This interpretation will be the tool that we use to prove the stability properties stated in Theorem 2.3.1. We recall it here.

Proposition (Proposition 2.2.4). *Let λ and ν be finite sequences of positive integers and let μ be a partition. Let $N \geq \ell(\lambda), \ell(\nu)$.*

The coefficient $b_{\lambda, \mu}^\nu$ is the cardinal of the set $Q_{\lambda, \mu}^\nu(N)$ of matrices $\mathcal{M} = (m_{i, T})$ with non-negative integer entries whose rows are indexed by the integers i between 1 and N , and whose columns are indexed by the semi-standard Young tableaux $T \in t(\mu, N)$ such that:

- ROW SUM CONDITION FOR \mathcal{M} : *the sum of the entries in the i^{th} row of \mathcal{M} is λ_i .*
- COLUMN SUM CONDITION FOR $\mathcal{M} \cdot \mathcal{P}_{\mu, N}$: *the sum of the entries in the j^{th} column of $\mathcal{M} \cdot \mathcal{P}_{\mu, N}$ is ν_j .*

In the plethysm $h_\lambda[s_\mu]$, we refer to μ as the inner partition, and to λ as the outer partition. We separate the stability properties into two groups, according to which partitions depend on n .

- In Section 2.5.1, we study the properties for which in the plethysm only the outer partition depends on n : $b_{\lambda+(n), \mu}^{\nu+n \cdot \mu}$ and $b_{\lambda+(n), \mu}^{\nu+(|\mu| \cdot n)}$.
- In Section 2.5.2, we study the properties for which in the plethysm only the inner partition depends on n : $b_{\lambda, \mu+(n)}^{\nu+(|\lambda| \cdot n)}$ and $b_{\lambda, \mu+n \cdot \pi}^{\nu+n \cdot |\lambda| \cdot \pi}$.

2.5.1. The stability properties for the outer partition

The proofs presented in this section are based on the study of the matrix $\mathcal{P}_{\mu,N}$ which keeps the information related to the weight of the semi-standard Young tableaux in $t(\mu, N)$.

We introduce more notation before starting with the proofs. For any sequence of integers $\alpha = (\alpha_1, \dots, \alpha_N)$, α^+ denotes the sequence of cumulative sums, described by $\alpha^+ = (\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{N-1})$. We also set $\|\alpha\|_N = \sum_{j=1}^N (N-j) \cdot \alpha_j$, which satisfies that

$$\|\mu\|_N + \sum_{j=1}^{\ell(\mu)} j \cdot \mu_j = N \cdot |\mu|. \tag{2.21}$$

The stability property of the h -plethysm coefficients $b_{\lambda+(n),\mu}^{\nu+n\cdot\mu}$

We prove the following theorem.

Theorem 2.5.1. *Let μ be a partition, and λ and ν be finite sequences of integers. The sequence with general term $b_{\lambda+(n),\mu}^{\nu+n\cdot\mu}$ is constant when*

$$n \geq \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j - |\lambda| \cdot \left(\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i \right) - \lambda_1.$$

Proof. Fix N such that $N \geq \max\{\ell(\lambda), \ell(\mu), \ell(\nu), 1\}$. By Proposition 2.2.4,

$$b_{\lambda+(n),\mu}^{\nu+n\cdot\mu} = \text{Card } Q_{\lambda+(n),\mu}^{\nu+n\cdot\mu}(N).$$

Set $E(n)$ for $Q_{\lambda+(n),\mu}^{\nu+n\cdot\mu}(N)$. Let T_1 be the semi-standard Young tableau in $t(\mu, N)$ whose i^{th} row is filled with occurrences of i , for each i . For instance, for $\mu = (6, 4, 3, 1)$, we have that T_1 is the following tableau

1	1	1	1	1	1
2	2	2	2		
3	3	3			
4					

Then, the first row of the matrix $\mathcal{P}_{\mu,N}$ is exactly $\rho(T_1) = \mu$.

Consider the following injection

$$\begin{aligned} \varphi_n : \quad E(n) &\longrightarrow E(n+1) \\ \mathcal{M} = (m_{iT}) &\longmapsto \mathcal{M}' = (m'_{iT}), \end{aligned}$$

where $m'_{1,T_1} = m_{1,T_1} + 1$ and $m'_{i,T_j} = m_{i,T_j}$, for $i, j \neq 1$.

We contend that φ_n is also surjective for n big enough. The map φ_n is surjective if and only if for all $\mathcal{M}' = (m'_{iT}) \in E(n+1)$, the entry m'_{1,T_1} is non-zero. Then, in order to prove surjectivity, we will show that $m'_{1,T_1} > 0$. Let $\mathcal{M}' = (m'_{iT}) \in E(n+1)$. Note that among all semi-standard Young tableaux in $t(\mu, N)$, T_1 is the unique one with maximum weight for the *dominance ordering* [Mac95, I.1]. Then,

$$\begin{cases} \|\rho(T)\|_N \leq \|\mu\|_N - 1 & \text{if } T \neq T_1, \\ \|\rho(T_1)\|_N = \|\mu\|_N & \text{if } T = T_1. \end{cases} \quad (2.22)$$

Looking at the column sum conditions for $\mathcal{M}' \cdot \mathcal{P}_{\mu, N}$ in Proposition 2.2.4, for any j , we have

$$\sum_{i,T} m_{iT} \cdot \rho_j(T) = \nu_j + (n+1) \cdot \mu_j.$$

Consider the cumulative sums of the columns:

$$\sum_j \sum_{i,T} m_{iT} \cdot \rho_j^+(T) = \sum_j [\nu_j^+ + (n+1) \cdot \mu_j^+].$$

We can write this as

$$\sum_{i,T} m_{iT} \cdot \|\rho(T)\|_N = \|\nu\|_N + (n+1) \cdot \|\mu\|_N.$$

We isolate the factor corresponding to T_1 from the other semi-standard Young tableaux to obtain that

$$\begin{aligned} \|\nu\|_N + (n+1) \cdot \|\mu\|_N &= \sum_i m_{iT_1} \cdot \|\rho(T_1)\|_N + \sum_i \sum_{T \neq T_1} m_{iT} \cdot \|\rho(T)\|_N \leq \\ &\leq \sum_i m_{iT_1} \cdot \|\mu\|_N + \sum_i \sum_{T \neq T_1} m_{iT} \cdot (\|\mu\|_N - 1), \end{aligned}$$

where, in the inequality, we apply the bounds (2.22). Now, we extract the factor $(\|\mu\|_N - 1)$, getting that

$$\|\nu\|_N + (n+1) \cdot \|\mu\|_N \leq (\|\mu\|_N - 1) \cdot \left(\sum_i m_{iT_1} + \sum_{i, T \neq T_1} m_{iT} \right) + \sum_i m_{iT_1}.$$

By the row sum conditions for \mathcal{M} in Proposition 2.2.4, each element m_{iT_1} is less than or equal to λ_i . Therefore,

$$\sum_{i \neq 1} m_{iT_1} \leq |\lambda| - \lambda_1 = \bar{|\lambda|}. \quad (2.23)$$

Using that $\sum_{i, T} m_{iT} = |\lambda| + n + 1$ and (2.23), we have that

$$\begin{aligned} \|\nu\|_N + (n+1) \cdot \|\mu\|_N &\leq (\|\mu\|_N - 1) \cdot \left(\sum_{i, T} m_{iT} \right) + \sum_{i \neq 1} m_{iT_1} + m_{1T_1} \leq \\ &\leq (\|\mu\|_N - 1) \cdot (|\lambda| + n + 1) + m_{1T_1} + \bar{|\lambda|}. \end{aligned}$$

This inequality simplifies as $\|\nu\|_N + (n+1) - \|\mu\|_N \cdot |\lambda| + \lambda_1 \leq m_{1T_1}$. Using (2.21), we get that

$$N \cdot |\nu| - \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j + (n+1) - N \cdot |\mu| \cdot |\lambda| + |\lambda| \cdot \left(\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i \right) + \lambda_1 \leq m_{1T_1}.$$

Since $|\lambda| \cdot |\mu| = |\nu|$, we obtain that

$$|\lambda| \cdot \left(\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i \right) - \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j + (n+1) + \lambda_1 \leq m_{1T_1}.$$

Therefore, $m_{1T_1} > 0$ as soon as

$$n \geq \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j - |\lambda| \cdot \left(\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i \right) - \lambda_1.$$

□

The relation between the coefficients $b_{\lambda\mu}^\nu$ and $a_{\lambda,\mu}^\nu$ in (2.2.2) shows that the stability for the coefficients $b_{\lambda\mu}^\nu$ implies the stability for the coefficients $a_{\lambda\mu}^\nu$.

Corollary 2.5.2 (Property Q1). *The sequence with general term $a_{\lambda+(n),\mu}^{\nu+n\cdot\mu}$ stabilizes.*

The bound for the coefficients $a_{\lambda+(n),\mu}^{\nu+n\cdot\mu}$ is presented in Section 2.6.

The stability property of the h -plethysm coefficients $b_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)}$

We use the same strategy for the coefficients $b_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)}$.

Theorem 2.5.3. *Let μ be a partition and λ and ν be finite sequences of integers. The sequence of general term*

$$b_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)} = \langle h_{\lambda+(n)}[s_\mu], h_{\nu+(\mu\cdot n)} \rangle$$

stabilizes. It has limit zero whenever $\ell(\mu) > 1$.

The case in which μ has one part is a particular case of Theorem 2.5.1. We include it in the proof because we obtain a better bound for stability.

Proof. Let N be an integer bigger than or equal to the lengths of λ and ν , and 1. By Proposition 2.2.4,

$$b_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)} = \text{Card } Q_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)}(N).$$

Set $E(n) = Q_{\lambda+(n),\mu}^{\nu+(\mu\cdot n)}(N)$.

Consider μ such that $\ell(\mu) > 1$. We will prove that $E(n)$ is empty for n big enough. The elements of $E(n)$ are matrices satisfying the row sum conditions for \mathcal{M} and the column sum conditions for $\mathcal{M} \cdot \mathcal{P}_{\mu,N}$ stated in Proposition 2.2.4. These conditions can be written as a system of equations. For example, the first column sum condition for $\mathcal{M} \cdot \mathcal{P}_{\mu,N}$ is

$$\nu_1 + |\mu| \cdot n = \sum_{i,T} m_{i,T} \cdot \rho_1(T). \tag{2.24}$$

Then, $E(n)$ is the set of solutions of this system. We want to show that this system has no solution, when n is sufficiently large. Consider the set of semi-standard Young tableaux $t(\mu, N)$. By the increasing condition of the columns of the semi-standard Young tableaux, the boxes labelled with 1's are in the first row, for any semi-standard Young tableau $T \in t(\mu, N)$. Then, $\rho_1(T) \leq \mu_1$. If we use this bound in (2.24), we obtain that

$$\nu_1 + |\mu| \cdot n \leq \mu_1 \cdot \left(\sum_{i,T} m_{i,T} \right) = \mu_1 \cdot (|\lambda| + n).$$

We conclude that for $n > \frac{\mu_1 \cdot |\lambda| - \nu_1}{|\mu| - \mu_1}$, the system has no solution. Thus, $E(n)$ is the empty set, and $b_{\lambda+(n),\mu}^{\nu+(\mu \cdot n)} = 0$ for $n > \frac{\mu_1 \cdot |\lambda| - \nu_1}{|\mu| - \mu_1}$.

For $\ell(\mu) \leq 1$, we write $\mu = (m)$, for some positive integer m . We will exhibit a bijection from $E(n)$ to $E(n+1)$, when n is big enough.

Let T_1 be the semi-standard Young tableau in $t(\mu, N)$ which is filled just with ones:

$$\boxed{1 \mid 1 \mid \cdots \mid 1 \mid 1 \mid 1}$$

We define the map φ as in Theorem 2.5.1:

$$\begin{aligned} \varphi_n : \quad E(n) &\longrightarrow E(n+1) \\ \mathcal{M} = (m_{iT}) &\longmapsto \mathcal{M}' = (m'_{iT}) \end{aligned}$$

where $m'_{1,T_1} = m_{1,T_1} + 1$ and all other coefficients of \mathcal{M}' are equal to the corresponding elements of \mathcal{M} .

The map φ_n is injective. To prove that it is also surjective, we show that for all $\mathcal{M}' \in E(n+1)$, $m'_{1,T_1} > 0$, provided that n is big enough. Comparing with the proof of Theorem 2.5.1, we do not need to consider the cumulative sums. It is enough to estimate the number of 1's in each semi-standard Young tableau $T \in t(\mu, N)$:

$$\begin{cases} \rho_1(T_1) = m, \\ \rho_1(T) \leq m - 1 \quad \text{if } T \neq T_1. \end{cases} \quad (2.25)$$

We consider the first column sum condition for $\mathcal{M}' \cdot \mathcal{P}_{\mu,N}$ of Proposition 2.2.4

$$\nu_1 + (n+1) \cdot m = \sum_{i,T} m_{iT} \cdot \rho_1(T).$$

Isolating the summand corresponding to T_1 from the others and applying (2.25), we obtain that

$$\begin{aligned} \nu_1 + (n+1) \cdot m &= \sum_{i,T \neq T_1} m_{iT} \cdot \rho_1(T) + \sum_i m_{iT_1} \cdot \rho_1(T_1) \leq \\ &\leq (m-1) \cdot \sum_{i,T \neq T_1} m_{iT} + m \cdot \sum_i m_{iT_1}. \end{aligned}$$

By (2.23) and the fact that $\sum_{i,T} m_{iT} = |\lambda| + n + 1$, we get that

$$\nu_1 + (n+1) \cdot m \leq (m-1) \cdot (|\lambda| + n + 1) + m_{1T_1} + |\bar{\lambda}|.$$

This simplifies as $\nu_1 \leq m \cdot |\lambda| - (n + 1) + m_{1T_1} - \lambda_1$. Thus, $m_{1T_1} > 0$ as soon as $n \geq |\bar{\nu}| - \lambda_1 - 1$.

□

The relation of the plethysm coefficients stated in Lemma 2.2.2 and Theorem 2.5.3 implies the following result.

Corollary 2.5.4 (Property P1). *The sequence of general term*

$$a_{\lambda+(n),\mu}^{\nu+(|\mu| \cdot n)} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(|\mu| \cdot n)} \rangle$$

stabilizes. The limit is zero when $\ell(\mu) > 1$.

2.5.2. The stability properties for the inner partition

For the properties where the inner partition of the plethysm changes with n , we follow a different strategy. Consider the combinatorial interpretation given in Proposition 2.2.4 for the coefficients $b_{\lambda,\mu+(n)}^{\nu+(|\lambda| \cdot n)}$ and $b_{\lambda,\mu+n \cdot \pi}^{\nu+n \cdot |\lambda| \cdot \pi}$. Then, in both cases, the set of semi-standard Young tableaux changes when n grows. That is why we will define an auxiliary map from the set of semi-standard Young tableaux associated to n to the set of semi-standard Young tableaux associated to $n + 1$.

The stability property of the h -plethysm coefficients $b_{\lambda,\mu+(n)}^{\nu+(|\lambda| \cdot n)}$

Theorem 2.5.5. *Let μ be a partition and λ and ν be finite sequences of integers. The following h -plethysm coefficients*

$$b_{\lambda,\mu+(n)}^{\nu+(|\lambda| \cdot n)} = \langle h_\lambda[s_{\mu+(n)}], h_{\nu+(|\lambda| \cdot n)} \rangle$$

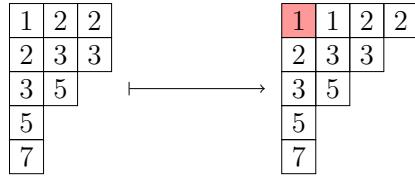
stabilize.

Notation. For each semi-standard Young tableau T , we set M_T for the column of \mathcal{M} associated to the semi-standard Young tableau T and $|M_T|$ for the sum of the entries of M_T . Then, the condition $\sum_{i,T} m_{iT} = |\lambda|$ becomes $\sum_T |M_T| = |\lambda|$ and each column sum condition for $\mathcal{M} \cdot \mathcal{P}_{\mu,N}$ in Proposition 2.2.4 is written as $\sum_i |M_T| \cdot \rho_j(T) = \nu_j$.

Proof. Let N be an integer bigger than or equal to the lengths of λ and ν , and 1. By Proposition 2.2.4,

$$b_{\lambda, \mu+(n)}^{\nu+(|\lambda| \cdot n)} = \text{Card } Q_{\lambda, \mu+(n)}^{\nu+(|\lambda| \cdot n)}(N).$$

Set $E(n) = Q_{\lambda, \mu+(n)}^{\nu+(|\lambda| \cdot n)}(N)$. We consider the set of semi-standard Young tableaux for n , $t(\mu + (n))$, and for $n + 1$, $t(\mu + (n + 1))$. Let us define the auxiliary injective map $\tau_n : t(\mu + (n), N) \rightarrow t(\mu + (n + 1), N)$. We define $\tau_n(T)$ as the semi-standard Young tableau obtained from T by adding one box labelled by one in the first row and pushing the original first row of T to the right. For instance,



The elements of $E(n + 1)$ are matrices whose columns are indexed by the semi-standard Young tableaux in $t(\mu + (n + 1), N)$. These semi-standard Young tableaux can be in $\text{Im}(\tau_n)$ or not. According to this classification of the semi-standard Young tableaux that indexed the columns of the elements of $E(n + 1)$, we define $\varphi_n : E(n) \rightarrow E(n + 1)$ in the following way: for any matrix $\mathcal{M} = (M_T) \in E(n)$, indexed by $T \in t(\mu + (n), N)$, we have that $\varphi_n(\mathcal{M}) \in E(n + 1)$ is the matrix such that $\varphi_n(\mathcal{M})_T = M_{\tau_n^{-1}(T)}$, if $T \in \text{Im}(\tau_n)$, and $\varphi_n(\mathcal{M}) = \bar{0}$, if $T \notin \text{Im}(\tau_n)$.

The map φ_n is injective. We check surjectivity. Let $\mathcal{M}' = (M'_T) \in E(n + 1)$. We set

$$S = \sum_{T \in \text{Im}(\tau_n)} |M'_T| \quad \text{and} \quad S^c = \sum_{T \notin \text{Im}(\tau_n)} |M'_T|.$$

Then, $S + S^c = |\lambda|$. The map φ_n is surjective if and only if $S^c = 0$. Since $m_{i,T} \geq 0$, it follows that the columns indexed by semi-standard Young tableaux not in the image of τ_n are zero. We restate the first column sum condition in Proposition 2.2.4 for $\mathcal{M}' \cdot \mathcal{P}_{\mu, N}$ as follows

$$\nu_1 + (n + 1) \cdot |\lambda| = \sum_{T \in \text{Im}(\tau_n)} |M_T| \cdot \rho_1(T) + \sum_{T \notin \text{Im}(\tau_n)} |M_T| \cdot \rho_1(T). \quad (2.26)$$

We bound the number of ones: for any $T \in t(\mu + (n + 1), N)$, $\tau_n^{-1}(T)$ is defined if there are more than μ_2 ones in the first row. In this case, the maximum number of ones is $\mu_1 + n + 1$ because there are not ones in any other column different from the

first one. Thus,

$$\begin{cases} \rho_1(T) \leq \mu_1 + n + 1 & \text{if } T \in \text{Im}(\tau_n), \\ \rho_1(T) \leq \mu_2 & \text{if } T \notin \text{Im}(\tau_n). \end{cases}$$

Using these bounds in (2.26), we get that

$$\nu_1 + (n + 1) \cdot |\lambda| \leq \sum_{T \in \text{Im}(\varphi_n)} |M_T| \cdot (\mu_1 + n + 1) + \sum_{T \notin \text{Im}(\varphi_n)} |M_T| \cdot \mu_2 \leq (\mu_1 + n + 1) \cdot S + \mu_2 \cdot S^c.$$

Since $S^c = |\lambda| - S$, reordering last inequality, we get that $S^c \leq \frac{\mu_1 \cdot |\lambda| - \nu_1}{\mu_1 + n + 1 - \mu_2}$. Therefore, $S^c = 0$, as soon as $n \geq \mu_1 \cdot (|\lambda| - 1) + \mu_2 - \nu_1$. \square

As a consequence of Theorem 2.5.5, we obtain the corresponding property for the $a_{\lambda^{(n)}, \mu^{(n)}}^{\nu^{(n)}}$ coefficients using (2.2.2).

Corollary 2.5.6 (Property R1). *The sequence of plethysm coefficients $a_{\lambda^{(n)}, \mu}^{\nu + (|\lambda| \cdot n)}$ stabilizes.*

The stability property of the h -plethysm coefficients $b_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi}$

The strategy for the coefficients $b_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi}$ is the same than in Theorem 2.5.5. First, we define a map τ_n between the sets of semi-standard Young tableaux, and a map φ_n between the sets of matrices. We will introduce a new combinatorial object, the *Gelfand–Tsetlin patterns*. Using Gelfand–Tsetlin patterns, we will bound the weight of the semi-standard Young tableaux in Lemma 2.5.10. Finally, we will use those bounds to prove that φ_n is surjective.

Theorem 2.5.7. *Let μ and π be partitions. Let λ and ν be positive integer sequences. Then, the following sequence stabilizes*

$$b_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi} = \langle h_\lambda[s_{\mu + n \cdot \pi}], h_{\nu + n \cdot |\lambda| \cdot \pi} \rangle.$$

Proof. Let N be an integer bigger than or equal to the lengths of λ and ν , and 1. By Proposition 2.2.4,

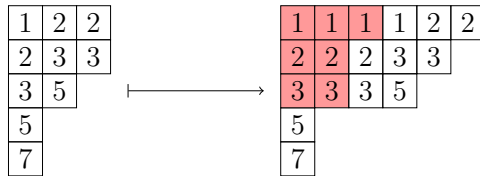
$$b_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi} = \text{Card } Q_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi}(N).$$

Set $E(n) = Q_{\lambda, \mu + n \cdot \pi}^{\nu + n \cdot |\lambda| \cdot \pi}(N)$. The elements of $E(n)$ are matrices whose columns are indexed by the semi-standard Young tableaux in $t(\mu + n \cdot \pi, N)$. These semi-standard Young tableaux have different shape than the semi-standard Young

tableaux indexing the elements of $E(n + 1)$. We define the following injective map

$$\begin{aligned} \tau_n : t(\mu + n \cdot \pi, N) &\longrightarrow t(\mu + (n + 1) \cdot \pi, N) \\ T &\longrightarrow \tau_n(T), \end{aligned}$$

where $\tau_n(T)$ is obtained from T adding in the left side the semi-standard Young tableau of shape π , which has π_i boxes filled with i 's in the i^{th} row, and pushing the original rows of T to the right. For instance,



The elements of $E(n + 1)$ are matrices whose columns are indexed by the semi-standard Young tableaux in $t(\mu + (n + 1), N)$. These semi-standard Young tableaux can be in $\text{Im}(\tau_n)$ or not. According to this, we define $\varphi_n : E(n) \longrightarrow E(n + 1)$ in the following way: $\varphi_n(\mathcal{M})_T = M_{\tau_n^{-1}(T)}$, if $T \in \text{Im}(\tau_n)$, and $\varphi_n(\mathcal{M}) = \bar{0}$, if $T \notin \text{Im}(\tau_n)$.

The map φ_n is injective. To check the surjectivity, we introduce another combinatorial object, the *Gelfand-Tsetlin patterns*.

Definition 2.5.8. A *Gelfand-Tsetlin pattern* is a triangular array, G , of non-negative integers, say

$$\begin{array}{cccccc} x_{n1} & & x_{n2} & & \cdots & & x_{n,n-1} & & x_{nn} \\ & \ddots & & \ddots & & \ddots & & \ddots & \\ & & x_{31} & & x_{32} & & x_{33} & & \\ & & & x_{21} & & x_{22} & & & \\ & & & & x_{11} & & & & \end{array}$$

such that $x_{i+1,j+1} \leq x_{i,j} \leq x_{i+1,j}$, for all $1 \leq j \leq i \leq n$, when all three numbers are defined.

Theorem 2.5.9. [Sta99, Section 7.10] *There exists a bijection between semi-standard Young tableaux and Gelfand-Tsetlin patterns. The bijection is defined in the following way: let T be a semi-standard Young tableau of shape μ and weight β . We associate to T the Gelfand-Tsetlin $G = (x_{ij})$ such that x_{ij} is the number of entries in the j^{th} row of T that are less than or equal to i .*

Note that the elements of the last row of G will be $x_{ki} = \mu_i$, and that $\sum_j x_{ij} = \beta_1 + \dots + \beta_i$. Moreover, these two conditions characterize the Gelfand–Tsetlin pattern G . For instance,

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \hline 3 & 5 & & \\ \hline 5 & 6 & & \\ \hline \end{array} \longmapsto \begin{array}{cccccc} 4 & 4 & 2 & 2 & 0 & 0 \\ & 4 & 4 & 2 & 1 & 0 \\ & & 4 & 4 & 1 & 0 \\ & & & 4 & 3 & 1 \\ & & & & 3 & 1 \\ & & & & & 1 \end{array}$$

We denote by $GF(T)$ the Gelfand-Tsetlin pattern associated to T , and by $GF(\mu)$ the set of Gelfand-Tsetlin patterns associated to semi-standard Young tableaux of shape μ .

The map defined between the sets of semi-standard Young tableaux, τ_n , induces a map between the Gelfand-Tsetlin patterns in the following way:

$$\begin{aligned} \tau'_n : GF(\mu + n \cdot \pi) &\longrightarrow GF(\mu + (n + 1) \cdot \pi) \\ GF(T) = (x_{ij}) &\longrightarrow GF(\tau_n(T)) = (x_{ij} + \pi_j) = (y_{ij}) \end{aligned}$$

We have the following Lemma, which proof is included at the end of this one.

Lemma 2.5.10. *For each i , there exists a constant c_i such that if T is a semi-standard Young tableau of $t(\mu + n \cdot \pi, N)$ for which the number of i 's in the i^{th} row is less than or equal to $\mu_i + n \cdot \pi_i - c_i$, for some i , then $M_T = \bar{0}$. In fact, $c_i > \sum_{j=1}^i (|\lambda| \cdot \mu_j - \nu_j)$.*

Let us check that φ_n is surjective. Consider $\mathcal{M}' \in E(n + 1)$. The map φ_n is surjective if and only if $M'_T = \bar{0}$ for all the columns indexed by semi-standard Young tableaux $T \notin \text{Im}(\tau_n)$.

We proceed by *reductio ad absurdum*. Suppose that there exists $T \notin \text{Im}(\tau_n)$ such that $M'_T \neq \bar{0}$. Then, by Lemma 2.5.10, for all i , the number of i 's in the i^{th} row is bigger than $\mu_i + (n + 1) \cdot \pi_i - c_i$, for the constants c_i of the lemma. We refer to this property of T as (\star) . This semi-standard Young tableau T has associated a Gelfand–Tsetlin pattern, $GF(T) = (y_{ij})$. We define the pre-image of $GF(T)$, $(\tau')^{-1}(GF(T)) = (x_{ij})$, by setting $x_{ij} = y_{ij} - \pi_j$. Then, we set $\tau_n^{-1}(T)$ to the associated semi-standard Young tableau to $(\tau')^{-1}(GF(T))$. If $(\tau')^{-1}(GF(T))$ is well define, then $\tau_n^{-1}(T)$ exists, and we get a contradiction with the initial assumption.

We check that $(\tau')^{-1}(GF(T))$ defined as above is a Gelfand–Tsetlin pattern in $GF(\mu + n \cdot \pi)$.

- To show that $x_{i+1,j} \geq x_{ij}$, we use directly the definition of the preimage and that (y_{ij}) is a Gelfand–Tsetlin pattern.
- We show that $x_{ij} \geq x_{i+1,j+1}$.
 If $\pi_j = \pi_{j+1}$, we have that $x_{ij} \geq x_{i+1,j+1}$ directly from the fact that (y_{ij}) is a Gelfand–Tsetlin pattern. If $\pi_j > \pi_{j+1}$, we need to show that $y_{ij} - \pi_j \geq y_{i+1,j+1} - \pi_{j+1}$. We consider a lower bound for y_{ij} and an upper bound for $y_{i+1,j+1}$. For y_{ij} , we know that it is the number of $\{1, \dots, i\}$ in the j^{th} row. Then, at least, y_{ij} is the number of j 's in the j^{th} row and, applying (\star) , we get the following bound:

$$y_{ij} \geq \mu_j + (n + 1) \cdot \pi_j - c_j. \tag{2.27}$$

For $y_{i+1,j+1}$, we will use the general upper bound saying that, at most, we have as many numbers as boxes in the $(j + 1)^{\text{th}}$ row of $\mu + (n + 1) \cdot \pi$:

$$y_{i+1,j+1} \leq \mu_{j+1} + (n + 1) \cdot \pi_{j+1}. \tag{2.28}$$

Putting together both bounds, (2.27) and (2.28), we get that $x_{ij} \geq x_{i+1,j+1}$ as soon as $n \geq \frac{\mu_{j+1} - \mu_j + c_j}{\pi_j - \pi_{j+1}}$.

- We show that $x_{ij} \geq 0$.
 Using the other inequalities, it is enough to check it for the elements $x_{ii} = y_{ii} - \pi_i$. If $\pi_i = 0$, then $x_{ii} = y_{ii} \geq 0$, because (y_{ij}) is a Gelfand–Tsetlin pattern. Suppose $\pi_i \neq 0$. In the i^{th} row, there is no numbers from $\{1, \dots, i - 1\}$. Then, y_{ii} is exactly the number of i 's in the i^{th} row and this means that $y_{ii} \geq \mu_i + (n + 1) \cdot \pi_i - c_i$. Then, we get that $x_{ii} \geq 0$ as soon as $n \geq \frac{c_i - \mu_i}{\pi_i}$.

Thus, the theorem is proved. In order to finish the proof, we prove Lemma 2.5.10.

Lemma. *For each i , there exists a constant c_i such that if T is a semi–standard Young tableau of $t(\mu + n \cdot \pi, N)$ for which the number of i 's in the i^{th} row is less than or equal to $\mu_i + n \cdot \pi_i - c_i$, for some i , then $M_T = \bar{0}$. In fact, $c_i > \sum_{j=1}^i (|\lambda| \cdot \mu_j - \nu_j)$.*

Proof. Suppose that there exists T_0 such that for all i , the number of i 's in the i^{th} row is less than or equal to $\mu_i + n\pi_i - c_i$ and $M_{T_0} \neq \bar{0}$. We fix j such that $1 \leq j \leq N$. For any semi–standard Young tableau T of $t(\mu + n \cdot \pi, N)$, there cannot be j 's in any k^{th} row, for $k > j$. Then, the number of $\{1, 2, \dots, j\}$ is less than or equal to

the number of boxes in the first j rows of $\mu + n \cdot \pi$, and we get that

$$\rho_1(T) + \cdots + \rho_i(T) \leq (\mu_1 + n \cdot \pi_1) + \cdots + (\mu_i + n \cdot \pi_i). \quad (2.29)$$

For T_0 , we refine this bound using the hypothesis of the lemma

$$\rho_1(T_0) + \cdots + \rho_i(T_0) \leq (\mu_1 + n \cdot \pi_1) + \cdots + (\mu_i + n \cdot \pi_i) - c_i. \quad (2.30)$$

Now, we consider the first i column sum conditions in Proposition 2.2.4 for $\mathcal{M} \cdot \mathcal{P}_{\mu+n\cdot\pi, N}$. Its sum can be restated as

$$\sum_{j=1}^i (\nu_j + n \cdot |\lambda| \cdot \pi_j) = \sum_{T \neq T_0} |M_T| \cdot \sum_{j=1}^i \rho_j(T) + |M_{T_0}| \cdot \sum_{j=1}^i \rho_j(T_0).$$

Using the estimates (2.29) and (2.30), we obtain that

$$\begin{aligned} \sum_{j=1}^i (\nu_j + n \cdot |\lambda| \cdot \pi_j) &= \sum_{T \neq T_0} |M_T| \cdot \sum_{j=1}^i (\mu_j + n \cdot \pi_j) + |M_{T_0}| \cdot \left(\sum_{j=1}^i (\mu_j + n \cdot \pi_j) - c_i \right) \leq \\ &\leq \sum_{j=1}^i (\mu_j + n \cdot \pi_j) \cdot (|\lambda| - |M_{T_0}|) + \sum_{j=1}^i (\mu_j + n \cdot \pi_j) \cdot |M_{T_0}| - |M_{T_0}| \cdot c_i. \end{aligned}$$

Reorganizing last inequality, we get that $|M_{T_0}| \cdot c_i \leq \sum_{j=1}^i (|\lambda| \cdot \mu_j - \nu_j)$. Therefore, it is enough to consider $c_i > \sum_{j=1}^i (|\lambda| \cdot \mu_j - \nu_j)$ in order to obtain a contradiction, as we wanted. \square

\square

As before, the stability property corresponding to the plethysm coefficients $a_{\lambda, \mu+n\cdot\pi}^{\nu+n\cdot|\lambda|\cdot\pi}$ is a consequence of Theorem 2.5.7 and Proposition 2.2.2.

Corollary 2.5.11 (Property R2). *The sequence of general term $a_{\lambda, \mu+n\cdot\pi}^{\nu+n\cdot|\lambda|\cdot\pi}$ stabilizes.*

2.6. Bounds

In this section we shorten the bounds obtained by Brion in [Bri93], the bounds provided in Section 2.4 by our adaptation of the results of Carré and Thibon published in [CT92], and the bounds obtained in Section 2.5 with combinatorial tools.

The following result summarizes the bounds obtained by Brion in [Bri93], and the bounds we obtain in the proofs of Theorem 2.4.5 and Theorem 2.4.4.

Proposition 2.6.1.

(P1) In Theorem 2.4.5, it is proved that the sequence of plethysm coefficients

$$a_{\lambda+(n),\mu}^{\nu+(\mu \cdot n)} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(\mu \cdot n)} \rangle$$

is identically zero for $n \geq \left\lfloor \frac{|\nu|}{|\mu|} - \frac{\nu_1}{|\mu|} \right\rfloor$, when $\ell(\mu) > 1$. For $\ell(\mu) \leq 1$, the sequence is constant for $n \geq \lambda_2 + |\nu| - |\lambda| - \nu_1$.

(Q1) In [Bri93, Theorem, Section 3.1], it is proved that the sequence of general term

$$a_{\lambda+(n),\mu}^{\nu+n \cdot \mu} = \langle s_{\lambda+(n)}[s_\mu], s_{\nu+n \cdot \mu} \rangle$$

is constant for

$$n \geq \lambda_2 - |\lambda| + \sum_{\substack{i < N \\ \mu_i \neq \mu_{i+1}}} \left(\sum_{j=1}^i |\lambda| \cdot \mu_j - \nu_j \right).$$

(R1) In Theorem 2.4.4, it is proved that the sequence with general term

$$a_{\lambda,\mu+(n)}^{\nu+(\lambda \cdot n)} = \langle s_\lambda[s_{\mu+(n)}], s_{\nu+(\lambda \cdot n)} \rangle$$

is constant for $n \geq |\lambda| \cdot \mu_1 + \mu_2 - \nu_1 - \left\lfloor \frac{\nu_1}{|\lambda|} \right\rfloor$.

(R2) In [Bri93, Corollary 1, Section 2.6], it is proved that the sequence of general term

$$a_{\lambda,\mu+n \cdot \pi}^{\nu+n \cdot \lambda \cdot \pi} = \langle s_\lambda[s_{\mu+n \cdot \pi}], s_{\nu+n \cdot \lambda \cdot \pi} \rangle$$

is constant when, for all j such that $\pi_j \neq \pi_{j+1}$, $n \geq \frac{\mu_{j+1} - \mu_j + c_j}{\pi_j - \pi_{j+1}}$, where we set $c_j = |\lambda| \cdot (\mu_1 + \dots + \mu_j) - (\nu_1 + \dots + \nu_j)$.

Proof. First, we analyse the bounds related to the results of Thibon and Carré, (P1) and (R1), and then the bounds of the sequences (R2) and (Q1).

(P1) In Theorem 2.4.5 we proved that the coefficients of

$$g(z) = \sum_{n,m \in \mathbb{Z}} \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(m)} \rangle z^m$$

are identically zero for $n \geq \frac{|\nu| \cdot |\mu|}{|\mu|} - \nu_1$, when $\ell(\mu) > 1$. On the other case, if $\mu = (p)$, then they are constant for $n \geq \lambda_2 \cdot p + |\nu| \cdot (p - 1) - \nu_1 \cdot p$. We are interested in the coefficients of

$$G(z) = \sum_{n \in \mathbb{Z}} \langle s_{\lambda+(n)}[s_\mu], s_{\nu+(|\mu| \cdot n)} \rangle z^n.$$

Otherwise, when $m \neq |\mu| \cdot n$, we know that the coefficients are zero. Since $g(z) = G(z^{|\mu|})$, we have to divide the bounds associated to $g(z)$ by $|\mu|$ and we obtain the bounds stated in the proposition.

(R1) In Theorem 2.4.4 we proved that the coefficients of

$$f(z) = \sum_{n,m \in \mathbb{Z}} \langle s_\lambda[s_{\mu+(n)}], s_{\nu+(m)} \rangle z^m$$

are constant for $n \geq |\lambda| \cdot (\mu_2 + |\lambda| \cdot \mu_1 - \nu_1 - 1) - \nu_1$. In fact, for $m \neq |\lambda| \cdot n$, the coefficients are zero. The coefficients with $m = |\lambda| \cdot n$ have the following generating function

$$F(z) = \sum_{n \in \mathbb{Z}} \langle s_\lambda[s_{\mu+(n)}], s_{\nu+(|\lambda| \cdot n)} \rangle z^n.$$

Since $f(z) = F(z^{|\lambda|})$, the coefficients of $F(z)$ are constant for $n \geq \mu_2 + |\lambda| \cdot \mu_1 - \nu_1 - \lfloor \frac{\nu_1}{|\lambda|} \rfloor$.

(R2) In [Bri93, Corollary 1, Section 2.6], we find directly that the sequence is stable when

$$n \geq \frac{\mu_{i+1} - \mu_i + |\lambda| \cdot (\mu_1 + \dots + \mu_i) - \nu_1 - \dots - \nu_i}{\pi_j - \pi_{j+1}}, \quad (2.31)$$

for every $i \in \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$, where we express $\pi = (p_1^{a_1}, p_2^{a_2}, \dots)$ with $p_1 > p_2 > p_3 > \dots > 0$. Since we consider the maximum of the bounds, we can consider the bounds in (2.31) for all i .

(Q1) In [Bri93, Theorem, Section 3.1], Brion stated in the framework of representation theory for algebraic groups in general and for the general linear group

in particular that the bound for the sequence (Q1). He uses the following notation

Recall that $|\lambda| \cdot \mu - \nu$ is a sum of simple roots whenever V_ν occurs in the G -module $S_\lambda V_\mu$. We denote by $\| |\lambda| \cdot \mu - \nu \|$ the sum of coefficients of $|\lambda| \cdot \mu - \nu$ on all simple roots which are not orthogonal to μ .

Then, Brion proved that the sequence is constant for $n \geq \lambda_2 - |\lambda| + \| |\lambda| \cdot \mu - \nu \|$.

For GL_N , the simple roots are exactly $R_i = (0, \dots, 0, \overbrace{1, -1}^{i, i+1}, 0, \dots, 0)$, for $1 \leq N - 1$. Moreover, they are orthogonal to μ if and only if $\mu_i \neq \mu_{i+1}$ and $i < N$. We set $|\lambda| \cdot \mu - \nu = (y_1, y_2, \dots, y_N)$, with $N = \max\{\ell(\lambda), \ell(\mu), \ell(\nu)\}$. Then, the equation $\sum_{i=1}^{N-1} a_i \cdot R_i = |\lambda| \cdot \mu - \nu$ implies the following system

$$\begin{cases} a_1 = y_1 \\ a_2 - a_1 = y_2 \\ \vdots \\ a_{N-1} - a_{N-2} = y_{N-1} \end{cases}$$

Thus, we consider the set of indices $I = \{i : \mu_i \neq \mu_{i+1} \text{ and } i < N\}$, and we get that $\| |\lambda| \cdot \mu - \nu \| = \sum_{i \in I} (\sum_{j=1}^i |\lambda| \cdot \mu_j - \nu_j)$.

□

We have obtained the following bounds for the h -plethysm coefficients, $b_{\lambda\mu}^\nu$, in Theorems 2.5.1, 2.5.3, 2.5.5 and 2.5.7.

Corollary 2.6.2.

1. Consider the sequence $\{b_{\lambda+(n),\mu}^{\nu+(n \cdot |\mu|)}\}$. Then,
 - When $\ell(\mu) = 1$, the sequence is constant for $n > |\bar{\nu}| - \lambda_1 - 1$.
 - When $\ell(\mu) > 1$, the coefficients are zero, for $n \geq \frac{\mu_1 \cdot |\lambda| - \nu_1}{|\bar{\mu}|}$.
2. The sequence $\{b_{\lambda+(n),\mu}^{\nu+n \cdot \mu}\}$ is constant for $n \geq \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j - |\lambda| \cdot (\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i) - \lambda_1$.
3. The sequence $\{b_{\lambda,\mu+(n)}^{\nu+(|\lambda| \cdot n)}\}$ is constant for $n \geq \mu_1 \cdot (|\lambda| - 1) + \mu_2 - \nu_1$.
4. The sequence $\{b_{\lambda,\mu+n \cdot \pi}^{\nu+n \cdot |\lambda| \cdot \pi}\}$ is constant for $n \geq \frac{\mu_{j+1} - \mu_j + c_j}{\pi_j - \pi_{j+1}}$, for all j , where $\pi_j \neq \pi_{j+1}$ and the c_j are the constants $c_j = \sum_{i=1}^j (|\lambda| \cdot \mu_i - \nu_i)$.
 If $\pi_j = \pi_{j+1}$ for all j , the sequence is constant when $n \geq \frac{c_j - \mu_j}{\pi_j}$.

Corollary 2.6.3. *The following list summarizes the bounds for the stability properties (P1), (Q1), (R1) and (R2).*

(P1) Consider the sequence $\{a_{\lambda+(n),\mu}^{\nu+(n\cdot|\mu|)}\}$. Then,

- When $\ell(\mu) = 1$, the sequence is constant for $n > |\bar{\nu}| - \lambda_1 - 1$.
- When $\ell(\mu) > 1$, the coefficients are zero, for $n \geq \frac{\mu_1 \cdot |\lambda| - \nu_1}{|\bar{\mu}|}$.

(Q1) The sequence $\{a_{\lambda+(n),\mu}^{\nu+n\cdot\mu}\}$ is constant for $n \geq \sum_{j=1}^{\ell(\nu)} j \cdot \nu_j - |\lambda| \cdot \left(\sum_{i=1}^{\ell(\mu)} i \cdot \mu_i\right) - \lambda_1$.

(R1) The sequence $\{a_{\lambda,\mu+(n)}^{\nu+(|\lambda|\cdot n)}\}$ is constant for $n \geq \mu_1 \cdot (|\lambda| - 1) + \mu_2 - \nu_1$.

(R2) The sequence $\{a_{\lambda,\mu+n\cdot\pi}^{\nu+n\cdot|\lambda|\cdot\pi}\}$ is constant for $n \geq \frac{\mu_{j+1} - \mu_j + c_j}{\pi_j - \pi_{j+1}}$, for all j , where $\pi_j \neq \pi_{j+1}$ and the c_j are constants satisfying $c_j > \sum_{i=1}^j (|\lambda| \cdot \mu_i - \nu_i)$.

If $\pi_j = \pi_{j+1}$ for all j , the sequence is constant when $n \geq \frac{c_j - \mu_j}{\pi_j}$.

Proof. By Lemma 2.2.2, the bound for the plethysm coefficients is the maximum of the bound for the h -plethysm coefficients associated, $b_{\lambda+\omega\sigma,\mu}^{\nu+\omega(\tau)}$, with $\sigma \in \mathfrak{S}_N$ and $\tau \in \mathfrak{S}_{N'}$, for $\ell(\lambda) \leq N$ and $\ell(\nu) \leq N'$. The following lemma presents several properties which imply that the maximum of the bounds stated for the h -plethysm coefficients in Corollary 2.6.2 is exactly the bound stated in the Corollary 2.6.3.

Lemma 2.6.4. *We have the following properties:*

1. $\min_{\tau \in \mathfrak{S}_{N'}} \{\omega_1(\tau)\} = 0$.
2. $|\lambda + \omega(\sigma)| = |\lambda|$, for any $\sigma \in \mathfrak{S}_N$.
3. $\min_{\tau \in \mathfrak{S}_{N'}} \{\sum_{i=1}^j \omega_i(\tau)\} = 0$.
4. $\min_{\tau \in \mathfrak{S}_{N'}} \{\|\omega(\tau)\|\} = 0$.

Proof. 1. Since τ is a permutation of $\mathfrak{S}_{N'}$ and $\omega_1(\tau) = \tau(1) - 1$, we have that $\omega_1(\tau) \in \{0, 1, \dots, N - 1\}$. Then. $\min_{\tau \in \mathfrak{S}_{N'}} \{\omega_1(\tau)\} = 0$.

2. We have that

$$|\lambda + \omega(\sigma)| = \sum_{j=1}^N (\lambda_j + \underbrace{\sigma(j) - j}_{\omega_j(\sigma)}) = \sum_{j=1}^N \lambda_j + \sum_{j=1}^N \sigma(j) - \sum_{j=1}^N j = |\lambda|,$$

where we use that σ is a permutation of \mathfrak{S}_N , and therefore, $\sum_{j=1}^N \sigma(j)$ and $\sum_{j=1}^N j$ are equal.

3. We achieve the minimum when τ is the identity permutation, for which we have that $\sum_{i=1}^j \omega_i(\tau) = 0$.

Suppose that the minimum is achieved by other permutation. Then, $\omega_1(\tau) - 1 > 0$, and for $j = 1$, we have that $\sum_{i=1}^j \omega_i(\tau) > 0$. This leads to a contradiction with our initial assumption.

4. We have that $\|\omega(\tau)\| = \sum_{j=1}^N (N + 1 - j) \cdot \omega_j(\tau)$. The minimum is achieved by considering τ the identity permutation.

□

Let us show how these properties prove Corollary 2.6.3. We present it proving the bound for the property (R1). Consider the plethysm coefficients of (R1), $a_{\lambda, \mu+(n)}^{\nu+(\lambda|n)}$. We want to know the maximum of the bound associated to $b_{\lambda+\omega(\sigma), \mu+(n)}^{\nu+\omega(\tau)+(\lambda|n)}$, i.e.

$$\max_{\tau, \sigma} \{ \mu_1 \cdot (|\lambda + \omega(\sigma)| - 1) + \mu_2 - \nu_1 - \omega_1(\tau) \},$$

where $\sigma \in \mathfrak{S}_N$ and $\tau \in \mathfrak{S}_{N'}$, with $N \geq \ell(\lambda)$ and $N' \geq \ell(\nu)$. By the second property in Lemma 2.6.4, we get that

$$\max_{\tau, \sigma} \{ \mu_1 \cdot (|\lambda + \omega(\sigma)| - 1) + \mu_2 - \nu_1 - \omega_1(\tau) \} = \max_{\tau, \sigma} \{ \mu_1 \cdot (|\lambda| - 1) + \mu_2 - \nu_1 - \omega_1(\tau) \}.$$

Moreover, since $\omega_1(\tau)$ appears with negative sign, the maximum becomes a minimum, and by the fourth property in Lemma 2.6.4,

$$\max_{\tau, \sigma} \{ \mu_1 \cdot (|\lambda + \omega(\sigma)| - 1) + \mu_2 - \nu_1 - \omega_1(\tau) \} = \max_{\tau, \sigma} \{ \mu_1 \cdot (|\lambda| - 1) + \mu_2 - \nu_1 \}.$$

This maximum does not depend on the permutations σ and τ . In fact, it is exactly the same bound that for the h -plethysm coefficients $b_{\lambda, \mu+(n)}^{\nu+(\lambda|n)}$.

The proofs for the plethysm coefficients of (P1), (Q1), and (R2) are analogous.

□

Chapter 3.

Study of some families of reduced Kronecker coefficients

Notwithstanding their importance, the Kronecker coefficients are rather poorly understood. Recently, they have taken special attraction for researchers of different areas. Using methods from geometry [Man15, Ste14, SS15], symmetric functions [PP14, PP15a, IP15], or enumerative combinatorics [Val99, Val14, Val09], several families of stable sequences of Kronecker coefficients have been discovered.

Nevertheless, the first stability phenomenon of the Kronecker coefficients was observed by Murnaghan in 1938. This is how the reduced Kronecker coefficients come into view. The reduced Kronecker coefficients englobe a large family of Kronecker coefficients: all those Kronecker coefficients whose indexing partitions have their first part large enough.

The study presented in this chapter focusses on the reduced Kronecker coefficients in order to study the rate of growth of the Kronecker coefficients. Unexpectedly, we also obtain nice combinatorial interpretation for the reduced Kronecker coefficients.

We start digging deeper in the definition of the reduced Kronecker coefficients. As we have said, the reduced Kronecker coefficients were introduced by Murnaghan in 1938, [Mur38], through the Kronecker product.

Theorem 3.0.5 (Murnaghan's Theorem, [Mur38] [Mur56]). *There exists a family of non-negative integers $\{\bar{g}_{\alpha\beta}^\gamma\}$, indexed by triples of partitions (α, β, γ) , such that, for α and β fixed, only many terms $\bar{g}_{\alpha\beta}^\gamma$ are non-zero, and for all $n \geq 0$,*

$$s_{\alpha[n]} * s_{\beta[n]} = \sum_{\gamma} \bar{g}_{\alpha\beta}^\gamma s_{\gamma[n]},$$

where $\alpha[n] = (n - |\alpha|, \alpha_1, \alpha_2, \dots)$. Moreover, the coefficient $\bar{g}_{\alpha\beta}^\gamma$ vanishes unless the weights of the three partitions fulfil the inequalities:

$$|\alpha| \leq |\beta| + |\gamma|, \quad |\beta| \leq |\alpha| + |\gamma|, \quad |\gamma| \leq |\alpha| + |\beta|.$$

We follow Klyachko, and we call the coefficients $\bar{g}_{\alpha\beta}^\gamma$ *reduced Kronecker coefficients*. On the other hand, Kirillov calls them *extended Littlewood–Richardson numbers*, because they coincide with the Littlewood–Richardson coefficients when $|\alpha| + |\beta| = |\gamma|$, [Mur55, Lit58].

Example 6 (Murnaghan’s example). Consider $\alpha = \beta = (1)$. Then, for all $n \geq 0$,

$$s_{(n-1,1)} \star s_{(n-1,1)} = s_{(n)} + s_{(n-1,1)} + s_{(n-2,2)} + s_{(n-2,1,1)}$$

For $n \geq 4$, this is the expansion of the Kronecker product of two Schur functions in the Schur basis. Nevertheless, the identity is still true for $n < 4$, considering the definition of the Schur function s_λ given by the Jacobi–Trudi formula:

· For $n = 0$, $s_{(-1,1)} \star s_{(-1,1)} = s_0 + s_{(-1,1)} + s_{(-2,2)} + s_{(-2,1,1)}$, with

$$\begin{aligned} s_{(-1,1)} &= \det \begin{pmatrix} h_{-1} & h_0 \\ h_0 & h_1 \end{pmatrix} = -h_0^2 = -1, \\ s_{(-2,2)} &= \det \begin{pmatrix} h_{-2} & h_1 \\ h_{-1} & h_2 \end{pmatrix} = 0, \\ s_{(-2,1,1)} &= \det \begin{pmatrix} h_{-2} & h_0 & h_{-1} \\ h_{-1} & h_1 & h_0 \\ h_0 & h_2 & h_1 \end{pmatrix} = h_0^2 = 1. \end{aligned}$$

Then, $s_{(-1,1)} \star s_{(-1,1)} = 1 + (-1) + 0 + 1 = 1$.

· For $n = 1$, $s_{(0,1)} \star s_{(0,1)} = s_1 + s_{(0,1)} + s_{(-1,2)} + s_{(-1,1,1)}$, with

$$\begin{aligned} s_{(0,1)} &= \det \begin{pmatrix} h_0 & h_0 \\ h_1 & h_1 \end{pmatrix} = 0, \\ s_{(-1,2)} &= \det \begin{pmatrix} h_{-1} & h_1 \\ h_0 & h_2 \end{pmatrix} = -h_1 h_0 = -h_1 = -s_1, \end{aligned}$$

$$s_{(-1,1,1)} = \det \begin{pmatrix} h_{-1} & h_0 & h_{-1} \\ h_0 & h_1 & h_0 \\ h_1 & h_2 & h_1 \end{pmatrix} = 0.$$

Then, $s_{(0,1)} \star s_{(0,1)} = s_1 + 0 + (-s_1) + 0 = 0$.

• For $n = 2$, $s_{(1,1)} \star s_{(1,1)} = s_2 + s_{(1,1)} + s_{(0,2)} + s_{(0,1,1)}$, with

$$\begin{aligned} s_{(0,2)} &= \det \begin{pmatrix} h_0 & h_1 \\ h_1 & h_2 \end{pmatrix} = -\det \begin{pmatrix} h_1 & h_2 \\ h_0 & h_1 \end{pmatrix} = -s_{(1,1)}, \\ s_{(0,1,1)} &= \det \begin{pmatrix} h_0 & h_0 & h_{-1} \\ h_1 & h_1 & h_0 \\ h_2 & h_2 & h_1 \end{pmatrix} = 0, \end{aligned}$$

Then, $s_{(1,1)} \star s_{(1,1)} = s_2 + s_{(1,1)} + (-s_{(1,1)}) = s_2$.

• For $n = 3$, $s_{(2,1)} \star s_{(2,1)} = s_3 + s_{(2,1)} + s_{(1,2)} + s_{(1,1,1)}$, with

$$s_{(1,2)} = \det \begin{pmatrix} h_1 & h_1 \\ h_2 & h_2 \end{pmatrix} = 0.$$

Then, $s_{(2,1)} \star s_{(2,1)} = s_3 + s_{(2,1)} + s_{(1,1,1)}$.

In [Thi91] we can find a proof of Murnaghan's theorem. We include a proof using vertex operators at the end of this chapter, Section 3.7.

Note that the reduced Kronecker coefficients inherit the symmetry of the Kronecker coefficients: $\bar{g}_{\alpha\beta}^\gamma$ is symmetric in all three labels α , β and γ .

The reduced Kronecker coefficients are interesting objects of their own right. In fact, we can recover the Kronecker coefficients $g_{\alpha\beta}^\gamma$ from the reduced Kronecker coefficients $\bar{g}_{\alpha\beta}^\gamma$, using a result of E. Briand, R. Orellana, and M. Rosas presented in [BOR11]. Moreover, A. N. Kirillov and A. Klyachko, in [Kly04] and [Kir04] conjectured that the reduced Kronecker coefficients satisfy the saturation hypothesis.

In this chapter, we present a study of four families of reduced Kronecker coefficients.

- ▷ **Family 1** $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$
- ▷ **Family 2** $\bar{g}_{(k^a), (k^b)}^{(k)}$

- ▷ **Family 3** $\bar{g}_{((k+i)^a), (k^b)}^{(k)}$
- ▷ **Family 4** $\bar{g}_{(k^b), (k+i, k^a)}^{(k)}$

In Section 3.2 we present explicit formulas of the piecewise quasipolynomial which describes the coefficients of Family 1 when λ , μ and ν are partitions of length at most 1. Families 2, 3 and 4 are studied in Sections 3.3, 3.4, and 3.5, respectively. In these cases, their study is more complete: we give the generating function of the families, as well as their descriptions in terms of plane partitions and quasipolynomials. We also analyse the saturation hypothesis. In Section 3.6, we translate the results obtained in terms of Kronecker coefficients and we analyse the corresponding families. Finally, Section 3.7 is dedicated to the study of the reduced Kronecker coefficients with the vertex operators introduced in Chapter 2.

3.1. Previous results

In this section, we present results related to reduced Kronecker coefficients and Kronecker coefficients that we will need to prove the results stated in this chapter.

3.1.1. The reduced Kronecker coefficients as Kronecker coefficients

Given three partitions α , β and γ , the sequence $\{g_{\alpha[n]\beta[n]}^{\gamma[n]}\}_n$ is eventually constant. The reduced Kronecker coefficient $\bar{g}_{\alpha\beta}^\gamma$ can be defined as the stable value of this sequence. Therefore, there exists a positive integer N such that for $n \geq N$,

$$\bar{g}_{\alpha\beta}^\gamma = g_{\alpha[n]\beta[n]}^{\gamma[n]}.$$

The point at which the expansion of the Kronecker product $s_{\alpha[n]} * s_{\beta[n]}$ stabilizes is denoted by $stab(\alpha, \beta)$. In [BOR11], E. Briand, R. Orellana, and M. Rosas prove that

$$stab(\alpha, \beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1. \tag{3.1}$$

The symmetry of the reduced Kronecker coefficients, together with (3.1), implies the following bound for N .

Corollary 3.1.1. *Consider three partitions α , β and γ such that, for $n \geq N$, $\bar{g}_{\alpha\beta}^\gamma = g_{\alpha[n]\beta[n]}^{\gamma[n]}$. Then, $N \leq \min\{stab(\alpha, \beta), stab(\alpha, \gamma), stab(\beta, \gamma)\}$.*

3.1.2. Kronecker tableaux

In [BO07], C. Ballantine and R. Orellana introduce the notion of the Kronecker tableaux to give a combinatorial description of a special kind of Kronecker coefficients. These will be one of the combinatorial tools we will use in this chapter.

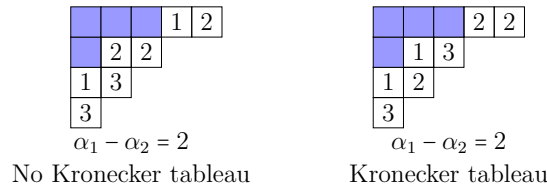
Definition 3.1.2. An α -lattice permutation is a sequence of integers such that in every part of the sequence the number of occurrences of i plus α_i is bigger than or equal to the number of occurrences of $i + 1$ plus α_{i+1} .

Definition 3.1.3. A Kronecker tableau is a semi-standard Young tableau T of shape λ/α and type ν/α , with $\alpha \subset \lambda \cap \nu$, whose reverse reading word is an α -lattice permutation, and such that

1. $\alpha_1 = \alpha_2$, OR
2. $\alpha_1 > \alpha_2$ AND
 - a) The number of 1's in the second row of λ/α is exactly $\alpha_1 - \alpha_2$.
 - b) The number of 2's in the first row of λ/α is exactly $\alpha_1 - \alpha_2$.

We refer to this last condition as α -condition. We denote by $k_{\alpha\nu}^\lambda$ the number of Kronecker tableaux of shape λ/α and type ν/α , with $\alpha \subset \lambda \cap \nu$.

For instance, consider $\lambda = (5, 3, 2, 1)$, $\nu = (5, 4, 2)$ and $\alpha = (3, 1)$. Then, on the left, there is an example of a semi-standard Young tableau that is not a Kronecker tableau, because it does not satisfy the α -condition, and on the right, there is an example of a Kronecker tableau:



The semi-standard Young tableau on the left is not a Kronecker tableau because it does not satisfy the α -condition. Note that in both cases, the reverse reading word is a $(3, 1)$ -lattice permutation.

The following theorem gives us a combinatorial interpretation of a special family of Kronecker coefficients in terms of Kronecker tableaux.

Theorem 3.1.4 (C. Ballantine, R. Orellana, [BO07]). *Let n and p be positive integers such that $n \geq 2p$. Let $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ and ν be partitions of n .*

- (a) *If $\lambda_1 \geq 2p - 1$, the multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ equals $\sum_{\substack{\alpha \vdash p \\ \alpha \subset \lambda \cap \nu}} k_{\alpha\nu}^\lambda$.*

(b) If $\ell(\lambda) \geq 2p - 1$, the multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ equals $\sum_{\substack{\alpha+p \\ \alpha \in \lambda' \cap \nu'}} k_{\alpha\nu'}^{\lambda'}$, where λ' denotes the conjugate partition of λ .

The cases (a) and (b) are not exclusive: consider $n = 4p$ and $\lambda = (2p, p, \dots, p)$, with $\ell(\lambda) = 2p + 1$. Then, λ is a partition of n and whether $\lambda_1 \geq 2p - 1$ and $\ell(\lambda) \geq 2p - 1$. Both computations give the same value. For our computations, we will always use the case (a).

3.2. Family 1: three-row partitions

Consider the partitions $\mu = (a)$, $\nu = (b)$ and $\lambda = (c)$, and Family 1 of reduced Kronecker coefficients $\left\{ \bar{g}_{k\mu, k\nu}^{k\lambda+(i)} \right\}_{k, i \geq 0}$.

For $i = 0$, due to Vergne and Baldoni, [BV15], the *stretching* reduced Kronecker coefficients $\bar{g}_{k\mu, k\nu}^{k\lambda}$, for any partitions λ, μ and ν , are given by a quasipolynomial depending on k, λ, μ and ν . Recently, Briand, Rattan and Rosas in [BRR16] also shows that the quasipolynomial is linear of period 2. For $i > 0$, the reduced Kronecker coefficients $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$, for any partitions λ, μ and ν , are given by piecewise quasipolynomials depending on k, λ, μ and ν , [Man15, MS99, BMS13].

In this section, we present explicit formulas for the reduced Kronecker coefficients $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$ with λ, μ and ν partitions of length at most 1. We start showing some examples.

Notation. Let $[a]$ be the largest integer not greater than a .

Example 7. Consider the family of reduced Kronecker coefficients $\bar{g}_{(k), (k)}^{(k+i)}$. We have the following table of coefficients depending on the values of k and i .

Table 3.1.: $\lambda = (1), \mu = (1)$ and $\nu = (1)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i=0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
i=1	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
i=2	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
i=3	0	0	0	1	1	2	2	3	3	4	4	5	5	6	6
i=4	0	0	0	0	1	1	2	2	3	3	4	4	5	5	6

The coefficients in each diagonal are stable. In fact, for the j^{th} diagonal, they are identically $\lfloor \frac{j}{2} \rfloor + 1$, with $j \geq 0$.

Example 8. Consider the family of reduced Kronecker coefficients $\bar{g}_{(k),(k)}^{(i)}$.

Table 3.2.: $\lambda = \emptyset, \mu = (1)$ and $\nu = (1)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
i= 0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
i= 1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
i= 2	0	1	2	2	2	2	2	2	2	2	2	2	2	2
i= 3	0	0	1	2	2	2	2	2	2	2	2	2	2	2
i= 4	0	0	1	2	3	3	3	3	3	3	3	3	3	3
i= 5	0	0	0	1	2	3	3	3	3	3	3	3	3	3

We observe in the Table 3.2 that after some initial zeros, the rows grow linearly until they stabilize to a non-zero value. In this case, the j^{th} row stabilizes to $\lfloor \frac{j}{2} \rfloor + 1$, with $j \geq 0$.

Example 9. Consider the family of reduced Kronecker coefficients $\bar{g}_{(2k),(k)}^{(i)}$. We have the following table:

Table 3.3.: $\lambda = \emptyset, \mu = (1)$ and $\nu = (2)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
i= 0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
i= 1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
i= 2	0	1	1	0	0	0	0	0	0	0	0	0	0	0
i= 3	0	1	1	1	0	0	0	0	0	0	0	0	0	0
i= 4	0	0	2	1	1	0	0	0	0	0	0	0	0	0
i= 5	0	0	1	2	1	1	0	0	0	0	0	0	0	0

Note that the stable value is in this case zero.

We introduce the definition of quasipolynomials.

Definition 3.2.1. We say that $f : \mathbb{N} \rightarrow \mathbb{Z}$ is a *quasipolynomial* if there exist a period m and polynomials $p_i \in \mathbb{Z}[t]$ such that $f(t) = p_i(t)$ for $t \equiv i \pmod m$.

Theorem 3.2.2. Fix $\mu = (a)$, $\nu = (b)$, and $\lambda = (c)$. Then, the value of the reduced Kronecker coefficient $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$ is given by a piecewise linear quasipolynomial. Moreover, we compute the piecewise linear quasipolynomials depending on the partitions:

1. For $\mathbf{a} + \mathbf{b} = \mathbf{c}$: we have $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = 0$, except for $i = 0$, in which case is 1, for all k .
2. For $\mathbf{a} + \mathbf{b} \neq \mathbf{c}$: we split in more cases.

a) For $\mathbf{a} > \mathbf{c}$:

$$\boxed{a \leq b + c}$$

$$\bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = \begin{cases} 0 & \text{if } 0 \leq k < \frac{i}{a+b-c}, \\ \left\lfloor \frac{k(a+b-c)-i}{2} \right\rfloor + 1 & \text{if } \frac{i}{a+b-c} \leq k < \frac{i}{a-c}, \\ \left\lfloor \frac{k(b+c-a)+i}{2} \right\rfloor + 1 & \text{if } k \geq \frac{i}{a-c}. \end{cases}$$

$$\boxed{a > b + c}$$

$$\bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = \begin{cases} 0 & \text{if } 0 \leq k < \frac{i}{a+b-c}, \\ \left\lfloor \frac{k(a+b-c)-i}{2} \right\rfloor + 1 & \text{if } \frac{i}{a+b-c} \leq k < \frac{i}{a-c}, \\ \left\lfloor \frac{k(b+c-a)+i}{2} \right\rfloor + 1 & \text{if } \frac{i}{a-c} \leq k \leq \frac{i}{a-b-c}, \\ 0 & \text{if } k > \frac{i}{a-b-c}. \end{cases}$$

b) For $\mathbf{a} = \mathbf{c}$:

$$\bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = \begin{cases} 0 & \text{if } 0 \leq k < \frac{i}{b}, \\ \left\lfloor \frac{bk-i}{2} \right\rfloor + 1 & \text{if } k \geq \frac{i}{b}. \end{cases}$$

c) For $\mathbf{a} < \mathbf{c}$:

$$\boxed{b + a \leq c} \quad \text{We have } \bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = 0, \text{ except for } i = 0 \text{ and } k = 0, \text{ in which case is 1.}$$

$$\boxed{b + a \geq c}$$

$$\bar{g}_{k\mu, k\nu}^{k\lambda+(i)} = \begin{cases} 0 & \text{if } 0 \leq k < \frac{i}{a+b-c}, \\ \left\lfloor \frac{k(a+b-c)-i}{2} \right\rfloor + 1 & \text{if } k \geq \frac{i}{a+b-c}. \end{cases}$$

Note that once we fix the partitions λ , μ and ν , the regions described by the pairs (k, i) depending on the piecewise quasipolynomials are closed polyhedral subcones called chambers, and that each chamber is given by a set of inequations.

Proof. The reduced Kronecker coefficients $\bar{g}_{k\mu, k\nu}^{k\lambda+(i)}$ can be translated into Kronecker coefficients as $g_{(N-k\cdot a, k\cdot a), (N-k\cdot b, k\cdot b)}^{(N-k\cdot c-i, k\cdot c+i)}$, for N large enough. The theorem follows from [Ros01, Corollary 5], together with the symmetry of the Kronecker coefficients. \square

The examples included at the beginning of the subsection summarize the different kinds of stability properties: stability along the diagonals, and stability of the rows with zero and non-zero stable value.

3.3. Results for Family 2

In this section we study Family 2 of reduced Kronecker coefficients $\left\{ \bar{g}_{(k^a), (k^b)}^{(k)} \right\}_{k \geq 0}$.

The table 3.4 shows first cases for $a = b$.

Table 3.4.: Family 2: Case $b = a$, for $a = 0, \dots, 6$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
a= 0	1	0	0	0	0	0	0	0	0	0	0	0	0	A000007
a= 1	1	1	2	2	3	3	4	4	5	5	6	6	7	A008619
a= 2	1	1	3	4	7	9	14	17	24	29	38	45	57	A008763
a= 3	1	1	3	5	9	13	22	30	45	61	85	111	150	A001993
a= 4	1	1	3	5	10	15	26	38	60	85	125	172	243	A070557
a= 5	1	1	3	5	10	16	28	42	68	100	151	215	312	A070558
a= 6	1	1	3	5	10	16	29	44	72	108	166	241	357	A070559

The references indicated in the right side of each row are references taken out from the *On-Line Encyclopedia of Integer Sequences*, <https://oeis.org/>.

In this section we show the generating function of the sequence of reduced Kronecker coefficients of Family 2, once we fix a . Using the generating function, we show the connection between Family 2 of reduced Kronecker coefficients and plane partitions fitting in a specific rectangle. As a consequence of this relation, we also give a description of Family 2 in terms of quasipolynomials. At the end of the section, we prove that the coefficients of the Family 2 satisfy the *saturation hypothesis*.

3.3.1. Generating function for Family 2

Theorem 3.3.1 (L. Colmenarejo and M. Rosas, [CR15]). *Fix integers $a \geq b \geq 0$. Consider Family 2 of reduced Kronecker coefficients, $\left\{ \bar{g}_{(k^a), (k^b)}^{(k)} \right\}_{k \geq 0}$.*

1. *If $a = b$, the generating function for the reduced Kronecker coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ is*

$$\mathcal{F}_{a,a} = \frac{1}{(1-x)(1-x^2)^2 \dots (1-x^a)^2(1-x^{a+1})}.$$

2. *If $a = b + 1$, then $\bar{g}_{(k^a), (k^b)}^{(k)} = 1$ for all $k \geq 0$. That is $\mathcal{F}_{b+1,b} = \frac{1}{1-x}$.*
3. *If $a > b + 1$, $\bar{g}_{(k^a), (k^b)}^{(k)} = 0$, except for $k = 0$, in which case is 1.*

Proof. The reduced Kronecker coefficients that we are considering can be written in terms of Kronecker coefficients as $\bar{g}_{(k^a), (k^b)}^{(k)} = g_{(N-ak, k^a), (N-bk, k^b)}^{(N-k, k)}$, for N big enough, i.e. for $N \geq N_0$ for some N_0 . Using (3.1) and Corollary 3.1.1, we can consider $N_0 = (3+a) \cdot k$,

$$\bar{g}_{(k^a), (k^b)}^{(k)} = g_{(3k, k^a), ((a-b+3)k, k^b)}^{((a+2)k, k)}. \tag{3.2}$$

Once we have the description as Kronecker coefficients, Theorem 3.1.4 of R. Orellana and C. Ballantine gives us an interpretation of these Kronecker coefficients in terms of Kronecker tableaux: $\bar{g}_{(k^a), (k^b)}^{(k)}$ equals the number of Kronecker tableaux with type $((a-b+3)k, k^b)/\alpha$ and shape $(3k, k^a)/\alpha$, where α is a partition of k with $\ell(\alpha) \leq a+1$.

We look over what happens in each of the three cases occurring in the statement of the theorem, from bottom to top:

- 3. If $a > b + 1$, and $k = 0$, we only have the empty Kronecker tableau. For $k \geq 1$, suppose that there is at least one Kronecker tableau of shape $(3k, k^a)/\alpha$ and type $((a - b + 3)k, k^b)/\alpha$, where α is a partition of k with $\ell(\alpha) \leq a + 1$. Then, we always find a column of height a or $a + 1$ (depending on the first part of α) that we cannot fill, because we only have $b + 1 < a$ possible numbers. We conclude that, $\bar{g}_{(k^a), (k^b)}^{(k)} = 0$, except for $k = 0$, in which case is 1.
- 2. If $a = b + 1$, then we count the Kronecker tableau with shape $(3k, k^{b+1})/\alpha$, type $(4k, k^b)/\alpha$, and $\alpha \vdash k$.

For $k = 0$, we only have the empty Kronecker tableau. For $k \geq 1$, we consider $\alpha = (k)$ and we have the following Kronecker tableau:

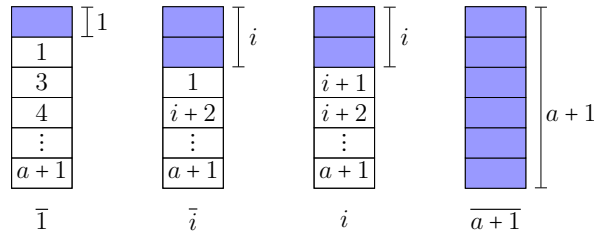
	k	k	k
		1	1
$b + 1$	1		
	2		
	⋮		
	b		
	$b + 1$		

For the partitions $\alpha \neq (k)$, note that there is, at least, one column of height $b + 2$ that we cannot fill because we do not have enough different numbers. Then, $\bar{g}_{(k^a), (k^b)}^{(k)} = 1$ for all $k \geq 0$.

- 1. The function $\mathcal{F}_{a,a}$ is the generating function of the coloured partitions with parts in $\mathcal{B}_a = \{\bar{1}, 2, \bar{2}, \dots, a, \bar{a}, \overline{a+1}\}$, i.e. weakly decreasing sequences of integers with parts in \mathcal{B}_a such that the parts are ordered by saying that $\bar{1} < \bar{2} < 2 < \bar{3} < \dots$ and such that both parts i and \bar{i} have weight i . To prove Theorem 3.3.1 we establish a bijection between coloured partitions with parts in \mathcal{B}_a , and Kronecker tableaux with shape $(3k, k^a)/\alpha$ and type $(3k, k^a)/\alpha$, where α is a partition of k with $\ell(\alpha) \leq a + 1$. This will imply that $\mathcal{F}_{a,a}$ is also the generating function for the reduced Kronecker coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$.

The bijection is defined by the following algorithm. To a coloured partition of k with parts in \mathcal{B}_a , β , we associate a Kronecker tableau $T(\beta)$ as follows.

First, we identify each element of \mathcal{B}_a to a column of height $a + 1$:



for $i \in \{2, 3, \dots, a - 1, a\}$.

Note that it is always possible to order the columns corresponding to the parts of β in such a way that we obtain a semi-standard Young tableau. We denote by m_i the number of times that the part $i \in \mathcal{B}_a$ appears in β . Then, the column i appears m_i times in the semi-standard Young tableau that we are building. We read the partition α from our semi-standard Young tableau by counting the number of blue boxes in each row: $\alpha_{a+1} = m_{\overline{a+1}}$, $\alpha_i = \alpha_{i+1} + m_i + m_{\bar{i}}$ for $i = 2, \dots, a$, and $\alpha_1 = \alpha_2 + m_{\bar{1}}$.

These columns correspond to the first columns on the left-hand side of $T(\beta)$. We build the rest of $T(\beta)$ of shape $(3k, k^a)/\alpha$ as follows: complete i^{th} row with \boxed{i} boxes, for $i = 2, \dots, a + 1$, and complete first row with the remaining numbers of the type $(3k, k^a)/\alpha$ in weakly increasing order from left to right.

For instance, the Kronecker tableau corresponding to $\lambda = \nu = (9, 3, 3, 3)$ and $\alpha = (2, 1)$, obtained by our algorithm taking $a = 3$ and $\beta = (2, \bar{1})$, is

		1	1	1	1	1	1	2	3
	1	2							
3	3	3							
4	4	4							

Let us see that the map defined by this algorithm described above is well-defined. For that, we have to check that $T(\beta)$ is a Kronecker tableau.

- By construction, $T(\beta)$ is a semi-standard Young tableau of shape $(3k, k^a)/\alpha$ and type $(3k, k^a)/\alpha$, where α is the sequence defined by the blue boxes.
- The sequence α defined using β is a partition of k .

Since $\alpha_i = \alpha_{i+1} + m_i + m_{\bar{i}}$, for $i = 2, \dots, a$, and $\alpha_2 = \alpha_1 + m_{\bar{1}}$, α is weakly increasing. We check that the sum of its parts is k . We express α in terms

of m_i , with $i \in \mathcal{B}_a$,

$$\begin{aligned}\alpha_{a+1} &= m_{\overline{a+1}}, \\ \alpha_i &= \sum_{l=i}^{a+1} m_{\overline{l}} + \sum_{l=i}^a m_l, \quad \text{for } i = 2, \dots, a,\end{aligned}$$

$$\alpha_1 = \sum_{l=1}^{a+1} m_{\overline{l}} + \sum_{l=2}^a m_l.$$

Then,

$$\begin{aligned}|\alpha| &= \sum_{i=1}^{a+1} \alpha_i = m_{\overline{a+1}} + \sum_{i=2}^a \left(\sum_{l=i}^{a+1} m_{\overline{l}} + \sum_{l=i}^a m_l \right) + \sum_{l=1}^{a+1} m_{\overline{l}} + \sum_{l=2}^a m_l = \\ &= m_{\overline{a+1}} + \sum_{l=2}^a (l-1) \cdot m_{\overline{l}} + (a-1) \cdot m_{\overline{a+1}} + \sum_{l=2}^a (l-1) \cdot m_l + m_{\overline{1}} + \sum_{l=2}^a m_{\overline{l}} + m_{\overline{a+1}} + \sum_{l=2}^a m_l = \\ &= \sum_{l=1}^{a+1} l \cdot m_{\overline{l}} + \sum_{l=2}^a l \cdot m_l = k.\end{aligned}$$

Last equality becomes from the fact that β is a coloured partition of k .

- We check that $(\#1)_{R1} \geq k - \alpha_1$. This guarantees that there is no incongruence in the columns of height $a + 1$.

Since $(\#1)_{R1} = 3k - \alpha_1 - \sum_{i=1}^a m_{\overline{i}}$, $(\#1)_{R1} \geq k - \alpha_1$ if and only if $2k - \sum_{i=1}^a m_{\overline{i}} \geq 0$. Since β is a coloured partition of k , we can express

$$k = \sum_{i=1}^{a+1} i \cdot m_{\overline{i}} + \sum_{i=2}^a i \cdot m_i.$$

Then,

$$2k - \sum_{i=1}^a m_{\overline{i}} = 2(a+1) \cdot m_{\overline{a+1}} + \sum_{i=1}^a (2a-1) \cdot m_{\overline{i}} + 2 \sum_{i=2}^a m_i \geq 0.$$

- The reverse reading word is an α -lattice permutation.

The reverse reading word of $T(\beta)$ is of the form:

$$\begin{aligned}(\#a+1)_{R1} \dots (\#1)_{R1} (\#2)_{R2} (\#1)_{R2} \dots \\ \dots (\#i)_{Ri} (\#1)_{Ri} \dots (\#a+1)_{Ra+1} (\#1)_{Ra+1} \dots\end{aligned}$$

where we regroup boxes with the same label in the same row. Furthermore, we only need to check the inequalities at the end of each group. We check the inequalities from left to right in the reverse reading word.

- At the level of the first row, we check that $\alpha_{i+1} + (\#i + 1)_{R1} \leq \alpha_i$: for $i = 2, \dots, a$,

$$\alpha_{i+1} + (\#i + 1)_{R1} = \alpha_{i+1} + m_{\bar{i}} \leq \alpha_{i+1} + m_i + m_{\bar{i}} = \alpha_i.$$

For $i = 1$,

$$(\#2)_{R1} + \alpha_2 = \alpha_2 + m_{\bar{1}} = \alpha_1. \tag{3.3}$$

- At the level of the second row, we check that

$$\alpha_2 + (\#2)_{R1} + (\#2)_{R2} \leq \alpha_1 + (\#1)_{R1}.$$

The left-hand side is exactly k , because it sums the total number of $\boxed{2}$ plus α_2 . By the previous item, $(\#1)_{R1} \geq k - \alpha_1$.

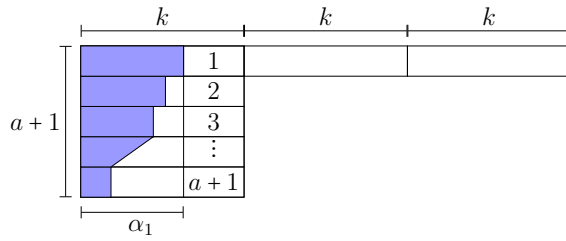
- At the other levels, fix the j^{th} row. Since the box \boxed{i} , with $i \neq 1$, only appears in the first row and in the i^{th} row, we only need to check that $\alpha_{j+1} + (\#j + 1)_{R1} + (\#j + 1)_{Rj} \leq \alpha_j + (\#j)_{R1}$, for $j = 2, \dots, a$. Both sides of this inequality are equal to k , because they count exactly the total number of \boxed{i} boxes plus α_i , for $i = j$ and $j + 1$.

- We check the α -condition.

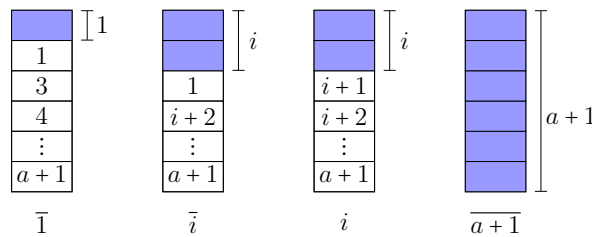
For $\alpha_1 = \alpha_2$, we do not need to check anything. For $\alpha_1 > \alpha_2$, by (3.3) and that in $T(\beta)$ there are no $\boxed{2}$ box in the second row, we get that $(\#1)_{R2} = \alpha_1 - \alpha_2 = (\#2)_{R1}$.

By construction, the map is injective. We show that it is also surjective.

Consider a Kronecker tableau, T , of shape $(3k, k^a)/\alpha$ and type $(3k, k^a)/\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_{a+1})$ a partition of k . We need to figure out its corresponding coloured partition β of k with parts in \mathcal{B}_a . Summarizing all the information we have about T for the moment, we can say that the Kronecker tableau T has the following form



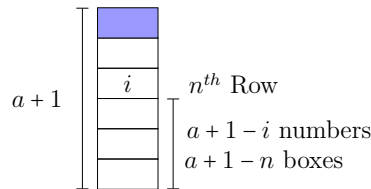
Excluding all columns of height 1 and of height $a + 1$ with no blue boxes, we claim that the following list summarizes all possible columns that appear in the remaining part of the Kronecker tableau:



Let us prove why there are no other kinds of columns of height $a + 1$ with blue boxes:

- (i) For $i = 2, \dots, a + 1$, the box \boxed{i} cannot appear in any row $n \leq i - 1$.

Suppose that there is a column of the form



Since T is a semi-standard Young tableau, we cannot fill the $a + 1 - n$ boxes with different numbers because we only have $a + 1 - i$ possible numbers.

- (ii) For $i = 2, \dots, a$, the box \boxed{i} cannot appear in any row $n \geq i + 1$.

We prove it by induction from a to 2. Looking at the end of the reverse reading word $(\#a + 1)_{R1} + \alpha_{a+1} \leq \alpha_a + (\#a)_{R1} + (\#a)_{Ra}$. The left-hand side of the equation is exactly k and in total there are $k - \alpha_{a+1}$ $\boxed{a + 1}$ boxes. Then, there are no more boxes than in the first row and in the a^{th} row.

Let us see that there cannot be \boxed{i} boxes in the $(i + 1)^{th}$ row, assuming that in any n^{th} row, with $n \geq i + 2$, there are only $\boxed{1}$ and \boxed{n} boxes. The

part of the reverse reading word corresponding to the $(i + 1)^{th}$ row says that

$$(\#i + 1)_{R1} + (\#i + 1)_{Ri+1} + \alpha_{i+1} \leq \alpha_i + (\#i)_{R1} + (\#i)_{Ri}.$$

Applying the induction hypothesis, left-hand side is exactly k . Then, we have that $(\#i)_{R1} + (\#i)_{Ri} \geq k - \alpha_i$. Since there are $k - \alpha_i$ \boxed{i} boxes in total, we also have that $(\#i)_{R1} + (\#i)_{Ri} \leq k - \alpha_i$. Thus, there are no \boxed{i} boxes in any row different from the first one and the i^{th} .

(iii) There are no columns of the form

1]	1
2		
3		
4		
⋮		
$a + 1$		

By the α -condition, $\alpha_1 = \alpha_2$ and there are no columns with only one blue box, or $\alpha_1 > \alpha_2$ and, in this case, $(\#1)_{R2} = (\#2)_{R1} = \alpha_1 - \alpha_2$

To define β , we denote by n_i , with $i \in \mathcal{B}_a$, the number of occurrences of the column associated to i in our Kronecker tableau. Then, $\beta := (\overline{1}^{n_1} \overline{2}^{n_2} 2^{n_2} \dots \overline{a + 1}^{n_{a+1}})$. The coloured partition β is a partition of k because the sum of its parts is exactly the sum of all blue boxes in T , which corresponds to the partition α :

$$k = \sum_{i=1}^{a+1} i \cdot n_i + \sum_{i=1}^a i \cdot n_i.$$

The map defined by the algorithm is a bijection and Theorem 3.3.1 is proved. □

The generating function obtained for the reduced Kronecker coefficients of Family 2 shows one of their stability properties.

Corollary 3.3.2. *Fix a positive integer k . Then, the reduced Kronecker coefficient $\overline{g}_{(k^a), (k^a)}^{(k)}$ stabilizes at $a = k$.*

Notation. We set $[x^k]f(x)$ for the coefficient of x^k in the series expansion of $f(x)$.

Proof. By Theorem 3.3.1, the general term of the sequence is

$$g_{(k^a), (k^a)}^{(k)} = [x^k] \frac{1}{(1-x)(1-x^2)^2 \dots (1-x^a)^2(1-x^{a+1})}.$$

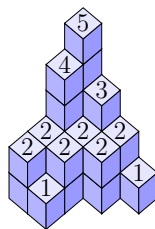
Once $a = k$, the factors $(1 - x^l)$, with $l = 1, \dots, a$ appear always with the same exponent (for the case $l \geq 2$, the exponent is 2 and for $l = 1$, the exponent is 1). Therefore, the reduced Kronecker coefficients stabilize. \square

3.3.2. Plane partitions: combinatorial interpretation of Family 2

In this subsection we establish a link between the family of reduced Kronecker coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ and plane partitions.

Definition 3.3.3. A *plane partition* is a finite subset \mathcal{P} of positive integer lattice points, $\{(i, j, k)\} \subset \mathbb{N}^3$, such that if (r, s, t) lies in \mathcal{P} and if (i, j, k) satisfies that $1 \leq i \leq r$, $1 \leq j \leq s$ and $1 \leq k \leq t$, then (i, j, k) also lies in \mathcal{P} . We denote by $\mathcal{B}(r, s, t)$ the set of plane partitions fitting in a $r \times s$ rectangle and with biggest part less or equal to t .

As an illustration we present a plane partition of 26 in $\mathcal{B}(4, 4, 5)$,



In [Mac04], P. MacMahon presents the generating function of the plane partitions in $\mathcal{B}(r, s, t)$.

Theorem 3.3.4 (P. MacMahon, [Mac04]). *The generating function for plane partitions in $\mathcal{B}(r, s, t)$ is*

$$pp_t(x; r, s) = \prod_{i=1}^r \prod_{j=1}^s \prod_{k=1}^t \frac{1 - x^{i+j+t-1}}{1 - x^{i+j+t-2}}.$$

The following lemma shows other formula for the generating function of the plane partitions fitting in a $r \times s$ rectangle. In this case, the largest part of a partition of n is exactly n .

Proposition 3.3.5. *Let $u = \min(r, s)$ and $v = \max(r, s)$. Then, the generating function for the plane partitions fitting inside a $r \times s$ rectangle is*

$$\prod_{n=u}^v \left(\frac{1}{1 - x^n} \right)^u \cdot \prod_{m=1}^{u-1} \left(\frac{1}{1 - x^m} \right)^m \left(\frac{1}{1 - x^{v+m}} \right)^{u-m}. \tag{3.4}$$

We need the following lemma.

Lemma 3.3.6. *For all $l \geq 1$,*

$$\frac{1 - x^{l+k}}{1 - x^l} = 1 + x^l + x^{2l} + \dots + x^{\lfloor \frac{k}{l} \rfloor l} + \mathcal{O}(k+1).$$

Notation. We set $\mathcal{O}(k)$ for the terms of degree bigger than or equal to k .

Proof of Lemma 3.3.6. We have that

$$\frac{1 - x^{l+k}}{1 - x^l} = (1 - x^{l+k}) \cdot (1 + x^l + x^{2l} + \dots + \underbrace{x^{\lfloor \frac{k}{l} \rfloor l}}_{\text{degree } \leq k} + \underbrace{x^{(\lfloor \frac{k}{l} \rfloor + 1)l}}_{\text{degree } > k} + \mathcal{O}(k+2)).$$

Then, $\frac{1 - x^{l+k}}{1 - x^l} = 1 + x^l + x^{2l} + \dots + x^{\lfloor \frac{k}{l} \rfloor l} + \underbrace{x^{(\lfloor \frac{k}{l} \rfloor + 1)l} + \dots}_{\text{degree } \geq k+1} - \underbrace{x^{l+k} - x^{2l+k} - \dots}_{\text{degree } \geq k+1}$, and the lemma follows. \square

Proof of Proposition 3.3.5. We will show that

$$[x^k]pp_k(x; r, s) = [x^k] \prod_{j=u}^v \left(\frac{1}{1 - x^j} \right)^u \cdot \prod_{i=1}^{u-1} \left(\frac{1}{1 - x^i} \right)^i \left(\frac{1}{1 - x^{v+i}} \right)^{u-i}.$$

Note that

$$\begin{aligned} pp_k(x; r, s) &= \prod_{i=1}^r \prod_{j=1}^s \prod_{l=1}^k \frac{1 - x^{i+j+l-1}}{1 - x^{i+j+l-2}} = \\ &= \prod_{i=1}^r \prod_{j=1}^s \frac{(1 - x^{i+j})}{(1 - x^{i+j-1})} \frac{(1 - x^{i+j+1})}{(1 - x^{i+j})} \dots \frac{(1 - x^{i+j+k-2})}{(1 - x^{i+j+k-3})} \frac{(1 - x^{i+j+k-1})}{(1 - x^{i+j+k-2})} = \\ &= \prod_{i=1}^r \prod_{j=1}^s \frac{1 - x^{i+j+k-1}}{1 - x^{i+j-1}}. \end{aligned} \quad (3.5)$$

Expanding $pp_k(x; r, s)$ according to the values of $i + j$ and applying Lemma 3.3.6 to each factor, we obtain that the coefficient of x^k is exactly the number of ways to write k as a sum of parts in \mathcal{A} , where \mathcal{A} is the following set:

- For $l = 1, \dots, u$, there are l different kind of l 's in \mathcal{A} .
- For $l = u + 1, \dots, v$, there are u different kind of l 's in \mathcal{A} .
- For $l = v + 1, \dots, v + u - 1$, there are $u - l$ different kind of l 's in \mathcal{A} .
- Each part labelled with l , whether kind it is, has weight l .

□

Writing the generating function of the reduced Kronecker coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ as in Proposition 3.3.5, we obtain the following result.

Theorem 3.3.7 (L. Colmenarejo and M. Rosas, [CR15]). *The reduced Kronecker coefficient $\bar{g}_{(k^a), (k^a)}^{(k)}$ counts the number of plane partitions of k fitting inside a $2 \times a$ rectangle.*

3.3.3. Family 2 in terms of quasipolynomials

Theorem 3.3.8 (L. Colmenarejo and M. Rosas, [CR15]). *Let $\mathcal{F}_a = \mathcal{F}_{a,a}$ be the generating function for the reduced Kronecker coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ stated in Theorem 3.3.1. Let ℓ be the least common multiple of $1, 2, \dots, a, a + 1$.*

1. *The generating function \mathcal{F}_a can be written as*

$$\mathcal{F}_a = \frac{P_a(x)}{(1 - x^\ell)^{2a}},$$

where $P_a(x)$ is a product of cyclotomic polynomials. Moreover, we have that $\deg(P_a(x)) = 2\ell a - (a + 2)a < 2a\ell - 1$.

2. *The polynomial P_a is the generating function for coloured partitions with parts in $\{1, 2, \bar{2}, 3, \bar{3}, \dots, a, \bar{a}, a + 1\}$, where parts j and \bar{j} appear with multiplicity less than or equal to ℓ/j times.*
3. *The coefficients of P_a are positive and palindrome, but in general are not a concave sequence.*
4. *The coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ are described by a quasipolynomial in k of degree $2a - 1$ and period dividing ℓ . In fact, we have checked that the period is exactly ℓ for $a \leq 10$.*
5. *The coefficients $\bar{g}_{(k^a), (k^a)}^{(k)}$ satisfy a formal reciprocity law $x^{a(a+2)}\mathcal{F}_a(x) = \mathcal{F}_a\left(\frac{1}{x}\right)$.*

Proof. 1. We define $P_a(x)$ as

$$P_a(x) = \frac{(1 - x^\ell)^{2a}}{(1 - x)(1 - x^2)^2 \dots (1 - x^a)^2(1 - x^{a+1})}.$$

Then, the generating function \mathcal{F}_a can be written as $\mathcal{F}_a = \frac{P_a(x)}{(1 - x^\ell)^{2a}}$.

Let Φ_i be the i^{th} cyclotomic polynomial. From the well-known identity $(x^n - 1) = \prod_{i|n} \Phi_i$, we express \mathcal{F}_a and $(1 - x^l)^{2a}$ as product of cyclotomic polynomials. The cyclotomic polynomials appearing in \mathcal{F}_a also appear in $(1 - x^l)^{2a}$, with exponent at least equal to their exponent in \mathcal{F}_a . Then, P_a is a polynomial and it can be written as a product of cyclotomic polynomials. Moreover, $\deg(\mathcal{F}_a) = a(a + 2)$, and then, $\deg(P_a) = 2al - a(a + 2)$.

2. By (1),

$$P_a(x) = \frac{(1 - x^l)^{2a}}{(1 - x)(1 - x^2)^2 \dots (1 - x^a)^2(1 - x^{a+1})}.$$

Separating this expression into factors and studying each one, we have that, for $i = 2, \dots, a$,

$$\begin{aligned} \frac{1 - x^l}{1 - x} &= 1 + x + x^2 + \dots + x^{l-1}, \\ \frac{(1 - x^l)^2}{(1 - x^i)^2} &= \left[1 + (x^i) + (x^i)^2 + \dots + (x^i)^{\lfloor \frac{l}{i} \rfloor} \right]^2, \\ \frac{1 - x^l}{1 - x^{a+1}} &= 1 + (x^{a+1}) + (x^{a+1})^2 + \dots + (x^{a+1})^{\lfloor \frac{l}{a+1} \rfloor}. \end{aligned}$$

This is exactly the combinatorial interpretation for the coefficients of P_a appearing in Theorem 3.3.8 because each factor corresponds to the part j or \bar{j} and each one appears at most $\lfloor \frac{l}{j} \rfloor$ times.

3. By (2), the coefficients of P_a are positive and they are also palindrome because Φ_1 does not appear in P_a . In the following example we observe that they are not a concave sequence: if we consider the case $a = 2$, after a few calculations, we obtain that the polynomial P_2 is

$$\begin{aligned} P_2(x) = x^{16} + x^{15} + 3x^{14} + 4x^{13} + 7x^{12} + 9x^{11} + 10x^{10} + 13x^9 + \\ + 12x^8 + 13x^7 + 10x^6 + 9x^5 + 7x^4 + 4x^3 + 3x^2 + x + 1. \end{aligned}$$

4. This follows using Proposition 4.13 of [BS16].

5. It follows by computing $\mathcal{F}_a\left(\frac{1}{x}\right)$.

□

Example 10. We express P_a as a product of cyclotomic polynomials. For instance, $P_2 = \Phi_2^2 \Phi_3^3 \Phi_6^4$ and $P_3 = \Phi_2^3 \Phi_3^4 \Phi_4^5 \Phi_6^6 \Phi_{12}^6$.

Example 11. The coefficients $\bar{g}_{(k^2),(k^2)}^{(k)}$ are given by the quasipolynomial of degree 3 and period 6.

$$\bar{g}_{(k^2),(k^2)}^{(k)} = \begin{cases} 1/72(k+6)(k^2+6k+12) & \text{if } k \equiv 0 \pmod{6} \\ 1/72(k+5)(k^2+7k+4) & \text{if } k \equiv 1 \pmod{6} \\ 1/72(k+4)(k^2+8k+16) & \text{if } k \equiv 2 \pmod{6} \\ 1/72(k+3)(k^2+9k+12) & \text{if } k \equiv 3 \pmod{6} \\ 1/72(k+2)(k^2+10k+28) & \text{if } k \equiv 4 \pmod{6} \\ 1/72(k+1)(k^2+11k+28) & \text{if } k \equiv 5 \pmod{6} \end{cases}$$

These quasipolynomials are computed applying the binomial identity to expand $(1-x^6)^4$, and then grouping the monomials in $P_2 = \Phi_2^2 \Phi_3^3 \Phi_6^4$ according to their degree mod 6. For this, we write each number as $n = 2k + r$, with $r \in \{0, \dots, 5\}$, and we write the result in terms of the variable k .

3.3.4. Saturation hypothesis

We start defining what it is the *saturation hypothesis*.

Definition 3.3.9. Let us denote by $\{C(\alpha^1, \dots, \alpha^n)\}$ any family of coefficients depending on the partitions $\alpha^1, \dots, \alpha^n$. The family $\{C(\alpha^1, \dots, \alpha^n)\}$ satisfies the *saturation hypothesis* if the conditions $C(\alpha^1, \dots, \alpha^n) > 0$ and $C(s \cdot \alpha^1, \dots, s \cdot \alpha^n) > 0$ for all $s > 1$ are equivalent, where $s \cdot \alpha = (s \cdot \alpha_1, s \cdot \alpha_2, \dots)$.

For the family of the Littlewood–Richardson coefficients, Allen Knutson and Terence Tao in [KT99] proved that they satisfy the saturation hypothesis. Remarkably, the Kronecker coefficients do not satisfy the saturation hypothesis. In [BOR09], E. Briand, R. Orellana, and M. Rosas give the following counterexample: $g_{(n,n),(n,n)}^{(n,n)}$ is 1, if n is even, and 0 otherwise. They also prove that even a weaker version of Mulmuley’s hypothesis is not satisfied by the Kronecker coefficients.

For the reduced Kronecker coefficients, in [CHM07], Christandl, Harrow, and Mitchison show that if $\bar{g}_{\alpha\beta}^\gamma \neq 0$ and $\bar{g}_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \neq 0$, then $\bar{g}_{\hat{\alpha}+\alpha, \hat{\beta}+\beta}^{\hat{\gamma}+\gamma} \neq 0$. This implies that if $\bar{g}_{\alpha\beta}^\gamma \neq 0$, then $\bar{g}_{n\alpha, n\beta}^{n\gamma} \neq 0$ for all $n > 0$. A. N. Kirillov and A. Klyachko, in [Kly04] and [Kir04], have conjectured that the converse also holds.

We show that the of reduced Kronecker coefficients of Family 2 satisfy the saturation hypothesis using the combinatorial interpretation in terms of plane partitions given in Theorem 3.3.7.

Theorem 3.3.10. *The saturation hypothesis holds for the coefficients $\bar{g}_{(k^a),(k^a)}^{(k)}$. In fact, $\bar{g}_{((sk)^a),((sk)^a)}^{(sk)} > 0$ for all $s \geq 1$. Moreover, the sequences of coefficients obtained by, either fixing k or a , and then letting the other parameter grow are weakly increasing.*

Proof. By Theorem 3.3.7, $\bar{g}_{((sk)^a),((sk)^a)}^{(sk)}$ counts the number of plane partitions of sk fitting in a $2 \times a$ rectangle. Consider the plane partition with only one part, \boxed{sk} . There always exists this plane partition, whether it is $s \geq 1$. Then, $\bar{g}_{((sk)^a),((sk)^a)}^{(sk)} > 0$ for any $s \geq 1$.

To prove that the coefficients are weakly increasing, we show that any plane partition of k fitting in a $2 \times a$ rectangle can be identify with a plane partition of $k + 1$ fitting in the same rectangle. Consider a plane partition of k fitting in a $2 \times a$ rectangle. We identify this plane partition with the plane partition of $k + 1$ fitting in the same rectangle. The identification is as follows:

$$\begin{array}{|c|c|c|c|} \hline \overbrace{\hspace{1.5cm}}^a & & & \\ \hline \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,a} \\ \hline \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,a} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline \overbrace{\hspace{1.5cm}}^a & & & \\ \hline \alpha_{11} + 1 & \alpha_{12} & \cdots & \alpha_{1,a} \\ \hline \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,a} \\ \hline \end{array}$$

Then, the set of plane partitions of $k + 1$ fitting in a $2 \times a$ rectangle has at least as many elements as the set of plane partitions of k fitting in the same rectangle. \square

3.4. Results for Family 3

Theorem 3.3.1 and its proof give us a method that can be applied to other families. We apply it to Family 3 of reduced Kronecker coefficients $\left\{ \bar{g}_{((k+i)^b),(k^a)}^{(k)} \right\}_{k \geq 0}$, for the different values of a and b .

For $b = a$, we consider the reduced Kronecker coefficients $\bar{g}_{((k+i)^a),(k^a)}^{(k)}$. We observe the following phenomenon: after some initial zeros, the sequence defined by the non-zero reduced Kronecker coefficients is independent of i and it is equal to the sequence defined by Family 2. Let us see this phenomenon with an example. The following table shows the first cases for $a = b = 2$.

Table 3.5.: Family 3: case $a = b = 2$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
i= 0	1	1	3	4	7	9	14	17	24	29	38	45	57	66	81	93	111
i= 1	0	0	0	1	1	3	4	7	9	14	17	24	29	38	45	57	66
i= 2	0	0	0	0	0	0	1	1	3	4	7	9	14	17	24	29	38
i= 4	0	0	0	0	0	0	0	0	0	1	1	3	4	7	9	14	17
i= 5	0	0	0	0	0	0	0	0	0	0	0	0	1	1	3	4	7

Note that the sequence defined by each row is exactly the third row, when $a = 2$, in Table 3.4.

As a consequence, the results obtained for Family 2 also hold for Family 3, once we shift the initial sequence of zeros.

3.4.1. Generating function for Family 3

Theorem 3.4.1. Fix positive integers a and b . Let $d = \frac{a(a+1)}{2}$. Consider Family 3 of reduced Kronecker coefficients, $\{\bar{g}_{((k+i)^a)(k^b)}^{(k)}\}_{k,i \geq 0}$. Then,

1. For $b = a$, the generating function for the reduced Kronecker coefficients $\bar{g}_{((k+i)^a)(k^a)}^{(k)}$ with $k \geq d \cdot i$ is exactly the function defined in Theorem 3.3.1, $\mathcal{F}_{a,a}$. Otherwise, for $k < d \cdot i$, the coefficients are zero.
2. For $b = a + 1$, the generating function of the reduced Kronecker coefficients $\bar{g}_{(k+i)(k,k)}^{(k)}$ is $\mathcal{F}_{1,2} = \frac{x^i}{1-x}$, and it is $\mathcal{F}_{a+1,a} = 1$, for $a \geq 2$.
3. For the other cases, the coefficients are zero except for $i = k = 0$, in which case is 1.

Proof. Since the case $i = 0$ is included in Family 2, we suppose $i \geq 1$.

3. Using Murnaghan’s theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients

$$\bar{g}_{(k^b),((k+i)^a)}^{(k)} = g_{(n-bk,k^b),(n-a(k+i),(k+i)^a)}^{(n-k,k)} \tag{3.6}$$

with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, $N \geq \min\{(b+3)k, (a+3)k + (a+1)i\}$. We need to use different bounds for N , depending on whether $b \leq a-1$ or $b \geq a+2$, in order to have that the sequences indexing the Kronecker coefficients are partitions. Otherwise, we could not apply Theorem 3.1.4 to obtain a combinatorial interpretation of them.

For $b \leq a-1$, we take $N = (a+3)k + (a+1)i$. Substituting it in (3.6), we can express our reduced Kronecker coefficients as

$$\overline{g}_{(k^b), ((k+i)^a)}^{(k)} = g_{((a+3-b)k+(a+1)i, k^b), (3k+i, (k+i)^a)}^{((a+2)k+(a+1)i, k)}$$

By Theorem 3.1.4, they count the Kronecker tableaux of shape $(3k+i, (k+i)^a)/\alpha$ and type $((a+3-b)k+(a+1)i, k^b)/\alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq b+1$.

Since α is partition of k , $\alpha_1 \leq k$, and there are always i columns of height $a+1$ and we only have $b+1 < a+1$ different numbers. Then, we cannot fill these columns. Thus, for $b \leq a-1$, there is no possible Kronecker tableau.

For $b \geq a+2$, we consider N big enough in order to have that $(N-bk, k^b)$ and $(N-a(k+i), (k+i)^a)$ are partitions. By Theorem 3.1.4, these coefficients count the Kronecker tableaux of shape $(N-bk, k^b)/\alpha$ and type $(N-a(k+i), (k+i)^a)/\alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq a+1$.

If $\alpha = (k)$, there are k columns of height $b+1$ with one blue box. We have to fill each column with b different numbers, and we only have $a+1$. Moreover, if $\alpha \neq (k)$, there is at least one column of height $b+1$ with no blue boxes. We conclude that there is no possible Kronecker tableau.

2. Using Murnaghan's theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients $\overline{g}_{(k^{a+1}), ((k+i)^a)}^{(k)} = g_{(n-(a+1)k, k^{a+1}), (n-a(k+i), (k+i)^a)}^{(n-k, k)}$, with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, $N \geq \min\{(a+4)k, (a+3)k + (a+1)i\}$. Take $N = (a+3)k + (a+1)i$. Then,

$$\overline{g}_{(k^{a+1}), ((k+i)^a)}^{(k)} = g_{((2k+(a+1)i, k^{a+1}), (3k+i, (k+i)^a)}^{((a+2)k+(a+1)i, k)}$$

For $a=1$ and $b=2$, we count the Kronecker tableaux with shape $(2k+2i, k, k)/\alpha$ and type $(3k+i, k+i)/\alpha$, with α a partition of k and $\ell(\alpha) \leq 2$. First, note that $\alpha = (k)$. Otherwise, there is a column of height 3 that we cannot fill only with $\boxed{1}$ and $\boxed{2}$ boxes. Now, for $\alpha = (k)$, there is only the following possible Kronecker tableau

k	k	i	i
	1	1	2
1			
2			

By the condition of the reverse reading word, $(\#2)_{R1} \leq \alpha_1$, and $i \leq k$. Therefore, again because of this condition, there is no other possible semi-standard Young tableau.

For $b = a + 1$ and $a \geq 2$, $\alpha = (k)$ for the same reason. Then, we have $k + i$ $a + 1$ boxes and the $a + 1^{th}$ row has length k . Again by the condition of the reverse reading word, there is no possible Kronecker tableau.

- For $a = b$, by Murnaghan's theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients

$$\bar{g}_{(k^a), ((k+i)^a)}^{(k)} = g_{(n-ak, k^a), (n-a(k+i), (k+i)^a)}^{(n-k, k)} \tag{3.7}$$

with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, $N \geq \min\{(a + 3)k, (a + 3)k + (a + 1)i\}$. We need to use different bounds for N , depending on whether the values of k and i , in order to have that the sequences indexing the Kronecker coefficients are partitions. Otherwise, we could not apply Theorem 3.1.4 to obtain a combinatorial interpretation.

For $k < \frac{(a+1)i}{2}$, we take $N = (a+3)k + (a+1)i$. Then, applying Theorem 3.1.4, we interpret them in terms of Kronecker tableaux: $\bar{g}_{((k+i)^a)(k^b)}^{(k)}$ equals the number of Kronecker tableaux of shape $(3k+i, (k+i)^a)/\alpha$ and type $(3k+(a+1)i, k^a)/\alpha$, with α a partition of k with $\ell(\alpha) \leq a + 1$. These Kronecker tableaux have the following form

k	i	k	k
	1		
	\vdots		
	a		
	$a + 1$		
$\underbrace{\hspace{1.5cm}}_{\alpha_{a+1}}$			

As we observe in the semi-standard Young tableau drawn above, there are always $k + i - \alpha_1$ columns of height $a + 1$ that are filled with j 's in the j^{th} row, for $j = 1, \dots, a + 1$. We only have $(\alpha_1 - \alpha_j - i)$ remaining j 's numbers to put on the semi-standard Young tableau. If $i > \alpha_1 - \alpha_j$, then we do not have enough numbers to fill those $k + i - \alpha_1$ columns of height $a + 1$ and no blue boxes.

Suppose $i \leq \alpha_1 - \alpha_j$, for all $j = 1, \dots, a + 1$. The condition over the reverse reading word implies that for $j \neq 1$, the \boxed{j} box only appears in the first and in the j^{th} row. This implies that in the j^{th} row there are only $\boxed{1}$ and \boxed{j} boxes. For the $\boxed{1}$ boxes, there are at least $k - \alpha_j - (\alpha_1 - i - \alpha_j)$ of them in the j^{th} row. Then, for $j = 1, \dots, a + 1$, $(\#1)_{Rj} \geq k + i - \alpha_1$.

In order to avoid the columns $\begin{matrix} \boxed{1} \\ \boxed{1} \end{matrix}$, $\alpha_j - \alpha_{j+1} \geq k + i - \alpha_1$. This implies that $\alpha_j - \alpha_{j+1} \geq 1$, and that $k = |\alpha| \geq a \cdot i$, which is a contradiction because we are in the case $k < \frac{(a+1)i}{2}$. We conclude that for $k < \frac{(a+1)i}{2}$, the reduced Kronecker coefficient is zero. This explains the first zeros of the sequence of reduced Kronecker coefficients, once we fix i . The rest of the initial zeros are explained in the other case.

For $k \geq \frac{(a+1)i}{2}$, we take $N = (a + 3)k$. Then, applying Theorem 3.1.4, we interpret them in terms of Kronecker tableaux: $\bar{g}_{((k+i)^a)(k^b)}^{(k)}$ equals the number of Kronecker tableaux of shape $(3k, k^a)/\alpha$ and type $(3k - ai, (k + i)^a)/\alpha$, with α a partition of k with $\ell(\alpha) \leq a + 1$. Let us see a general idea about how are these Kronecker tableaux. First, we observe that for $j \in \{2, \dots, a + 1\}$ there should be $k + i - \alpha_j$ \boxed{j} boxes in total and in the j^{th} row there are only $k - \alpha_j$ boxes. Using the condition of the reverse reading word, we realize that for all $j \in \{1, \dots, a\}$, $\alpha_j - \alpha_{j+1} \geq i$. Then, we can fill i boxes in the first row with \boxed{j} . In particular, this implies that $a \leq \ell(\alpha) \leq a + 1$. Setting these conditions over all partitions, for $k < \frac{a(a+1)i}{2} = d \cdot i$ there is no valid partition. The first valid positive integer is $k = d \cdot i$ with the partition $\alpha = (ai, (a - 1)i, \dots, 2a, a)$. That is why we have the rest of initial zeros.

Let $k \geq d \cdot i$. First, consider the translation $k \mapsto k + d \cdot i$, that makes the initial zeros disappear. Therefore, we consider the reduced Kronecker coefficients

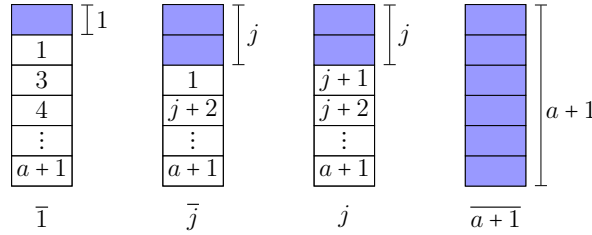
$$\left\{ \bar{g}_{((k+(d+1)i)^a), ((k+d \cdot i)^a)}^{(k+d \cdot i)} \right\}_{k, i \geq 0}. \tag{3.8}$$

One more time, we express them in terms of Kronecker coefficients using the Murnaghan's theorem and taking $N = (a + 3)d + (a + 1)di$. Applying Theorem 3.1.4, our reduced Kronecker coefficients count the number of Kronecker tableaux of shape $\lambda/\alpha = (3k + d \cdot i, (k + d \cdot i)^a)/\alpha$, type $\nu/\alpha = (3k + (d - a)i, (k + (d + 1)i)^a)/\alpha$ and associated partition $\alpha \vdash k + d \cdot i$, with $\ell(\alpha) \leq a + 1$.

To prove that their generating function is $\mathcal{F}_{a,a}$ we proceed as in the proof of Theorem 3.3.1: we establish a bijection between coloured partitions of k with

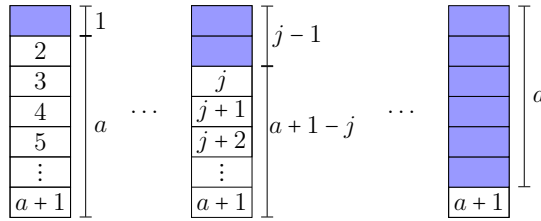
parts in $\mathcal{B}_a = \{\overline{1}, 2, \overline{2}, \dots, a, \overline{a}, \overline{a+1}\}$, and Kronecker tableaux of shape λ/α , type ν/α and associated partition $\alpha \vdash k + d \cdot i$, with $\ell(\alpha) \leq a + 1$.

The bijection is defined by the following algorithm. To a coloured partition of k with parts in \mathcal{B}_a , we associate a Kronecker tableau $T(\beta)$ as follows. First, we identify each element of \mathcal{B}_a to a column of height $a + 1$. It is the same identification than for Family 2 in Theorem 3.3.1:



for $j \in \{2, 3, \dots, a - 1, a\}$. If we write β as $(\overline{1}^{m_1} \overline{2}^{m_2} 2^{m_2} \dots \overline{a+1}^{m_{a+1}})$, then m_i will denote the number of times that the column i appears in the semi-standard Young tableau that we are building.

We continue adding the following columns, i times each one:



for $j \in \{3, \dots, a\}$. Note that it is always possible to order the columns corresponding to the parts of β and these last extra columns in such a way that we obtain a semi-standard Young tableau.

We read the partition α from our semi-standard Young tableau by counting the number of blue boxes in each row: $\alpha_{a+1} = m_{\overline{a+1}}$, $\alpha_j = \alpha_{j+1} + m_j + m_{\overline{j}} + i$, for $j = 2, \dots, a$, and $\alpha_1 = \alpha_2 + m_{\overline{1}} + i$. These columns are the left columns of $T(\beta)$. We build the rest of $T(\beta)$ as follows: complete j^{th} row with \boxed{j} boxes, for $j = 2, \dots, a + 1$, and complete the first row with the remaining numbers of $(3k + (3n - a)i, (k + (d + 1)i)^a) / \alpha$ in weakly increasing order from left to right.

For instance, take $a = 3$, $k = 6$ and $i = 1$ and consider $\beta = (\overline{4}, \overline{2}) \vdash 6$. Applying the algorithm defined above, we obtain the following Kronecker tableau of shape $(36, 12, 12, 12) / \alpha$ and type $(33, 13, 13, 13) / \alpha$, with $\alpha = (5, 4, 2, 1) \vdash 12$:

- The reverse reading word is an α -lattice permutation.

The reverse reading word of $T(\beta)$ is of the form:

$$(\#a+1)_{R1} \cdots (\#1)_{R1} (\#2)_{R2} (\#1)_{R2} \cdots \\ \cdots (\#i)_{Ri} (\#1)_{Ri} \cdots (\#a+1)_{Ra+1} (\#1)_{Ra+1},$$

where we regroup boxes with the same label in the same row. Furthermore, we only need to check the inequalities at the end of each group. We check them from left to right in the reverse reading word.

- At the level of the first row, we check that $\alpha_{j+1} + (\#h+1)_{R1} \leq \alpha_h$: for $j = 2, \dots, a$,

$$\alpha_{j+1} + (\#j+1)_{R1} = \alpha_{j+1} + m_{\bar{j}} \leq \alpha_{j+1} + m_j + m_{\bar{j}} + i = \alpha_j.$$

For $i = 1$, $(\#2)_{R1} + \alpha_2 = \alpha_2 + m_{\bar{1}} \leq \alpha_2 + m_{\bar{1}} + i = \alpha_1$.

- At the level of the second row, we check that

$$\alpha_2 + (\#2)_{R1} + (\#2)_{R2} \leq \alpha_1 + (\#1)_{R1}.$$

The left-hand side is exactly $k + d \cdot i$, because it sums the total number of $\boxed{2}$ plus α_2 . Finally, by the previous item, $(\#1)_{R1} \geq k + d \cdot i - \alpha_1$.

- At the other levels, fix the j^{th} row. The box \boxed{l} , with $l \neq 1$, only appears in the first row and in the l^{th} row. Let us check that

$$\alpha_{j+1} + (\#j+1)_{R1} + (\#j+1)_{Rj} \leq \alpha_j + (\#j)_{R1},$$

for $j = 2, \dots, a$. Both sides of this inequality are equal, because they are exactly the total number of \boxed{i} boxes plus α_i , for $i = j$ and $j + 1$, i.e. $k + d \cdot i$.

- We check the α -condition.

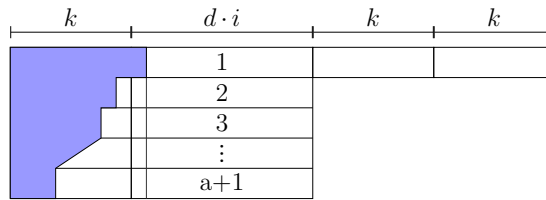
For $\alpha_1 = \alpha_2$, we do not need to check anything. For $\alpha_1 > \alpha_2$, we have that

$$(\#2)_{R1} = \underbrace{k + (d+1)i - \alpha_2}_{\text{total}} - \underbrace{(k + di - \alpha_2 - m_{\bar{1}})}_{2^{\text{nd}} \text{ row}} = i - m_{\bar{1}} = \alpha_1 - \alpha_2$$

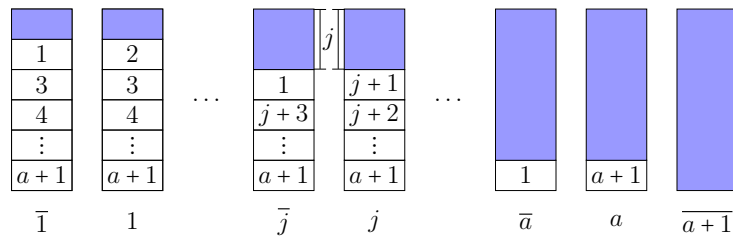
Therefore, $(\#2)_{R1} = \alpha_1 - \alpha_2$.

Then, the result of the algorithm is a Kronecker tableau and the map is well-defined. By the algorithm, the map is injective. We will show that it is also surjective.

Consider a Kronecker tableau, T , such that it has shape $(3k + di, (k + di)^a) / \alpha$, type $(3k + (d - a)i, (k + (d + 1)i)^a) / \alpha$ and associated partition $\alpha \vdash k + di$, with $\ell(\alpha) \leq a + 1$. We will define its corresponding coloured partition β of k with parts in \mathcal{B}_a . The Kronecker tableau T has the following form



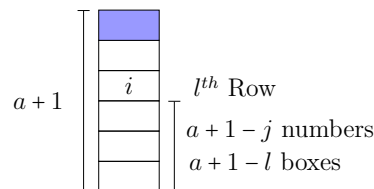
Excluding the columns of height 1 and $a + 1$ with no blue boxes, we claim that the following list summarizes all possible columns that appear in the remaining part of T .



We denote by n_j , with $n \in \mathcal{B}_a \cup \{1\}$, the number of columns of kind j that appear in T .

First, we prove that there are no other kind of columns of height $a + 1$ with blue boxes.

- (i) For $j = 2, \dots, a + 1$, the box \boxed{j} cannot appear in any row $l \leq j - 1$.
 Suppose that there is a column of the form



Since T is a semi-standard Young tableau, we cannot fill the $a + 1 - l$ boxes with different numbers because we only have $a + 1 - j$ possible numbers.

- (ii) For $j = 2, \dots, a$, the box \boxed{j} cannot appear in any row $l \geq j + 1$. We prove it by induction from a to 2 . Looking at the end of the reverse reading word

$$(\#a + 1)_{R1} + \alpha_{a+1} \leq \alpha_a + (\#a)_{R1} + (\#a)_{Ra}.$$

The left-hand side is exactly $k + (d+1)i$ and in total there are $k + (d+1)i - \alpha_a$ \boxed{a} boxes. Then, there are no more boxes than in the first row and in the a^{th} row.

Let us see that there cannot be \boxed{j} boxes in the $(j + 1)^{\text{th}}$ row, assuming that in any l^{th} row, with $l \geq j + 2$, there are only $\boxed{1}$ and \boxed{l} boxes. The part of the reverse reading word corresponding to the $(j + 1)^{\text{th}}$ row says that

$$(\#j + 1)_{R1} + (\#j + 1)_{Rj+1} + \alpha_{j+1} \leq \alpha_j + (\#j)_{R1} + (\#j)_{Rj}.$$

Applying the induction hypothesis, left-hand side is exactly $k + (d + 1)i$. Then, $(\#j)_{R1} + (\#j)_{Rj} \geq k + (d + 1)i - \alpha_j$. Since there are $k + (d + 1)i - \alpha_j$ \boxed{i} boxes in total, we also have that $(\#j)_{R1} + (\#j)_{Rj} \leq k + (d + 1)i - \alpha_j$. Thus, there are no \boxed{j} boxes in any row different from the first one and the j^{th} .

Let us see that there are, at least, i columns of type j , for $j = 1, \dots, a$. In the first row, by the condition of the reverse reading word, $(\#j)_{R1} \leq \alpha_j - \alpha_{j+1} = n_j + n_{\bar{j}}$. We can also count the $(\#j)_{R1}$ as the total number of j 's minus the number of j 's in the j^{th} row. Since in the j^{th} row there are only $\boxed{1}$ and \boxed{j} boxes, we count $(\#j)_{Rj}$ as the length of the j^{th} row minus $n_{\bar{j}}$. Then,

$$(\#j)_{R1} = \underbrace{k + (d + 1)i - \alpha_j}_{\text{total}} - \underbrace{(k + di - \alpha_j - m_{\bar{j}})}_{j^{\text{th}} \text{ row}} = i + m_{\bar{j}} \leq m_j + m_{\bar{j}},$$

which implies that $m_j \geq i$. Moreover, $\alpha_1 - \alpha_2 \geq i \neq 0$, and we are always in the case $\alpha_1 > \alpha_2$ of the α -condition. Furthermore, the case $(\#1)_{R2} = \alpha_1 - \alpha_2$ is not possible. Otherwise, there are $k + di - \alpha_1$ $\boxed{2}$ in the second row and $i + \alpha_1 - \alpha_2$ $\boxed{2}$ in the first row, which is a contradiction with the fact that $(\#2)_{R1} \leq \alpha_1 - \alpha_2$, by the condition of the reverse reading word. In particular, for $j = 1$, $m_j = i$.

We define the coloured partition β of k , with parts in \mathcal{B}_a .

$$\beta = (\overline{1}^{n_1} \overline{2}^{n_2} \overline{2}^{n_2-i} \overline{3}^{n_3} \overline{3}^{n_3-i} \dots \overline{a}^{n_a} \overline{a}^{n_a-i} \overline{a+1}^{n_{a+1}}).$$

If we denote by m_j , with $j \in \mathcal{B}_a$, the number of j in the sequence β , then $m_{\bar{j}} = n_{\bar{j}}$, for $j = 1, \dots, a+1$, and $m_j = n_j - i$ j parts, for $j = 2, \dots, a$.

Finally, we check that the sequence β is a partition of k . By the definition of β ,

$$|\beta| = \sum_{j=1}^{a+1} j \cdot n_{\bar{j}} + \sum_{j=2}^a j \cdot (n_j - i) = \sum_{j=1}^{a+1} j \cdot n_{\bar{j}} + \sum_{j=2}^a j \cdot n_j - i \cdot \frac{a(a-1)}{2}.$$

Since α is a partition of $k + di$ and it is also the total number of blue boxes, we have that

$$k + di = \sum_{j=1}^{a+1} j \cdot n_{\bar{j}} + \sum_{j=1}^a j \cdot n_j = \sum_{j=1}^{a+1} j \cdot n_{\bar{j}} + \sum_{j=2}^a j \cdot n_j + i.$$

Recalling that $d = \frac{a(a+1)}{2}$, $|\beta| = k + di - i - i \cdot \frac{a(a-1)}{2} = k$. This is the end of the proof. □

3.4.2. Consequences

In Theorem 3.4.1 we have seen the relation between the reduced Kronecker coefficients $\bar{g}_{(k^b), (k^a)}^{(k)}$ and $\bar{g}_{((k+i)^b), (k^a)}^{(k)}$. For $b = a$, once we fix i and we set $d = \frac{a(a+1)}{2}$, the reduced Kronecker coefficients $\bar{g}_{((k+(d+1)i)^a), ((k+d-i)^a)}^{(k+d-i)}$ equals to $\bar{g}_{(k^a), (k^a)}^{(k)}$, for $k \geq 0$. Thus, we can describe these reduced Kronecker coefficients of Family 3 in terms of reduced Kronecker coefficients of Family 2, and not depending on i . This implies that any result obtained for Family 2 of reduced Kronecker coefficients, is also satisfied by the shifted Family 3. We summarize the results in the following corollaries.

We start with the combinatorial interpretation in terms of plane partitions.

Corollary 3.4.2. *The reduced Kronecker coefficient $\bar{g}_{((k+i)^a), (k^a)}^{(k)}$ for $k \geq \frac{a(a+1)}{2}$ counts the number of plane partitions of k fitting inside a $2 \times a$ rectangle.*

We continue with the saturation hypothesis.

Corollary 3.4.3. *The reduced Kronecker coefficients $\bar{g}_{((k+(d+1)i)^a), ((k+d-i)^a)}^{(k+d-i)}$ satisfy the saturation hypothesis. Moreover, the sequences of coefficients obtained by fixing a and letting k grow, are weakly increasing.*

Note that the sequence once we fix k and we let a grow is not weakly increasing, because the number of initial zeros depends on a .

We finish this subsection with the analogue of the *Item 4* of Theorem 3.3.8.
Corollary 3.4.4. *The coefficients $\bar{g}_{((k+i)^a),(k^a)}^{(k)}$, for $k \geq \frac{a(a+1)}{2}$, are described by a quasipolynomial on k of degree $2a - 1$ and period dividing the least common multiple of $1, 2, \dots, a, a + 1$.*

3.5. Results for Family 4

The fourth family of reduced Kronecker coefficients that we include in this study is $\left\{ \bar{g}_{(k^b)(k+i,k^a)}^{(k)} \right\}_{k,i \geq 0}$. For this family, the most interesting case is when $b = a + 1$.

The table 3.6 shows what happens for the cases $a = 2$ and $b = 3$.

Table 3.6.: Family 4: case $a = 2$ and $b = 3$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	OEIS
i= 0	1	1	3	4	7	9	14	17	24	29	38	45	57	66	A008763
i= 1	0	1	2	4	7	11	16	23	31	41	53	67	83	102	A000601
i= 2	0	0	1	2	5	8	14	20	30	40	55	70	91	112	A006918
i= 3	0	0	0	1	2	5	9	15	23	34	47	64	84	108	A014126
i= 4	0	0	0	0	1	2	5	9	16	24	37	51	71	93	
i= 5	0	0	0	0	0	1	2	5	9	16	25	38	54	75	A175287
i= 6	0	0	0	0	0	0	1	2	5	9	16	25	39	55	

If we look at the sequence formed by the numbers in bold, we see that each diagonal stabilizes. In this case, we give the generating function of the *stable value of the diagonals*. Using this generating function, we can study the sequence of reduced Kronecker coefficients formed by the stable values of the diagonals as we do for Family 2: their relation with plane partitions, their interpretation in terms of quasipolynomials and saturation hypothesis.

3.5.1. Generating function for Family 4

Theorem 3.5.1. *Fix positive integers a and b , and consider the sequence of reduced Kronecker coefficients $\left\{ \bar{g}_{(k^b)(k+i,k^a)}^{(k)} \right\}_{k,i \geq 0}$.*

1. For $b = a$, the reduced Kronecker coefficient $\bar{g}_{(k^b)(k+i,k^a)}^{(k)}$ is zero, except for $i = 0$, in which case is 1, for all k .
2. For $b = a + 1$, the stable value of the j^{th} diagonal corresponds to the reduced Kronecker coefficients $\bar{g}_{(k^a),(2k-j,k^{a-1})}^{(k)}$ for $k \geq 2j$, and their generating function is

$$\mathcal{G}_a = \frac{1}{(1-x)^2(1-x^2)^3 \dots (1-x^{a-1})^3(1-x^a)^2(1-x^{a-1})}.$$

Particularly, when $a = 1$, $\mathcal{G}_1 = \frac{1}{(1-x)(1-x^2)}$, and when $a = 2$, there are no terms like $(1-x^j)^3$.

3. For $b = a + 2$, the reduced Kronecker coefficient $\bar{g}_{(k^b)(k+i,k^a)}^{(k)}$ is 1 for $k \geq i$, and 0 otherwise.
4. For all other cases, the coefficients are zero except for $i = k = 0$, in which case they are 1.

Proof. Since the case $i = 0$ is included in Family 2, we suppose $i \geq 1$.

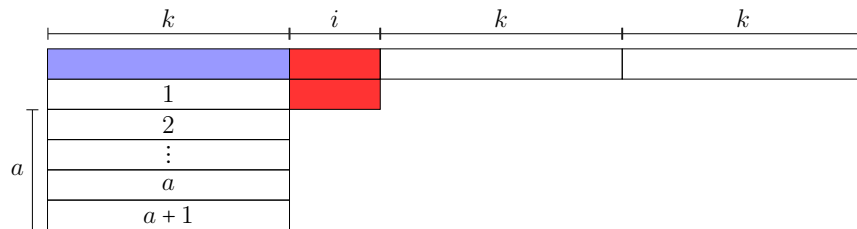
1. For $a = b$, by Murnaghan's theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients $\bar{g}_{(k^a),(k+i,k^a)}^{(k)} = g_{(n-ak,k^a),(n-(a+1)k-i),k+i,k^a}^{(n-k,k)}$, with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, $N \geq \min\{(a + 3)k, (a + 4)k + 2i\}$. We take $N = (a + 4)k + 2i$. Then, our reduced Kronecker coefficients can be written as

$$\bar{g}_{(k^a),(k+i,k^a)}^{(k)} = g_{(4k+2i,k^a),(3k+i,k+i,k^a)}^{((a+3)k+2i,k)}.$$

By Theorem 3.1.4, they count the Kronecker tableaux of shape $(3k + i, k + i, k^a) / \alpha$ and type $(4k + 2i, k^a) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq a + 1$.

If $\alpha \neq (k)$, there is at least one column of height $a + 2$ that we cannot fill. Then, $\alpha = (k)$, and we have the following Kronecker tableau:



Note that the columns of height $a + 2$ with one blue box cannot be filled in a different way. Furthermore, after filling these columns, all the remaining numbers according to the type are 1. This implies that we cannot fill the red columns, and that there are no possible Kronecker tableaux.

3. For $b = a + 2$, by Murnaghan’s theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients $\bar{g}_{(k^{a+2}), (k+i, k^a)}^{(k)} = g_{(n-(a+2)k, k^{a+2}), (n-(a+1)k-i, k+i, k^a)}^{(n-k, k)}$, with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, we can take $N = (a + 4)k + 2i$. Then, our reduced Kronecker coefficients can be written as

$$\bar{g}_{(k^{a+2}), (k+i, k^a)}^{(k)} = g_{(2k+2i, k^{a+2}), (3k+i, k+i, k^a)}^{((a+3)k+2i, k)}$$

By Theorem 3.1.4, they count the Kronecker tableaux of shape $(2k + 2i, k^{a+2}) / \alpha$ and type $(3k + i, k + i, k^a) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq a + 2$.

If $\alpha \neq (k)$, there is at least one column of height $a + 3$ that we cannot fill. Then, $\alpha = (k)$. In fact, we have the following Kronecker tableau

	k	k	i	i
		1	1	2
$a + 2$	1			
	2			
	\vdots			
	a			
	$a + 1$			

This is the only possible Kronecker tableau, because of the type of the semi-standard Young tableau and the condition of the reverse reading word. Moreover, the reverse reading word condition implies that $(\#2)_{R1} \leq \alpha_1$, and that $i \leq k$. Otherwise, there is no possible Kronecker tableau and the reduced Kronecker coefficient is zero.

4. By Murnaghan’s theorem, we can express the reduced Kronecker coefficients as Kronecker coefficients

$$\bar{g}_{(k^b), (k+i, k^a)}^{(k)} = g_{(n-bk, k^b), (n-(a+1)k-i, k+i, k^a)}^{(n-k, k)}, \tag{3.9}$$

with $n \geq N$, for some N .

By (3.1) and Corollary 3.1.1, $N \geq \min\{(b + 3)k, (a + 4)k + 2i\}$. We need to use different bounds for N , depending on whether $b < a$ or $b > a + 2$, in order to have that the sequences indexing the Kronecker coefficients are partitions.

Otherwise, we could not apply Theorem 3.1.4 to obtain a combinatorial interpretation.

For $a > b$, we take $N = (a + 4)k + 2i$. Then, we can express our reduced Kronecker coefficients as

$$\bar{g}_{(k^b), (k+i, k^a)}^{(k)} = g_{((a+4-b)k+2i, k^b), (3k+i, k+i, k^a)}^{((a+3)k+2i, k)}.$$

By Theorem 3.1.4, they count the Kronecker tableaux of shape $(3k + i, k + i, k^a) / \alpha$ and type $((a + 4 - b)k + 2i, k^b) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq b + 1$. There is always a column of height $a + 1$ or $a + 2$, depending on whether $\alpha_1 = k$ or $\alpha_1 < k$, that we cannot fill. Then, there is no possible Kronecker tableau for all these cases.

For $b > a + 2$, we consider N big enough in order to have that $(N - bk, k^b)$ and $(N - (a + 1)k - i, k + i, k^a)$ are partitions. By Theorem 3.1.4, these coefficients count the Kronecker tableaux of shape $(N - bk, k^b) / \alpha$ and type $(N - (a + 1)k - i, k + i, k^a) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \geq a + 2$. Since $N - bk \geq k \geq \alpha_1$, for any α , there is at least one column of height b or $b + 1$ that we cannot fill with $a + 2$ different numbers. Then, there are no possible Kronecker tableaux.

2. For $b = a + 1$, we look at the element of the j^{th} diagonal. Fix positive integers a and j . The j^{th} diagonal is describe by the coefficients $\bar{g}_{(k^a), (2k-j, k^{a-1})}^{(k)}$, with $j = k - i \geq 0$ and $k \geq 2j$.

By (3.1) and Corollary 3.1.1, we can take $N = (a + 4)k$, and these coefficients can be written as

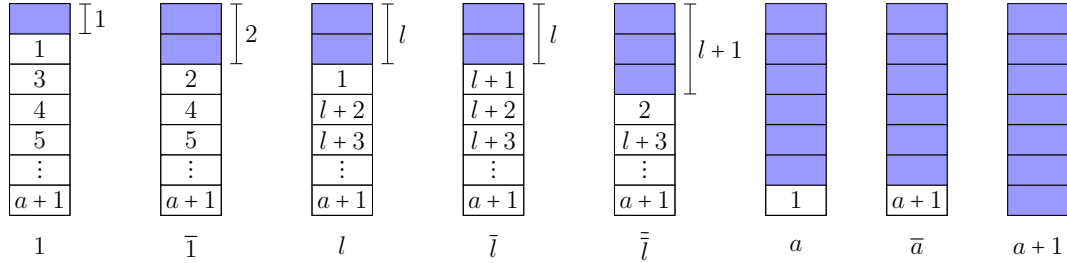
$$\bar{g}_{(k^a), (2k-j, k^{a-1})}^{(k)} = g_{(3k, k^a), (2k+j, 2k-j, k^{a-1})}^{((a+2)k, k)}.$$

Applying Theorem 3.1.4, these coefficients count the Kronecker tableaux of shape $(3k, k^a) / \alpha$ and type $(2k + j, 2k - j, k^{a-1}) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq a + 1$.

The function \mathcal{G}_a is the generating function of the coloured partitions with parts in $\mathcal{C}_a = \{1, \bar{1}, 2, \bar{2}, \bar{\bar{2}}, \dots, a - 1, \overline{a - 1}, \overline{\overline{a - 1}}, a, \bar{a}, a + 1\}$, with the following order $\bar{1} < 1 < \bar{\bar{2}} < \bar{2} < 2 < \dots < a - 1 < \bar{a} < a < a + 1$, and for which the parts i, \bar{i} and $\bar{\bar{i}}$ have weight i .

To prove that Theorem 3.5.1 holds, we give a bijective map between coloured partitions with parts in \mathcal{C}_a and Kronecker tableaux of shape $(3k, k^a) / \alpha$ and type $(2k + j, 2k - j, k^{a-1}) / \alpha$, with $\alpha \vdash k$ and $\ell(\alpha) \leq a + 1$.

The bijection is defined by the following algorithm: to a coloured partition β of j with parts in \mathcal{C}_a , we associate a Kronecker tableau $T(\beta)$ as follows. First, we identify each element of \mathcal{C}_a with a column of height $a + 1$:



for $l = 2, \dots, a - 1$. The partition α of k is defined by counting all blue boxes on the Kronecker tableau $T(\beta)$. If we consider $\beta = (\bar{1}^j)$, we have $2j$ blue boxes. Then, $k \geq 2j$. We consider partitions α of k . Note that for $\beta = (1^j)$, there are only j blue boxes, which is not enough to obtain α . That is why, the next step is to add as many columns as blue boxes we need in order to obtain a partition of k . Until that moment, we have the following number of blue boxes

$$\sum_{l=1}^{a+1} l \cdot m_l + \sum_{l=2}^a l \cdot m_{\bar{l}} + 2m_{\bar{1}} + \sum_{l=2}^{a-1} (l + 1) \cdot m_{\bar{l}}.$$

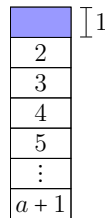
We also know that β is a coloured partition of j ,

$$j = \sum_{l=1}^{a+1} l \cdot m_l + \sum_{l=1}^a l \cdot m_{\bar{l}} + \sum_{l=2}^{a-1} l \cdot m_{\bar{l}}. \tag{3.10}$$

Let us define m_0 as k minus the number of blue boxes that we already have. Then,

$$m_0 = k - j - m_{\bar{1}} - \sum_{l=2}^{a-1} m_{\bar{l}}.$$

We add m_0 columns of height $a + 1$ equal to the following column



Denote by m_l , with $l \in \mathcal{C}_a$, the number of times that the part l appears in β . Then, α is defined by counting all the blue boxes:

$$\begin{aligned} \alpha_{a+1} &= m_{a+1}, \\ \alpha_l &= \alpha_{l+1} + m_l + m_{\bar{l}} + m_{\overline{l-1}}, \quad \text{for } l = 3, \dots, a, \\ \alpha_2 &= \alpha_3 + m_2 + m_{\bar{2}} + m_{\bar{1}}, \\ \alpha_1 &= \alpha_2 + m_1 + m_0. \end{aligned} \tag{3.11}$$

Note that there exists only a way to order all these columns in such a way that they form a semi-standard Young tableau. These columns correspond to the first columns on the left side of $T(\beta)$. The rest of $T(\beta)$ is built as follows: the l^{th} row is filled with l , for $l = 2, \dots, a + 1$, and the first row is filled with the remaining numbers of the type $(2k + j, 2k - j, k^{a-1})/\alpha$ in weakly increasing order from left to right.

For instance, take $a = 3$, $j = 3$ and the coloured partition $\beta = (\bar{2}, 1)$. Since $k \geq 2j$, we take $k = 7$. Then, the corresponding Kronecker tableau obtained by our algorithm is

						1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	4	
	1	2	2	2	2	2																
	3	3	3	3	3	3																
	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

Let us check that $T(\beta)$ is a Kronecker tableau, and hence, that the map is well-defined.

- By construction, $T(\beta)$ is a semi-standard Young tableau of shape $(3k + i, k^a)/\alpha$ and type $(2k + j, 2k - j, k^{a-1})/\alpha$, where α is a partition of k and $\ell(\alpha) \leq a + 1$.
- The sequence α defined by counting the blue boxes is a partition of k . By the recurrence (3.11) that describes α , the sequence is clearly a partition. Let us see that the sum of its parts is k . We express α in terms of m_l , with $l \in \mathcal{C}_a \cup \{0\}$.

$$\begin{aligned} \alpha_{a+1} &= m_{a+1}, \\ \alpha_l &= \sum_{k=l}^{a+1} m_k + \sum_{k=l}^a m_{\bar{k}} + \sum_{k=l}^{a-1} m_{\overline{k-1}}, \quad \text{for } l = 3, \dots, a, \\ \alpha_2 &= \sum_{k=2}^{a+1} m_k + \sum_{k=1}^a m_{\bar{k}} + \sum_{k=2}^{a-1} m_{\overline{k}}, \end{aligned}$$

and

$$\alpha_1 = k - j + \sum_{k=1}^{a+1} m_k + \sum_{k=2}^a m_{\bar{k}},$$

where we use directly the definition of m_0 to compute α_1 . Then,

$$\begin{aligned} \sum_{l=1}^{a+1} \alpha_l &= m_{a+1} + \sum_{l=3}^a \left(\sum_{k=l}^{a+1} m_k + \sum_{k=l}^a m_{\bar{k}} + \sum_{k=l}^{a-1} m_{\overline{\overline{k-1}}} \right) + \sum_{k=2}^{a+1} m_k + \sum_{k=1}^a m_{\bar{k}} + \sum_{k=2}^{a-1} m_{\bar{k}+} \\ &+ k - j + \sum_{k=1}^{a+1} m_k + \sum_{k=2}^a m_{\bar{k}} = k - j + \sum_{l=1}^{a+1} l \cdot m_l + \sum_{l=1}^a l \cdot m_{\bar{l}} + \sum_{l=2}^{a-1} l \cdot m_{\bar{l}} = k. \end{aligned}$$

- We need to check also that $(\#1)_{R1} \geq k - \alpha_1$. Otherwise, we have a column that we cannot fill.

We count $(\#1)_{R1}$ as the total number of 1's minus the $\boxed{1}$ boxes in all rows different from the first one, i.e. $(\#1)_{R1} = 2k + j - \alpha_1 - \sum_{l=1}^a m_l$. Then, by (3.10), $k + j - \sum_{l=1}^a m_l \geq 0$.

- The reverse word is an α -lattice permutation.

The reverse reading word of $T(\beta)$ is of the form

$$\begin{aligned} &(\#a+1)_{R1} \dots (\#1)_{R1} (\#2)_{R2} (\#1)_{R2} \dots \\ &\dots (\#l)_{Rl} (\#2)_{Rl} (\#1)_{Rl} \dots (\#a+1)_{Ra+1} (\#2)_{Ra+1} (\#1)_{Ra+1} \cdot \end{aligned}$$

We proceed to check that this sequence is an α -lattice permutation following the sequence from left to right.

- At the level of the first row, we check that $\alpha_{l+1} + (\#l+1)_{R1} \leq \alpha_l$.

For $l = 3, \dots, a$, we have that $(\#l+1)_{R1} = (\#1)_{Rl+1} + (\#2)_{Rl+1}$. We take the count as the total number of $(l+1)$'s minus the number of $\boxed{l+1}$ boxes of the $(l+1)^{th}$. Then,

$$\alpha_{l+1} + (\#l+1)_{R1} = \alpha_{l+1} + m_l + m_{\overline{\overline{l-1}}} \leq \alpha_{l+1} + m_l + m_{\overline{\overline{l-1}}} + m_{\bar{l}} = \alpha_l.$$

For $l = 2$, counting in the same way, we have that $(\#3)_{R1} = m_2 + m_{\bar{1}}$. Then,

$$\alpha_3 + (\#3)_{R1} = \alpha_2 + m_2 + m_{\bar{1}} \leq \alpha_2 + m_2 + m_{\bar{2}} + m_{\bar{1}} = \alpha_2.$$

For $l = 1$, we count $(\#2)_{R1}$.

$$(\#2)_{R1} = \underbrace{2k - j - \alpha_2}_{\text{total}} - \underbrace{(m_0 + k - \alpha_1)}_{2^{\text{nd}} \text{ row}} - \underbrace{m_{\bar{1}} - \sum_{l=2}^{a-1} m_{\bar{l}}}_{\text{other rows}} = \alpha_1 - \alpha_2. \quad (3.12)$$

Then, $\alpha_2 + (\#2)_{R1} = \alpha_1$.

– At the level of the second row, we check that $\alpha_2 + (\#2)_{R1} + (\#2)_{R2} = \alpha_1 + (\#1)_{R1}$.

By (3.12), we only need to check that $(\#2)_{R2} \leq (\#1)_{R1}$. Since $(\#2)_{R2} = m_0 + k - \alpha_1$ and $(\#1)_{R1} \geq 3k - \alpha_1$, by the definition of m_0 , we have that $2k - m_0 = k + j + \sum_{l=2}^{a-1} m_{\bar{l}} + m_{\bar{1}} \geq 0$. This implies that

$$(\#2)_{R2} = m_0 + k - \alpha_1 \leq 3k - \alpha_1 \leq (\#1)_{R1}.$$

– At the level of the l^{th} row, for $l = 3, \dots, a$, we have that

$$\alpha_{l+1} + (\#l+1)_{R1} + (\#l+1)_{Rl+1} = \alpha_l + (\#l)_{R1} + (\#l)_{Rl},$$

since the left-hand side is exactly the total number of $(l+1)$ plus α_{l+1} and the right-hand side is exactly the total number of l plus α_l . Thus, both sides are equal to k .

For $l = 2$, we check that $\alpha_3 + (\#3)_{R1} + (\#3)_{R3} \leq \alpha_2 + (\#2)_{R1} + (\#2)_{R2}$. The left-hand side is exactly the total number of 3's plus α_3 , i.e. k . The right-hand side is $k + m_0$, using (3.12) and that $(\#2)_{R2} = m_0 + k - \alpha_1$.

Finally, we check all the inequalities related to the boxes $\boxed{2}$ and $\boxed{1}$ involving more rows than the first and the second ones, i.e. for $s = 3, \dots, a$, we check that $\alpha_2 + \sum_{l=1}^s (\#2)_{Rl} \leq \alpha_1 + \sum_{l=1}^{s-1} (\#1)_{Rl}$. We have that

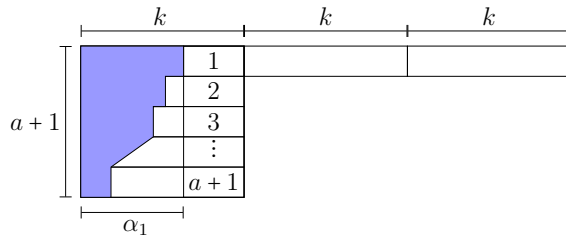
$$\begin{aligned} \sum_{l=1}^s (\#2)_{Rl} &= 2k - j - \alpha_2 - \sum_{l=s+1}^{a+1} (\#2)_{Rl} = 2k - j - \alpha_2 - \sum_{l=s}^{a-1} m_{\bar{l}}, \\ \sum_{l=1}^{s-1} (\#1)_{Rl} &= 2k + j - \alpha_1 - \sum_{l=s}^{a+1} (\#1)_{Rl} = 2k + j - \alpha_1 - \sum_{l=s}^a m_l. \end{aligned}$$

Then, the inequality $\alpha_2 + \sum_{l=1}^s (\#2)_{Rl} \leq \alpha_1 + \sum_{l=1}^{s-1} (\#1)_{Rl}$ follows from (3.10) and the fact that $2j + \sum_{l=s}^{a-1} m_{\bar{l}} - \sum_{l=s}^a m_l \geq 0$.

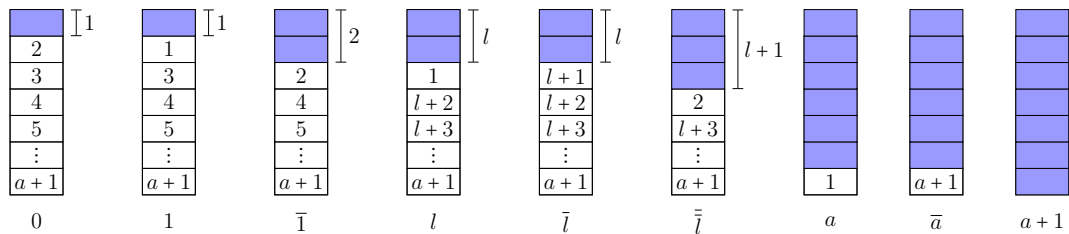
- For the α -condition, if $\alpha_1 = \alpha_2$, we have nothing to prove, and if $\alpha_1 > \alpha_2$, we have that $(\#2)_{R1} = \alpha_1 - \alpha_2$ by (3.12).

Then, the semi-standard Young tableau $T(\beta)$ defined by the algorithm is a Kronecker tableau and the map is well-defined and injective.

Finally, we show that the map is also surjective. Consider a Kronecker tableau T with shape $\lambda/\alpha = (3k, k^a)/\alpha$, type $\nu/\alpha = (2k + j, 2k - j, k^{a-1})/\alpha$ and α a partition of k , with $\ell(\alpha) \leq a + 1$. We will define the associated coloured partition β of j with parts in \mathcal{C}_a . We start studying our initial Kronecker tableau, T . It has the following form



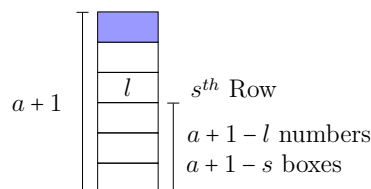
Excluding all columns of height 1 and of height $a + 1$ with no blue boxes, we claim that the following list summarizes all possible columns that appear in the remaining part of T .



for $l = 2, \dots, a - 1$.

Let us prove that there are no other kinds of columns of height $a + 1$ with blue boxes:

- For $l = 3, \dots, a + 1$, the box \boxed{l} cannot appear in any row $s \leq l - 1$. Suppose that there is a column of the form



Since T is a semi-standard Young tableau we cannot fill the $a + 1 - s$ boxes of this column with different numbers, because we only have $a + 1 - l$ possibilities.

- For $l = 3, \dots, a$, the box \boxed{l} cannot neither appear in any row $s \geq l + 1$. We proceed by induction from a to 3.

Look at the end of the reverse reading word

$$(\#a + 1)_{R1} + \alpha_{a+1} \leq \alpha_a + (\#a)_{R1} + (\#a)_{Ra}.$$

The left-hand side is exactly k and there are $k - \alpha_a$ \boxed{a} boxes in total. Then, there are no more boxes than in the first row and in the a^{th} row. Let us see that there cannot be \boxed{l} boxes in the $(l + 1)^{\text{th}}$ row, assuming that in any s^{th} row, with $s \geq l + 2$, there are only $\boxed{1}$ and \boxed{s} boxes. The part of the reverse reading word corresponding to the $(l + 1)^{\text{th}}$ row says that

$$(\#l + 1)_{R1} + (\#l + 1)_{Rl+1} + \alpha_{l+1} \leq \alpha_l + (\#l)_{R1} + (\#l)_{Rl}.$$

Applying the induction hypothesis, the left-hand side is exactly k . Then, $(\#l)_{R1} + (\#l)_{Rl} \geq k - \alpha_l$. Since there are $k - \alpha_l$ \boxed{l} boxes in total, we also have that $(\#l)_{R1} + (\#l)_{Rl} \leq k - \alpha_l$, and there are no \boxed{l} boxes in any row different from the first and the l^{th} ones.

- The boxes $\boxed{1}$ and $\boxed{2}$ can appear in all the rows, since there are $2k + j - \alpha_1$ $\boxed{1}$ boxes and $2k - j - \alpha_2$ $\boxed{2}$ boxes. Both amounts are bigger than $k - \alpha_l$ for all $l = 3, \dots, a + 1$, and there is no contradiction with the condition of the reverse reading word.

To define β , we denote by n_l , with $l \in \mathcal{C}_a \cup \{0\}$, the number of occurrences of the column l in T . Then,

$$\beta := \left(1^{n_1} \overline{1}^{n_{\overline{1}}} 2^{n_2} \overline{2}^{n_{\overline{2}}} \overline{2}^{n_{\overline{2}}} \dots \overline{a-1}^{n_{\overline{a-1}}} a^{n_a} \overline{a}^{n_{\overline{a}}} a + 1^{n_{a+1}} \right).$$

We finish the proof showing that the sequence β is a coloured partition of j . We separate into cases due to the α -condition:

- If $\alpha_1 = \alpha_2$, there are no $\boxed{2}$ boxes in the first row and $n_1 = n_0 = 0$. Then, counting the $\boxed{2}$ boxes, we get that

$$2k - j - \alpha_2 = \underbrace{k - \alpha_1}_{2^{\text{nd}} \text{ row}} + \underbrace{n_{\bar{1}}}_{3^{\text{rd}} \text{ row}} + \underbrace{\sum_{l=2}^{a+1} n_{\bar{l}}}_{\text{other rows}}. \quad (3.13)$$

Since α is a partition of k and it is defined by the blue boxes, we have also that

$$k = n_0 + n_1 + 2n_{\bar{1}} + (a+1)n_{a+1} + \sum_{l=2}^a l \cdot n_l + \sum_{l=2}^a l \cdot n_{\bar{l}} + \sum_{l=2}^{a-1} (l+1)n_{\bar{l}}. \quad (3.14)$$

Substituting (3.14) in (3.13), we have that $j = \sum_{l=1}^{a+1} l \cdot n_l + \sum_{l=2}^a l \cdot n_{\bar{l}} + \sum_{l=2}^{a-1} l \cdot n_{\bar{l}}$.

- For $\alpha_1 - \alpha_2 > 0$, first we show that we cannot have $(\#1)_{R2} = \alpha_1 - \alpha_2$. For this, we count the number of $\boxed{1}$ boxes, for $l = 1, \dots, a+1$, in the first row:

$$\begin{aligned} (\#1)_{R1} &= 2k + j - \alpha_1 - (\alpha_1 - \alpha_2) - \sum_{s=1}^a n_s, \\ (\#2)_{R1} &= k - j + \alpha_1 - \alpha_2 - n_0 - n_{\bar{1}} - \sum_{s=2}^{a-1} n_{\bar{s}}, \\ (\#3)_{R1} &= n_2 + n_{\bar{1}} \\ (\#l)_{R1} &= n_{l-1} + n_{\overline{l-2}} \quad \text{for } l = 4, \dots, a, \\ (\#a+1)_{R1} &= n_a + n_{\overline{a-1}}. \end{aligned}$$

Then, their sum is $3k - \alpha_1 + n_1$, since there are $3k - \alpha_1$ boxes in total in the first row, $n_1 = 0$ and there are no $\boxed{1}$ boxes in the second row. Then, by the α -condition, $(\#2)_{R1} = \alpha_1 - \alpha_2 > 0$, and we obtain that

$$\alpha_1 - \alpha_2 = 2k - j - \alpha_2 - ((k - \alpha_1) + n_0) - n_{\bar{1}} - \sum_{l=2}^{a-1} n_{\bar{l}},$$

which simplifies as $k - j - n_0 - n_{\bar{1}} - \sum_{l=2}^{a-1} n_{\bar{l}} = 0$. By (3.13), we get that

$$j = \sum_{l=1}^{a+1} l \cdot n_l + \sum_{l=2}^a l \cdot n_{\bar{l}} + \sum_{l=2}^{a-1} l \cdot n_{\bar{l}}.$$

This is the end of the proof. □

3.5.2. Plane partitions: combinatorial interpretation of Family 4

Let us denote by $\bar{g}_a(j) = \bar{g}_{(k^a)(2k-j, k^{a-1})}^{(k)}$, with $k \geq 2j$. In other words $\bar{g}_a(j)$ is the stable value of the j^{th} diagonal associated to the reduced Kronecker coefficients $\bar{g}_{(k+i, k^{a-1}), (k^a)}^{(k)}$ appearing in Theorem 3.5.1.

Let us see the relation of $\bar{g}_a(j)$ with the plane partitions.

Theorem 3.5.2. *We have the following combinatorial interpretation of the reduced Kronecker coefficients $\bar{g}_a(j)$:*

$$\bar{g}_a(j) = \sum_{l=0}^j \# \left\{ \begin{array}{l} \text{plane partitions of } l \\ \text{in } 3 \times (a-1) \text{ rectangle} \end{array} \right\} \# \left\{ \begin{array}{l} \text{plane partitions of } j-l \\ \text{in } 2 \times 1 \text{ rectangle} \end{array} \right\}.$$

Proof. The case $a = 1$ is included in the Corollary 3.4.2. Fix $a \geq 2$. Consider the generating function of the plane partitions fitting inside a $3 \times (a-1)$ rectangle:

$$\mathcal{H}_a = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^3 \dots (1-x^{a-1})^3(1-x^a)^2(1-x^{a+1})}.$$

By Theorem 3.5.1, the generating function of the reduced Kronecker coefficients $\bar{g}_a(j)$ is \mathcal{G}_a , which is related with \mathcal{H}_a by

$$\mathcal{H}_a = (1-x)(1-x^2)\mathcal{G}_a. \tag{3.15}$$

Then, we can express the coefficients appearing in the expansion for \mathcal{G}_a in terms of the coefficients of \mathcal{H}_a . Let $\mathcal{G}_a = \sum_n q_n x^n$ be the expansion for \mathcal{G}_a and $\mathcal{H}_a = \sum_n r_n x^n$ the corresponding for \mathcal{H}_a . Then, r_n is the number of plane partitions of n fitting inside a $3 \times (a-1)$ rectangle. Expanding both sides of (3.15) and identifying coefficients, we obtain the following recursive relation

$$\begin{aligned} r_0 &= q_0, \\ r_1 &= q_1 - q_0, \\ r_2 &= q_2 - q_1 - q_0, \\ r_n &= q_n - q_{n-1} - q_{n-2} + q_{n-3}, \quad \text{for all } n \geq 3. \end{aligned} \tag{3.16}$$

We can now express q_n in terms of the coefficients r_n , $m < n$. The statement is as follows.

Lemma 3.5.3. *With the same notation as above,*

$$q_n = \sum_{m=0}^n \left(\left\lfloor \frac{n-m}{2} \right\rfloor + 1 \right) r_m. \tag{3.17}$$

Proof by induction on n . Using the description in (3.16), we have that: $q_0 = r_0$, $q_1 = r_1 + r_0$ and $q_2 = r_2 + r_1 + 2r_0$. We proceed by induction: assume that the lemma holds for q_m , for all $m \leq n$. We show that the identity holds for q_{n+1} . The recursive relation shown in (3.16) for $n + 1$ is $r_{n+1} = q_{n+1} - q_n - q_{n-1} + q_{n-2}$. Applying the induction hypothesis to q_n , q_{n-1} , and q_{n-2} , we get that

$$q_{n+1} = r_{n+1} + \sum_{m=0}^n \left(\left\lfloor \frac{n-m}{2} \right\rfloor + 1 \right) r_m + \sum_{m=0}^{n-1} \left(\left\lfloor \frac{n-1-m}{2} \right\rfloor + 1 \right) r_m - \sum_{m=0}^{n-2} \left(\left\lfloor \frac{n-2-m}{2} \right\rfloor + 1 \right) r_m.$$

Collecting together the terms in r_m ,

$$q_{n+1} = r_{n+1} + r_n + 2r_{n-1} + \sum_{m=0}^{n-2} \left(\left\lfloor \frac{n-m}{2} \right\rfloor + 1 + \left\lfloor \frac{n-1-m}{2} \right\rfloor + 1 - \left\lfloor \frac{n-2-m}{2} \right\rfloor - 1 \right) r_m.$$

Since

$$\left\lfloor \frac{n-m}{2} \right\rfloor - \left\lfloor \frac{n-m-2}{2} \right\rfloor = 1 \quad \text{and} \quad \left\lfloor \frac{n-m-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-m+1}{2} \right\rfloor,$$

the coefficients of the sum are exactly of the form

$$\left\lfloor \frac{n-m+1}{2} \right\rfloor + 1.$$

Furthermore, the coefficients of r_{n+1} , r_n and r_{n-1} have also this form (considering $m = n + 1$, n and $n - 1$, respectively). \square

Finally, note that the coefficients that appears in Lemma 3.5.3 count the number of plane partitions fitting inside a 2×1 rectangle and the theorem follows. \square

3.5.3. Family 4 in terms of quasipolynomials

In Theorem 3.5.1 we computed the generating function for the reduced Kronecker coefficients $\bar{g}_{(k+i, k^{a-1}), (k^a)}^{(k)}$. We now consider the resulting implications of this calculation.

Theorem 3.5.4. *Let \mathcal{G}_a be the generating function for the reduced Kronecker coefficients $\bar{g}_a(j)$ stated in Subsection 3.5.2.*

Let ℓ be the least common multiple of $1, 2, \dots, a, a + 1$.

1. The generating function \mathcal{G}_a can be written as

$$\mathcal{G}_a = \frac{Q_a(x)}{(1 - x^\ell)^{3a-1}},$$

where $Q_a(x)$ is a product of cyclotomic polynomials. Moreover, we have that $\deg(Q_a(x)) = \ell(3a - 1) - \frac{3}{2}(a^2 + a) < \ell(3a - 1) - 1$.

2. The polynomial Q_a is the generating function for coloured partitions with parts in $\{1, \bar{1}, 2, \bar{2}, \bar{\bar{2}}, \dots, a - 1, \overline{a - 1}, \overline{\overline{a - 1}}, a, \bar{a}, a + 1\}$, where parts j, \bar{j} and $\bar{\bar{j}}$ appear with multiplicity at most $\frac{\ell}{j}$ times.

3. The coefficients of Q_a are positive and palindromic.

4. The coefficients $\bar{g}_a(j)$ are described by a quasipolynomial in j of degree $3a - 2$ and period dividing ℓ . In fact, we have checked that the period is exactly l for a less than 7.

5. The coefficients $\bar{g}_a(j)$ satisfy a formal reciprocity law $x^{\frac{3}{2}(a^2+a)}\mathcal{G}_a(x) = \mathcal{G}_a\left(\frac{1}{x}\right)$.

Proof. 1. We define $Q_a(x)$ as

$$Q_a(x) = \frac{(1 - x^\ell)^{3a-1}}{(1 - x)^2(1 - x^2)^3 \dots (1 - x^{a-1})^3(1 - x^a)^2(1 - x^{a+1})}.$$

Then, the generating function \mathcal{G}_a can be written as

$$\mathcal{G}_a = \frac{Q_a(x)}{(1 - x^\ell)^{3a-1}}.$$

From the well-known identity $(x^n - 1) = \prod_{i|n} \Phi_i$, we express \mathcal{G}_a and $(1 - x^\ell)^{3a-1}$ as a product of cyclotomic polynomials. The cyclotomic polynomials appearing in \mathcal{G}_a also appear in $(1 - x^\ell)^{3a-1}$, with at least equal exponent. Then, Q_a is a polynomial and it can be written as a product of cyclotomic polynomials. Moreover, $\deg(\mathcal{G}_a) = \frac{3}{2}a(a + 1)$, and $\deg(Q_a) = l(a - 1) - \frac{3}{2}a(a + 1)$.

2. By (1),

$$Q_a(x) = \frac{(1 - x^\ell)^{3a-1}}{(1 - x)^2(1 - x^2)^3 \dots (1 - x^{a-1})^3(1 - x^a)^2(1 - x^{a+1})}.$$

Separating this expression into factors and studying each one, we have that, for $i = 2, \dots, a - 1$,

$$\begin{aligned} \left(\frac{1-x^l}{1-x}\right)^2 &= [1+x+x^2+\dots+x^{l-1}]^2, \\ \left(\frac{1-x^l}{1-x^i}\right)^3 &= [1+(x^i)+(x^i)^2+\dots+(x^i)^{\lfloor \frac{l}{i} \rfloor}]^3, \\ \left(\frac{1-x^l}{1-x^a}\right)^2 &= [1+(x^a)+(x^a)^2+\dots+(x^a)^{\lfloor \frac{l}{a} \rfloor}]^2, \\ \frac{1-x^l}{1-x^{a+1}} &= 1+(x^{a+1})+(x^{a+1})^2+\dots+(x^{a+1})^{\lfloor \frac{l}{a+1} \rfloor}. \end{aligned}$$

This is exactly the combinatorial interpretation for the coefficients of Q_a of Theorem 3.5.4 because each factor corresponds to the part i, \bar{i} or $\bar{\bar{i}}$, and each one appears at most $\frac{l}{i}$ times.

3. By (2), the coefficients of Q_a are positive and they are also palindrome because Φ_1 does not appear in Q_a .
4. This follows using Proposition 4.13 of [BS16].
5. It follows by computing $\mathcal{G}_a\left(\frac{1}{x}\right)$.

□

Let us see some examples.

Example 12. We express Q_a as a product of cyclotomic polynomials. For instance, $Q_2 = \Phi_2^3 \Phi_3^4 \Phi_6^5$ and $Q_3 = \Phi_2^4 \Phi_3^6 \Phi_4^7 \Phi_6^8 \Phi_{12}^8$.

Example 13. The coefficients $\bar{\bar{g}}_2(j)$ are given by the quasipolynomial of degree 4 and period 6.

$$\bar{\bar{g}}_2(j) = \begin{cases} 1/288(j+6)(j^3+12j^2+40j+48) & \text{if } j \equiv 0 \pmod{6} \\ 1/288(j+5)(j^3+13j^2+47j+35) & \text{if } j \equiv 1 \pmod{6} \\ 1/288(j+4)(j^3+14j^2+56j+64) & \text{if } j \equiv 2 \pmod{6} \\ 1/288(j+3)(j^3+15j^2+67j+49) & \text{if } j \equiv 3 \pmod{6} \\ 1/288(j+2)(j^3+16j^2+80j+128) & \text{if } j \equiv 4 \pmod{6} \\ 1/288(j+1)(j^3+17j^2+95j+175) & \text{if } j \equiv 5 \pmod{6} \end{cases}$$

3.5.4. Saturation hypothesis

Finally, for Family 4, we have the corresponding result concerning the stable values of the diagonals, $\overline{\overline{g}}_a(j)$.

Corollary 3.5.5. *The reduced Kronecker coefficients $\overline{\overline{g}}_a(j)$ satisfy the saturation hypothesis. In fact, $\overline{\overline{g}}_a(sj) > 0$ for all $s \geq 1$, where $\overline{\overline{g}}_a(sj)$ denotes the associated reduced Kronecker coefficient with its three partitions multiplied by s . Moreover, the sequences of coefficients obtained by either fixing i or a , and then letting k grow, are weakly increasing.*

Proof. By Theorem 3.5.2, $\overline{\overline{g}}_a(j)$ has the following combinatorial description in terms of plane partitions:

$$\overline{\overline{g}}_a(j) = \sum_{l=0}^j \# \left\{ \begin{array}{l} \text{plane partitions of } l \\ \text{in } 3 \times (a-1) \text{ rectangle} \end{array} \right\} \# \left\{ \begin{array}{l} \text{plane partitions of } j-l \\ \text{in } 2 \times 1 \text{ rectangle} \end{array} \right\}.$$

There always exists a pair of plane partitions with a plane partition of $l \in \{0, \dots, sj\}$, fitting in a $3 \times (a-1)$ rectangle, and a plane partition of $sj-l$ fitting in a 2×1 rectangle: for $s \geq 1$, we consider the following pair $\boxed{sl} \quad \boxed{s(j-l)}$, with $l \in \{0, \dots, j\}$.

Indeed, any pair of plane partitions associated to j defines a pair of plane partitions associated to $j+1$. Let us consider a pair of plane partitions, where the first partition is any plane partition of $l \in \{0, \dots, j\}$ fitting in a $3 \times (a-1)$ rectangle and the second is any plane partition of $j-l$ fitting in a 2×1

$\overbrace{\hspace{1.5cm}}^{a-1}$				$\overbrace{\hspace{0.5cm}}^1$
α_{11}	α_{12}	\dots	$\alpha_{1,a-1}$	β_1
α_{21}	α_{22}	\dots	$\alpha_{2,a-1}$	β_2
α_{31}	α_{32}	\dots	$\alpha_{3,a-1}$	

If we add one to the first element of the first plane partition, $\alpha_{11} + 1$, we obtain a plane partition of $l' = l + 1 \in [0, \dots, j + 1]$ fitting in a $3 \times (a-1)$ rectangle. The other plane partition can be seen as a plane partition of $j + 1 - l' = j - l$ that fits in a 2×1 rectangle. □

3.6. Consequences for the Kronecker coefficients

Our original aim is trying to understand the rate of growth experienced by the Kronecker coefficient as we increase the sizes of its rows. By the classical stabil-

ity phenomena for the Kronecker coefficients discovered by Murnaghan, we can transcribe our results in terms of reduced Kronecker coefficients as results about Kronecker coefficients.

First, we state the most interesting cases.

Corollary 3.6.1 (Family 2, Kronecker coefficients version). *Consider the Kronecker coefficients of the form $g_{(n-a \cdot k, k^a), (n-a \cdot k, k^a)}^{(n-k, k)}$. Then, for $n \geq (a + 3) \cdot k$,*

(i) *Their generating function is*

$$\mathcal{F}_a = \frac{1}{(1-x)(1-x^2)^2 \dots (1-x^a)^2(1-x^{a+1})}.$$

(ii) *These coefficients count the number of plane partitions of k fitting inside a $2 \times a$ rectangle.*

(iii) *These coefficients can be described by a quasipolynomial of degree $2a - 1$ and period dividing l (least common multiple of $1, 2, \dots, a, a + 1$).*

Proof. Using (3.1) and the Corollary 3.1.1, we have that, for $n \geq (a + 3) \cdot k$,

$$g_{(n-a \cdot k, k^a), (n-a \cdot k, k^a)}^{(n-k, k)} = \bar{g}_{(k^a), (k^a)}^{(k)}.$$

Then, we apply the previous results: Theorem 3.3.1 for (i), Theorem 3.3.7 for (ii) and Theorem 3.3.8 for (iii). □

For Family 4, we have the corresponding result.

Corollary 3.6.2 (Family 4, Kronecker coefficients version). *Consider the Kronecker coefficients of the form $g_{(n-a \cdot k, k^a), (n-(a+1) \cdot k + j, 2k-j, k^{a-1})}^{(n-k, k)}$. For $k \geq 2j$ and $n \geq (a + 3)k$,*

(i) *The generating function of these coefficients is*

$$\mathcal{G}_a = \frac{1}{(1-x)^2(1-x^2)^3 \dots (1-x^{a-1})^3(1-x^a)^2(1-x^{a+1})}.$$

(ii) *These coefficients have the following combinatorial interpretation in terms of plane partitions*

$$\bar{\bar{g}}_a(j) = \sum_{l=0}^j \# \left\{ \begin{array}{l} \text{plane partitions of } l \\ \text{in } 3 \times (a-1) \text{ rectangle} \end{array} \right\} \# \left\{ \begin{array}{l} \text{plane partitions of } j-l \\ \text{in } 2 \times 1 \text{ rectangle} \end{array} \right\}.$$

(iii) *These coefficients can be described by a quasipolynomial of degree $3a - 2$ and period dividing l (least common multiple of $1, 2, \dots, a, a + 1$).*

Proof. We recall that we denote by $\bar{g}_a(j)$ the stable value of the j^{th} diagonal associated to the reduced Kronecker coefficients $\bar{g}_{(k+i, k^{a-1}), (k^a)}^{(k)}$. By (3.1) and Corollary 3.1.1, the Kronecker coefficients that we are considering are exactly these stable values. For $k \geq 2j$ and $n \geq (a + 3)k$, $g_{(n-k, k), (n-(a+1) \cdot k + j, 2k-j, k^{a-1})}^{(n-k, k)} = \bar{g}_a(j)$. Then, we apply the previous results: Theorem 3.5.1 for (i), Theorem 3.5.2 for (ii) and Theorem 3.5.4 for (iii). \square

We finish this section with some observations about the rate of growth of the Kronecker coefficients. Murnaghan observed that the sequences obtained by adding cells to the first parts of the partitions indexing a Kronecker coefficients are eventually constant. In [BRR16], E. Briand, A. Rattan, and M. Rosas show that fixed three partitions, the Kronecker coefficients indexed by them stabilize when we increase these partitions with n new boxes in their first row and n new boxes in their first column. They also show that the resulting sequence obtained by increasing the sizes of the second rows (keeping the first one very long in comparison) of the partitions indexing the Kronecker coefficients are described by a linear quasipolynomial of period 2.

An interesting question is then to describe what happens when we add cells to arbitrary rows of the partitions indexing a Kronecker (and reduced Kronecker) coefficient. The results presented here show several cases where we know what happens. For example, for $a = 1$, the families 2, 3 and 4 are described by a linear quasipolynomial of period 2, as is predicted in the work of Briand, Rattan and Rosa, [BRR16]. In fact, also Family 1 is described by linear quasipolynomial of period 2, depending on the values of the indexing partitions. When $a = 2$, the sequence corresponding to Family 2 is described by a quasipolynomial of degree 3 and the one corresponding to Family 4 is described by a quasipolynomial of degree 4. For $a = 3$, the sequences are described by quasipolynomial of degree 5 and 7 respectively, and so on.

Finally, we remark that the sequences described by both families, Families 2 and 4, when we increase the parameter a are weakly increasing, and bounded. These sequences correspond to increasing the sizes of the columns.

3.7. The reduced Kronecker coefficients and the vertex operators

In this section, we present a study of the reduced Kronecker coefficients using vertex operators. In the first subsection, vertex operators provide a different proof of Murnaghan's theorem. In the second subsection, we give a description of the reduced Kronecker coefficients in terms of Littlewood–Richardson coefficients using the vertex operators.

3.7.1. Proof of Murnaghan's theorem

We recall the Murnaghan's Theorem

Theorem (Murnaghan's Theorem). *There exists a family of non-negative integers $\{\bar{g}_{\alpha\beta}^\gamma\}$, indexed by triples of partitions (α, β, γ) , such that, for α and β fixed, only many terms $\bar{g}_{\alpha\beta}^\gamma$ are non-zero, and for all $n \geq 0$, $s_{\alpha[n]} \star s_{\beta[n]} = \sum_{\gamma} \bar{g}_{\alpha\beta}^\gamma s_{\gamma[n]}$, where $\alpha[n] = (n - |\alpha|, \alpha_1, \alpha_2, \dots)$.*

Proof of Murnaghan's Theorem. Decompose the Kronecker product in terms of the Schur basis:

$$s_{\alpha[n]} \star s_{\beta[n]} = \sum_{\gamma} g_{\alpha[n], \beta[n]}^{\gamma[n]} s_{\gamma[n]}.$$

Using 1.3.10, these coefficients also appear when we consider the Schur function on the product of alphabets

$$s_{\gamma[n]}[XY] = \sum_{\alpha, \beta} g_{\alpha[n], \beta[n]}^{\gamma[n]} s_{\alpha[n]}[X] \cdot s_{\beta[n]}[Y]$$

For n small, the sequences $\alpha[n]$ and $\beta[n]$ have negative parts. This implies that many of the Schur functions $s_{\alpha[n]}[X]$ and $s_{\beta[n]}[Y]$ are zero. However, if we consider n big enough and we compute the coefficients $g_{\alpha[n], \beta[n]}^{\gamma[n]}$, then the expression is also valid for small values of n . We extract the coefficients using the scalar product:

$$g_{\alpha[n], \beta[n]}^{\gamma[n]} = \langle s_{\alpha[n]}[X] \cdot s_{\beta[n]}[Y], s_{\gamma[n]}[XY] \rangle.$$

Consider the generating function of these coefficients

$$f(z) = \sum_{n \in \mathbb{Z}} g_{\alpha[n], \beta[n]}^{\gamma[n]} \cdot z^n.$$

We get that

$$\begin{aligned} f(z) &= \sum_{n \in \mathbb{Z}} \langle s_{\alpha[n]}[X] \cdot s_{\beta[n]}[Y], s_{\gamma[n]}[XY] \rangle z^n = \\ &= \sum_{n \in \mathbb{Z}} \langle s_{(n-|\alpha|, \alpha)}[X] \cdot s_{(n-|\beta|, \beta)}[Y], s_{(n-|\gamma|, \gamma)}[XY] \rangle z^n. \end{aligned}$$

Since $g_{\lambda\mu}^{\nu} = 0$ if λ , μ and ν are not partitions of same integer, and $s_{(n-|\alpha|, \alpha)} = 0$ if $n < |\alpha|$, we can replace the sum over n by three different sums. We also introduce the factor z^n inside the scalar product in the right-hand side:

$$f(z) = \left\langle \sum_{i=0}^{\infty} s_{(i-|\alpha|, \alpha)}[X] \sum_{j=0}^{\infty} s_{(j-|\beta|, \beta)}[Y], \sum_{k=0}^{\infty} s_{(k-|\gamma|, \gamma)} s_{\gamma}[XY] \cdot z^k \right\rangle.$$

Applying Lemma 2.4.3 to each sum, with $z = 1$ on the left-hand side, we get that

$$f(z) = \left\langle \sigma_1(X) \cdot s_{\alpha}[X-1] \cdot \sigma_1(Y) \cdot s_{\beta}[Y-1], z^{|\gamma|} \cdot \sigma_z[XY] \cdot s_{\gamma} \left[XY - \frac{1}{z} \right] \right\rangle.$$

By Proposition 2.4.2, we obtain

$$\begin{aligned} f(z) &= \left\langle s_{\alpha}[X-1] s_{\beta}[Y-1], z^{|\gamma|} \cdot \sigma_z[(X+1)(Y+1)] \cdot s_{\gamma} \left[(X+1)(Y+1) - \frac{1}{z} \right] \right\rangle = \\ &= \left\langle s_{\alpha}[X-1] s_{\beta}[Y-1], z^{|\gamma|} \cdot \sigma_z[XY + X + Y] \underbrace{\sigma_z[1]} \cdot s_{\gamma} \left[(X+1)(Y+1) - \frac{1}{z} \right] \right\rangle = \\ &= \frac{1}{1-z} \underbrace{\left\langle s_{\alpha}[X-1] s_{\beta}[Y-1], z^{|\gamma|} \cdot \sigma_z[XY + X + Y] \cdot s_{\gamma} \left[(X+1)(Y+1) - \frac{1}{z} \right] \right\rangle}_{P(z)}. \end{aligned}$$

Let us see that $P(z)$ is a Laurent polynomial. In the right-hand side of the scalar product, we have the series $\sigma_z[XY + X + Y] = \sum h_k[XY + X + Y] \cdot z^k$. For $k > |\alpha| + |\beta|$ and for any symmetric function f in X and Y , $f \cdot h_k[XY + X + Y]$ has no homogeneous terms of total degree $\leq |\alpha| + |\beta|$. Then, the scalar product with $s_{\alpha}[X-1] \cdot s_{\beta}[Y-1]$ is zero. We can truncate the series $\sigma_z[XY + X + Y]$ to

$$\sum_{k=0}^{|\alpha|+|\beta|} h_k[XY + X + Y] \cdot z^k$$

Thus, $P(z)$ is a Laurent polynomial of degree at most $|\alpha| + |\beta| + |\gamma|$. We conclude that the generating function of the coefficients $g_{\alpha[n]\beta[n]}^{\gamma[n]}$ can be written as $f(z) = \frac{P(z)}{1-z}$, with $P(z)$ a Laurent polynomial of degree at most $|\alpha| + |\beta| + |\gamma|$. If we write $f(z)$

as a series, we get that

$$f(z) = \sum_{n \in \mathbb{N}} g_{\alpha[n], \beta[n]}^{\gamma[n]} \cdot z^n = P(z) \cdot \sum_{n \geq 0} z^n.$$

This implies that the general term $g_{\alpha[n], \beta[n]}^{\gamma[n]}$ is stationary and we have the statement of Murnaghan's Theorem. \square

If we continue analysing the polynomial $P(z)$, we obtain the following result proved by Brion, [Bri93].

Proposition 3.7.1 (Brion's formula). *Let α , β , and γ three partitions. Then,*

$$\bar{g}_{\alpha\beta}^{\gamma} = \langle s_{\alpha}[X] \cdot s_{\beta}[Y], \sigma_1[XY] \cdot s_{\gamma}[XY + X + Y] \rangle.$$

Moreover, the sequence of general term $g_{\alpha[n], \beta[n]}^{\gamma[n]}$ is constant for $n \geq |\alpha| + |\beta| + \gamma_1$.

Proof. We can write $P(z)$ as

$$P(z) = \left\langle D_{\sigma_z(X)} s_{\alpha}[X] \cdot D_{\sigma_z(Y)} s_{\beta}[Y], z^{|\gamma|} \cdot \sigma_z[XY + X + Y] \cdot s_{\gamma} \left[(X+1)(Y+1) - \frac{1}{z} \right] \right\rangle.$$

Using that $D_{\sigma_z(X)} s_{\lambda}[X] = \sum_{i=0}^{\lambda_1} s_{\lambda/(i)}[X] \cdot z^i$, and that $\sigma_z[XY] = \sum_{k \geq 0} h_k[XY] \cdot z^k$, we have that

$$P(z) = \sum_{i,j,k} \left\langle s_{\alpha/(i)}[X] \cdot s_{\beta/(j)}[Y], h_k[XY] s_{\gamma} \left[(X+1)(Y+1) - \frac{1}{z} \right] \right\rangle z^{i+j+k+|\gamma|}$$

Looking at each summand, the degree in X of the left-hand side is $|\alpha| - i$. Moreover, the right-hand side has no homogeneous component of degree less than k . Then, each summand contributes to the sum only if $k \leq |\alpha| - i$. We also bound the degree of z in each summand by $|\alpha| + |\gamma| + \beta_1$. We specialize at $z = 1$, and we obtain that the stable value of the coefficients $g_{\alpha n, \beta[n]}^{\gamma[n]}$ is

$$\bar{g}_{\alpha\beta}^{\gamma} = P(1) = \langle s_{\alpha}[X] \cdot s_{\beta}[Y], \sigma_1[XY] \cdot s_{\gamma}[XY + X + Y] \rangle.$$

\square

3.7.2. Description in terms of Littlewood–Richardson coefficients

Let us describe another approach for the reduced Kronecker coefficients with one partition equal to (k) .

Littlewood observed that the reduced Kronecker coefficients coincide with the Littlewood–Richardson coefficients when $|\alpha| + |\beta| = |\gamma|$, [Mur55, Lit58]. In general, for any α, β partitions, by Proposition 3.7.1

$$\bar{g}_{\alpha\beta}^{(k)} = \langle s_\alpha[X]s_\beta[Y], \sigma[XY]h_k[XY + X + Y] \rangle.$$

We consider the decomposition

$$h_k[XY + X + Y] = \sum_{p+q+r=k} h_p[XY] \cdot h_q[X] \cdot h_r[Y].$$

By Proposition 1.3.8, we get that $h_k[XY + X + Y] = \sum s_\omega[X] \cdot s_\omega[Y] \cdot h_q[X] \cdot h_r[Y]$, with $p + q + r = k$. Since $\sigma[XY] = \sum_\varphi s_\varphi[X] \cdot s_\varphi[Y]$, we have that

$$\begin{aligned} \sigma[XY]h_k[XY + X + Y] &= \sum_{|\omega|+q+r=k} s_\varphi[X] \cdot s_\varphi[Y] \cdot s_\omega[X] \cdot s_\omega[Y] \cdot h_q[X] \cdot h_r[X] = \\ &= \sum_{|\omega|+q+r=k} (s_\varphi \cdot s_\omega \cdot h_q)[X] \cdot (s_\varphi \cdot s_\omega \cdot h_r)[Y]. \end{aligned} \quad (3.18)$$

The Littlewood–Richardson coefficients $c_{\alpha\beta}^\gamma$ can be generalized to define the coefficients, $c_{\alpha\beta\gamma}^\phi$, that appear when we consider the product of three Schur functions and we decompose the result in the Schur basis: $s_\alpha \cdot s_\beta \cdot s_\gamma = \sum_\phi c_{\alpha\beta\gamma}^\phi s_\phi$. That it is exactly what appears in (3.18):

$$\sigma[XY]h_k[XY + X + Y] = \sum_{|\omega|+q+r=k} \left(\sum_\phi c_{\varphi\omega(q)}^\phi s_\phi[X] \right) \left(\sum_\psi c_{\varphi\omega(r)}^\psi s_\psi[Y] \right).$$

Considering the scalar product with $s_\alpha[X] \cdot s_\beta[Y]$, the only terms that are non-zero are those for which $\phi = \alpha$ and $\psi = \beta$. Then, the reduced Kronecker coefficients that we are considering can be described as

$$\bar{g}_{\alpha\beta}^{(k)} = \sum_{|\omega|+q+r=k} c_{\varphi\omega(q)}^\alpha c_{\varphi\omega(r)}^\beta.$$

There is a way to describe these coefficients in terms of the usual Littlewood–Richardson coefficients: consider the product $(s_\alpha \cdot s_\beta) \cdot s_\gamma$ and express the first

product in terms of Littlewood-Richardson coefficients. Then,

$$(s_\alpha \cdot s_\beta) \cdot s_\gamma = \left(\sum_{\phi} c_{\alpha\beta}^{\phi} \cdot s_{\phi} \right) \cdot s_\gamma = \sum_{\phi} c_{\alpha\beta}^{\phi} \cdot (s_{\phi} \cdot s_\gamma).$$

If we express again the product $s_{\phi} \cdot s_\gamma$ in terms of Littlewood-Richardson coefficients, then

$$(s_\alpha \cdot s_\beta) \cdot s_\gamma = \sum_{\phi, \theta} c_{\alpha\beta}^{\phi} \cdot c_{\phi\gamma}^{\theta} \cdot s_{\theta}. \quad (3.19)$$

Thus,

$$c_{\alpha\beta\gamma}^{\theta} = \sum_{\phi} c_{\alpha\beta}^{\phi} c_{\phi\gamma}^{\theta}$$

In short, we obtain the following result.

Proposition 3.7.2. *The reduced Kronecker coefficients $\bar{g}_{\alpha\beta}^{(k)}$ in terms of Littlewood-Richardson coefficients are described as*

$$\bar{g}_{\alpha\beta}^{(k)} = \sum_{|\omega|+q+r=k} \sum_{\phi, \phi'} \left(c_{\varphi\omega}^{\phi} c_{\phi(q)}^{\alpha} \right) \cdot \left(c_{\varphi\omega}^{\phi'} c_{\phi'(q)}^{\beta} \right)$$

For $\alpha = \beta$, since $|\alpha| + |\beta| = |\gamma|$, we have the following list of conditions: $|\phi| = |\phi'|$, $q = r$, $|\omega| + 2r = k$, $|\varphi| + k - r = |\alpha|$, and φ and ω should be contained in α .

This proves the following result.

Corollary 3.7.3. *For $\alpha = \beta$ in Proposition 3.7.2, we have*

$$\bar{g}_{\alpha\alpha}^{(k)} = \sum_{\substack{|\omega|=k-2r \\ |\varphi|=|\alpha|-k+r \\ \varphi, \omega \subseteq \alpha}} \sum_{\phi, \phi'} \left(c_{\varphi\omega}^{\phi} c_{\phi(r)}^{\alpha} \right) \cdot \left(c_{\varphi\omega}^{\phi'} c_{\phi'(r)}^{\alpha} \right).$$

As a consequence of this description, we prove the saturation hypothesis for the reduced Kronecker coefficients $\bar{g}_{\alpha\alpha}^{(k)}$.

Corollary 3.7.4. *Let α be any partition. If $\bar{g}_{\alpha\alpha}^{(k)} > 0$, then $\bar{g}_{s\alpha, s\alpha}^{(sk)} > 0$, for all $s \geq 1$.*

Proof. By the Corollary 3.7.3,

$$\bar{g}_{\alpha\alpha}^{(k)} = \sum_{\substack{|\omega|=k-2r \\ |\varphi|=|\alpha|-k+r \\ \varphi, \omega \subseteq \alpha}} \sum_{\phi, \phi'} \left(c_{\varphi\omega}^{\phi} c_{\phi(r)}^{\alpha} \right) \cdot \left(c_{\varphi\omega}^{\phi'} c_{\phi'(r)}^{\alpha} \right).$$

The condition $\bar{g}_{\alpha\alpha}^{(k)} > 0$ means that there exist ω, φ, ϕ and ϕ' partitions and an integer r such that $|\omega| = 2k - r, |\varphi| = |\alpha| - k + r, \varphi, \omega \subseteq \alpha, |\phi| = |\phi'|$ and all the coefficients are positive

$$c_{\varphi\omega}^{\phi}, c_{\phi(r)}^{\alpha}, c_{\varphi\omega}^{\phi'}, c_{\phi'(r)}^{\alpha} > 0$$

By the saturation hypothesis for the Littlewood–Richardson coefficients, for any $s \geq 1$,

$$c_{s\varphi, s\omega}^{s\phi}, c_{s\phi, (sr)}^{s\alpha}, c_{s\varphi, s\omega}^{s\phi'}, c_{s\phi'(sr)}^{s\alpha} > 0$$

Then, we can describe the reduced Kronecker coefficient $\bar{g}_{s\alpha, s\alpha}^{(sk)}$, for all $s \geq 1$, as

$$\bar{g}_{s\alpha, s\alpha}^{(sk)} = \sum_{\substack{|\bar{\omega}|=sk-2\bar{r} \\ |\bar{\varphi}|=|s\alpha|-sk+\bar{r} \\ \bar{\varphi}, \bar{\omega} \subseteq s\alpha}} \sum_{\bar{\phi}, \bar{\phi}'} \left(c_{\bar{\varphi}\bar{\omega}}^{\bar{\phi}} c_{\bar{\phi}(\bar{r})}^{s\alpha} \right) \cdot \left(c_{\bar{\varphi}\bar{\omega}}^{\bar{\phi}'} c_{\bar{\phi}'(\bar{r})}^{s\alpha} \right).$$

For $\bar{\varphi} = s\varphi, \bar{\omega} = s\omega, \bar{\phi} = s\phi, \bar{\phi}' = s\phi'$ and $\bar{r} = sr$, this product is non-zero, and $\bar{g}_{s\alpha, s\alpha}^{(sk)} > 0$. □

Appendix A.

Representation theory

This appendix is a short introduction to representation theory of groups, taking special attention to the symmetric groups and the general linear groups. The representations of the symmetric group \mathfrak{S}_n is used to define irreducible representation of the general linear groups, as well as the wreath product group and its representations. More details can be found in [Sag01, FH91, Sun96, Mac80].

A.1. Representations

A *representation* of a group G on a complex vector space V is a group homomorphism $\rho: G \rightarrow GL(V)$ from G to the general linear group of V , $GL(V)$. We usually call V itself a representation of G . We also suppress the symbol ρ , and we write $g \cdot v$ or gv for $\rho(g)(v)$. The map ρ provides to V the structure of a G -module. The *dimension* of the representation is the dimension of V .

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under the group action. A representation V is called *irreducible* if there is no proper non-zero invariant subspace W of V . In Example 21 we show that both concepts are different, i.e. there exist irreducible representations with non-trivial subrepresentations.

The representations can be constructed from other representations.

- Let V and W be two representations of G . Then, the direct sum, $V \oplus W$, and the tensor product, $V \otimes W$, are also representations, via $g(v \oplus w) = gv \oplus gw$ and $g(v \otimes w) = gv \otimes gw$, respectively.
- For a representation V , the n^{th} tensor power $V^{\otimes n}$ is again a representation of G , and the exterior powers $\wedge^n(V)$ and symmetric powers $Sym^n(V)$ are subrepresentations of it.

· The dual $V^* = \text{Hom}(V, \mathbb{C})$ of V is also a representation via $\rho^*(g) = {}^t\rho(g^{-1})$.

A.2. Representation theory of finite groups

For any finite group, we consider the following *basic representations*.

Example 14. Let X be any finite set. Consider the action of G on the left on X , i.e. $G \rightarrow \text{Aut}(X)$ is a homomorphism to the permutation group of X . Then, we associate the *permutation representation*: Let V be the vector space with basis $\{e_x : x \in X\}$, and let G acts on V by

$$g \cdot \sum a_x e_x = \sum a_x e_{gx}.$$

Example 15. The *regular representation* R , corresponds to the left action of G on itself. Equivalently, R is the space of complex-valued functions on G , where an element $g \in G$ acts on a function α by $g\alpha(h) = \alpha(g^{-1}h)$.

Proposition A.2.1 (Schur's lemma for finite groups). *If V and W are irreducible representations of a finite group G and $\varphi : V \rightarrow W$ is a G -module homomorphism, then*

1. *Either φ is an isomorphism, or $\varphi = 0$.*
2. *If $V = W$, then $\varphi = \lambda \cdot \text{Id}$, for some $\lambda \in \mathbb{C}$ and Id the identity.*

As a consequence of the Schur's lemma, we know that the number of irreducible representations is finite.

Proposition A.2.2. *The number of irreducible representations of G is equal to the number of conjugacy classes of G .*

The decomposition into irreducible representations is given by *Maschke's theorem*.

Proposition A.2.3 (Maschke's Theorem). *For any representation V of a finite group G , there exists a decomposition*

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations and the a_i are their multiplicities. The decomposition of V into a direct sum of the k factors is unique, up to isomorphism.

Notation. We usually denote the decomposition into irreducible as

$$V \cong a_1 V_1 \oplus \cdots \oplus a_k V_k,$$

where the multiplicities a_i denotes how many times that the irreducible representation V_i appears on it.

For finite groups, the representations can be identified with matrices.

Definition A.2.4. A *matrix representation of a finite group G* is a group homomorphism $X : G \rightarrow GL_d(\mathbb{C})$. Equivalently, to each $g \in G$ is assigned a matrix of dimension $d \times d$ with coefficients in \mathbb{C} , such that $X(\epsilon)$ is the identity matrix, and $X(gh) = X(g)X(h)$, for all $g, h \in G$.

Using this definition, we can state the Maschke's theorem in terms of matrix representations.

Corollary A.2.5. *Let G be a finite group and let X be a matrix representation of G of dimension $d > 0$. Then, there is a fixed matrix T such that every matrix $X(g)$, $g \in G$, has the form*

$$TX(g)T^{-1} = \begin{pmatrix} X^{(1)}(g) & 0 & \dots & 0 \\ 0 & X^{(2)}(g) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X^{(k)}(g) \end{pmatrix},$$

where each $X^{(i)}$ is an irreducible matrix representation of G .

The following example illustrates that there exist irreducible representations with non-trivial subrepresentations. It can be found in [Sag01, Section 5].

Example 16. Let \mathbb{R}^+ be the positive real numbers, which are a group under multiplication. Consider the matrix representation defines as follows: for each $r \in \mathbb{R}^+$,

$$X(r) = \begin{pmatrix} 1 & \log r \\ 0 & 1 \end{pmatrix}.$$

The subspace $W = \{(c, 0)^t : c \in \mathbb{C}\} \subset \mathbb{C}^2$ is invariant under the action of G . Then, \mathbb{C}^2 must decompose as a direct sum of W and another one-dimensional submodule. By the matrix version of Maschke's theorem, Corollary A.2.5, there exists a fixed matrix T such that

$$TX(r)T^{-1} = \begin{pmatrix} x(r) & 0 \\ 0 & y(r) \end{pmatrix}$$

for all $r \in \mathbb{R}^+$. Thus $x(r)$ and $y(r)$ must be the eigenvalues of $X(r)$, which are both 1. But then,

$$X(r) = T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T = \begin{pmatrix} x(r) & 0 \\ 0 & y(r) \end{pmatrix}$$

for all $r \in \mathbb{R}^+$, which is impossible.

This means that the matrix representation $X(r)$ is irreducible and has W as non-trivial subrepresentation.

For a finite group G , the group algebra $\mathbb{C}G$ associated to it is an object that for all intents and purpose can completely replace the group G itself. Any statement about the representations of G has an exact equivalent statement about the group algebra. The underlying vector space of the group algebra of G is the vector space with basis $\{e_g\}$ corresponding to the elements of the group G , that is, the underlying vector space of the regular representation. We define the algebra structure on this vector space simply by $e_g \cdot e_h = e_{gh}$.

By a representation of the algebra $\mathbb{C}G$ on a vector space V we mean simply an algebra homomorphism $\mathbb{C}G \rightarrow \text{End}(V)$. Thus, a representation V of $\mathbb{C}G$ is a left $\mathbb{C}G$ -module. Moreover, a representation $\rho: G \rightarrow \text{End}(V)$ extends by linearity to a map $\tilde{\rho}: \mathbb{C}G \rightarrow \text{End}(V)$. Then, the representations of $\mathbb{C}G$ correspond exactly to representations of G and the left $\mathbb{C}G$ -module given by $\mathbb{C}G$ itself corresponds to the regular representation. If $\{W_i\}$ are the irreducible representations of G , then we have the following isomorphism of algebras:

$$\mathbb{C}G \cong \bigoplus \text{End}(W_i).$$

A.3. Induced and restricted representations

Let H be a subgroup of a group G . Then, any representation V of G restricts to a representation of H trivially. This representation is denoted by $\text{Res}_H^G V$, or simple $\text{Res } V$. Even if V is an irreducible representation, this does not implies that $\text{Res } V$ will be an irreducible representation. In fact, the *classical branching rules* describe the multiplicity with which an irreducible representation of H occurs in the restriction of an irreducible representation of G to the subgroup H . Branching rules for several classical groups were determined by: Weyl in [Wey46] between successive unitary groups; Murnaghan in [Mur63] between successive special orthogonal groups and unitary symplectic groups; and Littlewood in [Lit06] from the unitary groups

to the unitary symplectic groups and special orthogonal groups. In Subsection A.5 we give an example of the branching rules for the restriction of irreducible representations of the symmetric group.

From any representation of a subgroup H , we can consider a representation of G . Consider V a representation of G , and $W \subseteq V$ a subspace which is H -invariant. For any g in G , the subspace $g \cdot W$ depends only on the left coset gH of g modulo H , since $gh \cdot W = g \cdot W$. Let us denote by $\sigma \cdot W$, with σ a coset in G/H , this subspace of V . We say that V is *induced* by W if every element in V can be written uniquely as a sum of elements in such translates of W , i.e.

$$V \cong \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

In this case, we write $V = \text{Ind}_H^G W = \text{Ind } V$. Moreover, given a representation W of H , such V exists and is unique, up to isomorphism.

Example 17. The permutation representation associated to the left action of G on G/H is induced from the trivial one-dimensional representation W of H . Here V has basis $\{e_\sigma : \sigma \in G/H\}$, and $W = \mathbb{C} \cdot e_H$, with H the trivial coset.

Example 18. The regular representation of G is induced from the regular representation of H . Here V has basis $\{e_g : g \in G\}$, whereas W has basis $\{e_h : h \in H\}$.

A.4. Characters

There is a remarkably effective tool for understanding the representations of a finite group of G , called *character theory*. If V is a representation of G , its *character* χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = \text{Tr}(g|_V),$$

where Tr is the trace of g on V . In particular, we have

$$\chi_V(hgh^{-1}) = \chi_V(g).$$

Therefore, χ_V is constant on the conjugacy classes of G and the character is a class function.

Proposition A.4.1. *Let V and W be representations of G . Then,*

$$\begin{aligned}\chi_{V \oplus W} &= \chi_V + \chi_W, \\ \chi_{V \otimes W} &= \chi_V \cdot \chi_W, \\ \chi_{V^*} &= \overline{\chi_V}, \\ \chi_{\wedge^2 V}(g) &= \frac{1}{2} [\chi_V(g)^2 - \chi_V(g^2)], \\ \chi_{\text{Sym}^2 V}(g) &= \frac{1}{2} [\chi_V(g)^2 + \chi_V(g^2)].\end{aligned}$$

Example 19. If V is the permutation representation associated to the action of a group G on a finite set X , then $\chi_V(g)$ is the number of elements of X fixed by g .

Since the character of a representation of a group G is a function on the set of conjugacy classes in G , the basic information about the irreducible representations of a group G is very useful.

Theorem A.4.2. *Let V and W be representations of G with character χ and ψ respectively. Then, $V \cong W$ if and only if $\chi(g) = \psi(g)$ for all $g \in G$. I.e. any representation V is determined, up to isomorphism, by its character χ_V .*

Let us consider $\mathbb{C}_{\text{class}}(G) = \{\text{class functions on } G\}$. Then, we can define the following Hermitian inner product on it:

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

In terms of this inner product, the characters of the irreducible representations of G are orthonormal. In fact, we have the following properties.

Proposition A.4.3. *A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$. Moreover, the multiplicity a_i of V_i in V is $a_i = \langle \chi_{V_i}, \chi_V \rangle$.*

For the induced representation $V = \text{Ind } W$, note that $g \in G$ maps σW to $g\sigma W$. Then, the trace is computed from those cosets σ with $g\sigma = \sigma$, i.e. $s^{-1}gs \in H$ for $s \in \sigma$. Therefore,

$$\chi_{\text{Ind } W}(g) = \sum_{g\sigma = \sigma} \chi_W(s^{-1}gs),$$

with $s \in \sigma$ arbitrary.

Theorem A.4.4 (Frobenius reciprocity). *If W is a representation of the subgroup $H \subseteq G$, and U is a representation of G , then*

$$\langle \chi_{\text{Ind } W}, \chi_U \rangle_G = \langle \chi_W, \chi_{\text{Res } U} \rangle_H.$$

Note that for W and U irreducible representations, Frobenius reciprocity says that the number of times that U appears in $\text{Ind } W$ is the same as the number of times that W appears in $\text{Res } U$.

A.5. Representations of the symmetric group

In this subsection, we state the irreducible representations of the symmetric group \mathfrak{S}_n . By the Proposition A.2.2, we already know that the number of irreducible representation of \mathfrak{S}_n is the number of conjugacy classes of \mathfrak{S}_n , which is the number of partitions of n .

Young diagrams can be used to describe projection operators for the regular representation, which will give the irreducible representations of \mathfrak{S}_n . Consider the canonical tableau of shape λ , with λ a partition of n . We can define two subgroups of the symmetric group \mathfrak{S}_n :

$$\begin{aligned} P &= P_\lambda = \{g \in \mathfrak{S}_n : g \text{ preserves each row}\}, \\ Q &= Q_\lambda = \{g \in \mathfrak{S}_n : g \text{ preserves each column}\}. \end{aligned}$$

In the group algebra $\mathbb{C}\mathfrak{S}_n$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \text{sgn}(g) \cdot e_g.$$

In order to understand what a_λ and b_λ do, note that if V is any vector space and \mathfrak{S}_n acts on the n^{th} tensor power $V^{\otimes n}$ by permuting factors, the image of the element $a_\lambda \in \mathbb{C}\mathfrak{S}_n \rightarrow \text{End}(V^{\otimes n})$ is the subspace

$$\text{Sym}^{\lambda_1} V \otimes \text{Sym}^{\lambda_2} V \otimes \cdots \otimes \text{Sym}^{\lambda_k} V \subset V^{\otimes n},$$

where the inclusion is obtained by grouping the factors of $V^{\otimes n}$ according to the rows of the Young tableaux. Similarly, the image of b_λ on the tensor power is



$$\bigwedge^{\mu_1} \otimes \bigwedge^{\mu_2} \otimes \cdots \otimes \bigwedge^{\mu_l} \subset V^{\otimes n},$$

where $\mu = \lambda'$. Finally, we set $c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_n$, which is called a *Young symmetrizer*. **Theorem A.5.1.** *The image of c_λ by the right multiplication on $\mathbb{C}\mathfrak{S}_n$ is an irreducible representation V_λ of \mathfrak{S}_n . In fact, every irreducible representation of \mathfrak{S}_n can be obtained in this way for a unique partition.*




This theorem gives a direct correspondence between the conjugacy classes in \mathfrak{S}_n and the irreducible representations of \mathfrak{S}_n .

Example 20. We show the representations of the first symmetric groups.






\mathfrak{S}_2

		<u>Character table</u>
Trivial, U	Alternating, U'	
		[1] [(12)]
		U 1 1
		U' 1 -1

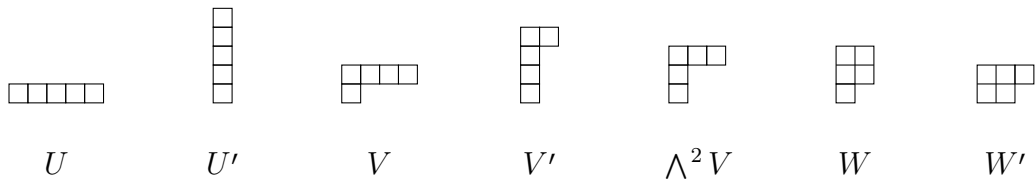
\mathfrak{S}_3

		<u>Character table</u>
Trivial, U	Alternating, U'	
		[1] [(12)] [(123)]
		U 1 1 1
Standard, V		U' 1 -1 1
		V 2 0 -1

\mathfrak{S}_4

					
U	U'	V	V'	W	
<u>Character table</u>					
	[1]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
Trivial, U	1	1	1	1	1
Alternating, U'	1	-1	1	-1	1
Standard, V	3	1	0	-1	-1
$V \otimes U' = V'$	3	-1	0	1	-1
W	2	0	-1	0	2

\mathfrak{S}_5



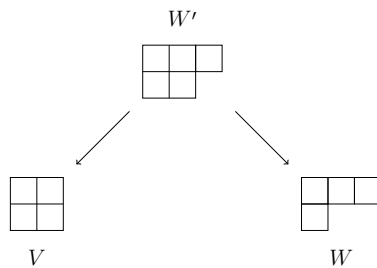
Character table

	[1]	[(12)]	[(123)]	[(1234)]	[(12345)]	[(12)(34)]	[(12)(345)]
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1

As we mentioned in the previous subsection, the restriction of an irreducible representation is not always irreducible. In fact, we have the following branching rule:

Proposition A.5.2. *The finite-dimensional irreducible representation of \mathfrak{S}_n associated to a partition μ is isomorphic to the direct sum of the irreducible representations of \mathfrak{S}_{n-1} associated to the partitions obtained from μ by removing a box from the Young diagram of μ .*

Example 21. Consider the irreducible representation of \mathfrak{S}_5 associated to $\mu = (3, 2)$, which we have denoted by W' above. We have the following options:



where V and W are irreducible representations of \mathfrak{S}_4 , also described above. Then, the restriction of W' to \mathfrak{S}_4 , $\text{Res } W'$, is isomorphic to the direct sum of the irreducible representations V and W of \mathfrak{S}_4 .

Let C_α denote the conjugacy class in \mathfrak{S}_n determined by a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, with $\sum j \cdot \alpha_j = n$, i.e. C_α consists on those permutations that have α_j j -cycles, for $1 \leq j \leq n$. Now, we introduce independent variables x_1, x_2, \dots, x_k , with k at least as large as the number of rows in the Young diagram of λ , $k \geq \ell(\lambda)$. We define the following polynomials:

$$\begin{aligned} p_j(x) &= x_1^j + x_2^j + \dots + x_k^j \quad \text{for } 1 \leq j \leq n, \\ \Delta(x) &= \prod_{i < j} (x_i - x_j). \end{aligned}$$

where x denotes the set of variables x_1, x_2, \dots, x_k . The polynomials p_j are the power sums symmetric functions.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, set $\ell_j = \lambda_j + k - j$, for $j = 1, 2, \dots, k$. Then $\ell = (\ell_1, \ell_2, \dots, \ell_k)$ is a strictly decreasing sequence of k non-negative integers.

Proposition A.5.3 (Frobenius Formula). *The character of V_λ evaluated on $g \in C_\alpha$, $\chi_\lambda(C_\alpha)$ or $\chi_\lambda(\alpha)$, is given by the coefficient of x^ℓ in the polynomial $\Delta(x) \cdot \prod_j p_j(x)^{\alpha_j}$.*

A.6. Schur functor and the representations of the general linear group

The correspondence between representations of symmetric groups and representations of general linear groups was given by Schur. First, we need to introduce the *Schur functor*. The symmetric group \mathfrak{S}_d acts on $V^{\otimes d}$, say on the right, by permuting the factors

$$(v_1 \otimes \dots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}.$$

This action commutes with the left action of $GL(V)$ and provides to $V^{\otimes d}$ of an \mathfrak{S}_d -module structure. For any partition λ of d , we consider the Young symmetrizer c_λ in $\mathbb{C}\mathfrak{S}_d$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda V$, which is again a representation of $GL(V)$. The function $V \rightsquigarrow \mathbb{S}_\lambda V$ is called the *Schur functor*. The functoriality means that a linear map $\varphi : V \rightarrow W$ of vector spaces determines a linear map $\mathbb{S}_\lambda(\varphi) : \mathbb{S}_\lambda V \rightarrow \mathbb{S}_\lambda W$, with $\mathbb{S}_\lambda(\varphi \circ \psi) = \mathbb{S}_\lambda(\varphi) \circ \mathbb{S}_\lambda(\psi)$ and $\mathbb{S}_\lambda(\text{Id}_V) = \text{Id}_{\mathbb{S}_\lambda V}$.

Consider the general linear group $GL_n\mathbb{C}$. Let $V = \mathbb{C}^n$ be the standard representation for $GL_n\mathbb{C}$. We also consider the representation $D_k := (\wedge^k V)^{\otimes n}$, for $k \geq 0$. These representations correspond to the one-dimensional representations of $GL_n\mathbb{C}$ given by the k^{th} power of the determinant. Then, we extend the definition to negative integers k .

For any index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length n , we consider $\lambda = (a_1 + a_2 + \dots + a_n, a_2 + \dots + a_n, \dots, a_{n-1} + a_n, a_n)$. We consider the Schur functors \mathbb{S}_λ applied to the standard representation V of $GL_n\mathbb{C}$. We denote this representation by Ψ_λ . Note that $\Psi_{\lambda_1+k, \dots, \lambda_n+k} = \Psi_\lambda \otimes D_k$. Likewise, this allows us to define Ψ_λ , for any λ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. In case that some of the λ_i are negative, we simply define $\Psi_\lambda = \Psi_{\lambda_1+k, \dots, \lambda_n+k} \otimes D_{-k}$, for any k large enough.

Proposition A.6.1. *Every irreducible complex representation of $GL_n\mathbb{C}$ is isomorphic to Ψ_λ for a unique index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$.*

A.7. The wreath product group and its representations

Let m and n be positive integers. The *wreath product of \mathfrak{S}_m with \mathfrak{S}_n* , $\mathfrak{S}_m[\mathfrak{S}_n]$, is defined to be the normaliser of the Young subgroup $\underbrace{\mathfrak{S}_n \times \dots \times \mathfrak{S}_n}_m$ in \mathfrak{S}_{mn} . The

wreath product is also denoted by $\mathfrak{S}_m \wr \mathfrak{S}_n$. The elements of $\mathfrak{S}_m[\mathfrak{S}_n]$ are denoted by $(x_1, \dots, x_m; \sigma)$, where $x_i \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_m$. A complete description of the elements of $\mathfrak{S}_m[\mathfrak{S}_n]$ can be found in [Sun96].

Consider W a representation of \mathfrak{S}_n and V a representation of \mathfrak{S}_m . The *wreath product module* $V[W]$ is the following representation of $\mathfrak{S}_m[\mathfrak{S}_n]$. As a vector space, $V[W]$ is the tensor product $W^{\otimes n} \otimes V$. The action of $\mathfrak{S}_m[\mathfrak{S}_n]$ on $V[W]$ is defined as follows: For $w_i \in W$, $1 \leq i \leq m$, and $v \in V$,

$$(x_1, \dots, x_m; \sigma) \cdot (w_1 \otimes \dots \otimes w_m \otimes v) = x_1 w_{\sigma^{-1}(1)} \otimes \dots \otimes x_m w_{\sigma^{-1}(m)} \otimes (\sigma \cdot v).$$

We can also consider the restricted and the induced representations. We described here the last ones. The embedding of $\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$ as a Young subgroup in $\mathfrak{S}_{m_1+m_2}$ induces an embedding of $\mathfrak{S}_{m_1}[\mathfrak{S}_n] \times \mathfrak{S}_{m_2}[\mathfrak{S}_n]$ as a subgroup of $\mathfrak{S}_{m_1+m_2}[\mathfrak{S}_n]$. Let V_i be a representation of \mathfrak{S}_{m_i} , with $i = 1, 2$, and W be a representation of \mathfrak{S}_n . Then, the following induced representations are isomorphic:

$$\text{Ind}_{\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}}^{\mathfrak{S}_{m_1+m_2}} (V_1 \otimes V_2)[W] \quad \text{and} \quad \text{Ind}_{\mathfrak{S}_{m_1}[\mathfrak{S}_n] \times \mathfrak{S}_{m_2}[\mathfrak{S}_n]}^{\mathfrak{S}_{m_1+m_2}[\mathfrak{S}_n]} (V_1[W] \otimes V_2[W]).$$

Appendix B.

The ring of symmetric functions

In this appendix we include an introduction to the theory of symmetric functions. We present Sym , the ring of symmetric functions, as the inverse limit of the graded rings of symmetric polynomials. We describe several bases of Sym , taking special interest on the Schur basis.

More details about symmetric functions can be found in [Mac95] and [Sta99], among others.

B.1. The symmetric functions

Let us consider the ring $\mathbb{Q}[x_1, \dots, x_n]$ of polynomials in independent variables x_1, \dots, x_n , with integer coefficients. The symmetric group \mathfrak{S}_n acts on it by permuting the variables. We say that a polynomial is *symmetric* if it is invariant under this action. The symmetric polynomials form a subring

$$Sym^n = \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}.$$

This ring is a graded ring by the degree of the homogeneous symmetric polynomials. Thus,

$$Sym^n = \bigoplus_{k \geq 0} Sym_k^n,$$

where Sym_k^n consists on the homogeneous symmetric polynomials of degree k , together with the zero polynomial.

Let λ be a partition of $\ell(\lambda) \leq n$. Then, the polynomial $m_\lambda(x_1, \dots, x_n) = \sum_{\alpha} x^\alpha$ summing over all distinct permutations α of λ is symmetric. In fact, they form a basis of Sym^n , $\{m_\lambda : \lambda \in \text{Par} \text{ and } \ell(\lambda) \leq n\}$.

The number of variables is usually irrelevant in the framework of symmetric functions, provided only that it is large enough. Actually, for the work we present in this thesis, it is more convenient to work with symmetric functions in infinite many variables. We consider $m \geq n$ and the homomorphism $\mathbb{Q}[x_1, \dots, x_m] \rightarrow \mathbb{Q}[x_1, \dots, x_n]$, which sends each of x_{n+1}, \dots, x_m to zero and the other x_i to themselves. Its restriction to Sym^m gives the homomorphism $\rho_{m,n} : Sym^m \rightarrow Sym^n$ that sends $m_\lambda(x_1, \dots, x_m)$ to $m_\lambda(x_1, \dots, x_n)$ if $\ell(\lambda) \leq n$, and to 0 otherwise. It follows that $\rho_{m,n}$ is surjective. We consider its restriction to Sym_k^m , $\rho_{m,n}^k : Sym_k^m \rightarrow Sym_k^n$, which is also an homomorphism for all $k \geq 0$ and $m \geq n$. These homomorphisms are always surjective and they are bijective for $m \geq n \geq k$. We now form the inverse limit on the \mathbb{Q} -modules Sym_k^n relative to the homomorphisms $\rho_{m,n}^k$

$$Sym_k = \varprojlim_n Sym_k^n.$$

Then, by definition, an element of Sym_k is a sequence $f = (f_n)_{n \geq 0}$, where each $f_n = f(x_1, \dots, x_n)$ is an homogeneous symmetric polynomial of degree k in x_1, \dots, x_n and, whenever $m \geq n$, $f_m(x_1, \dots, x_n, 0, \dots, 0) = f_n(x_1, \dots, x_n)$. Since $\rho_{m,n}^k$ is an isomorphism for $m \geq n \geq k$, the projection

$$\begin{aligned} \rho_n^k : Sym_k &\longrightarrow Sym_k^n \\ f &\longmapsto f_n, \end{aligned}$$

is an isomorphism for all $n \geq k$. This implies that $\{m_\lambda : \lambda \in \text{Par}(k)\}$ is a basis of Sym_k . Hence, Sym_k is a free \mathbb{Q} -module. We set

$$Sym = \bigoplus_{k \geq 0} Sym_k.$$

Therefore, Sym is the free \mathbb{Q} -module generated by the m_λ , for all $\lambda \in \text{Par}$. Moreover, we can consider the surjective homomorphisms

$$\rho_n = \bigoplus_{k \geq 0} \rho_n^k : Sym \longrightarrow Sym^n,$$

for each $n \geq 0$. Indeed, ρ_n is an isomorphism for $k \leq n$. Sym has a structure of a graded ring such that ρ_n are ring homomorphisms. The graded ring Sym defined as above is called *ring of symmetric functions* in countably many independent variables x_i .

Note that the elements of Sym , unlike those of Sym^n , are no longer polynomials: they are formal infinite sums of monomials. Other observation is that Sym is not the inverse limit of the rings Sym^n , in the category of rings, relative to the

homomorphisms $\rho_{m,n}$. Denote by \widehat{Sym} this other inverse limit. The infinite product $\prod(1 + x_i)$ belongs to \widehat{Sym} but not to Sym , since the elements of Sym are, by definition, finite sums of monomial symmetric functions m_λ . However, Sym is the inverse limit of Sym^n in the category of graded rings.

B.2. Bases of Sym

We have defined Sym using the monomial symmetric functions. There are other important bases for Sym . For a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$, we denote by $x^\alpha = x_1^{\alpha_1} \dots x_k^{\alpha_k}$. The following list summarizes them, including the monomial basis.

- *Monomial symmetric functions*

Given $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots) \vdash n$, we define $m_\lambda \in Sym^n$ by

$$m_\lambda = \sum_{\alpha} x^\alpha,$$

where the sum ranges over all distinct permutations α of the entries of the vector λ . For instance

$$\begin{aligned} m_0 &= 1, \\ m_k &= \sum_{i \geq 1} x_i^k, \\ m_{32} &= \sum_{i < j} (x_i^3 x_j^2 + x_i^2 x_j^3). \end{aligned}$$

- *Elementary symmetric functions*

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, we define

$$\begin{aligned} e_0 &= 1, \\ e_n &= m_{(1^n)} = \sum_{i_1 < \dots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}, \quad \text{for } n \geq 1, \\ e_n &= 0, \quad \text{for } n < 0, \\ e_\lambda &= e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}. \end{aligned}$$

For instance,

$$\begin{aligned} e_4 &= m_{(1111)} = \sum_{i_1 < \dots < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}, \\ e_{22} &= e_2 \cdot e_2 = m_{11} \cdot m_{11} = m_{22} + 2m_{211} + 6m_{1111}. \end{aligned}$$

· Complete homogeneous symmetric functions

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition. Then,

$$\begin{aligned} h_0 &= 1, \\ h_n &= \sum_{\mu \vdash n} m_\mu = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad \text{for } n \geq 1, \\ h_n &= 0, \quad \text{for } n < 0, \\ h_\lambda &= h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}. \end{aligned}$$

Thus, h_n is the sum of all monomials of degree n . For instance,

$$\begin{aligned} h_2 &= m_{11} + m_2 = \sum_{i \geq 1} x_i^2 + \sum_{i < j} x_i x_j, \\ h_{11} &= h_1 \cdot h_1 = \sum_{i \geq 1} x_i^2 + 2 \sum_{i < j} x_i x_j. \end{aligned}$$

· Power sums symmetric functions

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition. Then,

$$\begin{aligned} p_0 &= 1, \\ p_n &= m_n = \sum_{i \geq 1} x_i^n, \\ p_\lambda &= p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}. \end{aligned}$$

For instance,

$$p_{22} = p_2^2 = \sum_{i \geq 1} x_i^4 + \sum_{i < j} x_i^2 x_j^2.$$

Notation. The bases are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \dots)$. For simplifying the notation, we write the numbers corresponding to the partition as indices.

The definition of p_λ , h_λ , and e_λ can be extended to general sequences, not necessarily weakly decreasing. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be any sequence of integers. Then,

$$h_\alpha = h_{\alpha_1} \cdot h_{\alpha_2} \cdots h_{\alpha_k}.$$

In fact, h_α is equal to h_β , for β any permutation of the parts of α . Similarly, p_α and e_α are defined for any sequence of integers α .

Proposition B.2.1. *The collections $\{m_\lambda : \lambda \vdash n\}$, $\{e_\lambda : \lambda \vdash n\}$, $\{h_\lambda : \lambda \vdash n\}$, and $\{p_\lambda : \lambda \vdash n\}$ are all bases of Sym^n over \mathbb{Q} .*

In fact, the monomial basis, the complete homogeneous basis, and the elementary basis are bases of Sym^n over \mathbb{Z} .

Proposition B.2.2 (Fundamental Theorem of Symmetric Functions). *Sym can be seen as the \mathbb{Q} -algebra generated by $\{e_n : n \geq 1\}$, which are algebraically independent. I.e. $Sym = \mathbb{Q}[e_1, e_2, \dots]$ and any symmetric function can be expressed as a function in the elementary symmetric functions.*

From Theorem B.2.2 we can deduce that the complete homogeneous, the monomial and the power sums symmetric functions are also bases for Sym .

B.3. Generating Functions

Let z be an extra variable.

Proposition B.3.1. *The generating function of the complete homogeneous and the elementary bases are*

$$\begin{aligned}\sigma_z(X) &= \sum_{n \geq 0} h_n z^n = \prod_{x \in X} \frac{1}{1 - zx}, \\ \lambda_z(X) &= \sum_{n \geq 0} e_n z^n = \prod_{x \in X} (1 + zx).\end{aligned}$$

Moreover, from these expressions, we get that $\sigma_z(X) \cdot \lambda_{-z}(X) = 1$.

For the power sums basis, we denote by Ψ_z its generating function

$$\Psi_z(X) := \sum_{n \geq 1} p_n z^{n-1} = \sum_{n \geq 1} \frac{x_n}{1 - x_n z}.$$

We can also express it in terms of $\sigma_z(X)$ as

$$\Psi_z(X) = \frac{z \cdot \sigma'_z(X)}{\sigma_z(X)}.$$

B.4. Involution and scalar product

By Theorem B.2.2, any algebra endomorphism $f : Sym \rightarrow Sym$ is determined uniquely by its values $f(e_n)$, for $n \geq 1$.

Definition B.4.1. We define an endomorphism $\omega : Sym \rightarrow Sym$ by saying $\omega(e_n) = h_n$, for all $n \geq 1$. Since ω preserves multiplication, $\omega(e_\lambda) = h_\lambda$ for any partition λ .

Proposition B.4.2. *The endomorphism ω is an involution, i.e. $\omega^2 = 1$ (the identity automorphism). In fact, ω is called the involution of Sym .*

Proposition B.4.3. *Let $\lambda \vdash n$ and $\epsilon_\lambda = (-1)^{n-\ell(\lambda)}$. Then,*

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda.$$

We want to define over Sym a scalar product, $Sym \times Sym \rightarrow \mathbb{Q}$. Since Sym has the structure of vector space, we define the scalar product using the monomial and the complete homogeneous bases.

Definition B.4.4. We define a *scalar product* on Sym by requiring that $\{m_\lambda\}$ and $\{h_\mu\}$ are dual bases, i.e. $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$, for all partitions λ and μ .

Notation. We set $\delta_{\lambda,\mu}$ for the *Kronecker delta* of partitions. It takes value 1 if $\lambda = \mu$, and 0 otherwise.

Proposition B.4.5. *The scalar product defined above has the following properties:*

- *The scalar product respects the grading on Sym , i.e. if f and g are homogeneous symmetric functions with $\deg(f) \neq \deg(g)$, then $\langle f, g \rangle = 0$.*
- *The involution ω is an isometry: for all $f, g \in Sym$, $\langle \omega(f), \omega(g) \rangle = \langle f, g \rangle$.*
- *For the power sums basis, we have that*

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu},$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i} \cdot m_i!$, for the partition $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$. Hence, $\{p_\lambda\}$ is an orthogonal basis for Sym , but it is not an orthonormal basis.

B.5. Schur basis

In addition to the bases already introduced, there is another interesting basis: the Schur basis. Initially, the Schur functions are indexed by partitions. Here we introduce directly the Schur functions indexed by skew partitions, the *skew Schur functions*. They are a generalization of the Schur functions indexed by partitions, and they will be an important object along this thesis. For setting the combinatorial definition of the skew Schur functions, we start with a finite number of variables.

Definition B.5.1. Let $X = \{x_1, \dots, x_n\}$ be a finite alphabet, and λ/μ be a skew shape. The *skew Schur function* is defined by

$$s_{\lambda/\mu} = \sum_T x^T$$

summing over all semi-standard Young tableaux T of shape λ/μ , where x^T denote $x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \dots x_n^{\alpha_n(T)}$, for $\alpha(T) = (\alpha_1(T), \alpha_2(T), \dots, \alpha_n(T)) = \text{type}(T)$.

If $\mu = \emptyset$, then $\lambda/\mu = \lambda$ and $s_\lambda(X)$ is called the *Schur function* associated to λ . If λ/μ is not a skew shape, $s_{\lambda/\mu} = 0$.

From this combinatorial definition, we see that if $\ell(\lambda) < n$, then $s_\lambda(X) = 0$. Moreover, the number of variables will determinate the largest part that will appear in the semi-standard Young tableaux.

Example 22. The semi-standard Young tableaux of shape $(2, 2)$ with largest part at most 4 are given by

1 1 2 2	1 1 2 3	1 1 2 4	1 1 3 3	1 1 3 4	1 1 4 4	1 2 2 3	1 2 2 4	1 2 3 3	1 2 3 4
1 2 4 4	1 3 2 4	1 3 3 4	1 3 4 4	2 2 3 3	2 2 3 4	2 2 4 4	2 3 3 4	2 3 4 4	3 3 4 4

Using the definition for Schur functions, we have that

$$\begin{aligned}
 s_{22}(x_1, \dots, x_4) &= x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3^2 + x_1^2 x_3 x_4 + x_1^2 x_4^2 + x_1 x_2^2 x_3 + \\
 &+ x_1 x_2^2 x_4 + x_1 x_2 x_3^2 + x_1 x_2 x_3 x_4 + x_1 x_2 x_4^2 + x_1 x_2 x_3 x_4 + x_1 x_3^2 x_4 + \\
 &+ x_1 x_3 x_4^2 + x_2^2 x_3^2 + x_2^2 x_3 x_4 + x_2^2 x_4^2 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2 + x_3^2 x_4^2
 \end{aligned}$$

Thus,

$$s_{22}(x_1, \dots, x_4) = m_{22}(x_1, \dots, x_4) + m_{211}(x_1, \dots, x_4) + 2m_{1111}(x_1, \dots, x_4).$$

From the combinatorial definition it is not clear that s_λ is a symmetric function.

Proposition B.5.2. *For any skew shape λ/μ , the skew Schur functions $s_{\lambda/\mu}$ is a symmetric function. In fact, the Schur functions $s_\lambda(x_1, \dots, x_n)$, with $\ell(\lambda) \leq n$, form a \mathbb{Q} -basis of Sym^n .*

In the notation of Subsection B.1, we have that

$$\rho_{n+1,n}(s_\lambda(x_1, \dots, x_{n+1})) = s_\lambda(x_1, \dots, x_n).$$

Hence, for each partition λ , the polynomials $s_\lambda(x_1, \dots, x_n)$, as $n \rightarrow \infty$, define a unique element, $s_\lambda \in Sym$, homogeneous of degree $|\lambda|$.

By Proposition B.5.2, we have immediately.

Proposition B.5.3. *The set $\{s_\lambda : \lambda \in Par\}$ forms a \mathbb{Q} -basis of Sym and for each $k \geq 0$, the set $\{s_\lambda : \lambda \vdash k\}$ forms a \mathbb{Q} -basis of Sym_k .*

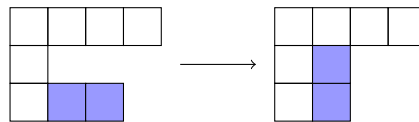
The Jacobi–Trudi formula, 1.3.13, can be used to define the Schur function $s_{\lambda/\mu}$ when λ and μ are sequences of integers, non–necessarily partitions. For instance, consider $\lambda = (4, 1, 3)$. Then,

$$s_{413} = \det \begin{pmatrix} h_4 & h_0 & h_1 \\ h_5 & h_1 & h_2 \\ h_6 & h_2 & h_3 \end{pmatrix} = h_{431} + h_{521} + h_{62} - h_{611} - h_{422} - h_{53} = -s_{422}.$$

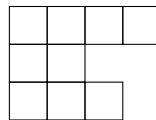
In fact, we have the following result.

Lemma B.5.4. *Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a sequence of integers. If there exist i and j such that $\alpha_{i+j} = \alpha_i + \alpha_j$, then $s_{\alpha} = 0$. If there is an α_i such that $\alpha_i \leq i - \ell(\alpha) - 1$, then $s_{\alpha} = 0$.*

This can be seen easily using Young diagrams: consider the Young diagram of the sequence $(4, 1, 3)$. As we see in the draw, we can rotate the blue block of two boxes and we obtain the partition $(4, 2, 2)$:



Thus, $s_{413} = -s_{422}$. However, if we consider the sequence $(4, 2, 3)$, there is no possible rotation of any block in the third row in order to obtain a partition:



Therefore, $s_{423} = 0$.

There are too many properties about the skew Schur functions. We summarize some of them here.

Proposition B.5.5. *The skew Schur functions satisfies the following properties:*

1. *The basis $\{s_{\lambda} \mid \lambda \in \text{Par}\}$ is an orthonormal basis for Sym .*
2. *For any $\lambda, \mu \in \text{Par}$, $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$. In particular, $\omega(s_{\lambda}) = s_{\lambda'}$.*
3. *For any $f \in \text{Sym}$, we have*

$$\langle f \cdot s_{\mu}, s_{\lambda} \rangle = \langle f, s_{\lambda/\mu} \rangle.$$

This means that the following linear transformations

$$\begin{array}{ll} D_\lambda: \text{Sym} & \longrightarrow \text{Sym} & M_\mu: \text{Sym} & \longrightarrow \text{Sym} \\ s_\mu & \longmapsto s_{\mu/\lambda} & f & \longmapsto s_\mu \cdot f \end{array}$$

are adjoint with respect to the scalar product. In particular,

$$\langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\nu, s_{\lambda/\mu} \rangle. \quad (\text{B.1})$$

This means that the skew Schur functions are the adjoint of the multiplication by a Schur function. Precisely, we define

$$\begin{array}{ll} s_\mu^\perp: \text{Sym} & \longrightarrow \text{Sym} \\ s_\lambda & \longmapsto s_\mu^\perp(s_\lambda) = s_{\lambda/\mu} \end{array}$$

4. We have that $s_\mu s_n = \sum_\lambda s_\lambda$, summing over all partitions λ such that λ/μ is an horizontal strip of size n (semi-standard Young tableaux that do not contain two boxes in the same column).

Appendix C.

Frobenius characteristic map

In this section we describe the relation between the representation theory and the theory of symmetric functions: the *Frobenius characteristic map*.

Consider $R_n = \mathbb{C}_{\text{class}}(\mathfrak{S}_n)$, the space of class functions on \mathfrak{S}_n . Then, there is an intimate connection between R_n and Sym_n . First of all, $\dim(R_n) = \dim(Sym_n) = \#\text{Par}(n)$. Therefore, they are isomorphic as vector spaces. We also have an inner product on R_n for which the irreducible characters on \mathfrak{S}_n form an orthonormal basis, and an inner product on Sym_n for which the Schur symmetric functions s_λ with $\lambda \vdash n$ form an orthonormal basis. We define a map to preserve these inner products.

Definition C.0.6. The *Frobenius characteristic map* is defined by

$$\begin{aligned} \text{ch}^n : R_n &\longrightarrow Sym_n \\ \chi &\longmapsto \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi(\mu) p_\mu, \end{aligned}$$

where $\chi(\mu)$ is the value of χ on the class μ , recalling that the classes of \mathfrak{S}_n are in bijection with the partitions of n . The map ch^n is linear. Furthermore, if we apply ch^n to the character of the irreducible representation of the symmetric group \mathfrak{S}_n , χ_λ , then $\text{ch}^n(\chi_\lambda) = s_\lambda$. Since ch^n takes one orthonormal basis to another, we immediately have that ch^n is an isometry between R_n and Sym_n .

Now consider $R = \bigoplus R_n$, which is isomorphic to Sym via the characteristic map $\text{ch} = \bigoplus \text{ch}^n$. But Sym also has the structure of graded algebra. To construct the corresponding product in R_n , we consider χ and ψ characters of \mathfrak{S}_n and \mathfrak{S}_m , respectively. We want to produce a character of \mathfrak{S}_{n+m} . The tensor product $\chi \otimes \psi$ gives us a character of $\mathfrak{S}_n \times \mathfrak{S}_m$, and induction gets us into the group we want. Therefore, we define a product on R by setting $\chi \cdot \psi = \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(\chi \otimes \psi)$, and extending it by bilinearity.

Theorem C.0.7. The map $\text{ch} : R \longrightarrow Sym$ is an isomorphism of algebras.

Summary

This thesis presents a recent study of two families of coefficients: the plethysm and the Kronecker coefficients. These two families arise from representation theory and theory of symmetric functions.

On one hand, in 1950, Foulkes observed some stability properties in sequences of plethysm coefficients depending on a parameter n : they are eventually constant for n large enough. These properties were proved in the 1990's by Carré and Thibon, using vertex operators and other arguments from the combinatorics of symmetric functions, and by Brion for algebraic groups in general (rather than just general linear groups) and using tools from geometric representation theory. We reproduce a detailed proof of the results proved by Carré and Thibon in order to obtain the bounds for which the coefficients are constant.

We also present a combinatorial interpretation of other plethysm coefficients, the h -plethysm coefficients, defined through the homogeneous complete basis. These coefficients are directly related with the plethysm coefficients by the *Jacobi-Trudi formula*. The combinatorial interpretation of the h -plethysm coefficients describes them as the number of integral points in a polytope depending on the partitions indexing the coefficients. Using this new interpretation, we provide a combinatorial proof for the stability properties already proved by Brion, and Carré and Thibon.

On the other hand, in 1938, Murnaghan observed a stability phenomenon in the Kronecker coefficients: the sequence of Kronecker coefficients whose indexing partitions have an increasing first part is eventually constant. The reduced Kronecker coefficients can be defined as the stable value of these sequence of Kronecker coefficients. The reduced Kronecker coefficients are interesting objects of their own right, and we can recover the Kronecker coefficients from them. We investigate what happen when we add boxes to other rows and columns of the partitions indexing reduced Kronecker coefficients. We present a study for four families.

For the first family of reduced Kronecker coefficients we give explicit formulas of the piecewise linear quasipolynomials of period 2 that define them, depending on the indexing partitions. The other three families have in common that one of the indexing partitions has only one part and that the other two partitions

are arbitrary large. For these three families, we present a complete study: the generation function for the reduced Kronecker coefficients and two combinatorial descriptions, one in terms of plane partitions fitting in a rectangle and the other as quasipolynomials, specifying the degree and the period. Moreover, we check that the *saturation hypothesis* holds for these three families of reduced Kronecker coefficients.

Another interesting approach for the reduced Kronecker coefficients is through the *vertex operators*. We include a proof of Murnaghan's Theorem using vertex operators. This proof provides a description of the reduced Kronecker coefficients obtained by Brion. Vertex operators are also used to give a description of the reduced Kronecker coefficients with one partition equals to (k) in terms of the Littlewood–Richardson coefficients.

Resumen

Esta tesis presenta el estudio de dos familias de coeficientes: los coeficientes del pletismo y los coeficientes de Kronecker. Ambas familias emergen de la teoría de representaciones y la teoría de funciones simétricas.

Por un lado, en 1950, Foulkes observó varias propiedades de estabilidad en sucesiones de coeficientes del pletismo dependientes de un parámetro n : las sucesiones son eventualmente constantes para n suficientemente grande. Estas propiedades fueron probadas en los 90 por Carré y Thibon, usando operadores vertex y otros argumentos combinatorios de funciones simétricas, y por Brion para grupos algebraicos en general (no solo para el grupo general lineal) y usando herramientas geométricas de la teoría de representaciones. Incluimos una prueba detallada de los resultados probados por Carré y Thibon con el fin de obtener las cotas para las que dichos coeficientes son constantes.

También presentamos una interpretación combinatoria de otros coeficientes del pletismo, los h -coeficientes del pletismo, definidos a partir de la base de funciones homogéneas. Estos coeficientes están relacionados directamente con los coeficientes del pletismo usuales mediante la *fórmula de Jacobi-Trudi*. Esta interpretación combinatoria de los h -coeficientes del pletismo los describe como el número de puntos enteros en un polítopo que depende de las particiones que indexan los coeficientes. Esta nueva interpretación nos permite dar una demostración combinatoria de las propiedades de estabilidad de Brion, Carré y Thibon.

Por otro lado, en 1938, Murnaghan observó un fenómeno de estabilidad en los coeficientes de Kronecker: la sucesión de coeficientes de Kronecker, cuyas particiones asociadas tienen una primera parte creciente, son eventualmente constantes. Los coeficientes de Kronecker reducidos se pueden definir como el valor estable de estas sucesiones de coeficientes de Kronecker. Los coeficientes de Kronecker reducidos son objetos interesantes en sí mismos, y podemos recuperar los coeficientes de Kronecker a partir de ellos. Nosotros investigamos qué ocurre cuando añadimos cajas a las filas y columnas de las particiones que indexan los coeficientes de Kronecker reducidos. Presentamos un estudio de cuatro familias.

Para la primera familia de coeficientes de Kronecker reducidos damos fórmulas explícitas de los quasipolinomios lineales de periodo 2 a trozos que los definen, dependiendo de las particiones asociadas. Las otras tres familias tienen en común que una de las particiones que las indexan tiene una sola parte, y que las otras dos particiones son arbitrariamente grandes. Para estas tres familias, presentamos un estudio completo: la función generatriz de los coeficientes de Kronecker reducidos y dos descripciones combinatorias, una en términos de particiones planas en un rectángulo y otra como quasipolinomios, especificando el periodo y el grado de los mismos. Además, comprobamos que la *hipótesis de saturación* se satisface para los coeficientes de Kronecker reducidos de estas tres familias.

Otro enfoque interesante para los coeficientes de Kronecker reducidos son los operadores vertex. Incluimos una prueba del teorema de Murnaghan usando operadores vertex. Esta prueba nos proporciona una descripción de los coeficientes de Kronecker reducidos obtenida por Brion. Los operadores vertex también son usados para dar una descripción de los coeficientes de Kronecker reducidos con una partición asociada de una sola parte en términos de los coeficientes de Littlewood–Richardson.

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