Programa de Doctorado "Matemáticas"

PhD Dissertation

Lagrangian submanifolds in complex space forms, the Maslov form in S-manifolds, generalized S-space forms and η-Einstein para-S manifolds.

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To my parents

Contents

PART I

INTRODUCTION

Acknowledgements

It was five years ago when I decided to start this project and, consequently, begin a completely new period in my life, full of aims, challenges and experiences to be fulfilled. Impossible to imagine how demanding and rewarding have they been!

Now, I must admit that I couldn't be here, writing this words, if you, mum and dad hadn't been an essential part of my life. You gave me this opportunity and now, I cannot find a better way, place and time to profoundly thank you. Thanks, grandparents, I really hope to have learnt something from your wisdom.

I would also like to thank you, Luis, because since that first Topology lesson, that now seems to be so far but at the same time so close, you showed me that any dream can become real by giving one hundred percent to get it. Thanks for walking with me from the very beginning until this end, which is nothing but the start point of a new stage.

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Thanks Carmen M., Diana, Loli, M. Ángeles, Pilar, Rosa, Rocío and Vero, for being with me and still continue making me laugh every time that we meet.

Thanks Pepa, I know that you are as glad as I am now.

Thanks Leo, Roberto, Robe and Alejandro for all your care, affection and advice, for being just my family.

Thanks Marisa, for being not only the best teacher but also the best friend possible, for knowing about S-manifolds as if you had studied Differential Geometry.

Resumen

En esta memoria presentamos cinco artículos que han sido publicados como resultado de la investigación que estamos realizando, parte de la cual se ha desarrollado en el Departamento de Geometría y Topología de la Universidad de Sevilla disfrutando de una beca predoctoral de la Fundación Cámara (1/10/2010 - 31/12/2013), bajo la dirección de los Drs. Luis M. Fernández Fernández y Pablo S. Alegre Rueda.

Dichos artículos se desarrollan en el ámbito de la Geometría Semi-Riemanniana y son los siguientes:

- A. Carriazo, J. Barrera, L.M. Fernández and A. Prieto-Martín, "The Maslov form in non-invariant slant submanifolds of S-space-forms". Ann. Mat. Pura Appl., **191** (2012), 803-818;
  DOI 10.1007/s10231-011-0207-0

- B.-Y. Chen and A. Prieto-Martín, "Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving $\delta(2,2)$". Diff. Geom. Appl., **30(1)** (2012), 107-123;
  DOI:10.1016/j.difgeo.2011.11.008

- B.-Y. Chen, A. Prieto-Martín and Xianfeng Wang, "Lagragian submanifolds in complex space forms satisfying an improved equality involving $\delta(2,2)$". Publ. Math. Debrecen, **82(1)** (2013), 193-217.

- A.M. Fuentes, L.M. Fernández and A. Prieto-Martín, "Generalized S-space forms". Publ. Inst. Math. (Beograd) N.S., **94(108)** (2013), 151-161;
  DOI:10.2998/PIM1308151P

- L.M. Fernández and A. Prieto-Martín, "On η-Einstein para-S-manifolds". Bull. Mal. Math. Sci. So. (2015);
  DOI 10.1007/s40840-015-0156-7

Abstract

In this memory we present five papers which have been published as a result of our researching, part of which has been done at the Department of Geometry and Topology of the University of Seville, thanks to a predoctoral grant of "Fundación Cámara" (1/10/2010 - 31/12/2013) under the supervision of Drs. Luis M. Fernández Fernández and Pablo S. Alegre Rueda.

These papers are developed in the subject of Semi-Riemannian Geometry being them the following:

- A. Carriazo, J. Barrera, L.M. Fernández and A. Prieto-Martín, "The Maslov form in non-invariant slant submanifolds of S-space-forms". Ann. Mat. Pura Appl., **191** (2012), 803-818;
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- B.-Y. Chen and A. Prieto-Martín, "Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving $\delta(2,2)$". Diff. Geom. Appl., **30(1)** (2012), 107-123;
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- B.-Y. Chen, A. Prieto-Martín and Xianfeng Wang, "Lagrangian submanifolds in complex space forms satisfying an improved equality involving $\delta(2,2)$". Publ. Math. Debrecen, **82(1)** (2013), 193-217.

- A.M. Fuentes, L.M. Fernández and A. Prieto-Martín, "Generalized S-space forms". Publ. Inst. Math. (Beograd) N.S., **94(108)** (2013), 151-161;
  DOI:10.2998/PIM1308151P

- L.M. Fernández and A. Prieto-Martín, "On η-Einstein para-S-manifolds". Bull. Mal. Math. Sci. So. (2015);
  DOI 10.1007/s40840-015-0156-7

Introduction

This thesis consists of a compendium of papers, so, according to the current regulations for this kind of thesis, we divide it in two different parts.

The first one is an introduction with three different sections; Goals, where we set up our work historically, motivate its study and establish our objectives. In the second section we summarize our main results and in the third one we talk about some open problems.

The second part consists of five published papers:

- [**A1**] B.-Y. Chen and A. Prieto-Martín, "Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving $\delta(2,2)$". Diff. Geom. Appl., **30(1)** (2012), 107-123; DOI:10.1016/j.difgeo.2011.11.008

- [**A2**] B.-Y. Chen, A. Prieto-Martín and Xianfeng Wang, "Lagrangian submanifolds in complex space forms satisfying an improved equality involving $\delta(2,2)$". Publ. Math. Debrecen, **82(1)** (2013), 193-217.

- [**A3**] A. Carriazo, J. Barrera, L.M. Fernández and A. Prieto-Martín, "The Maslov form in non-invariant slant submanifolds of S-space-forms". Ann. Mat. Pura Appl., **191** (2012), 803-818; DOI 10.1007/s10231-011-0207-0

- [**A4**] A.M. Fuentes, L.M. Fernández and A. Prieto-Martín, "Generalized S-space forms". Publ. Inst. Math. (Beograd) N.S., **94(108)** (2013), 151-161; DOI:10.2998/PIM1308151P

- [**A5**] L.M. Fernández and A. Prieto-Martín, "On η-Einstein para-S-manifolds". Bull. Mal. Math. Sci. So. (2015); DOI 10.1007/s40840-015-0156-7

They can be classify in two blocks, the so called [A1] and [A2] are referred to the Submanifold Theory in Complex Space-forms and the rest correspond to the study of manifolds with an f-structure (in the sense of K.Yano [52]), in particular of those defined as S-manifolds by D.E. Blair [7].

1. Goals.

1.1 Papers [A1] and [A2]

The study of submanifolds of a differential manifold is, from the very beginning of Differential Geometry, one of the most studied topics and additionally, one of those which has produced more interesting results and applications. Furthermore, the study of submanifolds which present an homogeneous behavior with respect to the structure of the ambient manifold has become an interesting research subject.

In particular, if the ambient space is an almost-Hermitian manifold, submanifolds which present this behaviour with respect to the almost-complex structure J are widely studied. So that, we can consider the so called complex submanifolds, where JX is tangent for any tangent vector field X or the totally real submanifolds, where JX is normal for any tangent vector field X. Among these ones, Lagrangian submanifolds, whose dimension is the half dimension of the ambient manifold, play an specially important role. Both complex submanifolds and totally real submanifolds were generalized by B.-Y. Chen [20, 21] when he introduced the notion of slant submanifold, where the angle θ between J_pX_p and T_pM is constant for any tangent vector field X and any point $p \in M$. Complex submanifolds and totally real submanifolds are slant submanifolds with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively.

Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics. For instance, the systems of partial differential equations of Hamilton-Jacobi type lead to the study of Lagrangian submanifolds and foliations in the cotangent bundle. Furthermore, Lagrangian submanifolds play some important roles in supersymmetric field theories as well as in string theory.

In Differential Geometry, theorems which relate intrinsic and extrinsic curvatures always play important roles. Intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds and to establish relationships between them is one of the most fundamental problems in submanifolds theory. In this context, B.-Y. Chen [23, 24, 25] proved some basic inequalities for submanifolds of a real space-form. Corresponding inequalities have been obtained for different kinds of submanifolds (invariant, anti-invariant, slant) in ambient manifolds endowed with different kinds of structures (mainly, real, complex and Sasakian space-forms).

The famous Nash embedding theorem published in 1956 [43] was aiming for the opportunity to use extrinsic help in the study of (intrinsic) Riemannian geometry, since Riemannian manifolds could be regarded as Riemannian submanifolds. However, this hope had not been materialized yet according to [37]. The main reason for this was the lack of control of the extrinsic properties of the submanifolds by the known intrinsic invariants. In order to overcome such difficulties as well as to provide answers to an open question on minimal immersions, B.-Y. Chen [22], introduced in the early 1990s new types of Riemannian invariants, denoted by $\delta(n_1, \ldots, n_k)$. For an n-dimensional submanifold M^n in a real space form $\mathbf{R}^m(c)$ of constant sectional curvature c, he proved the following sharp general inequality,

$$\delta(n_1, \ldots, n_k) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\Big)c, \tag{1}$$

where H^2 is the squared mean curvature of M^n.

An immersion satisfying the equality case of inequality (1) at every point is called $\delta(n_1, \ldots, n_k)$*-ideal*. Roughly speaking, an ideal immersion is an immersion which produces the least possible amount of tension from the ambient space.

It is known that inequality (1) holds for Lagrangian submanifolds in complex space-forms of constant holomorphic sectional curvature $4c$ as well (cf. [26, 27, 30]). Also, B.-Y. Chen proved in [28, Theorem 1] that every ideal Lagrangian submanifold of a complex space form is a minimal submanifold. In this context, $\delta(2)$-ideal submanifolds in real and complex space-forms have been studied by many geometers since the invention of δ-invariants.

In 2000, it was proved by B.-Y. Chen [26] that every Lagrangian submanifold M^5 of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies

$$\delta(2,2) \leq \frac{25}{3} H^2 + 8c, \tag{2}$$

where $\delta(2,2)$ is a δ-invariant of M^5.

Furthermore, it was proved in [29] that every Lagrangian submanifold M^5 of a complex space form $\widetilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the following optimal inequality:

$$\delta(2,2) \leq \frac{25}{4} H^2 + 8c. \tag{3}$$

Thus, in papers [A1] and [A2] we completely classify Lagrangian submanifolds of complex space forms $M^5(4c)$, for $c = 0, 1, -1$, satisfying the equality case of the inequality (2) and we also classify Lagrangian submanifolds of $M^5(4c)$ satisfying the equality case of the optimal inequality (3).

1.2 Paper [A3]

A $(2m+s)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ endowed with an f-structure f (that is, a tensor field f of type (1,1) and rank $2m$ satisfying $f^3 + f = 0$ (see [52]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \dots, \xi_s$ on $\widetilde{M}$ (called *structure vector fields*) such that, if $\eta_1, \dots, \eta_s$ are the dual 1-forms of $\xi_1, \dots, \xi_s$, then

$$f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0;\ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha; \tag{4}$$

$$g(X,Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{5}$$

for any X, Y tangent to $\widetilde{M}$. From the definition, the metric g satisfies that

$$g(fX, Y) = -g(X, fY), \tag{6}$$

for any X, Y. Let F be the 2-form on $\widetilde{M}$ defined by $F(X,Y) = g(X, fY)$. Since f is of rank $2m$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$ and, particularly, $\widetilde{M}$ is orientable. The f-structure f is said to be *normal* if

$$[f,f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ denotes the Nijenhuis tensor of f.

A metric f-manifold is said to be a K-manifold [7] if it is normal and $dF = 0$. In a K-manifold $\widetilde{M}$, the structure vector fields are Killing vector fields [7]. Furthermore, a K-manifold is called an S-manifold if $F = d\eta_\alpha$, for any α. Note that, if $s = 0$, a K-manifold would correspond to a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold and an S-manifold is a Sasakian manifold. When $s \geq 2$, non-trivial examples can be found in [7, 38]. Moreover, the Riemannian connection $\widetilde{\nabla}$ of an S-manifold satisfies (see [7]), for any tangent vector fields X, Y and any $\alpha = 1, \dots, s$:

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{7}$$

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X). \tag{8}$$

A plane section π on a metric f-manifold $\widetilde{M}$ is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c and then, it is denoted by $\widetilde{M}(c)$. In such case, the curvature tensor field R of $M(c)$ satisfies [40]

$$R(X,Y,Z,W) = \sum_{\alpha,\beta=1}^{s} (g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W)+$$

$$+g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z))+$$

$$+\frac{c+3s}{4}(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW))+$$

$$+\frac{c-s}{4}(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)), \quad (9)$$

for any tangent vector fields X, Y, Z, W.

On the other hand, for totally real submanifolds of almost Hermitian manifolds, one can consider the so-called Maslov form, defined as the dual form of the vector field JH, being J the almost Hermitian structure and H the mean curvature vector of the submanifold, which has been widely studied (for example, [8, 18, 19, 48] can be consulted). Thus, in [48], it is proved that any Lagrangian submanifold of $\mathcal{C}^m$ has closed Maslov form and, moreover, that the well-known Whitney sphere is the only compact Lagrangian submanifold of $\mathcal{C}^m$ with conformal Maslov form.

However, there are not too many papers devoted to study the Maslov form in anti-invariant submanifolds of metric almost contact manifolds or, more in general, of metric f-manifolds, considering such form as the dual form of the vector field ϕH (resp. fH), where ϕ (resp., f) denotes the almost contact structure (resp., the f-structure).

One of our main goals of paper [A3] is to deal with non-invariant slant submanifolds of S-manifold. In such submanifolds, we define the Maslov form as the dual 1-form of the tangent component of the vector field fH and our purpose is to find conditions for it to be closed and conformal in the case of being the ambient S-manifold an S-space-form, that is, of having constant f-sectional curvature.

1.3 Paper [A4]

In Riemannian geometry, it is an interesting problem to analyze what kind of Riemannian manifolds may be determined by special pointwise expressions for their curvatures. For instance, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if (M, g) is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field R has the pointwise expression

$$R(X,Y)Z = \lambda \{g(X,Z)Y - g(Y,Z)X\},$$

where λ is a differentiable function on M, then M is a space of constant sectional curvature, that is, a real-space-form and λ is a constant function.

Further, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from the special form of the curvature tensor field for this structure too. Thus, an almost-Hermitian manifold (M, J, g) is said to be a **generalized complex-space-form** [51] if its curvature tensor satisfies

$$R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} +$$

$$+f_2 \{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}, \tag{10}$$

where f_1 and f_2 are differentiable functions on M. This name derives from the fact that, when M is a complex-space-form, that is, a Kaehlerian manifold of constant holomorphic curvature equal to c, the curvature tensor field of M satisfies (10) with $f_1 = f_2 = c/4$.

Since Sasakian-spaces-forms play a similar role in contact metric geometry to that of complex-space-forms in complex geometry, P. Alegre, D.E. Blair and A. Carriazo have defined and studied **generalized Sasakian-space forms** [1] as those almost-contact metric manifolds (M, ϕ, ξ, η, g) whose curvature tensor field satisfies

$$R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} +$$

$$+f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} +$$

$$+f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \tag{11}$$

f_1, f_2, f_3 being differentiable functions on M. If M is actually a Sasakian-space-form, that is a Sasakian manifold with constant ϕ-sectional curvature equal to c, then:

$$f_1 = \frac{c+3}{4}; \; f_2 = f_3 = \frac{c-1}{4}.$$

For these reasons, we consider that it is interesting to introduce a notion of generalized S-space-form on metric f-manifolds. We observe that this work was made in [15] for metric f-manifolds with two structure vector fields, giving some interesting examples. In paper [A4], we present the definition for any number of structure vector fields. To this end, we have followed the same procedure as in almost complex and almost contact cases, that is, we have substituted the constants in the expression of the curvature tensor field of an S-space-form (an S-manifold of constant f-sectional curvature) obtained in [40] by certain differentiable functions on the manifold. So, S-space-forms are natural examples of generalized S-space-forms. Furthermore, we check that C-space-forms are also generalized S-space-forms.

1.4 Paper [A5]

Nowadays, one of the topics which has many applications and it is related to some physical problems (the nice survey [20] can be consulted for more details)is the study of paracomplex structures. When, moreover, a compatible pseudo-Riemannian metric is considered, we have the para-Hermitian and para-Kaehler manifolds and their variants.

On the other hand, (almost) paracontact manifolds are semi-Riemannian manifolds which can be viewed as the odd dimensional counterpart of (almost) paracomplex manifolds. They were introduced by Sato in [49] and Kaneyuki and Williams in [39]. Recently, there seems to be an increasing interest in paracontact geometry and, in particular, in para-Sasakian manifolds, due to its links to more consolidated theory of para-Kaehler manifolds and to their role in geometry and mathematical physics (see, for instance, [31, 32, 33]).

Actually, the notion of almost paracontact structure is an analogue of that one of almost contact structure and is closely related to the almost product structure. In this context, Bucki and Miernowski defined in [11] the notion of an almost r-paracontact structure which generalizes almost paracontact structure in a similar way to f-structures of co-rank greater than one generalizes almost contact structures. They also started the study of almost r-paracontact manifolds equipped with a Riemannian compatible metric [9, 10, 42]

So, it is interesting to study what happens if instead of a Riemannian metric we consider a pseudo-Riemannian metric and this is the goal of paper [A5]. Zamkovoy in [53] has obtained a complete arrangement of all the theory in the case of paracontact manifolds and recently, Brunetti and Pastore have done a similar work in the context of indefinite globally framed f-manifolds in [12]. For these reasons, we want to introduce in this work the notion of para-S-manifold and begin the study of some of its properties. The name is chosen to point out that it is the analogue of S-manifolds introduced by Blair [7] in the setting of f-structures.

2. Main results.

In the paper [A1], we study Lagrangian submanifolds in a 5-dimensional complex space form $\tilde{M}^5(4c)$, where this inequality (2) is verified.

By definition, a Lagrangian submanifold M^5 in $\tilde{M}^5(4c)$ is $\delta(2,2)$-ideal if and only if it satisfies the equality sign of (2) identically. A $\delta(2,2)$-ideal submanifold in $\tilde{M}^5(4c)$ is called *proper* if it is not a $\delta(2)$-ideal Lagrangian submanifold in $\tilde{M}^5(4c)$.

Our purpose is firstly, to classify proper $\delta(2,2)$-ideal Lagrangian submanifolds in $\mathbf{C}^5, CP^5(4)$ and $CH^5(-4)$. Now, we present some of the main and most original theorems of this work.

For $\mathbf{C}^5$, we prove:

Theorem. ([A1] 5.1) *Let $L : M^5 \to \mathbf{C}^5$ be a Lagrangian immersion into the complex Euclidean 5-space $\mathbf{C}^5$. Then L is a proper $\delta(2,2)$-ideal Lagrangian immersion if and only if L is locally congruent to one of the following immersions:*

(1) *the direct product of an open interval I of the real line in $\mathbf{C}$ and two non-totally geodesic Lagrangian minimal immersions $\phi_i : M_i^2 \to \mathbf{C}^2\, (i = 1, 2)$, i.e.,*

$$L : I \times M_1^2 \times M_2^2 \to \mathbf{C} \times \mathbf{C}^2 \times \mathbf{C}^2;\ (t, p, q) \mapsto (t, \phi_1(p), \phi_2(q)) \tag{12}$$

(2) *a Lagrangian immersion defined by*

$$L : I \times M_1^2 \times_t M_2^2 \to \mathbf{C}^2 \times \mathbf{C}^3;\ (t, p, q) \mapsto (\phi(p), t\zeta(q)), \tag{13}$$

where $\phi : M_1^2 \to \mathbf{C}^2$ is a non-totally geodesic Lagrangian minimal immersion and $\zeta : M_2^2 \to S^5(1) \subset \mathbf{C}^3$ is a non-totally geodesic Legendrian minimal immersion of M_2^2 into $S^5(1)$.

For $CP^5(4)$, we have:

Theorem. ([A1] 6.1) *Let $L : M^5 \to CP^5(4)$ be a Lagrangian immersion. Then M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifold if and only if L is locally congruent to $\pi \circ \tilde{L}$, where $\pi : S^{11}(1) \to CP^5(4)$ is the Hopf fibration, $\tilde{L} : M^5 \to S^{11}(1) \subset \mathbf{C}^6$ is given by*

$$\tilde{L}(t, p, q) = (\cos t)\phi_1(p) + (\sin t)\phi_2(q), \quad t \in \mathbf{R}, \tag{14}$$

and $\phi_i : M_i^2 \to S^5(1) \subset \mathbf{C}^3\, (i = 1, 2)$ are non-totally geodesic Legendrian minimal immersions into the Sasakian $S^5(1)$.

Finally, for $CH^5(-4)$, we prove:

Theorem. ([A1] 7.1) *Let $L : M^5 \to CH^5(-4)$ be a Lagrangian immersion of M^5 into $CH^5(-4)$. Then M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifold of $CH^5(-4)$ if and only if L is locally congruent to $\pi \circ \tilde{L}$, where $\pi : H_1^{11}(-1) \to CH^5(-4)$ is the Hopf fibration and either*

(a) *$\tilde{L} : M^5 \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ is given by*

$$\tilde{L}(t,p,q) = (\cosh t)\phi(p) + (\sinh t)\psi(q), \quad t \in \mathbf{R}, \tag{15}$$

and $\phi : M_1^2 \to H^5(-1) \subset \mathbf{C}_1^3$ and $\psi : M_2^2 \to S^5(1) \subset \mathbf{C}^3$ are non-totally geodesic Legendrian minimal immersions into the Sasakian $H_1^5(-1)$ and $S^5(1)$, resp., or

(b) *$\tilde{L} : M^5 \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ is given by*

$$\begin{aligned}\tilde{L}(t,x,y,z,w) = \Big(&\sinh t + e^t(u(z,y) + v(z,w) - 1),\\ &\sinh t + e^t(u(x,y) + v(z,w)), e^t\psi_1(x,y), e^t\psi_2(z,w)\Big),\end{aligned} \tag{16}$$

$\psi_i : M_2^2 \to \mathbf{C}^2$ $(i = 1,2)$ are non-totally geodesic minimal Lagrangian immersions, u, v are complex-valued functions satisfying the following PDE systems, respectively:

$$\begin{cases} u_{xx} = \left\{(\ln E_1)_x + \dfrac{i}{E_1^2}\right\}u_x - (\ln E_1)_y u_y - E_1^2,\\ u_{xy} = (\ln E_1)_y u_x + \left\{(\ln E_1)_x - \dfrac{i}{E_1^2}\right\}u_y,\\ u_{yy} = -\left\{(\ln E_1)_x + \dfrac{i}{E_1^2}\right\}u_x + (\ln E_1)_y u_y - E_1^2, \end{cases}$$

$$\begin{cases} v_{zz} = \left\{(\ln E_2)_z + \dfrac{i}{E_2^2}\right\}v_z - (\ln E_2)_w v_w - E_2^2,\\ v_{zw} = (\ln E_2)_w v_z + \left\{(\ln E_2)_z - \dfrac{i}{E_z^2}\right\}v_w,\\ v_{ww} = -\left\{(\ln E_2)_z + \dfrac{i}{E_2^2}\right\}v_z + (\ln E_2)_w v_w - E_2^2, \end{cases}$$

and the metric tensors of M_1^2, M_2^2 are given respectively by

$$g_1 = E_1^2(dx^2 + dy^2), \quad g_2 = E_2^2(dz^2 + dw^2)$$

for some positive functions $E_1 = E_1(x,y)$ and $E_2 = E_2(z,w)$.

In addition, it was proved by B.-Y. Chen and F. Dillen in 2011 [30] that every Lagrangian submanifold M^5 of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the optimal inequality (3). So, in paper [A2], we also classify Lagrangian submanifolds of $M^5(4c)$ satisfying this improved inequality.

First, we get:

Theorem. ([A2] 6.1) *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $\mathbf{C}^5$. Then it is one of the following Lagrangian submanifolds:*

(a) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(b) *an H-umbilical Lagrangian submanifold of ratio 4;*

(c) *a Lagrangian submanifold defined by*

$$L(\mu, u_2, \ldots, u_n) = \frac{e^{\frac{4}{3} i \tan^{-1} \sqrt{\mu^3/(c^2-\mu^3)}}}{\sqrt{c^2\mu^{-1} - \mu^2} + i\mu} \phi(u_2, \ldots, u_n), \tag{17}$$

where c is a positive real number and $\phi(u_2, \ldots, u_n)$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$.

On the other hand, we obtain:

Theorem. ([A2] 7.1) *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CP^5(4)$. Then it is one of the following Lagrangian submanifolds:*

(1) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(2) *an H-umbilical Lagrangian submanifold of ratio 4;*

(3) *a Lagrangian submanifold defined by*

$$L(\mu, u_2, \ldots, u_4) = \frac{1}{c}\Big(\sqrt{\mu} e^{i\theta}\phi, e^{3i\theta}(\sqrt{c^2 - \mu^3 - \mu} - i\mu^{\frac{3}{2}})\Big), \tag{18}$$

where c is a positive real number, $\phi : N^4 \to S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$, and $\theta(\mu)$ satisfies

$$\frac{d\theta}{d\mu} = \frac{1}{2\sqrt{c^2\mu^{-1} - \mu^2 - 1}}. \tag{19}$$

Finally, we have:

Theorem. ([A2] 8.1) *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CH^5(-4)$. Then M is one of the following Lagrangian submanifolds:*

(i) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(ii) *an H-umbilical Lagrangian submanifold of ratio 4;*

(iii) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \dots, u_4) = \frac{1}{c}\left(\sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2, \dots, u_4), e^{-\mathrm{i}\theta}(\sqrt{\mu - \mu^3 - c^2} - \mathrm{i}\mu^{\frac{3}{2}})\right), \tag{20}$$

where c is a positive number, $\phi : N^4 \to H_1^9(-1) \subset \mathbf{C}_1^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CH^4(-4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 - c^2\mu^{-1}}$;

(iv) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \dots, u_4) = \frac{1}{c}\left(e^{-\mathrm{i}\theta}(\sqrt{\mu - \mu^3 + c^2} - \mathrm{i}\mu^{\frac{3}{2}}), \sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2, \dots, u_4)\right), \tag{21}$$

where c is a positive number, $\phi : N^4 \to S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CP^4(4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 + c^2\mu^{-1}}$;

(v) *a Lagrangian submanifold defined by*

$$\begin{aligned} L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - \mathrm{i}\sinh t}&\Bigg(2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi, \psi\rangle - \frac{1}{4}\right), \\ &\psi,\ 2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi, \psi\rangle + \frac{1}{4}\right)\Bigg), \end{aligned} \tag{22}$$

where $\psi(u_1, \dots, u_4)$ is a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $\mathbf{C}^4$ and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2\left\langle\psi_{u_j}, \mathrm{i}\psi\right\rangle$, $j = 1, 2, 3, 4$;

(vi) *a Lagrangian submanifold defined by*

$$\begin{aligned} L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - \mathrm{i}\sinh t}&\Bigg(2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi, \psi\rangle - \frac{1}{4}\right), \\ &\psi_1, \psi_2,\ 2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi, \psi\rangle + \frac{1}{4}\right)\Bigg), \end{aligned} \tag{23}$$

where $\psi = (\psi_1, \psi_2)$ is the direct product immersion of two non-totally geodesic Lagrangian minimal immersions $\psi_\alpha : N_\alpha^2 \to \mathbf{C}^2$, $\alpha = 1, 2$, and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2\left\langle\psi_{u_j}, \mathrm{i}\psi\right\rangle$, $j = 1, 2, 3, 4$.

Recall that, given a Riemannian manifold (M, g), a vector field X in M is said to be *closed* in M if the 1-form ω given by $\omega_X(Y) = g(X, Y)$ (the dual 1-form of X) is closed. Then, X is closed if and only if

$$g(Y, \nabla_Z X) = g(Z, \nabla_Y X), \tag{24}$$

for any vector fields Y, Z in M, where ∇ is denoting the Riemannian connection of M. On the other hand, X is called *conformal* in M (and the dual 1-form is also called *conformal* in M) if $L_X g = \rho g$, being ρ a differentiable function on M. A closed vector field X is conformal if and only if

$$\nabla_Y X = hY, \tag{25}$$

for any vector field Y in M, being h a differentiable function on M.

In paper [A3] we consider $(m+s)$-dimensional (being s the number of structure vector fields) non-invariant slant submanifolds of an S-space-form of dimension $2m+s$ and we prove the following two theorems:

Theorem. ([A3] 4.1) Let M^{m+s} be an $(m+s)$-dimensional S-slant submanifold of an S-space-form $M^{2m+s}(c)$ of dimension $2m+s$. Then, the Maslov form is closed if and only if $c = -3s$.

Theorem. ([A3] 4.2) *Let M^{m+s} be an anti-invariant submanifold of an S-space-form $M^{2m+s}(c)$ of dimension $2m+s$, tangent to the structure vector fields. Then, ω_H is closed if and only if $c = -3s$.*

From above theorems we prove the following topological obstruction to S-slant immersions as well as to anti-invariant immersions tangent to the structure vector fields into an S-space-form of constant f-sectional curvature $c = -3s$:

Theorem. ([A3] 4.3) *Let M^{m+s} be a compact simply-connected $(m+s)$-dimensional differentiable manifold. Then, M can not be immersed in any S-space-form $M^{2m+s}(-3s)$ of dimension $2m+s$ as a non-minimal anti-invariant submanifold tangent to the structure vector fields. Moreover, if m is even, M can not be immersed in such a S-space-form as a non-minimal S-slant submanifold either. In particular, if $m = 2$, M cannot be immersed in $M(-3s)$ as a non-minimal and non-invariant slant submanifold with no minimal points.*

Theorem. ([A3] 5.4) *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-manifold M^{2m+s} of dimension $2m+s$, tangent to the structure vector fields and such that its Maslov form is closed. Then, this Maslov form is conformal in M if and only if the mean curvature vector is parallel.*

Finally, in this paper we prove:

Theorem. ([A3] 5.5) *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(-3s)$ of dimension $2m+s$, tangent to the structure vector fields. If*

$$\sigma(X,Y) = \frac{m+s}{m+s+1}\{g(fX,fY)H - (\omega_H(X) + \frac{m+s+1}{m+s}\sum_{\alpha=1}^{s}\eta_\alpha(X))fY -$$

$$-(\omega_H(Y) + \frac{m+s+1}{m+s}\sum_{\alpha=1}^{s}\eta_\alpha(Y))fX\}, \tag{26}$$

for any tangent vector fields X, Y tangent to M, then the Maslov form of M is $\mathcal{L}$-conformal.

As we said before, generalized S-space-forms with two structure vector fields were defined in [15], giving some interesting examples. Now, in paper [A4], we present the definition for any number of structure vector fields. A metric f-manifold $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$ is said to be a *generalized S-space-form* if there exists a family of differentiable functions on M,

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\},$$

such that the curvature tensor field R of M satisfies

$$R = F_1R_1 + F_2R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}R_{\alpha\beta} + \sum_{1\leq\alpha<\beta\leq s} G_{\alpha\beta}\widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}R_{\alpha\beta\gamma}, \tag{27}$$

where

$$\begin{aligned}
R_1(X,Y,Z,W) =& g(X,W)g(Y,Z) - g(X,Z)g(Y,W);\\
R_2(X,Y,Z,W) =& F(X,W)F(Y,Z) - F(X,Z)F(Y,W)\\
&- 2F(X,Y)F(Z,W);\\
R_{\alpha\beta}(X,Y,Z,W) =& g(Y,W)\eta_\alpha(X)\eta_\beta(Z) - g(X,W)\eta_\alpha(Y)\eta_\beta(Z)\\
&+ g(X,Z)\eta_\alpha(Y)\eta_\beta(W) - g(Y,Z)\eta_\alpha(X)\eta_\beta(W);\\
\widetilde{R}_{\alpha\beta}(X,Y,Z,W) =& \eta_\alpha(X)\eta_\beta(Y)\eta_\beta(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\beta(Z)\eta_\alpha(W)\\
&+ \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\beta(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\beta(W);\\
R_{\alpha\beta\gamma}(X,Y,Z,W) =& \eta_\alpha(X)\eta_\beta(Y)\eta_\gamma(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\gamma(Z)\eta_\alpha(W)\\
&+ \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\gamma(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\gamma(W),
\end{aligned} \tag{28}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$.

So, S-space-forms are natural examples of generalized S-space-forms (see [40]). Furthermore, we check that C-space-forms are also generalized S-space-forms.

Now, let us suppose that M is a generalized S-space-form such that the distribution spanned by the structure vector fields is flat (for instance, if M is either a metric f-K-contact manifold or a K-manifold, see [34]).Then, we prove the following results:

Theorem. ([A4] 5.1) *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$, such that $\nabla\xi_\alpha = -f$, for any $\alpha = 1, \ldots, s$. Then, M is an S-manifold and*

$$\begin{aligned}
&F_1 = \frac{c+3s}{4}; F_2 = \frac{c-s}{4}; F_{\alpha\alpha} = \frac{c+3s}{4} - 1;\\
&F_{\alpha\beta} = -1\ (\alpha \neq \beta); G_{\alpha\beta} = \frac{c+3s}{4} - 2\ (\alpha < \beta);\\
&H_{\alpha\beta\gamma} = -1\ (\alpha \neq \beta \neq \gamma \neq \alpha),
\end{aligned}$$

where $\alpha, \beta, \gamma \in \{1, \ldots, s\}$ and $c = F_1 + 3F_2$. In particular, any generalized S-space-form with a metric f-K-contact-structure is an S-manifold.

Theorem. ([A4] 5.2) *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and with an underlying C-structure. Then*

$$F_1 = F_2 = F_{\alpha\alpha} = G_{\alpha\beta} = \frac{c}{4},\ \alpha < \beta;$$

$$F_{\alpha\beta} = H_{\alpha\beta\gamma} = 0,\ \alpha \neq \beta \neq \gamma \neq \alpha,$$

where $\alpha, \beta, \gamma \in \{1, \ldots, s\}$ and $c = F_1 + 3F_2$. Moreover, if $n > 1$, M is a C-space-form.

In paper [A5], we introduce the notion of para-S-manifold as follows:

Let M be a $(2n+s)$-dimensional smooth manifold. It is said to have an *almost para-f-structure* $(f, \eta_1, \ldots, \eta_s, \xi_1, \ldots, \xi_s)$ and it is called an *almost para-f-manifold* if it admits a tensor field f of type $(1,1)$, s global tangent vector fields $\xi_1, \ldots, \xi_s$, called the *structure vector fields* and s 1-forms $\eta_1, \ldots, \eta_s$, satisfying the following compatibility conditions:

- $f(\xi_\alpha) = 0$, $\eta_\alpha \circ f = 0$, $\alpha = 1, \ldots, s$;
- $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, s$;

- $f^2 = Id - \sum_{\alpha=1^s} \eta_\alpha \otimes \xi_\alpha$ and the eigendistributions of f corresponding to the eigenvalues 1 and -1, denoted by $\mathcal{D}^+$ and $\mathcal{D}^-$ respectively, have the same dimension equal to n.

If an almost para-f-manifold M admits a pseudo-Riemannian metric g such that

$$g(fX, fY) + g(X, Y) = \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{29}$$

for any $X, Y \in TM$, we say that M is a *metric almost para-f-manifold* and g is called a *compatible metric*.

On a metric almost para-f-manifold, we define a 2-form by $F(X, Y) = g(X, fY)$, for any $X, Y \in TM$. Moreover, an almost para-f-estructure is said to be *normal* if

$$[f, f](X, Y) = 2\sum_{\alpha=1}^{s} d\eta_\alpha(X, Y)\xi_\alpha,$$

where $[f, f]$ is denoting the Nijenhuis tensor of f

A para-K-manifold is a normal almost para-f-manifold such that $\mathrm{d}F = 0$. A para-S-manifold is a normal para-f-manifold. In these cases, the structures are called para-K-structure and para-S-structure, respectively.

In this context, we firstly prove the following theorem.

Theorem. ([A5] 2) *For $s \geq 2$ there are not Einstein para-S-manifolds.*

This motivates, as in the case of Sasakian geometry, to introduce the notion of η-Einstein para-S-manifold. We say that a para-S-manifold M is an η-Einstein manifold if its Ricci tensor field satisfies

$$Ric = ag + b\sum_{\alpha=1}^{s} \eta_\alpha \otimes \eta_\alpha + (a + b)\sum_{\alpha\neq\beta}^{s} \eta_\alpha \otimes \eta_\beta, \tag{30}$$

where a and b are differentiable functions on M.

Then, we obtain:

Theorem. ([A5] 3) *Let M be a $(2n + s)$-dimensional η-Einstein para-S-manifold. If we assume that the foliation generated by the structure vector fields is regular, then M projects onto an Einstein para-Kaehler manifold.*

Moreover, for ξ-conformally flatness, we get:

Theorem. ([A5] 4) *Let M be a $(2n + s)$-dimensional η-Einstein para-S-manifold with $n \geq 1$. Then:*

(i) If $s = 1$, that is, is M is a para-Sasakian manifold, M is ξ-conformally flat.

(ii) If $s = 2$, M is ξ-conformally flat if and only if $a = -4n$.

(iii) If $s > 2$, M cannot be ξ-conformally flat.

Theorem. ([A5] 5) *A para-Sasakian manifold M is ξ-conformally flat if and only if it is an η-Einstein manifold.*

Finally, for $s = 2$ we have:

Theorem. ([A5] 6) *Let M be a ξ-conformally flat para-S-manifold with two structure vector fields. Then, M is an η-Einstein manifold with $a = -4n$.*

3. Open problems.

As a consequence of all these results, we have now many interesting open-problems that we have started to work on.

- We have proven charaterization theorems for the Maslov form in certain submanifolds of S-spaces-forms to be closed. So now, we would like to know what happens when the ambient manifold is a generalized S-space-form.

- In 1985 J. Oubiña in [45] introduced a new class of contact metric manifolds, called trans-Sasakian manifolds. If there are smooth functions (α, β) on an almost contact metric manifold (M, ϕ, ξ, η, g) satisfying

  $$(\nabla\phi)(X,Y) = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X), \quad (31)$$

  then this is said to be a trans-Sasakian manifold, where $(\nabla\phi)(X,Y) = (\nabla_X\phi)Y - \phi\nabla_X Y$, $X, Y \in \mathcal{X}(M)$ and ∇ is the Levi-Civita connection with respect to the metric g.

  Moreover, A.M. Fuentes, in her PhD thesis [36], develops the concept of generalized S-space forms, giving some examples by using warped products. In theses examples, we observe that ∇f has an expression that seems to generalized (31), for $s \geq 1$. Therefore, we have also introduced the notion of trans-S-manifolds in [4], as a generalization of trans-Sasakian manifolds; so that now, trans-Sasakian manifolds are the particular case in which the f-metric manifold has one structure vector field. As particular cases of trans-S-manifolds we have S-manifolds and f-Kenmotsu manifolds.

In fact, we have found non-trivial examples which justify this new notion and we are now trying to prove some characterization theorems. We would also like to study the submanifolds of these manifolds, specially when the trans-S-manifold has an additional structure of generalized S-space-form and try to prove characterization theorems for the Maslov form to be close in this case as well.

- In relation to generalized S-space-forms, since the use of different geometrical constructions is a very important tool to obtain interesting non-trivial examples of them, we want to consider other metric changes, such as $\mathcal{D}$-homothetic and $\mathcal{D}$-conformal transformations. In the case $s = 1$, this work has been done in [2].

  Moreover, we think it is interesting to study submanifolds of generalized S-space-forms. We want to highlight the case of the almost semi-invariant submanifols, which has been recently studied in [3]. So, we would like to obtain some inequalities where the Ricci curvature and the Scalar curvature appear (not depending on the chosen metric).

  In addition, in [5, 6] it was proven that an S-manifold endowed with a semi-symmetric connection (metric or non-metric) is a generalized S-space-form of constant sectional curvature if and only if it is a generalized S-space-form with respect to the Levi-Civita connection. Therefore, we want to check if there exist generalized S-space-forms endowed with such a semi-symmetric connection and, if case, give examples. For the case $s = 1$, this work has already been done in [50].

- We also want to study para f-manifolds which are not para S-manifolds, giving new and interesting examples. In particular, topics as the behaviour of the curvature tensor fields or $\mathcal{D}$-homothetic transformations should be considered.

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PART II

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Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving $\delta(2,2)$ ☆

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ABSTRACT

Lagrangian submanifolds appear naturally in the context of classical mechanics. They play important roles in geometry as well as in physics. It was proved by B.-Y. Chen in (2000) [6] that every Lagrangian submanifold M^5 of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies

$$\delta(2,2) \leqslant \frac{25}{3}H^2 + 8c, \tag{A}$$

where H^2 is the squared mean curvature and $\delta(2,2)$ is a δ-invariant of M^5 (cf. Chen, 2000, 2011 [6,9]). The main purpose of this paper is to completely classify Lagrangian submanifolds of complex space forms $\tilde{M}^5(4c)$, $c = 0, 1, -1$, satisfying the equality case of the inequality (A) identically.

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1. Introduction

Let $\tilde{M}^n$ be a Kähler n-manifold endowed with the complex structure J and with a Kähler metric g. The Kähler 2-form ω is defined by $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. An isometric immersion $\psi : M^n \to \tilde{M}^n$ of a Riemannian n-manifold M^n into $\tilde{M}^n$ is called *Lagrangian* if $\psi^*\omega = 0$. Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics. For instance, the systems of partial differential equations of Hamilton–Jacobi type lead to the study of Lagrangian submanifolds and foliations in the cotangent bundle. Furthermore, Lagrangian submanifolds play some important roles in supersymmetric field theories as well as in string theory.

In differential geometry theorems which relate intrinsic and extrinsic curvatures always play important roles. Related with Nash's embedding theorem [16], the first author introduced in [3,4,6] a new type of Riemannian invariants, denoted by $\delta(n_1,\ldots,n_k)$. For an n-dimensional submanifold M^n in a real space form $R^m(c)$ of constant sectional curvature c, he proved the following sharp general inequality:

$$\delta(n_1,\ldots,n_k) \leqslant \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right)c, \tag{1.1}$$

where H^2 is the squared mean curvature of M^n.

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An immersion satisfying the equality case of inequality (1.1) at every point is called *$\delta(n_1,\dots,n_k)$-ideal*. Roughly speaking, an ideal immersion is an immersion which produces the least possible amount of tension from the ambient space.

It is known that inequality (1.1) holds for Lagrangian submanifolds in complex space forms of constant holomorphic sectional curvature $4c$ as well (cf. [6,9,10]). Also, the first author proved in [7, Theorem 1] that every ideal Lagrangian submanifold of a complex space form is a minimal submanifold. $\delta(2)$-ideal submanifolds in real and complex space forms have been studied by many geometers since the invention of δ-invariants (see [1] and [9, Chapter 20] for details).

For Lagrangian submanifolds in a 5-dimensional complex space form $\tilde M^5(4c)$, inequality (1.1) reduces to

$$\delta(2,2)\leqslant \frac{25}{3}H^2+8c. \tag{1.2}$$

By definition, a Lagrangian submanifold M^5 in $\tilde M^5(4c)$ is $\delta(2,2)$-ideal if and only if it satisfies the equality sign of (1.2) identically. A $\delta(2,2)$-ideal submanifold in $\tilde M^5(4c)$ is called *proper* if it is not a $\delta(2)$-ideal Lagrangian submanifold in $\tilde M^5(4c)$.

The main purpose of this paper is to classify proper $\delta(2,2)$-ideal Lagrangian submanifolds in $\mathbf{C}^5$, $CP^5(4)$ and $CH^5(-4)$.

2. Preliminaries

2.1. Basic formulas

Let $\tilde M^n(4c)$ denote a complete simply-connected Kähler n-manifold with constant holomorphic sectional curvature $4c$. Then it is well-known that $\tilde M^n(4c)$ is holomorphically isometric to the complex Euclidean n-plane $\mathbf{C}^n$, the complex projective n-space $CP^n(4c)$, or a complex hyperbolic n-space $CH^n(-4c)$ according to $c=0$, $c>0$ or $c<0$.

Let M^n be a Lagrangian submanifold of $\tilde M^n(4c)$. We denote the Levi-Civita connections of M and $\tilde M^n(4c)$ by ∇ and $\tilde\nabla$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [2])

$$\tilde\nabla_X Y=\nabla_X Y+h(X,Y), \tag{2.1}$$

$$\tilde\nabla_X \xi=-A_\xi X+D_X\xi, \tag{2.2}$$

for tangent vector fields X and Y and normal vector fields ξ, where h is the second fundamental form, A is the shape operator and D is the normal connection.

The second fundamental form h is related to the shape operator A by

$$\langle h(X,Y),\xi\rangle=\langle A_\xi X,Y\rangle.$$

The mean curvature vector $\vec H$ of M^n is defined by

$$\vec H=\frac{1}{n}\operatorname{trace}h,$$

and the *squared mean curvature* is given by $H^2=\langle\vec H,\vec H\rangle$.

For Lagrangian submanifolds, we have (cf. [9,12])

$$D_X JY=J\nabla_X Y, \tag{2.3}$$

$$A_{JX}Y=-Jh(X,Y)=A_{JY}X. \tag{2.4}$$

Formula (2.4) implies that $\langle h(X,Y),JZ\rangle$ is totally symmetric.

The equations of Gauss and Codazzi are given respectively by

$$\langle R(X,Y)Z,W\rangle=\langle A_{h(Y,Z)}X,W\rangle-\langle A_{h(X,Z)}Y,W\rangle+c\big(\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle\big), \tag{2.5}$$

$$(\nabla_X h)(Y,Z)=(\nabla_Y h)(X,Z) \tag{2.6}$$

for X,Y,Z,W tangent to M, where R is the curvature tensor of M^n and ∇h is defined by

$$(\nabla_X h)(Y,Z)=D_X h(Y,Z)-h(\nabla_X Y,Z)-h(Y,\nabla_X Z). \tag{2.7}$$

For an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM^n at a point $p\in M^n$, we put

$$h^i_{jk}=\langle h(e_j,e_k),Je_i\rangle,\quad i,j,k=1,\dots,n.$$

It follows from (2.4) that

$$h^i_{jk}=h^j_{ik}=h^k_{ij}. \tag{2.8}$$

2.2. Horizontal lift of Lagrangian submanifolds

The following link between Legendrian submanifolds and Lagrangian submanifolds is due to [17] (see [9, pp. 247–248]).

Case (i): $CP^n(4)$. Consider Hopf's fibration $\pi : S^{2n+1} \to CP^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at u is the orthogonal complement of $\mathrm{i}u$, $\mathrm{i} = \sqrt{-1}$, with respect to the metric on S^{2n+1} induced from the metric on $\mathbf{C}^{n+1}$. Let $\iota : N \to CP^n(4)$ be a Lagrangian isometric immersion. Then there is a covering map $\tau : \hat{N} \to N$ and a horizontal immersion $\hat{\iota} : \hat{N} \to S^{2n+1}$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Thus each Lagrangian immersion can be lifted locally (or globally if N is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $CP^n(4)$ is lifted to a minimal Legendrian submanifold of the Sasakian $S^{2n+1}(1)$.

Conversely, suppose that $f : \hat{N} \to S^{2n+1}$ is a Legendrian isometric immersion, then $\iota = \pi \circ f : N \to CP^n(4)$ is again a Lagrangian isometric immersion. Under this correspondence the second fundamental forms h^f and h^ι of f and ι satisfy $\pi_* h^f = h^\iota$. Moreover, h^f is horizontal with respect to π.

Case (ii): $CH^n(-4)$. We consider the complex number space $\mathbf{C}_1^{n+1}$ equipped with the pseudo-Euclidean metric:

$$g_0 = -dz_1\, d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j\, d\bar{z}_j.$$

Consider the anti-de Sitter spacetime

$$H_1^{2n+1}(-1) = \left\{z \in \mathbf{C}_1^{2n+1}\colon\ \langle z, z\rangle = -1\right\}$$

with the canonical Sasakian structure, where $\langle , \rangle$ is the induced inner product.

Put $T_z' = \{u \in \mathbf{C}^{n+1}\colon\ \langle u, z\rangle = 0\}$, $H_1^1 = \{\lambda \in \mathbf{C}\colon\ \lambda\bar{\lambda} = 1\}$. Then there is an H_1^1-action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(-1)$, the vector $\xi = -\mathrm{i}z$ is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\langle \xi, \xi\rangle = -1$. The quotient space $H_1^{2n+1}(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4 whose complex structure J is induced from the complex structure J on $\mathbf{C}_1^{n+1}$ via Hopf's fibration $\pi : H_1^{2n+1}(-1) \to CH^n(4c)$.

Just like case (i), suppose that $\iota : N \to CH^n(-4)$ is a Lagrangian immersion, then there is an isometric covering map $\tau : \hat{N} \to N$ and a Legendrian immersion $f : \hat{N} \to H_1^{2n+1}(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus every Lagrangian immersion into $CH^n(-4)$ can be lifted locally (or globally if N is simply-connected) to a Legendrian immersion into $H_1^{2n+1}(-1)$. In particular, Lagrangian minimal submanifolds of $CH^n(-4)$ are lifted to Legendrian minimal submanifolds of $H_1^{2n+1}(-1)$.

Conversely, if $f : \hat{N} \to H_1^{2n+1}(-1)$ is a Legendrian immersion, then $\iota = \pi \circ f : N \to CH^n(-4)$ is a Lagrangian immersion. Under this correspondence the second fundamental forms h^f and h^ι are related by $\pi_* h^f = h^\iota$. Also, h^f is horizontal with respect to π.

2.3. Existence and uniqueness theorem for Lagrangian minimal surfaces

We need the following theorem from [5, Corollary 3.6] for later use.

Theorem 2.1. *Let $L : M^2 \to \tilde{M}^2(4c)$ be a Lagrangian minimal immersion without totally geodesic points. Then with respect to a suitable isothermal coordinate system (x, y) we have*

(1) *the metric tensor of M^2 is given by $g = E^2(dx^2 + dy^2)$ such that E satisfies*

$$\Delta(\ln E) = \frac{2 - cE^6}{E^4}, \qquad \Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}, \tag{2.9}$$

(2) *the second fundamental form of L satisfies*

$$h(\partial_x, \partial_x) = \frac{J\partial_x}{E_1^2}, \qquad h(\partial_x, \partial_y) = -\frac{J\partial_y}{E_1^2}, \qquad h(\partial_y, \partial_y) = -\frac{J\partial_x}{E_1^2}. \tag{2.10}$$

Conversely, if $E = E(x, y)$ is a positive function defined on a simply-connected domain U of the 2-plane $\mathbf{R}^2$ satisfying (2.9) for some real number c, then up to rigid motions there exists a unique Lagrangian minimal immersion from $M^2 = (U, g)$, $g = E^2(dx^2 + dy^2)$, into a complete simply-connected complex space form $\tilde{M}^2(4c)$ whose second fundamental form satisfies (2.10).

By applying Theorem 2.1 and the link via Hopf's fibration given in Section 2.2, we have the following.

Corollary 2.1. *If E is a positive function defined on a simply-connected domain U of $\mathbf{R}^2$ satisfying (2.9) for $c = 1$ (respectively $c = -1$) then there exists a Legendrian minimal immersion from $M^2 = (U, g)$, $g = E^2(dx^2 + dy^2)$, into the Sasakian $S^5(1)$ (resp., the Sasakian $H_1^5(-1)$) whose second fundamental form satisfies*

$$h(\partial_x, \partial_x) = \frac{\phi(\partial_x)}{E_1^2}, \qquad h(\partial_x, \partial_y) = -\frac{\phi(\partial_y)}{E_1^2}, \qquad h(\partial_y, \partial_y) = -\frac{\phi(\partial_x)}{E_1^2}, \tag{2.11}$$

where ϕ is the $(1,1)$-tensor of $S^5(1)$ (resp., of $H_1^5(-1)$) induced from the complex structure on $\mathbf{C}^3$ (resp., on $\mathbf{C}_1^3$).

3. δ-invariants and fundamental inequalities

Let M^n be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of M^n associated with a plane section $\pi \subset T_pM^n$, $p \in M^n$. For any orthonormal basis $e_1, \ldots, e_n$ of T_pM^n, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{3.1}$$

Let L be an r-subspace of T_pM^n with $r \geqslant 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha, \beta \leqslant r. \tag{3.2}$$

For given integers $n \geqslant 3$ and $k \geqslant 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers satisfying

$$2 \leqslant n_1, \ldots, n_k < n \quad \text{and} \quad n_1 + \cdots + n_k \leqslant n.$$

Denote the union $\bigcup_{k \geqslant 1} \mathcal{S}(n,k)$ by $\mathcal{S}(n)$. For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the first author introduced in [6] the Riemannian invariant $\delta(n_1, \ldots, n_k)$ defined by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \quad p \in M^n, \tag{3.3}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_j = n_j$, $j = 1, \ldots, k$. The invariants $\delta(n_1, \ldots, n_k)$ and the scalar curvature τ are very much different in nature (cf. [8,9] for details).

The following fundamental relation between $\delta(n_1, \ldots, n_k)$ and the squared mean curvature H^2 was proved in [6].

Theorem A. *Let M^n be an n-dimensional submanifold in a real space form $R^m(c)$ of constant curvature c. Then for each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \ldots, n_k) \leqslant \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^k n_j(n_j-1)\right)c. \tag{3.4}$$

The equality case of inequality (3.4) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ at p such that the shape operator of M^n in $R^m(c)$ at p with respect to $\{e_1, \ldots, e_m\}$ takes the form:

$$A_r = \begin{bmatrix} \begin{matrix} A_1^r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k^r \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mu_r I \end{bmatrix}, \quad r = n+1, \ldots, m, \tag{3.5}$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying

$$\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \mu_r.$$

The same result holds for a Lagrangian submanifolds in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. More precisely, we have

Theorem B. *Let M^n be an n-dimensional Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then, for each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k) \leqslant \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^k n_j(n_j-1)\right)c. \tag{3.6}$$

The equality case of inequality (3.6) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ at p, such that the shape operators of M in $\tilde{M}^n(4c)$ at p with respect to $\{e_1, \ldots, e_m\}$ take the form of (3.5).

The following result was proved in [7] which extends a result in [10,11] on $\delta(2)$.

Theorem C. *Every Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ that satisfies the equality case of (3.6) identically for a k-tuple $(n_1, \ldots, n_k)$ is minimal.*

4. Some lemmas

Now we provide some lemmas to be used in the proofs of our main theorems.

Lemma 4.1. *A Lagrangian submanifold M^5 of a complex space form $\tilde{M}^5(4c)$ satisfies*

$$\delta(2,2) \leqslant \frac{25}{3}H^2 + 8c. \tag{4.1}$$

If the equality sign of (4.1) holds identically, then M^5 is a minimal submanifold. Moreover, the second fundamental form h of M^5 satisfies

$$\begin{aligned}
&h(e_1,e_1) = h^1_{11}Je_1 + h^2_{11}Je_2,\\
&h(e_1,e_2) = h^2_{11}Je_1 - h^1_{11}Je_2,\\
&h(e_2,e_2) = -h^1_{11}Je_1 - h^2_{11}Je_2,\\
&h(e_3,e_3) = h^3_{33}Je_3 + h^4_{33}Je_4,\\
&h(e_3,e_4) = h^4_{33}Je_3 - h^3_{33}Je_4,\\
&h(e_4,e_4) = -h^3_{33}Je_3 - h^4_{33}Je_4,\\
&h(e_i,e_j) = 0, \quad \textit{otherwise},
\end{aligned} \tag{4.2}$$

with respect a suitable orthonormal frame $\{e_1, \ldots, e_5\}$.

Proof. This is an immediate consequence of Theorems B and C. □

Assume that M^n is a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Let $p \in M^n$ and V be a d-dimensional subspace of T_pM^n. Denote by $\pi_V : T_pM^n \to V$ the orthogonal projection. For each $v \in V$, we define a symmetric endomorphism A^V_{Jv} on V by $A^V_{Jv} = \pi_V \circ A_{Jv}$, where A_{Jv} is the shape operator at Jv.

We need the following lemma from [7, Lemma 1].

Lemma 4.2. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ and V be a d-dimensional subspace of T_pM^n at some point $p \in M^n$. Then there exists an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_d\}$ of V such that*

$$(4.1) \qquad A^V_{J\varepsilon_1}\varepsilon_i = \lambda_i\varepsilon_i, \quad i = 1, \ldots, d,$$

where $\lambda_1, \ldots, \lambda_d$ satisfy $\lambda_1 \geqslant 2\lambda_j$, $j = 2, \ldots, d$; $\lambda_1 > \lambda_j$ for $j = 2, \ldots, d$.

Lemma 4.3. *Let M^5 be a $\delta(2,2)$-ideal Lagrangian submanifold of a complex space form $\tilde{M}^5(4c)$. Then there exists an orthonormal frame $\{e_1, \ldots, e_5\}$ such that*

$$\begin{aligned}
&h(e_1,e_1) = aJe_1, \qquad h(e_1,e_2) = -aJe_2, \qquad h(e_2,e_2) = -aJe_1,\\
&h(e_3,e_3) = bJe_3, \qquad h(e_3,e_4) = -bJe_4, \qquad h(e_4,e_4) = -bJe_3,\\
&h(e_i,e_j) = 0, \quad \textit{otherwise}
\end{aligned} \tag{4.3}$$

for some functions a and b.

Moreover, M^5 is proper $\delta(2,2)$-ideal if and only if $a, b \neq 0$.

Proof. By applying Lemma 4.2 to $V = \mathrm{Span}\{e_1, e_2\}$ and $V = \mathrm{Span}\{e_3, e_4\}$, we obtain (4.3) with respect to a suitable orthonormal frame $\{e_1, \ldots, e_5\}$ on M^5.

The second statement follows from the definition of proper $\delta(2,2)$-ideal submanifolds, (4.3) and Theorem A. □

From now on, we assume that M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifold of a complex space form $\tilde{M}^5(4c)$ and we shall always choose the orthonormal frame $\{e_1, \ldots, e_5\}$ satisfying (4.3). Since M^5 is proper $\delta(2,2)$-ideal, we have $a, b \neq 0$.

Let us put

$$\nabla_X e_i = \sum_{j=1}^{5} \omega_i^j(X), \quad X \in TM^5. \tag{4.4}$$

Then we have $\omega_i^j = -\omega_j^i$, $i, j = 1, \ldots, 5$.

Lemma 4.4. *Let M^5 be a proper $\delta(2,2)$-ideal Lagrangian submanifold of a complex space form $\tilde{M}^5(4c)$. Then we have*

$$e_1 a = -3a\mu, \qquad e_2 a = 3a\lambda, \qquad e_3 a = e_4 a = 0, \qquad e_5 a = a\alpha, \tag{4.5}$$

$$e_1 b = e_2 b = 0, \qquad e_3 b = -3b\eta, \qquad e_4 b = 3b\varphi, \qquad e_5 b = b\beta, \tag{4.6}$$

where α, β, λ, μ, φ, η are defined by

$$\begin{aligned} &\lambda = \omega_1^2(e_1), \qquad \mu = \omega_1^2(e_2), \qquad \varphi = \omega_3^4(e_3), \qquad \eta = \omega_3^4(e_4), \\ &\alpha = \omega_1^5(e_1) = \omega_2^5(e_2), \qquad \beta = \omega_3^5(e_3) = \omega_4^5(e_4). \end{aligned} \tag{4.7}$$

Moreover, we have $\omega_i^j(e_k) = 0$, $i, j, k \in \{1, \ldots, 5\}$, for those $\omega_i^j(e_k)$ which do not appear in (4.7).

Proof. This was done by performing long computations on Codazzi's equation via Lemma 4.3. □

By using (4.4) and Lemma 4.4 we obtain the following.

Lemma 4.5. *Under the hypothesis of Lemma 4.4, the Levi-Civita connection ∇ of M^5 satisfies*

$$\begin{aligned} &\nabla_{e_1} e_1 = \lambda e_2 + \alpha e_5, \qquad \nabla_{e_1} e_2 = -\lambda e_1, \qquad \nabla_{e_1} e_5 = -\alpha e_1, \\ &\nabla_{e_2} e_1 = \mu e_2, \qquad \nabla_{e_2} e_2 = -\mu e_1 + \alpha e_5, \qquad \nabla_{e_2} e_5 = -\alpha e_2, \\ &\nabla_{e_3} e_3 = \varphi e_3 + \beta e_5, \qquad \nabla_{e_3} e_4 = -\varphi e_3, \qquad \nabla_{e_3} e_5 = -\beta e_3, \\ &\nabla_{e_4} e_3 = \eta e_4, \qquad \nabla_{e_4} e_4 = -\eta e_3 + \beta e_5, \qquad \nabla_{e_4} e_5 = -\beta e_4, \\ &\nabla_{e_i} e_j = 0, \quad \textit{otherwise}. \end{aligned} \tag{4.8}$$

We put

$$T_0 = \mathrm{Span}\{e_5\}, \qquad T_1 = \mathrm{Span}\{e_1, e_2\}, \qquad T_2 = \mathrm{Span}\{e_3, e_4\}. \tag{4.9}$$

Lemma 4.6. *Under the hypothesis of Lemma 4.4, we have*

(a) *T_0 is a totally geodesic distribution, i.e. T_0 is integrable whose leaves are totally geodesic submanifolds*;
(b) *$T_0 \oplus T_1$ and $T_0 \oplus T_2$ are totally geodesic distributions*;
(c) *T_1 and T_2 are spherical distributions, i.e. T_1 and T_2 are integrable distributions and their leaves are totally umbilical submanifolds with parallel mean curvature vector.*

Proof. Since the distribution T_0 is of rank one, it is always integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ according to Lemma 4.5, the integral curves of e_5 are geodesics in M^5. Thus we have statement (a). Statement (b) follows easily from (4.8).

To prove statement (c), first we observe that $[e_1, e_2] \in T_1$ and $[e_3, e_4] \in T_2$ follow from (4.8). Thus T_1, T_2 are integrable. Also, it follows from (4.8) that the second fundamental form h_1 of a leave $\mathcal{L}_1$ of T_1 in M^5 is given by

$$h_1(X, Y) = \alpha g_1(X_1, Y_1) e_5, \quad X_1, Y_1 \in T\mathcal{L}_1, \tag{4.10}$$

where g_1 is the metric of $\mathcal{L}_1$. Moreover, from (4.8), we find

$$\nabla_{e_i} e_5 = -\alpha e_i, \quad i = 1, 2.$$

Thus we get

$$D^1_{e_1} e_5 = D^1_{e_2} e_5 = 0, \tag{4.11}$$

where D^1 denotes the normal connection of $\mathcal{L}_1$ in M^5. From the equation of Gauss and Lemma 4.3 we know that the curvature tensor R of M^5 satisfies

$$\langle R(e_1, e_2)e_1, e_j\rangle = 0, \quad j = 3, 4, 5. \tag{4.12}$$

Thus we derive from (4.12) and Lemma 4.5 that

$$\begin{aligned} 0 &\equiv R(e_1, e_2)e_1 \\ &\equiv \nabla_{e_1}(\mu e_2) - \nabla_{e_2}(\lambda e_2 + \alpha e_5) + \lambda \nabla_{e_1} e_1 + \mu \nabla_{e_2} e_1 \\ &\equiv -(e_2\alpha)e_5 \pmod{T_1}. \end{aligned} \tag{4.13}$$

Hence we find $e_2\alpha = 0$.

Similarly, by considering $R(e_2, e_1)e_2$, we also have $e_1\alpha = 0$. By combining these with (4.11), we conclude that $\mathcal{L}_1$ has parallel mean curvature vector in M^5. Consequently, T_1 is a spherical distribution.

Similarly, we also have $e_3\beta = e_4\beta = 0$. Moreover, we know that T_2 is a spherical distribution as well. Thus we obtain statement (c) of the lemma. □

Lemma 4.7. *Under the hypothesis of Lemma 4.4, the Lagrangian submanifold M^5 is a locally warped product $I \times_{\rho_1(t)} M_1^2 \times_{\rho_2(t)} M_2^2$, where t is function such that $e_5 = \partial_t$ (i.e., $e_5 = \frac{\partial}{\partial t}$), ρ_1 and ρ_2 are two positive functions in t and M_1^2, M_2^2 are Riemannian 2-manifolds.*

Proof. It follows from Lemma 4.6 and result of Hiepko [14] (see also [13]). □

Lemma 4.8. *Under the hypothesis of Lemma 4.4 and under the same notations as previous lemmas, we have*

$$e_j\alpha = e_j\beta = 0, \quad j = 1, 2, 3, 4, \tag{4.14}$$

$$e_3\lambda = e_4\lambda = e_3\mu = e_4\mu = 0, \qquad e_5\mu = \alpha\mu, \tag{4.15}$$

$$e_1\lambda = -e_2\mu, \tag{4.16}$$

$$e_1\alpha + 3e_5\mu = 3\alpha\mu, \qquad e_2\alpha - 3e_5\lambda = -3\alpha\lambda, \tag{4.17}$$

$$e_1\varphi = e_2\varphi = e_1\eta = e_2\eta = 0, \qquad e_5\eta = \beta\eta, \tag{4.18}$$

$$e_3\varphi = -e_4\eta, \tag{4.19}$$

$$e_3\beta + 3e_5\eta = 3\beta\eta, \qquad e_4\beta - 3e_5\varphi = -3\beta\varphi. \tag{4.20}$$

Proof. The equations $e_1\alpha = e_2\alpha = e_3\beta = e_4\beta = 0$ are already derived in the proof of Lemma 4.6. The other equations in (4.14)–(4.20) are obtained by applying (4.5), (4.6), (4.8) and the compatibility conditions:

$$[e_i, e_j]f = (\nabla_{e_i} e_j - \nabla_{e_j} e_i)f, \quad i, j = 1, \dots, 5,$$

for $f = a, b$. For instance, we find (4.16) from $[e_1, e_2]a = (\nabla_{e_1} e_2 - \nabla_{e_2} e_1)a$ via (4.5) and (4.8); and $e_3\lambda = 0$ from $[e_2, e_3]a = (\nabla_{e_2} e_3 - \nabla_{e_3} e_2)a$. □

It follows from (4.14) and $e_5 = \frac{\partial}{\partial t}$ in Lemma 4.7 that $\alpha = \alpha(t)$ and $\beta = \beta(t)$.

Lemma 4.9. *Under the hypothesis of Lemma 4.4, we may choose isothermal coordinate systems $\{x, y\}$ on M_1^2 and $\{z, w\}$ on M_2^2 such that the metric tensors g_1, g_2 of the Riemannian 2-manifolds M_1^2, M_2^2 in Lemma 4.7 are given respectively by*

$$g_1 = E_1^2\left(dx^2 + dy^2\right), \qquad g_2 = E_2^2\left(dz^2 + dw^2\right). \tag{4.21}$$

Proof. By using (4.5) in Lemma 4.4 and Lemma 4.5 we find

$$\left[a^{-\frac{1}{3}} e_1, a^{-\frac{1}{3}} e_2\right] = 0. \tag{4.22}$$

It follows from $e_3 a = e_4 a = 0$ and $e_5 a = a\alpha$ in Lemma 4.4 that

$$a = f e^{\int^t \alpha(t)\, dt} \tag{4.23}$$

for some function f defined on M_1^2. We conclude from (4.22) and (4.23) that there exists a coordinate system $\{x, y\}$ on M_1^2 with $\frac{\partial}{\partial x} = E_1 e_1$ and $\frac{\partial}{\partial y} = E_1 e_2$. Now, by putting $E_1 = f^{-\frac{1}{3}}(x, y)$, we obtain $g_1 = E_1^2(dx^2 + dy^2)$. After applying the same argument to M_2^2, we obtain a similar result for M_2^2. □

It follows from Lemmas 4.7 and 4.9 that there is a coordinate system $\{t, x, y, z, w\}$ on $M^5 = I \times_{\rho_1(t)} M_1^2 \times_{\rho_2(t)} M_2^2$ such that the metric tensor g of M^5 is given by

$$g = dt^2 + \rho_1^2(t)E_1^2(x,y)\big(dx^2+dy^2\big) + \rho_2^2(t)E_2^2(z,w)\big(dz^2+dw^2\big). \tag{4.24}$$

Lemma 4.10. *The Levi-Civita connection of the metric tensor (4.24) satisfies*

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=0,\\
&\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x}=\frac{\rho_1'}{\rho_1}\frac{\partial}{\partial x},\qquad \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial y}=\frac{\rho_1'}{\rho_1}\frac{\partial}{\partial y},\\
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=\frac{\partial(\ln E_1)}{\partial x}\frac{\partial}{\partial x}-\frac{\partial(\ln E_1)}{\partial y}\frac{\partial}{\partial y}-\rho_1\rho_1'E_1^2\frac{\partial}{\partial t},\\
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\frac{\partial(\ln E_1)}{\partial y}\frac{\partial}{\partial x}+\frac{\partial(\ln E_1)}{\partial x}\frac{\partial}{\partial y},\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=-\frac{\partial(\ln E_1)}{\partial x}\frac{\partial}{\partial x}+\frac{\partial(\ln E_1)}{\partial y}\frac{\partial}{\partial y}-\rho_1\rho_1'E_1^2\frac{\partial}{\partial t},\\
&\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial z}=\frac{\rho_2'}{\rho_2}\frac{\partial}{\partial z},\qquad \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial w}=\frac{\rho_2'}{\rho_2}\frac{\partial}{\partial w},\\
&\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z}=\frac{\partial(\ln E_2)}{\partial z}\frac{\partial}{\partial z}-\frac{\partial(\ln E_2)}{\partial w}\frac{\partial}{\partial w}-\rho_2\rho_2'E_2^2\frac{\partial}{\partial t},\\
&\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial w}=\frac{\partial(\ln E_2)}{\partial w}\frac{\partial}{\partial z}+\frac{\partial(\ln E_2)}{\partial z}\frac{\partial}{\partial w},\\
&\nabla_{\frac{\partial}{\partial w}}\frac{\partial}{\partial w}=-\frac{\partial(\ln E_2)}{\partial z}\frac{\partial}{\partial z}+\frac{\partial(\ln E_2)}{\partial w}\frac{\partial}{\partial w}-\rho_2\rho_2'E_2^2\frac{\partial}{\partial t},\\
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}=\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial w}=\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z}=\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial w}=0.
\end{aligned}$$

Proof. It follows from (4.24) and direct computation. □

Lemma 4.11. *Let M^5 be a proper $\delta(2,2)$-ideal Lagrangian submanifold of a complex space form $\tilde M^5(4c)$. The with respect to metric (4.24) the second fundamental form of M^5 satisfies*

$$\begin{aligned}
&h(\partial_x,\partial_x)=\frac{J\partial_x}{E_1^2},\qquad h(\partial_x,\partial_y)=-\frac{J\partial_y}{E_1^2},\qquad h(\partial_y,\partial_y)=-\frac{J\partial_x}{E_1^2},\\
&h(\partial_z,\partial_z)=\frac{J\partial_z}{E_2^2},\qquad h(\partial_z,\partial_w)=-\frac{J\partial_w}{E_2^2},\qquad h(\partial_w,\partial_w)=-\frac{J\partial_z}{E_2^2},\\
&h(\partial_x,\partial_z)=h(\partial_x,\partial_w)=h(\partial_y,\partial_z)=h(\partial_y,\partial_w)=0,\\
&h(\partial_x,\partial_t)=h(\partial_y,\partial_t)=h(\partial_z,\partial_t)=h(\partial_w,\partial_t)=0,
\end{aligned}\tag{4.25}$$

where $\partial_x=\frac{\partial}{\partial x}$, $\partial_y=\frac{\partial}{\partial y}$, etc.

Proof. It follows from the proof of Lemma 4.9 and (4.24) that

$$\partial_x=\rho_1E_1e_1,\qquad \partial_y=\rho_1E_1e_2,\qquad \partial_z=\rho_2E_2e_3,\qquad \partial_w=\rho_2E_2e_4,\qquad \partial_t=e_5. \tag{4.26}$$

By combining Lemma 4.4 and (4.25) we get

$$\begin{aligned}
&\partial_x(\ln a)=-3\rho_1E_1\mu,\qquad \partial_y(\ln a)=3\rho_1E_1\lambda,\qquad \partial_t(\ln a)=\alpha,\\
&\partial_z(\ln b)=-3\rho_2E_2\eta,\qquad \partial_w(\ln b)=3\rho_2E_2\varphi,\qquad \partial_t(\ln b)=\beta.
\end{aligned}\tag{4.27}$$

On the other hand, by applying (4.7), (4.8) and Lemma 4.10, we find

$$\begin{aligned}
&\alpha=-\partial_t(\ln\rho_1),\qquad \beta=-\partial_t(\ln\rho_2),\\
&\mu=\frac{\partial_x(\ln E_1)}{\rho_1E_1},\qquad \lambda=-\frac{\partial_y(\ln E_1)}{\rho_1E_1},\\
&\eta=\frac{\partial_z(\ln E_2)}{\rho_2E_2},\qquad \varphi=-\frac{\partial_w(\ln E_2)}{\rho_2E_2}.
\end{aligned}\tag{4.28}$$

Now, we obtain from (4.27), (4.28) and Lemma 4.4 that

$$\partial_x(\ln a) = \partial_x\big(\ln E_1^{-3}\big), \qquad \partial_y(\ln a) = \partial_y\big(\ln E_1^{-3}\big), \qquad \partial_t(\ln a) = \partial_t\big(\ln \rho_1^{-1}\big),$$
$$\partial_z(\ln b) = \partial_z\big(\ln E_2^{-3}\big), \qquad \partial_w(\ln b) = \partial_w\big(\ln E_2^{-3}\big), \qquad \partial_t(\ln b) = \partial_t\big(\ln \rho_2^{-1}\big). \tag{4.29}$$

Therefore, after combining (4.29) with $e_3a = e_4a = e_1b = e_2b = 0$ from Lemma 4.4, we obtain

$$a = \frac{c_1}{\rho_1 E_1^3}, \qquad b = \frac{c_2}{\rho_2 E_2^3} \tag{4.30}$$

for some real numbers $c_1, c_2 \neq 0$. Without loss of generality, we may choose $c_1 = c_2 = 1$ by rescaling E_1, E_2 if necessary. Consequently, we obtain (4.25) from (4.3) of Lemma 4.3, (4.26) and (4.30). □

5. Proper $\delta(2,2)$-ideal Lagrangian submanifolds in $\mathbf{C}^5$

First, we classify all proper $\delta(2,2)$-ideal Lagrangian submanifolds in $\mathbf{C}^5$.

Theorem 5.1. *Let $L : M^5 \to \mathbf{C}^5$ be a Lagrangian immersion into the complex Euclidean 5-space $\mathbf{C}^5$. Then L is a proper $\delta(2,2)$-ideal Lagrangian immersion if and only if L is locally congruent to one of the following immersions*:

(1) *the direct product of an open interval I of the real line in $\mathbf{C}$ and two non-totally geodesic Lagrangian minimal immersions $\phi_i : M_i^2 \to \mathbf{C}^2$ $(i = 1, 2)$, i.e.,*

$$L : I \times M_1^2 \times M_2^2 \to \mathbf{C} \times \mathbf{C}^2 \times \mathbf{C}^2; \quad (t, p, q) \mapsto \big(t, \phi_1(p), \phi_2(q)\big). \tag{5.1}$$

(2) *a Lagrangian immersion defined by*

$$L : I \times M_1^2 \times_t M_2^2 \to \mathbf{C}^2 \times \mathbf{C}^3; \quad (t, p, q) \mapsto \big(\phi(p), t\zeta(q)\big), \tag{5.2}$$

where $\phi : M_1^2 \to \mathbf{C}^2$ is a non-totally geodesic Lagrangian minimal immersion and $\zeta : M_2^2 \to S^5(1) \subset \mathbf{C}^3$ is a non-totally geodesic Legendrian minimal immersion of M_2^2 into $S^5(1)$.

Proof. Let $L : M^5 \to \mathbf{C}^5$ be a proper $\delta(2,2)$-ideal Lagrangian immersion. Then, by applying Lemma 4.10, we find

$$\big\langle R(\partial_x, \partial_z)\partial_z, \partial_x\big\rangle = -\rho_1\rho_2\rho_1'\rho_2' E_1^2 E_2^2. \tag{5.3}$$

On the other hand, we find from the equation of Gauss and Lemmas 4.3 and 4.9 that $\langle R(\partial_x, \partial_z)\partial_z, \partial_x\rangle = 0$. Combining this with (5.3) gives $\rho_1'\rho_2' = 0$. Hence either ρ_1 is constant or ρ_2 is constant. Without loss of generality, we may assume that ρ_1 is constant. Thus we may assume $\rho_1 = 1$ by rescaling E_1 if necessary.

Next, by computing $\langle R(\partial_z, \partial_t)\partial_t, \partial_z\rangle$ using Lemma 4.10, we find

$$\big\langle R(\partial_z, \partial_t)\partial_t, \partial_z\big\rangle = -\rho_2\rho_2'' E_1^2. \tag{5.4}$$

On the other hand, it follows from Lemma 4.3 and equation of Gauss that $\langle R(\partial_z, \partial_t)\partial_t, \partial_z\rangle = 0$. By combining this with (5.4), we get $\rho_2'' = 0$. Thus $\rho_2 = ct + k$ for some constant c, k, not simultaneous zero. Hence, after rescaling E_2 and applying a suitable translation in t if necessary, we have either $\rho_2 = t$ or $\rho_2 = 1$.

Case (i): $\rho_1 = \rho_2 = 1$. In this case, M^5 is the Riemannian product $I \times M_1^2 \times M_2^2$ of an open interval I and two Riemannian 2-manifolds M_1^2, M_2^2. Since the second fundamental form of M^5 in $\mathbf{C}^5$ is mixed-totally geodesic (i.e., $h(X, Y) = 0$ for any X, Y tangent to two different factors of $I \times M_1^2 \times M_2^2$), Moore's lemma [15] implies that $L : M^5 \to \mathbf{C}^5$ is the direct product of three immersions. Moreover, since L is Lagrangian whose second fundamental form satisfies (4.25), each of the three immersions are Lagrangian. Thus, we obtain case (1) of the theorem.

Case (ii): $\rho_1 = 1$ *and* $\rho_2 = t$. It follows from (4.24) that the metric tensor of M^5 is

$$g = dt^2 + E_1^2(x, y)\big(dx^2 + dy^2\big) + t^2 E_2^2(z, w)\big(dz^2 + dw^2\big). \tag{5.5}$$

Thus M^5 is the Riemannian product of a Riemannian 2-manifold M_1^2 and the warped product $N^3 := I \times_t M_2^2$. It follows from Lemma 4.11 that the second fundamental form of $M_1^2 \times N^3$ is mixed-totally geodesic. So, L is the direct product of a Lagrangian immersion $M_1^2 \to \mathbf{C}^2$ and a Lagrangian immersion of $N^3 \to \mathbf{C}^3$.

On the other hand, it follows from Lemmas 4.10, 4.11 and Gauss's formula that L satisfies

$$L_{xx} = \left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}L_x - (\ln E_1)_y L_y, \tag{5.6}$$

$$L_{xy} = (\ln E_1)_y L_x + \left\{(\ln E_1)_x - \frac{\mathrm{i}}{E_1^2}\right\}L_y, \tag{5.7}$$

$$L_{yy} = -\left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}L_x + (\ln E_1)_y L_y, \tag{5.8}$$

$$L_{zz} = \left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}L_z - (\ln E_2)_w L_w - tE_2^2 L_t, \tag{5.9}$$

$$L_{zw} = (\ln E_2)_w L_z + \left\{(\ln E_2)_z - \frac{\mathrm{i}}{E_2^2}\right\}L_w, \tag{5.10}$$

$$L_{ww} = -\left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}L_z + (\ln E_2)_w L_w - tE_2^2 L_t, \tag{5.11}$$

$$L_{xz} = L_{xw} = L_{yz} = L_{yw} = 0, \tag{5.12}$$

$$L_{xt} = L_{yt} = 0, \tag{5.13}$$

$$L_{zt} = \frac{L_z}{t}, \qquad L_{wt} = \frac{L_w}{t}, \tag{5.14}$$

$$L_{tt} = 0. \tag{5.15}$$

The compatibility condition of this PDE system is given by

$$\Delta_1(\ln E_1) = \frac{2}{E_1^4}, \qquad \Delta_1 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}, \tag{5.16}$$

$$\Delta_2(\ln E_2) = \frac{2 - E_2^6}{E_2^4}, \qquad \Delta_2 = \frac{\partial^2}{\partial^2 z} + \frac{\partial^2}{\partial^2 w}. \tag{5.17}$$

After solving Eqs. (5.12)–(5.15), we obtain

$$L = \phi(x, y) + t\zeta(z, w) \tag{5.18}$$

for some vector functions $\phi(x, y)$ and $\zeta(z, w)$.

By substituting (5.18) into (5.6)–(5.11), we find

$$\begin{cases} \phi_{xx} = \left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}\phi_x - (\ln E_1)_y \phi_y, \\ \phi_{xy} = (\ln E_1)_y \phi_x + \left\{(\ln E_1)_x - \frac{\mathrm{i}}{E_1^2}\right\}\phi_y, \\ \phi_{yy} = -\left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}\phi_x + (\ln E_1)_y \phi_y, \end{cases} \tag{5.19}$$

$$\begin{cases} \zeta_{zz} = \left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\zeta_z - (\ln E_2)_w \zeta_w - E_2^2 \zeta, \\ \zeta_{zw} = (\ln E_2)_w \zeta_z + \left\{(\ln E_2)_z - \frac{\mathrm{i}}{E_2^2}\right\}\zeta_w, \\ \zeta_{ww} = -\left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\zeta_z + (\ln E_2)_w L_w - E_2^2 \zeta. \end{cases} \tag{5.20}$$

The compatibility condition of systems (5.19) and (5.20) are given respectively by (5.16) and (5.17).

It follows from system (5.19) that $\phi : M_1^2 \to \mathbf{C}^2$ is a non-totally geodesic Lagrangian minimal immersion. Also, it follows from (5.17) and (5.20) that $\zeta : M_2^2 \to \mathbf{C}^3$ maps M_2^2 into $S^5(1) \subset \mathbf{C}^3$ as a non-totally geodesic Legendrian minimal submanifold (see Theorem 2.1 and Corollary 2.1). Therefore, we obtain case (2).

The converse can be verified by direct computation. $\square$

6. Proper $\delta(2,2)$-ideal Lagrangian submanifolds in $CP^5(4)$

Now, we classify proper $\delta(2,2)$-ideal Lagrangian submanifolds in CP^5.

Theorem 6.1. *Let $L: M^5 \to CP^5(4)$ be a Lagrangian immersion. Then M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifold if and only if L is locally congruent to $\pi \circ \tilde{L}$, where $\pi: S^{11}(1) \to CP^5(4)$ is the Hopf fibration, $\tilde{L}: M^5 \to S^{11}(1) \subset \mathbf{C}^6$ is given by*

$$\tilde{L}(t,p,q) = (\cos t)\phi_1(p) + (\sin t)\phi_2(q), \quad t \in \mathbf{R}, \tag{6.1}$$

and $\phi_i: M_i^2 \to S^5(1) \subset \mathbf{C}^3$ $(i = 1,2)$ are non-totally geodesic Legendrian minimal immersions into the Sasakian $S^5(1)$.

Proof. Let $L: M^5 \to CP^5(4)$ be a proper $\delta(2,2)$-ideal Lagrangian immersion. Then we may assume the metric tensor of M^5 is given by (4.24) (cf. Section 4). From Lemma 4.3 and Gauss' equation we find

$$\langle R(\partial_x,\partial_t)\partial_t,\partial_x\rangle = \rho_1^2.$$

On the other hand, by applying Lemma 4.10 we also find

$$\langle R(\partial_x,\partial_t)\partial_t,\partial_x\rangle = -\rho_1\rho_1''.$$

Hence $\rho_1'' + \rho_1 = 0$, which implies that $\rho_1 = r\cos(t+t_0)$ for some real numbers t_0 and $r > 0$. So we obtain $\rho_1 = \cos t$ after applying a suitable translation in t and a rescaling of E_1 if necessary. Similarly, we have $\rho_2 = \cos(t+t_0)$. Now, it follows from (4.28), Lemma 4.11, and the equation of Gauss that

$$\langle R(\partial_x,\partial_z)\partial_z,\partial_x\rangle = \cos^2 t\cos^2(t+t_0)E_1^2E_2^2. \tag{6.2}$$

On the other hand, it follows from Lemma 4.10 and the definition of R that

$$\langle R(\partial_x,\partial_z)\partial_z,\partial_x\rangle = -\sin t\cos t\sin(t+t_0)\cos(t+t_0)E_1^2E_2^2. \tag{6.3}$$

By combining (6.2) and (6.3) we find $\cos t_0 = 0$. Thus we may choose $t_0 = -\frac{\pi}{2}$ which gives $\cos(t+t_0) = \sin t$. Consequently, (4.24) becomes

$$g = dt^2 + (\cos^2 t)E_1^2(x,y)(dx^2+dy^2) + (\sin^2 t)E_2^2(z,w)(dz^2+dw^2). \tag{6.4}$$

Next, by applying Lemmas 4.10, 4.11, (6.4) and Gauss' formula, we obtain

$$\tilde{L}_{xx} = \left\{(\ln E_1)_x + \frac{i}{E_1^2}\right\}\tilde{L}_x - (\ln E_1)_y\tilde{L}_y + (\cos t)E_1^2(\sin t\tilde{L}_t - \cos t\tilde{L}), \tag{6.5}$$

$$\tilde{L}_{xy} = (\ln E_1)_y\tilde{L}_x + \left\{(\ln E_1)_x - \frac{i}{E_1^2}\right\}\tilde{L}_y, \tag{6.6}$$

$$\tilde{L}_{yy} = -\left\{(\ln E_1)_x + \frac{i}{E_1^2}\right\}\tilde{L}_x + (\ln E_1)_y\tilde{L}_y + (\cos t)E_1^2(\sin t\tilde{L}_t - \cos t\tilde{L}), \tag{6.7}$$

$$\tilde{L}_{zz} = \left\{(\ln E_2)_z + \frac{i}{E_2^2}\right\}\tilde{L}_z - (\ln E_2)_w\tilde{L}_w - (\sin t)E_2^2(\cos\tilde{L}_t + \sin t\tilde{L}), \tag{6.8}$$

$$\tilde{L}_{zw} = (\ln E_2)_w\tilde{L}_z + \left\{(\ln E_2)_z - \frac{i}{E_2^2}\right\}\tilde{L}_w, \tag{6.9}$$

$$\tilde{L}_{ww} = -\left\{(\ln E_2)_z + \frac{i}{E_2^2}\right\}\tilde{L}_z + (\ln E_2)_w\tilde{L}_w - (\sin t)E_2^2(\cos\tilde{L}_t + \sin t\tilde{L}), \tag{6.10}$$

$$\tilde{L}_{xz} = \tilde{L}_{xw} = \tilde{L}_{yz} = \tilde{L}_{yw} = 0, \tag{6.11}$$

$$\tilde{L}_{xt} = -\tan t\tilde{L}_x, \qquad \tilde{L}_{yt} = -\tan t\tilde{L}_y, \tag{6.12}$$

$$\tilde{L}_{zt} = \cot t\tilde{L}_z, \qquad \tilde{L}_{wt} = \cot t\tilde{L}_w, \tag{6.13}$$

$$\tilde{L}_{tt} = -\tilde{L}. \tag{6.14}$$

The compatibility conditions of system (6.5)–(6.14) are given by

$$\Delta_1(\ln E_1) = \frac{2-E_1^6}{E_1^4}, \qquad \Delta_2(\ln E_2) = \frac{2-E_2^6}{E_2^4}. \tag{6.15}$$

After solving (6.11)–(6.14), we get

$$\tilde{L} = (\cos t)\phi_1(x,y) + (\sin t)\phi_2(z,w) \tag{6.16}$$

for some $\mathbf{C}^6$-valued functions ϕ_1, ϕ_2. Substituting (6.16) into (6.5)–(6.10) yields

$$\begin{cases} (\phi_1)_{xx} = \left\{(\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2}\right\}(\phi_1)_x - (\ln E_1)_y(\phi_1)_y - E_1^2\phi_1, \\ (\phi_1)_{xy} = (\ln E_1)_y(\phi_1)_x + \left\{(\ln E_1)_x - \dfrac{\mathrm{i}}{E_1^2}\right\}(\phi_1)_y, \\ (\phi_1)_{yy} = -\left\{(\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2}\right\}(\phi_1)_x + (\ln E_1)_y(\phi_1)_y - E_1^2\phi_1, \end{cases} \tag{6.17}$$

$$\begin{cases} (\phi_2)_{zz} = \left\{(\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2}\right\}(\phi_2)_z - (\ln E_2)_w(\phi_2)_w - E_2^2\phi_2, \\ (\phi_1)_{zw} = (\ln E_2)_w(\phi_2)_z + \left\{(\ln E_2)_z - \dfrac{\mathrm{i}}{E_2^2}\right\}(\phi_2)_w, \\ (\phi_2)_{ww} = -\left\{(\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2}\right\}(\phi_2)_z + (\ln E_2)_w(\phi_2)_w - E_2^2\phi_2. \end{cases} \tag{6.18}$$

It follows from system (6.17) and the first equation in (6.15) that $\phi_1 : M_1^2 \to \mathbf{C}^3$ gives rises to a Legendrian minimal surface in the Sasakian $S^5(1) \subset \mathbf{C}^3$. Similarly, system (6.18) and the second equation in (6.15) imply that $\phi : M_2^2 \to \mathbf{C}^3$ gives a Legendrian minimal surface in $S^5(1)$ too. Now, because M^5 is proper $\delta(2,2)$-ideal, both Legendrian minimal submanifolds in $S^5(1)$ are non-totally geodesic.

The converse can be verified by direct long computation. □

The following provides a simple example of proper $\delta(2,2)$-ideal Lagrangian submanifold in $CP^5(4)$ associated with $E_1 = E_2 = 1$.

Example 6.1. Consider the map $\tilde{L} : M^5 \to \mathbf{C}^6$ defined by

$$\begin{aligned} \tilde{L} = \frac{1}{\sqrt{3}}\bigg(& e^{\mathrm{i}\sqrt{2}x}\cos t,\ \sqrt{2}e^{-\frac{\mathrm{i}x}{\sqrt{2}}}\cos t\cos\left(\frac{\sqrt{3}}{\sqrt{2}}y\right),\ \sqrt{2}e^{-\frac{\mathrm{i}x}{\sqrt{2}}}\cos t\sin\left(\frac{\sqrt{3}}{\sqrt{2}}y\right), \\ & e^{\mathrm{i}\sqrt{2}z}\sin t,\ \sqrt{2}e^{-\frac{\mathrm{i}z}{\sqrt{2}}}\sin t\cos\left(\frac{\sqrt{3}}{\sqrt{2}}w\right),\ \sqrt{2}e^{-\frac{\mathrm{i}z}{\sqrt{2}}}\sin t\sin\left(\frac{\sqrt{3}}{\sqrt{2}}w\right)\bigg). \end{aligned}$$

It is direct to verify that $\tilde{L}(M^5)$ lies in the unit hypersphere $S^{11}(1) \subset \mathbf{C}^6$ and that the composition $\pi \circ \tilde{L} : M^5 \to CP^5(4)$ is a proper $\delta(2,2)$-ideal Lagrangian submanifold of $CP^5(4)$.

7. Proper $\delta(2,2)$-ideal Lagrangian submanifolds in $CH^5(-4)$

Finally, we classify all proper $\delta(2,2)$-ideal Lagrangian submanifolds in CH^5.

Theorem 7.1. *Let $L : M^5 \to CH^5(-4)$ be a Lagrangian immersion of M^5 into $CH^5(-4)$. Then M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifold of $CH^5(-4)$ if and only if L is locally congruent to $\pi \circ \tilde{L}$, where $\pi : H_1^{11}(-1) \to CH^5(-4)$ is the Hopf fibration and either*

(a) *$\tilde{L} : M^5 \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ is given by*

$$\tilde{L}(t,p,q) = (\cosh t)\phi(p) + (\sinh t)\psi(q), \quad t \in \mathbf{R}, \tag{7.1}$$

and $\phi : M_1^2 \to H^5(-1) \subset \mathbf{C}_1^3$ and $\psi : M_2^2 \to S^5(1) \subset \mathbf{C}^3$ are non-totally geodesic Legendrian minimal immersions into the Sasakian $H_1^5(-1)$ and $S^5(1)$, resp., or

(b) *$\tilde{L} : M^5 \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ is given by*

$$\begin{aligned} \tilde{L}(t,x,y,z,w) = \big(&\sinh t + e^t\big(u(z,y) + v(z,w) - 1\big), \\ &\sinh t + e^t\big(u(x,y) + v(z,w)\big), e^t\psi_1(x,y), e^t\psi_2(z,w)\big), \end{aligned} \tag{7.2}$$

$\psi_i : M_i^2 \to \mathbf{C}^2$ $(i = 1,2)$ are non-totally geodesic minimal Lagrangian immersions, u, v are complex-valued functions satisfying the following PDE systems, respectively:

$$\begin{cases} u_{xx} = \left\{ (\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2} \right\} u_x - (\ln E_1)_y u_y - E_1^2, \\ u_{xy} = (\ln E_1)_y u_x + \left\{ (\ln E_1)_x - \dfrac{\mathrm{i}}{E_1^2} \right\} u_y, \\ u_{yy} = -\left\{ (\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2} \right\} u_x + (\ln E_1)_y u_y - E_1^2, \end{cases}$$

$$\begin{cases} v_{zz} = \left\{ (\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2} \right\} v_z - (\ln E_2)_w v_w - E_2^2, \\ v_{zw} = (\ln E_2)_w v_z + \left\{ (\ln E_2)_z - \dfrac{\mathrm{i}}{E_z^2} \right\} v_w, \\ v_{ww} = -\left\{ (\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2} \right\} v_z + (\ln E_2)_w v_w - E_2^2, \end{cases}$$

and the metric tensors of M_1^2, M_2^2 are given respectively by

$$g_1 = E_1^2\big(dx^2 + dy^2\big), \qquad g_2 = E_2^2\big(dz^2 + dw^2\big)$$

for some positive functions $E_1 = E_1(x, y)$ and $E_2 = E_2(z, w)$.

Proof. Let $L : M^5 \to CH^5(-4)$ be a proper $\delta(2,2)$-ideal Lagrangian immersion. Then we may assume that the metric tensor of M^5 is given by (4.24) according to Section 4. From Lemma 4.3 and Gauss' equation we find

$$\big\langle R(\partial_x, \partial_t)\partial_t, \partial_x\big\rangle = -\rho_1^2.$$

On the other hand, by applying Lemma 4.10 we also have

$$\big\langle R(\partial_x, \partial_t)\partial_t, \partial_x\big\rangle = -\rho_1 \rho_1''.$$

Hence $\rho_1'' = \rho_1$, which implies that

$$\rho_1 = r \cosh t + s \sinh t \tag{7.3}$$

for some real numbers r and s, not both zero.

If $s = 0$ (resp., $r = 0$, or $r = \pm s$), then (7.3) reduces $\rho_1 = r \cosh t$ (resp., $\rho_1 = s \sinh t$, or $\rho_1 = r e^{\pm t}$). If $r^2 > s^2$ (resp., $r^2 < s^2$), then (7.3) reduces to $\rho_1 = c \cosh(t + t_0)$ (resp., $\rho_1 = c \sinh(t + t_0)$) for some real numbers $c \neq 0$ and t_0. Thus without loss of generality, we may assume that ρ_1 is one of the functions: $\cosh t, \sinh t, e^t$, by applying a suitable translation and or reflection in t and a suitable rescaling of E_1 if necessary. Similarly, we may also assume that ρ_2 is one of functions: $\cosh(t + t_1), \sinh(t + t_1), e^{t+t_1}$, $t_1 \in \mathbf{R}$.

Case (i): $\rho_1 = \cosh t$ and $\rho_2 = \cosh(t + t_1)$. It follows from (4.28), Lemma 4.11 and the equation of Gauss that

$$\big\langle R(\partial_x, \partial_z)\partial_z, \partial_x\big\rangle = -\cosh^2 t \cosh^2(t + t_1) E_1^2 E_2^2. \tag{7.4}$$

On the other hand, it follows from Lemma 4.10 that

$$\big\langle R(\partial_x, \partial_z)\partial_z, \partial_x\big\rangle = -\sinh t \cosh t \sinh(t + t_1) \cosh(t + t_1) E_1^2 E_2^2. \tag{7.5}$$

By combining (7.4) and (7.5) we obtain $\cosh t_1 = 0$ which is impossible.

Case (ii): $\rho_1 = \cosh t$ and $\rho_2 = \sinh(t + t_1)$. By considering the two different expressions of $\langle R(\partial_x, \partial_z)\partial_z, \partial_x\rangle$ via Lemmas 4.10 and 4.11 in the same way as in case (i), we get $\sinh t_1 = 0$. Therefore, $t_1 = 0$ and (4.24) reduces to

$$g = dt^2 + \big(\cosh^2 t\big) E_1^2(x, y)\big(dx^2 + dy^2\big) + \big(\sinh^2 t\big) E_2^2(z, w)\big(dz^2 + dw^2\big). \tag{7.6}$$

Therefore, after applying Lemmas 4.10 and 4.11, (6.2) and Gauss' formula, we have

$$\tilde{L}_{xx} = \left\{ (\ln E_1)_x + \frac{\mathrm{i}}{E_1^2} \right\} \tilde{L}_x - (\ln E_1)_y \tilde{L}_y + (\cosh t) E_1^2 (\cosh t \tilde{L} - \sinh t \tilde{L}_t), \tag{7.7}$$

$$\tilde{L}_{xy} = (\ln E_1)_y \tilde{L}_x + \left\{ (\ln E_1)_x - \frac{\mathrm{i}}{E_1^2} \right\} \tilde{L}_y, \tag{7.8}$$

$$\tilde{L}_{yy} = -\left\{ (\ln E_1)_x + \frac{\mathrm{i}}{E_1^2} \right\} \tilde{L}_x + (\ln E_1)_y \tilde{L}_y + (\cosh t) E_1^2 (\cosh t \tilde{L} - \sinh t \tilde{L}_t), \tag{7.9}$$

$$\tilde{L}_{zz} = \left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\tilde{L}_z - (\ln E_2)_w \tilde{L}_w + (\sinh t)E_2^2(\sinh t\tilde{L} - \cosh t\tilde{L}_t), \tag{7.10}$$

$$\tilde{L}_{zw} = (\ln E_2)_w \tilde{L}_z + \left\{(\ln E_2)_z - \frac{\mathrm{i}}{E_2^2}\right\}\tilde{L}_w, \tag{7.11}$$

$$\tilde{L}_{ww} = -\left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\tilde{L}_z + (\ln E_2)_w \tilde{L}_w + (\sinh t)E_2^2(\sinh t\tilde{L} - \cosh t\tilde{L}_t), \tag{7.12}$$

$$\tilde{L}_{xz} = \tilde{L}_{xw} = \tilde{L}_{yz} = \tilde{L}_{yw} = 0, \tag{7.13}$$

$$\tilde{L}_{xt} = \tanh t\tilde{L}_x, \qquad \tilde{L}_{yt} = \tanh t\tilde{L}_y, \tag{7.14}$$

$$\tilde{L}_{zt} = \coth t\tilde{L}_z, \qquad \tilde{L}_{wt} = \coth t\tilde{L}_w, \tag{7.15}$$

$$\tilde{L}_{tt} = \tilde{L}. \tag{7.16}$$

The compatibility conditions of system (7.7)–(7.16) are given by

$$\Delta_1(\ln E_1) = \frac{2 + E_1^6}{E_1^4}, \qquad \Delta_2(\ln E_2) = \frac{2 - E_2^6}{E_2^4}. \tag{7.17}$$

After solving (7.13)–(7.16) we obtain

$$\tilde{L} = (\cosh t)\phi(x, y) + (\sinh t)\psi(z, w) \tag{7.18}$$

for some vector-valued functions ϕ, ψ. Substituting (7.18) into (7.7)–(7.12) gives

$$\begin{cases} \phi_{xx} = \left\{(\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2}\right\}\phi_x - (\ln E_1)_y\phi_y + E_1^2\phi, \\ \phi_{xy} = (\ln E_1)_y\phi_x + \left\{(\ln E_1)_x - \dfrac{\mathrm{i}}{E_1^2}\right\}\phi_y, \\ \phi_{yy} = -\left\{(\ln E_1)_x + \dfrac{\mathrm{i}}{E_1^2}\right\}\phi_x + (\ln E_1)_y\phi_y + E_1^2\phi, \end{cases} \tag{7.19}$$

$$\begin{cases} \psi_{zz} = \left\{(\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2}\right\}\psi_z - (\ln E_2)_w\psi_w - E_2^2\psi, \\ \psi_{zw} = (\ln E_2)_w\psi_z + \left\{(\ln E_2)_z - \dfrac{\mathrm{i}}{E_2^2}\right\}\psi_w, \\ r\psi_{ww} = -\left\{(\ln E_2)_z + \dfrac{\mathrm{i}}{E_2^2}\right\}\psi_z + (\ln E_2)_w\psi_w - E_2^2\psi. \end{cases} \tag{7.20}$$

It follows from (7.19) and the first equation in (7.17) that ϕ gives rises to a Legendrian minimal surface in $H_1^5(-1) \subset \mathbf{C}_1^3$. Similarly, (7.20) and the second equation in (7.17) imply that ψ gives rises to a Legendrian minimal surface in $S^5(1) \subset \mathbf{C}^3$. Now, because M^5 is a proper $\delta(2,2)$-ideal Lagrangian submanifolds in $CP^5(4)$, both Legendrian submanifolds are non-totally geodesic. Consequently, we obtain case (a) of the theorem.

Case (iii): $\rho_1 = \cosh t$ and $\rho_2 = e^{t+t_1}$. It follows from (4.28), Lemmas 4.10, 4.11, and the equation of Gauss that

$$\langle R(\partial_x, \partial_z)\partial_z, \partial_x\rangle = -(\cosh^2 t)e^{2t+2t_1}E_1^2E_2^2, \tag{7.21}$$

$$\langle R(\partial_x, \partial_z)\partial_z, \partial_x\rangle = -(\sinh t\cosh t)e^{2t+2t_1}E_1^2E_2^2, \tag{7.22}$$

which is impossible.

Case (iv): $\rho_1 = \sinh t$ and $\rho_2 = \sinh(t + t_1)$. Using the same arguments as in case (i), we find $\cosh t_1 = 0$, which is impossible.

Case (v): $\rho_1 = \sinh t$ and $\rho_2 = e^{t+t_1}$. By applying the same arguments as in case (iii), we get $\sinh t = \cosh t$, which is also impossible.

Case (vi): $\rho_1 = \sinh t$ and $\rho_2 = \cosh(t + t_1)$. As case (ii), this also gives case (a) of the theorem.

Case (vii): $\rho_1 = e^t$ and $\rho_2 = \cosh(t + t_1)$. Using the same arguments as in case (iii), we conclude that this case is impossible.

Case (viii): $\rho_1 = e^t$ and $\rho_2 = \sinh(t + t_1)$. This is impossible by applying the same arguments as in case (v).

Case (ix): $\rho_1 = e^t$ and $\rho_2 = e^{t+t_1}$. Since $\rho_2 = e^{t_1}e^t$, without loss of generality we may assume that the metric tensor of M^5 is given by

$$g = dt^2 + e^{2t}E_1^2(x,y)\big(dx^2+dy^2\big) + e^{2t}E_2^2(z,w)\big(dz^2+dw^2\big). \tag{7.23}$$

So, after applying Lemmas 4.10, 4.11, (6.2) and Gauss' formula, we obtain

$$\tilde L_{xx} = \left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}\tilde L_x - (\ln E_1)_y\tilde L_y + e^{2t}E_1^2(\tilde L - \tilde L_t), \tag{7.24}$$

$$\tilde L_{xy} = (\ln E_1)_y\tilde L_x + \left\{(\ln E_1)_x - \frac{\mathrm{i}}{E_1^2}\right\}\tilde L_y, \tag{7.25}$$

$$\tilde L_{yy} = -\left\{(\ln E_1)_x + \frac{\mathrm{i}}{E_1^2}\right\}\tilde L_x + (\ln E_1)_y\tilde L_y + e^{2t}E_1^2(\tilde L - \tilde L_t), \tag{7.26}$$

$$\tilde L_{zz} = \left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\tilde L_z - (\ln E_2)_w\tilde L_w + e^{2t}E_2^2(\tilde L - \tilde L_t), \tag{7.27}$$

$$\tilde L_{zw} = (\ln E_2)_w\tilde L_z + \left\{(\ln E_2)_z - \frac{\mathrm{i}}{E_2^2}\right\}\tilde L_w, \tag{7.28}$$

$$\tilde L_{ww} = -\left\{(\ln E_2)_z + \frac{\mathrm{i}}{E_2^2}\right\}\tilde L_z + (\ln E_2)_w\tilde L_w + e^{2t}E_2^2(\tilde L - \tilde L_t), \tag{7.29}$$

$$\tilde L_{xz} = \tilde L_{xw} = \tilde L_{yz} = \tilde L_{yw} = 0, \tag{7.30}$$

$$\tilde L_{xt} = \tilde L_x, \qquad \tilde L_{yt} = \tilde L_y, \qquad \tilde L_{zt} = \tilde L_z, \qquad \tilde L_{wt} = \tilde L_w, \tag{7.31}$$

$$\tilde L_{tt} = \tilde L. \tag{7.32}$$

The compatibility conditions of system (7.24)–(7.32) are given by

$$\Delta_1(\ln E_1) = \frac{2}{E_1^4}, \qquad \Delta_2(\ln E_2) = \frac{2}{E_2^4}. \tag{7.33}$$

After solving (7.30)–(7.32) we get

$$\tilde L = e^t\big(\tilde\phi(x,y) + \tilde\psi(z,w)\big) + c_0\sinh t \tag{7.34}$$

for some vector-valued functions $\tilde\phi, \tilde\psi$. Since $\tilde L$ maps M^5 into $H_1^{11}(-1) \subset \mathbf{C}_1^6$ as a Legendrian minimal submanifold, we find from (7.34) that

$$-1 = e^{2t}\langle\tilde\phi+\tilde\psi, \tilde\phi+\tilde\psi\rangle + \big(e^{2t}-1\big)\langle c_0, \tilde\phi+\tilde\psi\rangle + \langle c_0, c_0\rangle\sinh^2 t. \tag{7.35}$$

Thus we have

$$\langle c_0, c_0\rangle = 0, \qquad \langle c_0, \tilde\phi+\tilde\psi\rangle = 1, \qquad \langle\tilde\phi+\tilde\psi, \tilde\phi+\tilde\psi\rangle = -1. \tag{7.36}$$

It follows from the first equation in (7.36) that either $c_0 = 0$ or c_0 is a light-vector.

If $c_0 = 0$, it follows from (7.34) and (7.36) that

$$-1 = e^{2t}\langle\tilde\phi+\tilde\psi, \tilde\phi+\tilde\psi\rangle = -e^{2t},$$

which is impossible. Thus c_0 must be a light-like vector. Therefore we may put

$$c_0 = (1,1,0,0,0,0) \in \mathbf{C}_1^6. \tag{7.37}$$

Since $\langle c_0, \phi+\psi\rangle = 1$ from (7.36), in views of (7.34) we may also put

$$\tilde L = \big(\sinh t + e^t(f-1), \sinh t + e^t f, e^t\big(\psi_1(x,y) + \psi_2(z,w)\big)\big) \tag{7.38}$$

for some complex-valued functions f with $f_t = 0$ and some vector-valued functions ψ_1, ψ_2. It follows from (7.23) and (7.38) that

$$\big\langle(\psi_1)_x, (\psi_2)_z\big\rangle = \big\langle(\psi_1)_x, (\psi_2)_w\big\rangle = \big\langle(\psi_1)_y, (\psi_2)_z\big\rangle = \big\langle(\psi_1)_y, (\psi_2)_w\big\rangle = 0.$$

Thus, for simplicity we may put

$$\tilde{L}=\big(\sinh t+e^t(f-1),\sinh t+e^t f,e^t\psi_1(x,y),e^t\psi_2(z,w)\big). \tag{7.39}$$

From (7.39) we get

$$\tilde{L}-\tilde{L}_t=\big(-e^t,-e^t,0,0,0,0\big). \tag{7.40}$$

By substituting (7.39) into (7.24)–(7.29), we obtain

$$\begin{cases}(\psi_1)_{xx}=\left\{(\ln E_1)_x+\dfrac{\mathrm{i}}{E_1^2}\right\}(\psi_1)_x-(\ln E_1)_y(\psi_1)_y,\\ (\psi_1)_{xy}=(\ln E_1)_y(\psi_1)_x+\left\{(\ln E_1)_x-\dfrac{\mathrm{i}}{E_1^2}\right\}(\psi_1)_y,\\ (\psi_1)_{yy}=-\left\{(\ln E_1)_x+\dfrac{\mathrm{i}}{E_1^2}\right\}(\psi_1)_x+(\ln E_1)_y(\psi_1)_y,\end{cases} \tag{7.41}$$

$$\begin{cases}(\psi_2)_{zz}=\left\{(\ln E_2)_z+\dfrac{\mathrm{i}}{E_2^2}\right\}(\psi_2)_z-(\ln E_2)_w(\psi_2)_w,\\ (\psi_2)_{zw}=(\ln E_2)_w(\psi_2)_z+\left\{(\ln E_2)_z-\dfrac{\mathrm{i}}{E_2^2}\right\}(\psi_2)_w,\\ (\psi_2)_{ww}=-\left\{(\ln E_2)_z+\dfrac{\mathrm{i}}{E_2^2}\right\}(\psi_2)_z+(\ln E_2)_w(\psi_2)_w.\end{cases} \tag{7.42}$$

It follows from (7.39), (7.41) and (7.42) that $\psi_i:M_1^2\to\mathbf{C}^2$ $(i=1,2)$ are Lagrangian minimal. Since L is proper $\delta(2,2)$-ideal, both ϕ and ψ are non-totally geodesic.

In order to determine the function f in (7.38), we only need to consider the second components from (7.24)–(7.29). First, we know from (7.30) and (7.38) that $f=u(x,y)+v(z,w)$ for some complex-valued functions u, v. Thus (7.39) becomes

$$\tilde{L}=\big(\sinh t+e^t(u+v-1),\sinh t+e^t(u+v),e^t\phi(x,y),e^t\psi(z,w)\big). \tag{7.43}$$

Now, by substituting (7.43) into (7.24)–(7.29) and using (7.40), we find from the second components of (7.24)–(7.29) that

$$\begin{cases}u_{xx}=\left\{(\ln E_1)_x+\dfrac{\mathrm{i}}{E_1^2}\right\}u_x-(\ln E_1)_y u_y-E_1^2,\\ u_{xy}=(\ln E_1)_y u_x+\left\{(\ln E_1)_x-\dfrac{\mathrm{i}}{E_1^2}\right\}u_y,\\ u_{yy}=-\left\{(\ln E_1)_x+\dfrac{\mathrm{i}}{E_1^2}\right\}u_x+(\ln E_1)_y u_y-E_1^2,\end{cases} \tag{7.44}$$

$$\begin{cases}v_{zz}=\left\{(\ln E_2)_z+\dfrac{\mathrm{i}}{E_2^2}\right\}v_z-(\ln E_2)_w v_w-E_2^2,\\ v_{zw}=(\ln E_2)_w v_z+\left\{(\ln E_2)_z-\dfrac{\mathrm{i}}{E_z^2}\right\}v_w,\\ v_{ww}=-\left\{(\ln E_2)_z+\dfrac{\mathrm{i}}{E_2^2}\right\}v_z+(\ln E_2)_w v_w-E_2^2.\end{cases} \tag{7.45}$$

It is direct to verify that the compatibility condition of system (7.44) (resp., system (7.45)) is exactly the compatibility condition of (7.41) (resp., (7.42)). Hence, for any two given Lagrangian minimal surfaces ψ_1, ψ_2 in $\mathbf{C}^2$, there always exist solutions u and v of (7.44) and (7.45). Consequently, we obtain case (b) of the theorem.

The converse can be verified by direct long computation. □

Finally, we provide a simple example of type (b) proper $\delta(2,2)$-ideal Lagrangian submanifold in $CH^5(-4)$.

Example 7.1. Let $E=\sqrt{2}\cosh x$. Then E satisfies (2.9) with $c=0$. Hence there is a non-totally geodesic Lagrangian minimal immersion ψ_1 into $\mathbf{C}^2$ according to Theorem 2.1. In fact, up to congruences, $\psi_1(x,y)$ is given by

$$\left(2\sqrt{2}\cos\left(\frac{y}{2}\right)\left(\cosh\left(\frac{x}{2}\right)-\mathrm{i}\sinh\left(\frac{x}{2}\right)\right),2\sqrt{2}\sin\left(\frac{y}{2}\right)\left(\cosh\left(\frac{x}{2}\right)-\mathrm{i}\sinh\left(\frac{x}{2}\right)\right)\right).$$

Similarly, for $E = \sqrt{2\cosh z}$, there exists a non-totally geodesic Lagrangian minimal immersion ψ_2 into $\mathbf{C}^2$ such that $\psi_2(z, w)$ is given by

$$\left(2\sqrt{2}\cos\left(\frac{w}{2}\right)\left(\cosh\left(\frac{z}{2}\right) - \mathrm{i}\sinh\left(\frac{z}{2}\right)\right), 2\sqrt{2}\sin\left(\frac{w}{2}\right)\left(\cosh\left(\frac{z}{2}\right) - \mathrm{i}\sinh\left(\frac{z}{2}\right)\right)\right).$$

Also, it is easy to verify that $u = 4\mathrm{i}x - \cosh x$ and $v = 4\mathrm{i}z - \cosh z$ are solutions of systems (7.44) and (7.45), respectively. Thus if we define $\tilde{L} : M^5 \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ by

$$\begin{aligned}\tilde{L}(t, z, y, z, w) = \Bigg(&\sinh t + 4e^t(\mathrm{i}x + \mathrm{i}z - \cosh x - \cosh z) - e^t,\\
&\sinh t + 4e^t(\mathrm{i}x + \mathrm{i}z - \cosh x - \cosh z), 2\sqrt{2}e^t\cos\left(\frac{y}{2}\right)\left(\cosh\left(\frac{x}{2}\right) - \mathrm{i}\sinh\left(\frac{x}{2}\right)\right),\\
&2\sqrt{2}e^t\sin\left(\frac{y}{2}\right)\left(\cosh\left(\frac{x}{2}\right) - \mathrm{i}\sinh\left(\frac{x}{2}\right)\right), 2\sqrt{2}e^t\cos\left(\frac{w}{2}\right)\left(\cosh\left(\frac{z}{2}\right) - \mathrm{i}\sinh\left(\frac{z}{2}\right)\right),\\
&2\sqrt{2}e^t\sin\left(\frac{w}{2}\right)\left(\cosh\left(\frac{z}{2}\right) - \mathrm{i}\sinh\left(\frac{z}{2}\right)\right)\Bigg),\end{aligned}$$

then $\pi \circ \tilde{L} : M^5 \to CH^5(-4)$ is a proper $\delta(2, 2)$-ideal Lagrangian submanifold.

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Lagrangian submanifolds in complex space forms satisfying an improved equality involving $\delta(2,2)$

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Abstract. It was proved in [8], [9] that every Lagrangian submanifold M of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the following optimal inequality:

$$\delta(2,2) \leq \frac{25}{4}H^2 + 8c, \tag{A}$$

where H^2 is the squared mean curvature and $\delta(2,2)$ is a δ-invariant on M introduced by the first author. This optimal inequality improves a special case of an earlier inequality obtained in [B.-Y. Chen, Japan. J. Math. **26** (2000), 105–127].

The main purpose of this paper is to classify Lagrangian submanifolds of $\tilde{M}^5(4c)$ satisfying the equality case of the improved inequality (A).

1. Introduction

Let $\tilde{M}^n$ be a Kähler n-manifold with the complex structure J, a Kähler metric g and the Kähler 2-form ω. An isometric immersion $\psi : M \to \tilde{M}^n$ of a Riemannian n-manifold M into $\tilde{M}^n$ is called *Lagrangian* if $\psi^*\omega = 0$.

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Let $\tilde{M}^n(4c)$ denote a Kähler n-manifold with constant holomorphic sectional curvature $4c$, called a *complex space form*. A complete simply-connected complex space form $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean n-plane $\mathbf{C}^n$, the complex projective n-space $CP^n(4c)$, or a complex hyperbolic n-space $CH^n(4c)$ according to $c = 0$, $c > 0$ or $c < 0$, respectively.

B.-Y. Chen introduced in 1990s new Riemannian invariants $\delta(n_1, \dots, n_k)$. For any n-dimensional submanifold M in a real space form $R^m(c)$ of constant curvature c, he proved the following sharp general inequality (see [5], [7] for details):

$$\delta(n_1, \dots, n_k) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\Big)c. \tag{1.1}$$

For Lagrangian submanifolds in a complex space form $\tilde{M}^n(4c)$, we have

Theorem A. *Let M be an n-dimensional Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then inequality (1.1) holds for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$.*

The following result from [6] extends a result in [10] on $\delta(2)$.

Theorem B. *Every Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ is minimal if it satisfies the equality case of (1.1) identically.*

Theorem B was improved recently in [8], [9] to the following inequality.

Theorem C. *Let M be an n-dimensional Lagrangian submanifold of $\tilde{M}^n(4c)$. Then, for an $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $\sum_{i=1}^k n_i < n$, we have*

$$\delta(n_1, \dots, n_k) \leq \frac{n^2\big\{\big(n - \sum_{i=1}^k n_i + 3k - 1\big) - 6\sum_{i=1}^k (2+n_i)^{-1}\big\}}{2\big\{\big(n - \sum_{i=1}^k n_i + 3k + 2\big) - 6\sum_{i=1}^k (2+n_i)^{-1}\big\}} H^2 + \frac{1}{2}\Big\{n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\Big\}c. \tag{1.2}$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, \dots, e_n\}$ at p such that the second fundamental form h satisfies

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{\gamma_i} h^{\gamma_i}_{\alpha_i\beta_i} Je_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{N+1}, \quad \sum_{\alpha_i=1}^{n_i} h^{\gamma_i}_{\alpha_i\alpha_i} = 0,$$

$$h(e_{\alpha_i}, e_{\alpha_j}) = 0, \quad i \neq j; \qquad h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i} Je_{\alpha_i}, \quad h(e_{\alpha_i}, e_u) = 0,$$

$$h(e_{N+1},e_{N+1})=3\lambda Je_{N+1},\qquad h(e_{N+1},e_u)=\lambda Je_u,\quad N=n_1+\cdots+n_k,$$
$$h(e_u,e_v)=\lambda\delta_{uv}Je_{N+1},\quad i,j=1,\dots,k;\qquad u,v=N+2,\dots,n. \tag{1.3}$$

For simplicity, we call a Lagrangian submanifold of a complex space form $\delta(n_1,\dots,n_k)$-*ideal* (resp., *improved* $\delta(n_1,\dots,n_k)$-*ideal*) if it satisfies the equality case of (1.1) (resp., the equality case of (1.2)) identically.

For $k=2$ and $n_1=n_2=2$, Theorem C reduces to the following.

Theorem D. *Let M be a Lagrangian submanifold in a complex space form $\tilde M^5(4c)$ of constant holomorphic sectional curvature $4c$. Then we have*

$$\delta(2,2)\le\frac{25}{4}H^2+8c. \tag{1.4}$$

If the equality sign of (1.4) *holds identically, then with respect some suitable orthonormal frame $\{e_1,\dots,e_5\}$ the second fundamental form h satisfies*

$$\begin{aligned}
&h(e_1,e_1)=\alpha Je_1+\beta Je_2+\mu Je_5, && h(e_1,e_2)=\beta Je_1-\alpha Je_2,\\
&h(e_2,e_2)=-\alpha Je_1-\beta Je_2+\mu Je_5, && \\
&h(e_3,e_3)=\gamma Je_3+\delta Je_4+\mu Je_5, && h(e_3,e_4)=\delta Je_3-\gamma Je_4,\\
&h(e_4,e_4)=-\gamma Je_3-\delta Je_4+\mu Je_5, && h(e_5,e_5)=4\mu Je_5,\\
&h(e_i,e_5)=\mu Je_i,\ i\in\Delta; && h(e_i,e_j)=0,\ \textit{otherwise},
\end{aligned} \tag{1.5}$$

for some functions α, β, γ, δ, μ, where $\Delta=\{1,2,3,4\}$.

The classification of $\delta(2,2)$-ideal Lagrangian submanifolds in complex space forms $\tilde M^5(4c)$ is done in [13]. In this paper we classify improved $\delta(2,2)$-ideal Lagrangian submanifolds in $\tilde M^5(4c)$. The main results of this paper are stated as Theorem 6.1, Theorem 7.1 and Theorem 8.1.

2. Preliminaries

2.1. Basic formulas. Let $\tilde M^n(4c)$ denote a complete simply-connected Kähler n-manifold with constant holomorphic sectional curvature $4c$. Then $\tilde M^n(4c)$ is holomorphically isometric to the complex Euclidean n-plane $\mathbf{C}^n$, the complex projective n-space $CP^n(4c)$, or a complex hyperbolic n-space $CH^n(-4c)$ according to $c=0$, $c>0$ or $c<0$.

Let M be a Lagrangian submanifold of $\tilde{M}^n(4c)$. We denote the Levi–Civita connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [7])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.1}$$

for tangent vector fields X and Y and normal vector fields ξ, where h is the second fundamental form, A is the shape operator and D is the normal connection.

The second fundamental form and the shape operator are related by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

The mean curvature vector $\overrightarrow{H}$ of M is defined by $\overrightarrow{H} = \frac{1}{n}$ trace h and the *squared mean curvature* is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H}\rangle$.

For Lagrangian submanifolds, we have (cf. [7], [12])

$$D_X JY = J\nabla_X Y, \tag{2.2}$$

$$A_{JX} Y = -Jh(X,Y) = A_{JY} X. \tag{2.3}$$

Formula (2.3) implies that $\langle h(X,Y), JZ\rangle$ is totally symmetric.

The equations of Gauss and Codazzi are given respectively by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle \\ + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle),\end{aligned} \tag{2.4}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.5}$$

where R is the curvature tensor of M and ∇h is defined by

$$(\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.6}$$

For an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, we put

$$h^i_{jk} = \langle h(e_j, e_k), Je_i\rangle, \quad i,j,k = 1, \ldots, n.$$

It follows from (2.3) that $h^i_{jk} = h^j_{ik} = h^k_{ij}$.

2.2. δ-invariants. Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \dots, e_n$ of T_pM, the scalar curvature τ at p is $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$.

Let L be a r-subspace of T_pM with $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha,\ \beta \leq r. \tag{2.7}$$

For given integers $n \geq 3$, $k \geq 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of k-tuples $(n_1, \dots, n_k)$ of integers satisfying $2 \leq n_1, \dots, n_k < n$ and $\sum_{j=1}^{k} i \leq n$.

Put $\mathcal{S}(n) = \cup_{k\geq 1} \mathcal{S}(n,k)$. For each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the first author introduced in 1990s the Riemannian invariant $\delta(n_1, \dots, n_k)$ by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \quad p \in M, \tag{2.8}$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$ (cf. [7] for details).

2.3. Horizontal lift of Lagrangian submanifolds. The following link between Legendrian submanifolds and Lagrangian submanifolds is due to [16] (see also [7, pp. 247–248]).

Case (i): $CP^n(4)$. Consider Hopf's fibration $\pi : S^{2n+1} \to CP^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at u is the orthogonal complement of $\imath u$, $\imath = \sqrt{-1}$, with respect to the metric on S^{2n+1} induced from the metric on $\mathbf{C}^{n+1}$. Let $\iota : N \to CP^n(4)$ be a Lagrangian isometric immersion. Then there is a covering map $\tau : \hat{N} \to N$ and a horizontal immersion $\hat{\iota} : \hat{N} \to S^{2n+1}$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Thus each Lagrangian immersion can be lifted locally (or globally if N is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $CP^n(4)$ is lifted to a minimal Legendrian submanifold of the Sasakian $S^{2n+1}(1)$.

Conversely, suppose that $f : \hat{N} \to S^{2n+1}$ is a Legendrian isometric immersion. Then $\iota = \pi \circ f : N \to CP^n(4)$ is again a Lagrangian isometric immersion. Under this correspondence the second fundamental forms h^f and h^ι of f and ι satisfy $\pi_* h^f = h^\iota$. Moreover, h^f is horizontal with respect to π.

Case (ii): $CH^n(-4)$. We consider the complex number space $\mathbf{C}_1^{n+1}$ equipped with the pseudo-Euclidean metric: $g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j$.

Consider $H_1^{2n+1}(-1) = \{z \in \mathbf{C}_1^{2n+1} : \langle z, z\rangle = -1\}$ with the canonical Sasakian structure, where $\langle\ ,\ \rangle$ is the induced inner product.

Put $T'_z = \{u \in \mathbf{C}^{n+1} : \langle u, z\rangle = 0\}$, $H^1_1 = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. Then there is an H^1_1-action on $H^{2n+1}_1(-1)$, $z \mapsto \lambda z$ and at each point $z \in H^{2n+1}_1(-1)$, the vector $\xi = -\imath z$ is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\langle \xi, \xi\rangle = -1$. The quotient space $H^{2n+1}_1(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4 whose complex structure J is induced from the complex structure J on $\mathbf{C}^{n+1}_1$ via Hopf's fibration $\pi : H^{2n+1}_1(-1) \to CH^n(4c)$.

Just like case (i), suppose that $\iota : N \to CH^n(-4)$ is a Lagrangian immersion, then there is an isometric covering map $\tau : \hat{N} \to N$ and a Legendrian immersion $f : \hat{N} \to H^{2n+1}_1(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus every Lagrangian immersion into $CH^n(-4)$ an be lifted locally (or globally if N is simply-connected) to a Legendrian immersion into $H^{2n+1}_1(-1)$. In particular, Lagrangian minimal submanifolds of $CH^n(-4)$ are lifted to Legendrian minimal submanifolds of $H^{2n+1}_1(-1)$. Conversely, if $f : \hat{N} \to H^{2n+1}_1(-1)$ is a Legendrian immersion, then $\iota = \pi \circ f : N \to CH^n(-4)$ is a Lagrangian immersion. Under this correspondence the second fundamental forms h^f and h^ι are related by $\pi_* h^f = h^\iota$. Also, h^f is horizontal with respect to π.

Let h be the second fundamental form of M in $S^{2n+1}(1)$ (or in $H^{2n+1}_1(-1)$). Since $S^{2n+1}(1)$ and $H^{2n+1}_1(-1)$ are totally umbilical with one as its mean curvature in $\mathbf{C}^{n+1}$ and in $\mathbf{C}^{n+1}_1$, respectively, we have

$$\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) - \varepsilon L, \tag{2.9}$$

where $\varepsilon = 1$ if the ambient space is $\mathbf{C}^{n+1}$; and $\varepsilon = -1$ if it is $\mathbf{C}^{n+1}_1$.

3. *H*-umbilical Lagrangian submanifolds and complex extensors

3.1. H-umbilical Lagrangian submanifolds.

Definition 3.1. A non-totally geodesic Lagrangian submanifold of a Kähler n-manifold is called *H-umbilical* if its second fundamental form satisfies

$$\begin{aligned} &h(e_j, e_j) = \mu J e_n, \qquad && h(e_j, e_n) = \mu J e_j, \quad j = 1, \ldots, n-1,\\ &h(e_n, e_n) = \varphi J e_n, \qquad && h(e_j, e_k) = 0, \quad 1 \le j \ne k \le n-1, \end{aligned} \tag{3.1}$$

for some functions μ, φ with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$. If the ratio of $\varphi : \mu$ is a constant r, the H-umbilical submanifold is said to be of ratio r.

If $G : N^{n-1} \to \mathbb{E}^n$ is a hypersurface of a Euclidean n-space $\mathbb{E}^n$ and $\gamma : I \to \mathbf{C}^*$ is a unit speed curve in $\mathbf{C}^* = \mathbf{C} - \{0\}$, then we may extend $G : N^{n-1} \to \mathbb{E}^n$ to an immersion $I \times N^{n-1} \to \mathbf{C}^n$ by $\gamma \otimes G : I \times N^{n-1} \to \mathbf{C} \otimes \mathbb{E}^n = \mathbf{C}^n$, where $(\gamma \otimes G)(s,p) = F(s) \otimes G(p)$ for $s \in I$, $p \in N^{n-1}$. This extension of G via tensor product $\otimes$ is called the *complex extensor* of G via the *generating curve* γ.

H-umbilical Lagrangian submanifolds in complex space forms were classified in a series of papers by the first author (cf. [2], [3], [4]). In particular, the following two results were proved in [2].

Theorem E. *Let $\iota : S^{n-1} \subset \mathbb{E}^n$ be the unit hypersphere in $\mathbb{E}^n$ centered at the origin. Then every complex extensor of ι via a unit speed curve $\gamma : I \to \mathbf{C}^*$ is an H-umbilical Lagrangian submanifold of $\mathbf{C}^n$ unless γ is contained in a line through the origin (which gives a totally geodesic Lagrangian submanifold).*

Theorem F. *Let M be an H-umbilical Lagrangian submanifold of $\mathbf{C}^n$ with $n \geq 3$. Then M is either a flat space or congruent to an open part of a complex extensor of $\iota : S^{n-1} \subset \mathbb{E}^n$ via a curve $\gamma : I \to \mathbf{C}^*$.*

3.2. Legendre curves. A unit speed curve $z : I \to S^3(1) \subset \mathbf{C}^2$ (resp., $z : I \to H^3_1(-1) \subset \mathbf{C}^2_1$) is called *Legendre* if $\langle z', \mathrm{i}z\rangle = 0$. It was proved in [3] that a unit speed curve z in $S^3(1)$ (resp., in $H^3_1(-1)$) is Legendre if and only if it satisfies

$$z'' = \mathrm{i}\lambda z' - z \quad (\text{resp., } z'' = \mathrm{i}\lambda z' + z) \tag{3.2}$$

for a real-valued function λ. It is known in [3] that λ is the curvature function of z in $S^3(1)$ (resp., in $H^3_1(-1)$) (see also [1, Lemmas 3.1 and 3.2]).

3.3. *H*-umbilical submanifolds with arbitrary ratio. We provide a general method to construct H-umbilical Lagrangian submanifolds with any given ratio in $CP^n(4)$ via curves in $S^2\left(\frac{1}{2}\right)$ (resp., in $CH^n(-4)$ via curves in $H^2(-\frac{1}{2})$).

Proposition 3.2. *For any real number r there exist H-umbilical Lagrangian submanifolds of ratio r in $CP^n(4)$ and in $CH^n(-4)$.*

Proof. If $r = 2$ this was done in [3, Theorems 5.1 and 6.1]. If $r \neq 2$, H-umbilical Lagrangian submanifolds of ratio r can be constructed as follows:

Case (a): $CP^n(4)$. Let $S^2\left(\frac{1}{2}\right) = \left\{\mathbf{x} \in \mathbb{E}^3;\ \langle \mathbf{x}, \mathbf{x}\rangle = \frac{1}{4}\right\}$. The Hopf fibration π from $S^3(1)$ onto $S^2\left(\frac{1}{2}\right) \equiv CP^1(4)$ is given by (cf. [1])

$$\pi(z_1, z_2) = \left(z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)\right), \quad (z_1, z_2) \in S^3(1) \subset \mathbf{C}^2. \tag{3.3}$$

For a Legendre curve z in $S^3(1)$, the projection $\gamma_z = \pi \circ z$ is a curve in $S^2\big(\frac{1}{2}\big)$. Conversely, each curve γ in $S^2\big(\frac{1}{2}\big)$ gives rise to a horizontal lift $\tilde{\gamma}$ in $S^3(1)$ via π which is unique up to a factor $e^{i\theta}, \theta \in \mathbf{R}$. Notice that each horizontal lift of γ is a Legendre curve in $S^3(1)$. Moreover, since the Hopf fibration is a Riemannian submersion, each unit speed Legendre curve z in $S^3(1)$ is projected to a unit speed curve γ_z in $S^2\big(\frac{1}{2}\big)$ with the same curvature.

It was known in [3, Lemma 7.2] that, for a given H-umbilical Lagrangian submanifold of ratio $r \neq 2$ in $\tilde{M}^n(4c)$, the function μ in (3.1) satisfies

$$\mu\mu'' - \left(\frac{r-3}{r-2}\right)\mu'^2 + (r-2)\mu^2((r-1)\mu^2 + c) = 0. \tag{3.4}$$

If μ is a non-trivial solution of (3.4) with $c = 1$, then there is a unit speed curve γ in $S^2\big(\frac{1}{2}\big)$ whose curvature equals to $r\mu$. Let z be a horizontal lift of γ in $S^3(1)$. Then z is a unit speed Legendre curve satisfying $z''(x) = \mathrm{i}r\mu z'(x) - z(x)$ (cf. [3, Theorem 4.1] or [1, Lemma 3.1]).

Consider the map $\psi : M^5 \to S^{11}(1) \subset \mathbf{C}^6$ defined by

$$\psi(x, y_1, \ldots, y_5) = (z_1(x), z_2(x)y_1, \ldots, \ldots, z_2(x)y_5), \quad \sum_{j=1}^{5} y_j^2 = 1. \tag{3.5}$$

It follows from [3, Theorem 4.1 and Lemma 7.2] that $\pi \circ \psi$ is a H-umbilical Lagrangian submanifold of ratio r in $CP^n(4)$ such that

$$\begin{aligned} &h(e_j, e_j) = \mu J e_5, && h(e_j, e_n) = J e_j, \\ &h(e_n, e_n) = r\mu J e_n, && h(e_j, e_k) = 0, \quad 1 \le j \neq k \le n-1, \end{aligned} \tag{3.6}$$

with respect to suitable orthonormal frame $\{e_1, \ldots, e_5\}$.

Case (b): $CH^n(-4)$. For a non-trivial solution of (3.4) with $c = -1$, we can construct an H-umbilical Lagrangian submanifold of $CH^n(-4)$ via the Hopf fibration $\pi : H_1^3(-1) \to CH^1(-4) \equiv H^2\big(-\frac{1}{2}\big)$ in a similar way as case (a), where

$$\pi(z_1, z_2) = \left(z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 + |z_2|^2)\right), \quad (z_1, z_2) \in H_1^3(-1) \subset \mathbf{C}_1^2, \tag{3.7}$$

and $H^2\big(-\frac{1}{2}\big) = \big\{(x_1, x_2, x_3) \in \mathbb{E}_1^3 : x_1^2 - x_2^2 - x_3^2 = \frac{1}{4},\ x_1 \ge \frac{1}{2}\big\}$ is the model of the real projective plane of curvature -4. $\square$

3.4. Classification of H-umbilical submanifolds of ratio 4. The equation of Gauss and (3.1) imply that H-umbilical Lagrangian submanifolds of ratio $r \neq 4$ in complex space forms contain no open subsets of constant sectional curvature. Hence we conclude from [3, Theorems 4.1 and 7.1] and §3.3 the following results.

Lemma 3.3. *An H-umbilical Lagrangian submanifold M of ratio 4 in $CP^5(4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{11}(1) \to CP^5(4)$ is Hopf's fibration, $\psi : M \to S^{11}(1) \subset \mathbf{C}^6$ is given by*

$$\psi(t, y_1, \ldots, y_5) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \tag{3.8}$$

and $z = (z_1, z_2) : I \to S^3(1) \subset \mathbf{C}^2$ is a unit speed Legendre curve satisfying $z'' = 4\mathrm{i}\mu z' - z$, and μ is a nonzero solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 + 1) = 0$.

Let M be an H-umbilical Lagrangian submanifold in $CH^5(-4)$ satisfying (3.1). We may assume that μ is defined on an open interval $I \ni 0$. Since H-umbilical submanifolds of ratio 4 in $CH^5(-4)$ contain no open subsets of constant curvature, Theorems 4.2 and 9.1 of [3] and results in §3.3 imply the following classification of H-umbilical submanifolds of ratio 4 in $CH^5(-4)$.

Lemma 3.4. *An H-umbilical Lagrangian submanifold M of ratio 4 in $CH^5(-4)$ is congruent to an open part of $\pi \circ \psi$, where $\pi : H_1^{11}(-1) \to CH^5(-4)$ is Hopf's fibration and $\psi : M \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ is either one of*

$$\psi(t, y_1, \ldots, y_4) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \tag{3.9}$$

$$\psi(t, y_1, \ldots, y_4) = (z_1(t)\mathbf{y}, z_2(t)), \quad \{\mathbf{y} \in \mathbb{E}_1^5 : \langle \mathbf{y}, \mathbf{y} \rangle = -1\}, \tag{3.10}$$

where z is a unit speed Legendre curve in $H_1^3(-1)$ satisfying $z'' = 4\mathrm{i}\mu z' + z$ and μ is a non-trivial solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 - 1) = 0$; or ψ is

$$\begin{aligned}\psi(t, u_1, \ldots, u_4) = \sqrt{\mu} e^{\mathrm{i}\int_0^t \mu(t)dt} \Bigg(& 1 + \frac{1}{2}\sum_{j=1}^{4} u_j^2 - \mathrm{i}t + \frac{1}{2\mu} - \frac{1}{2\mu(0)}, \\ & \left(i\mu(0) - \frac{\mu'(0)}{2\mu(0)} \right) \left(\frac{1}{2}\sum_{j=1}^{4} u_j^2 - \mathrm{i}t + \frac{1}{2\mu} - \frac{1}{2\mu(0)} \right), u_1, \ldots, u_4 \Bigg),\end{aligned} \tag{3.11}$$

where $z = (z_1, z_2) : I \to H_1^3(-1) \subset \mathbf{C}_1^2$ is a unit speed Legendre curve and μ is a non-trivial solution of $\mu'^2 = 4\mu^2(1 - \mu^2)$.

Example. It is easy to verify that $\mu = \operatorname{sech} 2t$ is a non-trivial solution of $\mu'^2 = 4\mu^2(1-\mu^2)$. Using $\mu = \operatorname{sech} 2t$, (3.11) reduces to

$$\psi(t, u_1, \dots, u_4) = \frac{e^{\mathrm{i}\tan^{-1}(\tanh t)}}{\sqrt{\cosh 2t}} \left(\frac{1}{2} - \mathrm{i}t + \frac{1}{2}\sum_{j=1}^{4} u_j^2 + \frac{\cosh 2t}{2}, \right.$$
$$\left. t - \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{2}\sum_{j=1}^{4} u_j^2 + \frac{\mathrm{i}\cosh 2t}{2}, u_1, \dots, u_4 \right). \quad (3.12)$$

It is direct to verify that (3.12) satisfies $\langle \psi, \psi \rangle = -1$ and the composition $\pi \circ \psi$ gives rise to an H-umbilical Lagrangian submanifold of ratio 4 in $CH^5(-4)$.

4. Some lemmas

We need the following lemmas for the proof of the main theorems.

Lemma 4.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold of $\tilde{M}^5(4c)$. Then with respect to some orthonormal frame $\{e_1, \dots, e_5\}$ we have*

$$\begin{aligned}
&h(e_1,e_1) = aJe_1 + \mu Je_5, && h(e_1,e_2) = -aJe_2,\\
&h(e_2,e_2) = -aJe_1 + \mu Je_5, && h(e_3,e_3) = bJe_3 + \mu Je_5,\\
&h(e_3,e_4) = -bJe_4, && h(e_4,e_4) = -bJe_3 + \mu Je_5,\\
&h(e_i,e_5) = \mu Je_i,\ i \in \Delta, && h(e_5,e_5) = 4\mu Je_5,\\
&h(e_i,e_j) = 0,\ \textit{otherwise}.
\end{aligned} \quad (4.1)$$

Proof. Under the hypothesis, we have (1.5) with respect to an orthonormal frame $\{e_1, \dots, e_5\}$. Thus, after applying [6, Lemma 1] to $V = \operatorname{Span}\{e_1, e_2\}$ and $V = \operatorname{Span}\{e_3, e_4\}$, we obtain (4.1). □

Let us put

$$\nabla_X e_i = \sum_{j=1}^{5} \omega_i^j(X) e_j, \quad i = 1, \dots, 5,\ X \in TM^5. \quad (4.2)$$

Then $\omega_i^j = -\omega_j^i$, $i, j = 1, \dots, 5$.

If $\mu = 0$, then M is a minimal Lagrangian submanifold according (4.1). Such submanifolds in complex space forms $\tilde{M}^5(4c)$ have been classified in [13].

If $a = b = 0$ and $\mu \neq 0$, then M is an H-umbilical Lagrangian submanifold with ratio 4. Therefore, from now on we assume that $a, \mu \neq 0$.

Lemma 4.2. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies* (4.1) *with $a, b, \mu \neq 0$. Then we have*

$$\begin{aligned}
&\nabla_{e_1} e_1 = \frac{e_2 a}{3a} e_2 - \nu e_5, && \nabla_{e_1} e_2 = -\frac{e_2 a}{3a} e_1, && \nabla_{e_2} e_1 = -\frac{e_1 a}{3a} e_2,\\
&\nabla_{e_2} e_2 = \frac{e_1 a}{3a} e_1 - \nu e_5, && \nabla_{e_3} e_3 = \frac{e_4 b}{3b} e_4 - \nu e_5, && \nabla_{e_3} e_4 = -\frac{e_4 b}{3b} e_3,\\
&\nabla_{e_4} e_3 = -\frac{e_3 b}{3b} e_4, && \nabla_{e_4} e_4 = \frac{e_3 b}{3b} e_3 - \nu e_5, && \nabla_{e_i} e_5 = \nu e_i,\ i \in \Delta,\\
&\nabla_{e_k} e_j = 0, \quad \textit{otherwise},
\end{aligned} \tag{4.3}$$

with $\nu = \frac{1}{2} e_5(\ln \mu) = -e_5(\ln a) = -e_5(\ln b)$, where $\Delta = \{1, 2, 3, 4\}$. Moreover, we have

$$e_j \mu = 0, j \in \Delta, \quad e_1 b = e_2 b = e_3 a = e_4 a = 0. \tag{4.4}$$

Proof. This lemma is obtained from Codazzi's equations via Lemma 4.1 and (4.2) and long computations. $\square$

Lemma 4.3. *Under the hypothesis of Lemma 4.2, we have*

(a) *T_0 is a totally geodesic distribution, i.e. T_0 is integrable whose leaves are totally geodesic submanifolds;*

(b) *$T_0 \oplus T_1$ and $T_0 \oplus T_2$ are totally geodesic distributions;*

(c) *T_1 and T_2 are spherical distributions, i.e. T_1, T_2 are integrable distributions whose leaves are totally umbilical submanifolds with parallel mean curvature vector,*

where $T_0 = \mathrm{Span}\{e_5\}$, $T_1 = \mathrm{Span}\{e_1, e_2\}$ and $T_2 = \mathrm{Span}\{e_3, e_4\}$.

Proof. Since the distribution T_0 is of rank one, it is integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ by Lemma 4.2, the integral curves of e_5 are geodesics in M. Thus we have statement (a). Statement (b) follows easily from (4.3).

To prove statement (c), first we observe that $[e_1, e_2] \in T_1$ and $[e_3, e_4] \in T_2$ follow from (4.3). Thus T_1, T_2 are integrable. Also, it follows from (4.3) that the second fundamental form h_1 of a leaf $\mathcal{L}_1$ of T_1 in M is given by

$$h_1(X, Y) = -\nu g_1(X_1, Y_1) e_5, \quad X_1, Y_1 \in T\mathcal{L}_1, \tag{4.5}$$

where g_1 is the metric of $\mathcal{L}_1$. From (4.3) we obtain $\nabla_{e_i} e_5 = \nu e_i$, $i = 1, 2$. Thus $D^1_{e_1} e_5 = D^1_{e_2} e_5 = 0$, where D^1 is the normal connection of $\mathcal{L}_1$ in M. It follows from Gauss' equation and Lemma 4.1 that the curvature tensor R satisfies

$$\langle R(e_1, e_2) e_1, e_j \rangle = 0, \quad j = 3, 4, 5. \tag{4.6}$$

Thus (4.6) and Lemma 4.2 imply that $0 \equiv R(e_1, e_2) e_1 \equiv (e_2 \nu) e_5 \pmod{T_1}$. Hence $e_2 \nu = 0$. Similarly, by considering $R(e_2, e_1) e_2$, we also have $e_1 \alpha = 0$. After combining these with $D^1 e_5 = 0$, we conclude that $\mathcal{L}_1$ has parallel mean curvature vector in M. Hence T_1 is a spherical distribution. Similarly, T_2 is also a spherical distribution. Consequently, we obtain statement (c). □

Lemma 4.4. *Under the hypothesis of Lemma 4.2, M is locally a warped product $I \times_{\rho_1(t)} M_1^2 \times_{\rho_2(t)} M_2^2$, where t is function such that $e_5 = \partial_t$ (i.e., $e_5 = \frac{\partial}{\partial t}$), ρ_1 and ρ_2 are two positive functions in t and M_1^2, M_2^2 are Riemannian 2-manifolds.*

Proof. This lemma follows from Lemma 4.3 and a result of Hiepko [15] (see also [7, Theorem 4.4, p. 90]). □

Lemma 3.3 and (4.4) imply that μ depends only on t. Thus $\mu = \mu(t)$.

Lemma 4.5. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, b, \mu \neq 0$. Then we have $c = -\nu^2 - \mu^2 < 0$. Thus μ satisfies $\mu'(t)^2 = -4\mu^2(t)(c + \mu^2(t))$.*

Proof. Under the hypothesis, it follows from Gauss' equation and Lemma 4.1 that $\langle R(e_1, e_3)e_3, e_1\rangle = c + \mu^2$. On the other hand, the definition of curvature tensor and Lemma 4.2 imply that $\langle R(e_1, e_3)e_3, e_1\rangle = -\nu^2$. Thus $c = -\nu^2 - \mu^2 < 0$. By combining this with the definition of ν, we obtain the lemma. □

5. More lemmas

Next, we consider the case $a, \mu \neq 0$ and $b = 0$.

Lemma 5.1. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, \mu \neq 0$ and $b = 0$. Then we have*

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{e_2 a}{3a} e_2 + \frac{e_3 a}{a} e_3 + \frac{e_4 a}{3a} e_4 - \nu e_5, \\
\nabla_{e_1} e_2 &= -\frac{e_2 a}{3a} e_1 - 3\o_1^2(e_3) e_3 - 3\o_1^2(e_4) e_4, \\
\nabla_{e_1} e_3 &= -\frac{e_3 a}{a} e_1 + 3\o_1^2(e_3) e_2 + \o_3^4(e_1) e_4, \\
\nabla_{e_1} e_4 &= -\frac{e_4 a}{a} e_1 + 3\o_1^2(e_4) e_2 - \o_3^4(e_1) e_3, \\
\nabla_{e_2} e_1 &= -\frac{e_1 a}{3a} e_2 + 3\o_1^2(e_3) e_3 + \o_1^4(e_2) e_4, \\
\nabla_{e_2} e_2 &= \frac{e_1 a}{3a} e_1 + \frac{e_3 a}{a} e_3 + \frac{e_4 a}{a} e_4 - \nu e_5,
\end{aligned}$$

$$
\begin{aligned}
&\nabla_{e_2}e_3 = -3\o_1^2(e_3)e_1 - \frac{e_3a}{a}e_2 + \o_3^4(e_2)e_4,\\
&\nabla_{e_2}e_4 = -\o_1^4(e_2)e_1 - \frac{e_4a}{a}e_2 - \o_3^4(e_2)e_3,\\
&\nabla_{e_3}e_1 = \o_1^2(e_3)e_2, \qquad \nabla_{e_3}e_2 = -\o_1^2(e_3)e_1,\\
&\nabla_{e_3}e_3 = \o_3^4(e_3)e_4 - \nu e_5, \ \nabla_{e_3}e_4 = -\o_3^4(e_3)e_3,\\
&\nabla_{e_4}e_1 = \o_1^2(e_4)e_2, \qquad \nabla_{e_4}e_2 = -\o_1^2(e_4)e_1,\\
&\nabla_{e_4}e_3 = \o_3^4(e_4)e_4, \qquad \nabla_{e_4}e_4 = -\o_3^4(e_4)e_3 - \nu e_5,\\
&\nabla_{e_5}e_3 = \o_3^4(e_5)e_4, \qquad \nabla_{e_5}e_4 = -\o_3^4(e_5)e_5,\\
&\nabla_{e_i}e_5 = \nu e_i,\ i \in \Delta, \qquad \nabla_{e_k}e_j = 0,\ \textit{otherwise}.
\end{aligned}
\tag{5.1}
$$

with $\nu = \frac{1}{2}e_5(\ln\mu) = -e_5(\ln a)$. *Moreover, we have*

$$e_j\mu = 0, \quad j \in \Delta = \{1,2,3,4\}. \tag{5.2}$$

Proof. Follows from Codazzi's equations via Lemma 4.1 and (4.2). □

Lemma 5.2. *Under the hypothesis of Lemma 5.1, we have*

(i) T_0 *is a totally geodesic distribution;*

(ii) T_3 *is a spherical distribution,*

where $T_0 = \mathrm{Span}\{e_5\}$ *and* $T_3 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$.

Proof. Clearly, T_0 is integrable. Moreover, since $\nabla_{e_5}e_5 = 0$ by Lemma 5.1, integral curves of e_5 are geodesics in M^5. Thus statement (i) follows. To prove statement (ii), we observe that the integrability of T_3 follows from (5.1). Also, (5.1) implies that the second fundamental form $\hat{h}$ of a leaf $\mathcal{L}$ of T_3 in M^5 is given by $\hat{h}(X,Y) = -\nu\hat{g}(X,Y)e_5$ for $X, Y \in T\mathcal{L}$, where $\hat{g}$ is the metric of $\mathcal{L}$. Since $[e_j, e_5]\mu = 0$ by (5.1) and $e_j\mu = 0$, for $j \in \Delta$, we find $e_ie_5\mu - e_5e_i\mu = 2e_1\nu = 0$. Therefore T_3 is a spherical distribution. □

Lemma 5.3. *Under the hypothesis of Lemma 5.1, M is locally a warped product $I \times_{\rho(t)} N^4$, where t is function such that $e_5 = \frac{\partial}{\partial t}$ and ρ is a positive function in t and N^4 is a Riemannian 4-manifold.*

Proof. Follows from Lemma 5.2 and Hiepko's theorem. □

It follows from (5.2) and the definition of ν that $\mu = \mu(t)$ and $\nu = \nu(t)$.

Lemma 5.4. *Under the hypothesis of Lemma 5.1, we have*

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2 - c, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{5.3}$$

Proof. From Gauss' equation and (5.1) we find $\langle R(e_1, e_5)e_5, e_1\rangle = 3\mu^2 + c$. On the other hand, (5.1) of Lemma 5.1 yields $\langle R(e_1, e_5)e_5, e_1\rangle = -e_5\nu - \nu^2$. Thus we find the first equation of (5.3). The second one follows immediately from the definition of ν given in Lemma 5.1. $\square$

6. Improved $\delta(2,2)$-ideal Lagrangian submanifolds of $\mathbf{C}^5$

Theorem 6.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $\mathbf{C}^5$. Then it is one of the following Lagrangian submanifolds:*

(a) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(b) *an H-umbilical Lagrangian submanifold of ratio 4;*

(c) *a Lagrangian submanifold defined by*

$$L(\mu, u_2, \dots, u_n) = \frac{e^{\frac{4}{3}\mathrm{i}\tan^{-1}\sqrt{\mu^3/(c^2-\mu^3)}}}{\sqrt{c^2\mu^{-1} - \mu^2} + \mathrm{i}\mu}\,\phi(u_2, \dots, u_n), \tag{6.1}$$

where c is a positive real number and $\phi(u_2, \dots, u_n)$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$.

Proof. Assume that M is an improved $\delta(2,2)$-ideal Lagrangian submanifold in $\mathbf{C}^5$. Then there exists an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds. If $\mu = 0$, then M is a minimal $\delta(2,2)$-ideal Lagrangian submanifold. Thus, we obtain case (a). If $\mu \neq 0$ and $a = b = 0$, we obtain case (b).

Now, let us assume $a, \mu \neq 0$. Then Lemma 4.5 implies $b = 0$. So, by Lemmas 5.1 we have (5.1) and $e_j\mu = 0$, $j \in \Delta$. Further, by Lemma 5.3, M is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$. Moreover, 4.1 shows that the second fundamental form satisfies

$$\begin{aligned}
&h(e_1, e_1) = aJe_1 + \mu Je_5, \qquad h(e_1, e_2) = -aJe_2, \\
&h(e_2, e_2) = -aJe_1 + \mu Je_5, \\
&h(e_3, e_3) = h(e_4, e_4) = \mu Je_5, \\
&h(e_i, e_5) = \mu Je_i, \ i \in \Delta,
\end{aligned}$$

$$h(e_5,e_5)=4\mu Je_5, \qquad h(e_i,e_j)=0, \text{ otherwise.} \tag{6.2}$$

From Lemma 5.4 we have the following differential system:

$$\frac{d\nu}{dt}=-3\mu^2-\nu^2, \quad \frac{d\mu}{dt}=2\mu\nu. \tag{6.3}$$

Let $\varphi(t)$ be a function satisfying $\frac{d\varphi}{dt}=-4\mu$. Consider the map

$$\phi=e^{\mathrm{i}\varphi}e_5. \tag{6.4}$$

Then $\langle\phi,\phi\rangle=1$. It follows from $\nabla_{e_5}e_5=0$, $\frac{d\varphi}{dt}=-4\mu$ and (6.2) that $\tilde\nabla_{e_5}\phi=0$, where $\tilde\nabla$ is the Levi–Civita connection of $\mathbf{C}^5$. Thus ϕ is independent of t.

Let L denote the Lagrangian immersion of M in $\mathbf{C}^5$. Then (6.4) yields

$$e_5=L_t=e^{-\mathrm{i}\varphi}\phi(u_1,\dots,u_4), \tag{6.5}$$

where $u_1,\dots,u_4$ are local coordinates of N^4. For each $j\in\Delta$, we obtain from $\nabla_{e_j}e_5=\nu e_j$ of Lemma 5.1 and the first equation of (6.3) that

$$\phi_*(e_j)=\tilde\nabla_{e_j}\phi=e^{\mathrm{i}\varphi}\tilde\nabla_{e_j}e_5=e^{\mathrm{i}\varphi}(\nu+\mathrm{i}\mu)e_j. \tag{6.6}$$

Thus

$$\tilde\nabla_{e_j}(\phi_*(e_i))=e^{\mathrm{i}\varphi}(\nu+\mathrm{i}\mu)\tilde\nabla_{e_j}e_i. \tag{6.7}$$

In view of $\nabla_{e_j}e_5=\nu e_j$ and (6.2), we may put

$$\tilde\nabla_{e_i}e_j=\left(\sum_{k=1}^{4}\Gamma_{ij}^k+\mathrm{i}h_{ij}^k\right)e_k-(\nu-\mathrm{i}\mu)\delta_{ij}e_5, \quad i,j\in\Delta, \tag{6.8}$$

for some functions Γ_{ij}^k. Now, it follows from (6.4), (6.6), (6.7), and (6.8) that

$$\begin{aligned}\tilde\nabla_{e_j}(\phi_*(e_i))&=\sum_{\gamma=2}^{n}\left(\Gamma_{ij}^k+\mathrm{i}h_{ij}^k\right)\phi_*(e_k)-(\mu^2+\nu^2)\delta_{ij}\phi\\&=\sum_{\gamma=2}^{n}(\Gamma_{ij}^k+\mathrm{i}h_{ij}^k)\phi_*(e_k)-\langle\phi_*(e_i),\phi_*(e_j)\rangle\phi.\end{aligned} \tag{6.9}$$

Since M is a Lagrangian submanifold in $\mathbf{C}^5$, (6.4) and (6.6) show that $\mathrm{i}\phi$ is perpendicular to each tangent space of M. Hence ϕ is a horizontal immersion in the unit hypersphere $S^9(1)\subset\mathbf{C}^5$. Moreover, it follows from (6.9) that the second fundamental form of ϕ is the original second fundamental form of M

respect to to the second factor N^4 of the warped product $I \times_{\rho(t)} N^4$. Hence, ϕ is a minimal horizontal immersion in $S^9(1)$. Therefore, ϕ is a horizontal lift of a minimal Lagrangian immersion in $CP^4(4)$. Now, it follows from (6.2) that ϕ is a horizontal lift of a $\delta(2)$-ideal minimal Lagrangian submanifold of $CP^4(4)$.

By direct computation we find

$$\tilde{\nabla}_{e_\alpha}\left(L - \frac{e_5}{\nu + \mathrm{i}\mu}\right) = 0, \quad \alpha = 1, \ldots, 5. \tag{6.10}$$

Thus, by (6.4), up to translations the Lagrangian immersion L is

$$L = \frac{e^{-\mathrm{i}\varphi}}{\nu + \mathrm{i}\mu}\phi(u_1, \ldots, u_4), \tag{6.11}$$

where ϕ is a horizontal minimal immersion in $S^9(1)$ and ν, φ, μ satisfy

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2, \quad \frac{d\varphi}{dt} = -4\mu, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{6.12}$$

From (6.12) we find

$$\frac{d\nu}{d\mu} + \frac{\nu}{2\mu} = -\frac{3\mu}{2\nu}. \tag{6.13}$$

After solving (6.13) we get $\nu = \pm\sqrt{c^2\mu^{-1} - \mu^2}$ for some real number $c > 0$. Replacing e_5 by $-e_5$ if necessary, we have

$$\nu = \sqrt{c^2\mu^{-1} - \mu^2}. \tag{6.14}$$

It follows from (6.12) an (6.14) that $\varphi'(\mu) = -2/\sqrt{c^2\mu^{-1} - \mu^2}$. By solving the last equation we find $\varphi = -\frac{4}{3}\mathrm{i}\tan^{-1}\sqrt{\mu^3/(c^2 - \mu^3)} + c_0$ for some constant c_0. Therefore, we have the theorem after applying a suitable translation in μ. $\square$

Remark 6.2. Minimal $\delta(2,2)$-ideal Lagrangian submanifolds in complex space forms $\mathbf{C}^5$, CP^5 and CH^5 are classified in [13]. Also $\delta(2)$-ideal minimal Lagrangian submanifolds in CP^4 and CH^4 have been classified recently in [14].

Let $\gamma(t)$ be a unit speed curve in $\mathbf{C}^*$. We put

$$\gamma(t) = r(t)e^{i\theta(t)}, \quad \gamma'(t) = e^{i\varsigma(t)}. \tag{6.15}$$

The following result gives H-umbilical submanifolds of $\mathbf{C}^5$ with ratio 4.

Proposition 6.3. *If M is an H-umbilical Lagrangian submanifold of $\mathbf{C}^5$ of ratio 4, then M is an open part of a complex extensor $\gamma \otimes \iota$ of the unit hypersphere $\iota : S^4(1) \subset \mathbb{E}^5$ via a generating curve $\gamma : I \to \mathbf{C}^*$ whose curvature satisfies $\kappa = 4\theta'$.*

Proof. If M is an H-umbilical Lagrangian submanifold of $\mathbf{C}^5$ with ratio 4, then the second fundamental form satisfies

$$h(e_j, e_j) = \mu J e_5, \qquad h(e_j, e_5) = \mu J e_j, \quad j \in \Delta,$$
$$h(e_5, e_5) = 4\mu J e_5, \qquad h(e_j, e_k) = 0, \quad 1 \le j \ne k \le 4,$$

for a nonzero function μ. Thus Gauss' equation yields $K(e_1 \wedge e_5) = 3\mu^2$. Hence M is non-flat. Therefore, according to Theorem F, M is an open part of a complex extensor of $\iota : S^{n-1}(1) \subset \mathbb{E}^n$ via a generating curve $\gamma : I \to \mathbf{C}^*$. It follows from [2] that the functions φ and μ in (4.1) are related with the two angle functions ζ and θ by $\varphi = \zeta'(t) = \kappa$ and $\mu = \theta'(t)$. Thus whenever γ is a unit speed curve satisfying $\kappa = 4\theta'$, the complex extensor $\gamma \otimes \iota$ is an H-umbilical Lagrangian submanifold of ratio 4. Conversely, every H-umbilical Lagrangian submanifold of ratio 4 in $\mathbf{C}^n$ can be obtained in such way. □

7. Improved $\delta(2,2)$-ideal Lagrangian submanifolds of CP^5

Theorem 7.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CP^5(4)$. Then it is one of the following Lagrangian submanifolds:*

(1) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(2) *an H-umbilical Lagrangian submanifold of ratio 4;*

(3) *a Lagrangian submanifold defined by*

$$L(\mu, u_2, \dots, u_4) = \frac{1}{c}\Big(\sqrt{\mu} e^{\mathrm{i}\theta}\phi, e^{3\mathrm{i}\theta}\big(\sqrt{c^2 - \mu^3 - \mu} - \mathrm{i}\mu^{\frac{3}{2}}\big)\Big), \tag{7.1}$$

where c is a positive real number, $\phi : N^4 \to S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$, and $\theta(\mu)$ satisfies

$$\frac{d\theta}{d\mu} = \frac{1}{2\sqrt{c^2\mu^{-1} - \mu^2 - 1}}. \tag{7.2}$$

Proof. Under the hypothesis there is an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds. If $\mu = 0$, then M is a $\delta(2,2)$-ideal Lagrangian minimal submanifold. Thus we obtain case (1). If $\mu \ne 0$ and $a, b = 0$, then M is an H-umbilical Lagrangian submanifold of ratio 4, which gives case (2).

Next, assume that $a, \mu \ne 0$. Then Lemma 4.5 implies $b = 0$. So, by Lemmas 5.1 we obtain (5.1) and (5.2). Also, in this case M is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$ according to Lemma 5.3. From Lemma 4.1, we find

$$h(e_1, e_1) = aJe_1 + \mu J e_5, \qquad h(e_1, e_2) = -aJe_2,$$

$$\begin{aligned}
&h(e_2,e_2) = -aJe_1 + \mu Je_5, \\
&h(e_3,e_3) = h(e_4,e_4) = \mu Je_5, \quad h(e_5,e_5) = 4\mu Je_5, \\
&h(e_i,e_5) = \mu Je_i,\ i \in \Delta, \qquad h(e_i,e_j) = 0,\ \text{otherwise}.
\end{aligned} \tag{7.3}$$

By Lemma 5.4 we have the following ODE system:

$$\frac{d\nu}{dt} = -1 - \nu^2 - 3\mu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{7.4}$$

Let $\theta(t)$ be a function on M satisfying

$$\theta'(t) = \mu. \tag{7.5}$$

Let L denote the horizontal lift in $S^{11}(1) \subset \mathbf{C}^6$ of the Lagrangian immersion of M in $CP^5(4)$ via Hopf 's fibration. Consider the maps:

$$\xi = \frac{e^{-3\mathrm{i}\theta}(e_5 - (\nu + \mathrm{i}\mu)L)}{\sqrt{1+\mu^2+\nu^2}}, \quad \phi = \frac{e^{-\mathrm{i}\theta}(L + (\nu - \mathrm{i}\mu)e_5)}{\sqrt{1+\mu^2+\nu^2}}. \tag{7.6}$$

Then $\langle \xi, \xi \rangle = \langle \phi, \phi \rangle = 1$. From $\nabla_{e_j} e_5 = \nu e_j$, $j \in \Delta$, and (7.4), we find $\tilde{\nabla}_{e_j}\xi = 0$. Moreover, it follows from Lemma 5.1 and (7.3) that $\tilde{\nabla}_{e_5} e_5 = 4\mathrm{i}\mu e_5 - L$. Thus we also have $\tilde{\nabla}_{e_5}\xi = 0$. Hence ξ is a constant unit vector in $\mathbf{C}^6$. Similarly, we also have $\tilde{\nabla}_{e_5}\phi = 0$. So ϕ is independent of t. Therefore, by combining (7.6) we find

$$L = \frac{e^{\mathrm{i}\theta}\phi - e^{3\mathrm{i}\theta}(\nu - \mathrm{i}\mu)\xi}{\sqrt{1+\mu^2+\nu^2}}. \tag{7.7}$$

Since ϕ is orthogonal to $\xi, \mathrm{i}\xi$, after choosing $\xi = (0, \dots, 0, 1) \in \mathbf{C}^6$ we obtain

$$L = \frac{1}{\sqrt{1+\mu^2+\nu^2}}(e^{\mathrm{i}\theta}\phi, e^{3\mathrm{i}\theta}(\nu - \mathrm{i}\mu)) \tag{7.8}$$

It follows from (7.4) and (7.5) that

$$\frac{d\nu}{d\mu} = -\frac{1+\nu^2+3\mu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{7.9}$$

Solving the first differential equation in (7.9) gives

$$\nu = \pm\sqrt{c^2\mu^{-1} - \mu^2 - 1}, \quad c \in \mathbf{R}^+. \tag{7.10}$$

By replacing e_5 by $-e_5$ if necessary, we have $\nu = \sqrt{c^2\mu^{-1} - \mu^2 - 1}$. Consequently,

$$L = \frac{1}{c}\left(\sqrt{\mu}e^{\mathrm{i}\theta}\phi, e^{3\mathrm{i}\theta}\left(\sqrt{c^2 - \mu^3 - \mu} - \mathrm{i}\mu^{\frac{3}{2}}\right)\right), \tag{7.11}$$

It follows from (5.1), (7.3) and the second formula in (7.6) that

$$\hat{\nabla}_{e_j}\phi = \frac{ce^{-\mathrm{i}\theta}}{\sqrt{\mu}}e_j, \quad j \in \Delta. \tag{7.12}$$

Thus after applying (6.11) and (7.12) we derive that

$$\hat{\nabla}_{e_\beta}\hat{\nabla}_{e_\alpha}\phi = \sum_{\gamma=2}^{n}(\Gamma_{ij}^k + \mathrm{i}h_{ij}^k)\phi_*(e_k) - \langle\phi_*(e_i), \phi_*(e_j)\rangle\phi, \quad i, j \in \Delta. \tag{7.13}$$

Hence ϕ is a horizontal immersion in $S^9(1)$. Moreover, it follows from (7.13) that the second fundamental form of ϕ is a scalar multiple of the original second fundamental form of M restricted to the second factor of the warped product $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $S^9(1)$ of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal submanifold of $CP^4(4)$.

The converse is easy to verify. □

8. Improved $\delta(2,2)$-ideal Lagrangian submanifolds of CH^5

Theorem 8.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CH^5(-4)$. Then M is one of the following Lagrangian submanifolds:*

(i) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*

(ii) *an H-umbilical Lagrangian submanifold of ratio 4;*

(iii) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \ldots, u_4) = \frac{1}{c}\left(\sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2, \ldots, u_4), e^{-\mathrm{i}\theta}\left(\sqrt{\mu - \mu^3 - c^2} - \mathrm{i}\mu^{\frac{3}{2}}\right)\right), \tag{8.1}$$

where c is a positive number, $\phi : N^4 \to H_1^9(-1) \subset \mathbf{C}_1^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CH^4(-4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 - c^2\mu^{-1}}$;

(iv) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \ldots, u_4) = \frac{1}{c}\left(e^{-\mathrm{i}\theta}\left(\sqrt{\mu - \mu^3 + c^2} - \mathrm{i}\mu^{\frac{3}{2}}\right), \sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2, \ldots, u_4)\right), \tag{8.2}$$

where c is a positive number, $\phi : N^4 \to S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CP^4(4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 + c^2\mu^{-1}}$;

(v) *a Lagrangian submanifold defined by*

$$L(t,u_1,\dots,u_4)=\frac{1}{\cosh t-\mathrm{i}\sinh t}\left(2t+w+\mathrm{i}\left(\cosh 2t-\langle\psi,\psi\rangle-\frac14\right),\right.$$
$$\left.\psi,2t+w+\mathrm{i}\left(\cosh 2t-\langle\psi,\psi\rangle+\frac14\right)\right), \tag{8.3}$$

where $\psi(u_1,\dots,u_4)$ is a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $\mathbf{C}^4$ and up to a constant $w(u_1,\dots,u_4)$ is the unique solution of the PDE system: $w_{u_j}=2\langle\psi_{u_j},\mathrm{i}\psi\rangle$, $j=1,2,3,4$;

(vi) *a Lagrangian submanifold defined by*

$$L(t,u_1,\dots,u_4)=\frac{1}{\cosh t-\mathrm{i}\sinh t}\left(2t+w+\mathrm{i}\left(\cosh 2t-\langle\psi,\psi\rangle-\frac14\right),\right.$$
$$\left.\psi_1,\psi_2,2t+w+\mathrm{i}\left(\cosh 2t-\langle\psi,\psi\rangle+\frac14\right)\right), \tag{8.4}$$

where $\psi=(\psi_1,\psi_2)$ is the direct product immersion of two non-totally geodesic Lagrangian minimal immersions $\psi_\alpha: N^2_\alpha\to\mathbf{C}^2$, $\alpha=1,2$, and up to a constant $w(u_1,\dots,u_4)$ is the unique solution of the PDE system: $w_{u_j}=2\langle\psi_{u_j},\mathrm{i}\psi\rangle$, $j=1,2,3,4$.

Proof. Under the hypothesis there exists an orthonormal frame $\{e_1,\dots,e_5\}$ such that (4.1) holds.

Case (1) $\mu=0$. In this case, we obtain case (i) of the theorem.

Case (2): $\mu\neq0$ *and* $a,b=0$. In this case M is an H-umbilical Lagrangian submanifold with ratio 4, which gives case (ii).

Case (3): $\mu\neq0$ *and at least one of* a,b *is nonzero.* Without loss of generality, we may assume $a\neq0$ and $\mu>0$. We divide this into two cases.

Case (3.a): $a,\mu\neq0$ *and* $b=0$. By Lemmas 5.1 we obtain (5.1) and (5.2). Also, M is locally a warped product $I\times_{\rho(t)}N^4$ with $e_5=\partial_t$ according to Lemma 5.3. From Lemma 4.1 we find

$$\begin{aligned}
&h(e_1,e_1)=aJe_1+\mu Je_5, && h(e_1,e_2)=-aJe_2,\\
&h(e_2,e_2)=-aJe_1+\mu Je_5,\\
&h(e_3,e_3)=h(e_4,e_4)=\mu Je_5, && h(e_5,e_5)=4\mu Je_5,\\
&h(e_i,e_5)=\mu Je_i,\ i\in\Delta, && h(e_i,e_j)=0,\ \text{otherwise}.
\end{aligned}\tag{8.5}$$

Let L be a horizontal immersion of M in $H_1^{11}(-1) \subset \mathbf{C}_1^6$ of the Lagrangian immersion of M in $CH^5(-4)$ via Hopf 's fibration and $\theta(t)$ a function satisfying

$$\frac{d\theta}{dt} = \mu. \tag{8.6}$$

From Lemma 5.4 we obtain the following ODE system:

$$\frac{d\nu}{dt} = 1 - 3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{8.7}$$

It follows from (8.6) and (8.7) that

$$\frac{d\nu}{d\mu} = \frac{1 - 3\mu^2 - \nu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{8.8}$$

Solving the first differential equation in (8.8) gives $\nu = \pm\sqrt{1 - \mu^2 - k\mu^{-1}}$ for some real number k. By replacing e_5 by $-e_5$ if necessary, we find

$$\nu = \sqrt{1 - \mu^2 - k\mu^{-1}}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\sqrt{1 - \mu^2 - k\mu^{-1}}}. \tag{8.9}$$

It follows from (8.7) that $\frac{d}{dt}(1 - \mu^2 - \nu^2) = -2\nu(1 - \mu^2 - \nu^2)$. Since this equation for $y(t) = 1 - \mu^2 - \nu^2 = k\mu^{-1}$ has a unique solution for each given initial condition, each solution either vanishes identically or is nowhere zero.

Case (3.a.1): $\mu^2 + \nu^2 < 1$. In this case, (8.9) implies $k > 0$. Thus we may put $k = c^2$, $c > 0$. Consider the maps:

$$\eta = \frac{e^{-3\mathrm{i}\theta}(e_5 - (\nu + \mathrm{i}\mu)L)}{\sqrt{1 - \mu^2 - \nu^2}}, \quad \phi = \frac{e^{-\mathrm{i}\theta}((\nu - \mathrm{i}\mu)e_5 - L)}{\sqrt{1 - \mu^2 - \nu^2}}. \tag{8.10}$$

Then $\langle \eta, \eta \rangle = 1$ and $\langle \phi, \phi \rangle = -1$. From $\nabla_{e_j} e_5 = \nu e_j$, $j \in \Delta$, and (8.5), we obtain $\tilde{\nabla}_{e_j}\xi = 0$, where $\tilde{\nabla}$ is the Levi–Civita connection of $\mathbf{C}_1^6$. Lemma 5.1 and (8.5) give $\tilde{\nabla}_{e_5} e_5 = 4\mathrm{i}\mu e_5 + L$. Thus we find $\tilde{\nabla}_{e_5}\xi = 0$. So η is a constant unit vector. Also, we find $\tilde{\nabla}_{e_5}\phi = 0$. Hence ϕ is independent of t. From (8.10) we get

$$L = -\frac{e^{\mathrm{i}\theta}\phi + e^{-\mathrm{i}\theta}(\nu - \mathrm{i}\mu)\eta}{\sqrt{1 - \mu^2 - \nu^2}}. \tag{8.11}$$

Since ϕ is orthogonal to η, $\mathrm{i}\eta$ and η is a constant unit space-like vector, we conclude from (8.9) and (8.11) that L is congruent to (8.1). Next, by applying the same method of the proof of Theorem 7.1, we conclude that ϕ is a horizontal immersion in $H_1^9(-1)$ whose second fundamental form is a scalar multiple of the original second fundamental form restricted to the second factor of $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $H_1^9(-1)$ of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal submanifold of $CH^4(-4)$. This gives case (iii).

Case (3.a.2): $\mu^2+\nu^2>1$. In this case (8.8) implies $k<0$. Thus we may put $k=-c^2$, $c>0$. Now, we consider the maps:

$$\eta=\frac{e^{-3\mathrm{i}\theta}(e_5-(\nu+\mathrm{i}\mu)L)}{\sqrt{\mu^2+\nu^2-1}},\quad \phi=\frac{e^{-\mathrm{i}\theta}((\nu-\mathrm{i}\mu)e_5-L)}{\sqrt{\mu^2+\nu^2-1}} \tag{8.12}$$

instead. Then $\langle\phi,\phi\rangle=-\langle\eta,\eta\rangle=1$. By applying similar arguments as case (3.a.1), we know that η is a constant time-like vector and ϕ is independent of t and orthogonal to $\eta,\mathrm{i}\eta$. Moreover, we may prove that ϕ is a minimal Legendre immersion in $S^9(1)$. Therefore we have case (iv) after choosing $\eta=(1,0,\dots,0)$.

Case (3.a.3): $\mu^2+\nu^2=1$. In this case system (8.7) gives $\frac{d\nu}{dt}=2(\nu^2-1)$ and $\mu=\pm\sqrt{1-\nu^2}$. Solving these and applying a suitable translations in t, we find

$$\mu=\operatorname{sech}2t,\quad \nu=-\tanh 2t. \tag{8.13}$$

It follows from $\nabla_{e_5}e_5=0$, (8.5) and (8.13) that the horizontal lift L of the Lagrangian immersion of M in $CH^5(-4)\subset\mathbf{C}_1^6$ satisfies

$$L_{tt}-4\mathrm{i}(\operatorname{sech}2t)L_t-L=0. \tag{8.14}$$

Solving this second order differential equation gives

$$L=\frac{\phi(u_1,\dots,u_4)+B(u_1,\dots,u_4)(2t+\mathrm{i}\cosh 2t)}{\cosh t-\mathrm{i}\sinh t}, \tag{8.15}$$

where $\phi(u_1,\dots,u_4)$ and $B(u_1,\dots,u_4)$ are $\mathbf{C}_1^6$-valued functions.

On the other hand, it follows from Lemma 5.1, (8.5) and (8.13) that

$$L_{tu_j}=(\mathrm{i}\operatorname{sech}2t-\tanh 2t)L_{u_j},\quad j\in\Delta. \tag{8.16}$$

Substituting (8.15) into (8.16) shows that B is a constant vector ζ. Thus

$$L(t,u_1,\dots,u_4)=\frac{\phi(u_1,\dots,u_4)}{\cosh t-\mathrm{i}\sinh t}+\frac{(2t+\mathrm{i}\cosh 2t)}{\cosh t-\mathrm{i}\sinh t}\zeta, \tag{8.17}$$

Since $\langle L,L\rangle=-1$, (8.17) implies

$$-\cosh 2t=\langle\phi,\phi\rangle+\langle\phi,(4t+2\mathrm{i}\cosh 2t)\zeta\rangle+(4t^2+\cosh^2(2t))\langle\zeta,\zeta\rangle. \tag{8.18}$$

Since $\phi_t = 0$, by differentiating (8.18) with respect t we find

$$-\sinh 2t = 2t\langle \phi, \zeta\rangle + 2\sinh 2t\langle \phi, \mathrm{i}\zeta\rangle + (4t + \sinh 4t)\langle \zeta, \zeta\rangle. \tag{8.19}$$

We find from (8.19) at $t = 0$ that $\langle \phi, \zeta\rangle = 0$. Thus (8.19) gives

$$0 = \sinh 2t(1 + \langle \phi, \mathrm{i}\zeta\rangle) + (4t + \sinh 4t)\langle \zeta, \zeta\rangle. \tag{8.20}$$

Differentiating (8.20) gives $\langle \phi, \mathrm{i}\zeta\rangle = -\frac{1}{2} - 2\langle \zeta, \zeta\rangle$. Thus (8.17) yields $\langle \phi, \mathrm{i}\zeta\rangle = -\frac{1}{2}$ and $\langle \zeta, \zeta\rangle = 0$. Now, we find from (8.18) that $\langle \phi, \phi\rangle = 0$. Consequently we have

$$\langle \phi, \phi\rangle = \langle \zeta, \zeta\rangle = \langle \phi, \zeta\rangle = 0, \quad \langle \phi, \mathrm{i}\zeta\rangle = -\frac{1}{2}. \tag{8.21}$$

Since ζ is a constant light-like vector, we may put

$$\zeta = (1, 0, \dots, 0, 1), \quad \phi = (a_1 + \mathrm{i}b_1, \dots, a_6 + \mathrm{i}b_6). \tag{8.22}$$

It follows from (8.21) and (8.22) that $a_6 = a_1$ and $b_6 = b_1 + \frac{1}{2}$. Therefore

$$\phi = \left(a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2, \dots, a_1 + \mathrm{i}\left(b_1 + \frac{1}{2}\right)\right). \tag{8.23}$$

Now, by using $\langle \phi, \phi\rangle = 0$ and (8.23), we find $\psi = (a_2 + \mathrm{i}b_2, \dots, a_5 + \mathrm{i}b_5)$ and $b_1 = -\frac{1}{4} - \langle \psi, \psi\rangle$. Combining these with (8.23) yields

$$\phi = \left(w - \mathrm{i}\langle \psi, \psi\rangle - \frac{\mathrm{i}}{4}, \psi, w - \mathrm{i}\langle \psi, \psi\rangle + \frac{\mathrm{i}}{4}\right) \tag{8.24}$$

with $w = a_1$. It follows from (8.22) and (8.24) that $\langle \phi_{u_j}, \zeta\rangle = \langle \phi_{u_j}, \mathrm{i}\zeta\rangle = 0$. Thus, by applying $\langle L_{u_j}, \mathrm{i}L\rangle = 0$, $j \in \Delta$, we find from (8.17) that $\langle \phi_{u_j}, \mathrm{i}\phi\rangle = 0$.

On the other hand, (8.24) implies that

$$\langle \phi_{u_j}, \mathrm{i}\phi\rangle = -\frac{1}{2}w_{u_j} + \langle \psi_{u_j}, \mathrm{i}\psi\rangle \tag{8.25}$$

with $w_{u_j} = \frac{\partial w}{\partial u_j}$. Therefore w satisfies the PDE system: $w_{u_j} = 2\langle \psi_{u_j}, \mathrm{i}\psi\rangle$.

Now, we derive from (8.17), (8.22) and (8.23) that

$$\begin{aligned} L = \frac{1}{\cosh t - \mathrm{i}\sinh t}\Bigg(&2t + w + \mathrm{i}\left(\cosh 2t - \langle \psi, \psi\rangle - \frac{1}{4}\right), \\ &\psi, 2t + w + \mathrm{i}\left(\cosh 2t - \langle \psi, \psi\rangle + \frac{1}{4}\right)\Bigg). \end{aligned} \tag{8.26}$$

It follows from (8.26) that

$$L_{u_j} = \frac{1}{\cosh t - \mathrm{i}\sinh t}\Big(w_{u_j} - \mathrm{i}\langle\psi,\psi\rangle_{u_j}, \psi_{u_j}, w_{u_j} - \mathrm{i}\langle\psi,\psi\rangle_{u_j}\Big). \tag{8.27}$$

Thus we find $\langle\psi_{u_j},\psi_{u_k}\rangle = \cosh 2t\langle L_{u_j}, L_{u_k}\rangle$ which implies that ψ is an immersion in $\mathbf{C}^4$. Also, we find from (8.27) and $\langle L_{u_j}, \mathrm{i}L_{u_k}\rangle = 0$ that $\langle\psi_{u_j}, \mathrm{i}\psi_{u_k}\rangle = 0$. Thus ψ is a Lagrangian immersion. Now, by applying an argument similar to the last part of the proof of [11, Theorem 6.1], we conclude that

$$\psi_{u_j u_k} = \sum_{i=1}^{4}(\Gamma^i_{jk} + \mathrm{i}h^i_{jk})\phi_{u_i}, \quad j,k \in \Delta.$$

Therefore, according to (8.5), ψ is a $\delta(2)$-ideal minimal Lagrangian immersion in $\mathbf{C}^4$. Consequently, we obtain case (v) of the theorem.

Case (3.b): $a, b, \mu \neq 0$. We obtain case (vi) of the theorem by applying the same argument as case (3.a.3). $\square$

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The Maslov form in non-invariant slant submanifolds of S-space-forms

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The Maslov form in non-invariant slant submanifolds of S-space-forms

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Abstract We present a characterization theorem for the Maslov form in certain non-invariant slant submanifolds of S-space-forms to be closed and, from it, we deduce a topological obstruction for these types of non-invariant slant immersions. Moreover, we also give conditions for an anti-invariant submanifolds of an S-manifold, tangent to the structure vector fields, to have closed and conformal Maslov form.

Keywords S-space-forms · Slant submanifolds · Maslov form

Mathematics Subject Classification (2000) 53C25 · 53C40

1 Introduction

The study of submanifolds which present a homogeneous behavior with respect to the structure of the ambient manifold has become an interesting research subject. In particular, slant submanifolds, defined by B.-Y. Chen in complex geometry as a natural generalization of both holomorphic and totally real submanifolds [10,11], have this homogeneous behavior, and they can be considered in more general situations (see, for instance, [6,7,12,15]).

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On the other hand, for totally real submanifolds of almost Hermitian manifolds, one can consider the so-called Maslov form, defined as the dual form of the vector fields JH, being J the almost Hermitian structure and H the mean curvature vector of the submanifold, which has been widely studied (for example, [3,8,9,19] can be consulted). Thus, in [19], it is proved that any Lagrangian submanifold of $\mathbf{C}^m$ has closed Maslov form and, moreover, that the well-known Whitney sphere is the only compact Lagrangian submanifold of $\mathbf{C}^m$ with conformal Maslov form. However, there are not too many papers devoted to study the Maslov form in anti-invariant submanifolds of metric almost contact manifolds or, more in general, of metric f-manifolds, considering such form as the dual form of the vector field ϕH (resp. fH), where ϕ (resp. f) denotes the almost contact structure (resp. the f-structure). In fact, the more significative results can be found in [17,18] for integral submanifolds of Sasakian manifolds (that is, anti-invariant submanifolds normal to the structure vector field).

In the present paper, we deal with non-invariant slant submanifolds of S-manifold. These S-manifolds were introduced by D.E. Blair in [1] and, for manifolds endowed with a general f-structure, they play the role of the Kaehlerian manifolds in complex geometry and of the Sasakian manifolds in contact geometry. In such submanifolds, we define the Maslov form as the dual 1-form of the tangent component of the vector field fH, and our purpose is to find conditions for it to be closed and conformal in the case of being the ambient S-manifold an S-space-form, that is, of having constant f-sectional curvature.

To this end and after two preliminary sections containing basic notions of Riemannian submanifolds theory and some definitions and formulas concerning metric f-manifolds and their submanifolds for later use, in Sect. 4 we consider $(m+s)$-dimensional (being s the number of structure vector fields) non-invariant slant submanifolds of an S-space-form of dimension $2m+s$, and we prove that, in the particular cases of S-slant submanifolds and anti-invariant submanifolds tangent to the structure vector fields, the Maslov form is closed if and only if the constant f-sectional curvature equals to $-3s$ (this holds for $\mathbf{R}^{2m+s}$ with its usual structure of S-manifold [13]) and, as a consequence, we get a topological obstruction to S-slant immersions as well as to anti-invariant immersions tangent to the structure vector fields into an S-space-form of constant f-sectional curvature $c=-3s$. To obtain these results, we use special local orthonormal frames for the ambient S-space-form adapted to the structure of the submanifolds in each case, which cannot be deduced one from the other and thus, even though the mentioned final results are the same, the computations have to be done independently for the two cases.

Finally, in Sect. 5, we first prove that the Maslov form of an $(m+s)$-dimensional anti-invariant submanifold tangent to the structure vector fields of an S-manifold of dimension $2m+s$, if it is closed, is also conformal if and only if the mean curvature vector is parallel. Then, we introduce the more restrictive notion for the Maslov form to be $\mathcal{L}$-conformal, with $\mathcal{L}$ the distribution orthogonal to the structure vector fields, and present a sufficient condition for such type of submanifolds to have $\mathcal{L}$-conformal Maslov form when the ambient S-manifold has constant f-sectional curvature $c=-3s$, giving examples of submanifolds satisfying this condition.

2 Preliminaries

Let $(\widetilde{M}, g)$ a Riemannian manifold. A vector field X in $\widetilde{M}$ is said to be *closed* in $\widetilde{M}$ if the 1-form ω given by $\omega_X(Y)=g(X,Y)$ (the dual 1-form of X) is closed. Then, X is closed if

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and only if

$$g(Y, \widetilde{\nabla}_Z X) = g(Z, \widetilde{\nabla}_Y X), \tag{2.1}$$

for any vector fields Y, Z in $\widetilde{M}$, where $\widetilde{\nabla}$ is denoting the Riemannian connection of $\widetilde{M}$. On the other hand, X is called *conformal* in $\widetilde{M}$ (and the dual 1-form is also called *conformal* in $\widetilde{M}$) if $L_X g = \rho g$, being ρ a differentiable function on $\widetilde{M}$. A closed vector field X is conformal if and only if

$$\widetilde{\nabla}_Y X = fY, \tag{2.2}$$

for any vector field Y in $\widetilde{M}$, being f a differentiable function on $\widetilde{M}$.

Now, let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $\widetilde{M}$. Let g denote the metric tensor of $\widetilde{M}$ as well as the induced metric tensor on M. If ∇ denotes the Riemannian connection of M, the Gauss–Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{2.3}$$

for any vector fields X, Y (resp., V) tangent (resp., normal) to M, where D is the normal connection, σ is the second fundamental form of the immersion, and A_V is the Weingarten endomorphism associated with V. Then, A_V and σ are related by:

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{2.4}$$

The curvature tensor fields of ∇ and $\widetilde{\nabla}$ are denoted by R and $\widetilde{R}$, respectively. Then, $\widetilde{R}$ satisfies the Codazzi equation

$$(\widetilde{R}(X, Y)Z)^{\perp} = \left(\overline{\nabla}_X \sigma\right)\left(Y, Z\right) - \left(\overline{\nabla}_Y \sigma\right)\left(X, Z\right), \tag{2.5}$$

for any X, Y, Z tangent to M, where

$$\left(\overline{\nabla}_X \sigma\right)\left(Y, Z\right) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{2.6}$$

and $(\widetilde{R}(X, Y)Z)^{\perp}$ is denoting the normal component of $\widetilde{R}(X, Y)Z$.

The *mean curvature vector* H is defined by

$$H = \frac{1}{m}\operatorname{trace}\sigma = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i),$$

where $\dim M = m$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of tangent vector fields to M. This mean curvature vector is said to be *parallel* if $D_X H = 0$, for any vector field X tangent to M. The submanifold M is called *minimal* if H vanishes identically or, equivalently, if $\operatorname{trace} A_V = 0$, for any vector field V normal to M. Moreover, M is said to be *totally geodesic* in $\widetilde{M}$ if $\sigma \equiv 0$.

Next, we assume that $m \geq 2$. If $\dim(\widetilde{M}) = \widetilde{m}$, a local orthonormal basis of $\mathcal{X}(\widetilde{M})$

$$\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{\widetilde{m}}\}$$

can be chosen such that, restricted to M, the vector fields $e_1, \ldots, e_m$ are tangent to M and so, $e_{m+1}, \ldots, e_{\widetilde{m}}$ are normal to M. Let $\{\omega^1, \ldots, \omega^{\widetilde{m}}\}$ be the field of dual frames. Then, for any vector field X tangent to $\widetilde{M}$, it can be written that

$$\widetilde{\nabla}_X e_A = \sum_{B=1}^{\widetilde{m}} \omega_A^B(X) e_B, \tag{2.7}$$

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for any $A = 1, \ldots, \widetilde{m}$. The 1-forms ω_A^B, defined by the Eq. (2.7), are called the *connection forms* of M in $\widetilde{M}$, and they satisfy $\omega_B^A + \omega_A^B = 0$, for any $A, B = 1, \ldots, \widetilde{m}$. Moreover, the structure equations of $\widetilde{M}$ are given by

$$d\omega^A = -\sum_{B=1}^{\widetilde{m}} \omega_B^A \wedge \omega^B, \quad d\omega_B^A = -\sum_{C=1}^{\widetilde{m}} \omega_C^A \wedge \omega_B^C + \Omega_B^A, \tag{2.8}$$

where Ω_B^A are the so-called *curvature forms*, defined by

$$\Omega_B^A = \frac{1}{2} \sum_{C,D=1}^{\widetilde{m}} \widetilde{R}_{BCD}^A \omega^C \wedge \omega^D, \tag{2.9}$$

with $1 \leq A, B, C, D \leq \widetilde{m}$.

3 Submanifolds of metric f-manifolds

A $(2m+s)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ endowed with an f-structure f (that is, a tensor field f of type (1,1) and rank $2m$ satisfying $f^3 + f = 0$ (see [20]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on $\widetilde{M}$ (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0; \quad \eta_\alpha \circ f = 0; \quad f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha; \tag{3.10}$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{3.11}$$

for any X, Y tangent to $\widetilde{M}$. From the definition, the metric g satisfies that

$$g(fX, Y) = -g(X, fY), \tag{3.12}$$

for any X, Y. Let F be the 2-form on $\widetilde{M}$ defined by $F(X, Y) = g(X, fY)$. Since f is of rank $2m$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$ and, particularly, $\widetilde{M}$ is orientable. The f-structure f is said to be *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ denotes the Nijenhuis tensor of f.

A metric f-manifold is said to be a K-manifold [1] if it is normal and $\mathrm{d}F = 0$. In a K-manifold $\widetilde{M}$, the structure vector fields are Killing vector fields [1]. Furthermore, a K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$, for any α. Note that, if $s = 0$, a K-manifold would correspond to a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold and an S-manifold is a Sasakian manifold. When $s \geq 2$, non-trivial examples can be found in [1,13]. Moreover, the Riemannian connection $\widetilde{\nabla}$ of an S-manifold satisfies (see [1]), for any tangent vector fields X, Y and any $\alpha = 1, \ldots, s$:

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{3.13}$$

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X). \tag{3.14}$$

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A plane section π on a metric f-manifold $\widetilde{M}$ is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c and then, it is denoted by $\widetilde{M}(c)$. In such case, the curvature tensor field R of $M(c)$ satisfies [14]:

$$\begin{aligned} R(X, Y, Z, W) = \sum_{\alpha,\beta=1}^{s} &(g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) \\ &+ g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)) \\ &+ \frac{c+3s}{4}(g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)) \\ &+ \frac{c-s}{4}(F(X, W)F(Y, Z) - F(X, Z)F(Y, W) - 2F(X, Y)F(Z, W)), \end{aligned} \tag{3.15}$$

for any tangent vector fields X, Y, Z, W.

Next, let M be a isometrically immersed submanifold of a metric f-manifold $\widetilde{M}$. Given a differentiable function on $\widetilde{M}$, we also denote by F the composition $F \circ x$, where $x : M \longrightarrow \widetilde{M}$ is the corresponding immersion. For any vector field X tangent to M, we write

$$fX = TX + NX, \tag{3.16}$$

where TX and NX are the tangential and normal components of fX, respectively. The submanifold M is said to be *invariant* if N is identically zero, that is, if fX is tangent to M, for any vector field X tangent to M. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if fX is normal to M, for any X tangent to M.

Similarly, for any vector field V normal to M, we have

$$fV = tV + nV, \tag{3.17}$$

where tV (resp., nV) is the tangential component (resp., the normal component) of fV. From (3.12), by using (3.16) and (3.17), we get

$$g(TX, Y) = -g(X, TY) \tag{3.18}$$

and

$$g(NX, V) = -g(X, tV), \tag{3.19}$$

for any X, Y tangent to M and V normal to M. Moreover, if $\widetilde{M}$ is a S-manifold and the structure vector fields are tangent to M, from (2.3), (3.13) and (3.16), it is easy to show that

$$\nabla_X \xi_\alpha = -TX, \quad \sigma(X, \xi_\alpha) = -NX, \tag{3.20}$$

for any X tangent to M and any $\alpha = 1, \ldots, s$ and, in particular, since $f\xi_\alpha = 0$, for any α:

$$\sigma(\xi_\alpha, \xi_\beta) = 0, \quad \alpha, \beta = 1, \ldots, s. \tag{3.21}$$

Also, if we extend formula (3.14), taking into account (2.3), the tangent component gives

$$A_{NY}X = (\nabla_X T)Y - t\sigma(X, Y) - \sum_{\alpha=1}^{s} [g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X], \tag{3.22}$$

for any X, Y tangent to M, where $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$.

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Concerning the behavior of the second fundamental of a submanifold in a metric f-manifold, we know that the study of totally geodesic submanifolds of S-manifolds reduces to the study of invariant submanifolds (see [7]). It is necessary, then, to use a variation of this concept, more related to the structure, namely *totally f-geodesic* submanifolds, introduced by Ornea [16]. Thus, a submanifold of an S-manifold, tangent to the structure vector fields, is said to be a totally f-geodesic submanifold if the distribution $\mathcal{L}$ is totally geodesic, that is, if $\sigma(X, Y) = 0$, for any $X, Y \in \mathcal{L}$. Thus, from (3.20), the submanifold M is totally f-geodesic if and only if

$$\sigma(X, Y) = -\sum_{\alpha=1}^{s}(\eta_\alpha(X)NY + \eta_\alpha(Y)NX), \tag{3.23}$$

for any X, Y tangent to M. It is easy to show that a totally f-geodesic submanifold is minimal.

From now on, we will always suppose that all the structure vector fields are tangent to the submanifold M. Then, the distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$, and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \dots, s$ and if $X \in \mathcal{M}$, then $fX = 0$.

The submanifold M is said to be a *slant* submanifold if, for any $x \in M$ and any $X \in T_xM$, linearly independent on $\xi_1, \dots, \xi_s$, the angle between fX and T_xM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$ (see [4] for a general survey concerning slant submanifolds in different geometric structures). Moreover, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is not invariant nor anti-invariant is called a *proper* slant immersion, and the submanifold is said to be *proper slant*. If M is a non-anti-invariant θ-slant submanifold (that is, if $\theta \in [0, \pi/2)$), then it was proved in [12] that

$$(\overline{f}, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$$

is a metric f-structure on M, where $\overline{f} = (\sec\theta)T$, which implies that, if $\dim(M) = m + s$ then, m is even. Moreover, in a θ-slant submanifold of a metric f-manifold, we have [6]:

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y)), \tag{3.24}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y)), \tag{3.25}$$

for any vector fields X, Y tangent to the submanifold.

We say that a proper slant submanifold of an S-manifold is an *S-slant submanifold* if

$$(\nabla_X T)Y = \cos^2\theta\sum_{\alpha=1}^{s}(g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X), \tag{3.26}$$

for any tangent vector fields X, Y to M, where θ is the slant angle. Note that, if $X, Y \in \mathcal{L}$, then $(\nabla_X T)Y = (\nabla_Y T)X$. Furthermore, it has been proved [6] that every $(2 + s)$-dimensional proper slant submanifold of an S-manifold is an S-slant submanifold. Observe that $2 + s$ is the minimum possible dimension for a submanifold of an S-manifold, tangent to the structure vector fields, to be proper slant.

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4 Closed Maslov form

Let M^{m+s} be an $(m+s)$-dimensional slant submanifold, with slant angle θ, of an S-manifold $\widetilde{M}^{2m+s}$ of dimension $2m+s$. By following the analogy with totally real submanifolds of Kaehlerian manifolds, we define the *Maslov form* ω_H of M as the dual form of the vector field tH, that is,

$$\omega_H(X) = g(X, tH), \tag{4.27}$$

for any tangent vector field X to M. In this section, our goal is to give a characterization for ω_H to be closed when the ambient S-manifold is an S-space-form $\widetilde{M}(c)$. First, we consider the case of M being a proper slant submanifold. As we have already pointed out, then m has to be even and so, we can write $m = 2k$.

Now, we are going to define an special local frame for $\widetilde{M}$. Let e_1 be a unit tangent vector field of M, orthogonal to the structure vector fields. We put:

$$e_2 = (\sec\theta)Te_1, \quad e_{1*} = (\csc\theta)Ne_1, \quad e_{2*} = (\csc\theta)Ne_2.$$

Since $k \geq 1$, then, by using an induction procedure, for each $l = 1, \ldots, k-1$, we can choose a unit tangent vector field e_{2l+1} of M such that e_{2l+1} is normal to

$$\{e_1, e_2, \ldots, e_{2l-1}, e_{2l}, \xi_1, \ldots, \xi_s\}$$

and we put:

$$e_{2l+2} = (\sec\theta)Te_{2l+1}, e_{(2l+1)*} = (\csc\theta)Ne_{2l+1}, e_{(21+2)*} = (\csc\theta)Ne_{2l+2}.$$

Thus, by using (3.24) and (3.25), we have a local orthonormal frame of tangent vector fields of $\widetilde{M}$,

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1*}, \ldots, e_{m*}\},$$

such that $e_1, \ldots, e_m \in \mathcal{L}$ and $e_{1*}, \ldots, e_{m*}$ are normal to M. Furthermore, a direct computation gives:

$$Te_{2j-1} = (\cos\theta)e_{2j}, Te_{2j} = -(\cos\theta)e_{2j-1}, j = 1, \ldots, k; \tag{4.28}$$
$$Ne_i = (\sin\theta)e_{i*}, te_{i*} = -(\sin\theta)e_i, i = 1, \ldots, m; \tag{4.29}$$
$$ne_{(2j-1)*} = -(\cos\theta)e_{(2j)*}, ne_{(2j)*} = (\cos\theta)e_{(2j-1)*}, j = 1, \ldots, k. \tag{4.30}$$

We call such an orthonormal frame an *adapted slant frame* of M in $\widetilde{M}$.

Now, we define a canonical 1-form on M by:

$$\Theta = \sum_{i=1}^{m} \omega_i^{i*}. \tag{4.31}$$

We are going to compute $d\Theta$. By following the same line of reasoning as in [5], we have,

$$\omega_{2i}^{(2j)*} + \omega_{2i-1}^{(2j-1)*} = \omega_{2j}^{(2i)*} + \omega_{2j-1}^{(2i-1)*}, \tag{4.32}$$
$$\omega_{(2i)*}^{(2j)*} - \omega_{(2i-1)*}^{(2j-1)*} = \omega_{2i}^{2j} - \omega_{2i-1}^{2j-1}, \tag{4.33}$$
$$\omega_{2j}^{2i-1} - \omega_{(2j)*}^{(2j-1)*} = \omega_{2i}^{2j-1} - \omega_{(2i)*}^{(2j-1)*}, \tag{4.34}$$

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for any $i, j = 1, \ldots, k$. On the other hand, (2.9), (3.15), (4.28)–(4.30) and a straightforward computation, give

$$\begin{aligned}
\Omega_{2j}^{(2j)*} &= \frac{c-s}{4}\sin\theta\cos\theta(-\omega^{2j-1}\wedge\omega^{2j}+\omega^{(2j-1)*}\wedge\omega^{(2j)*}) \\
&\quad -\left(\frac{c+3s}{4}+\frac{c-s}{4}\sin^2\theta\right)\omega^{2j}\wedge\omega^{(2j)*}+\frac{c-s}{4}\cos^2\theta\omega^{2j-1}\wedge\omega^{(2j-1)*} \\
&\quad +\frac{c-s}{2}\sum_{p=1}^{k}\{\sin\theta\cos\theta(-\omega^{2p-1}\wedge\omega^{2p}+\omega^{(2p-1)*}\wedge\omega^{(2p)*}) \\
&\quad -\sin^2\theta(\omega^{2p}\wedge\omega^{(2p)*}+\omega^{2p-1}\wedge\omega^{(2p-1)*}\},
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
\Omega_{2j-1}^{(2j-1)*} &= \frac{c-s}{4}\sin\theta\cos\theta(-\omega^{2j-1}\wedge\omega^{2j}+\omega^{(2j-1)*}\wedge\omega^{(2j)*}) \\
&\quad -\left(\frac{c+3s}{4}+\frac{c-s}{4}\sin^2\theta\right)\omega^{2j-1}\wedge\omega^{(2j-1)*}+\frac{c-s}{4}\cos^2\theta\omega^{2j}\wedge\omega^{(2j)*} \\
&\quad +\frac{c-s}{2}\sum_{p=1}^{k}\{\sin\theta\cos\theta(-\omega^{2p-1}\wedge\omega^{2p}+\omega^{(2p-1)*}\wedge\omega^{(2p)*}) \\
&\quad -\sin^2\theta(\omega^{2p}\wedge\omega^{(2p)*}+\omega^{2p-1}\wedge\omega^{(2p-1)*}\},
\end{aligned} \tag{4.36}$$

for any $j = 1, \ldots, k$. By using these results, we can prove:

Lemma 1 *Let M^{m+s} be an $(m+s)$-dimensional $(m = 2k)$ proper slant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$. Then, the 1-form Θ satisfies*

$$\begin{aligned}
d\Theta &= -\sin^2\theta\frac{(m+1)c-s(m-3)}{2}\sum_{j=1}^{k}(\omega^{2j-1}\wedge\omega^{(2j-1)*}+\omega^{2j}\wedge\omega^{(2j)*}) \\
&\quad -\sin\theta\cos\theta\frac{(m+1)c-s(m-3)}{2}\sum_{j=1}^{k}(\omega^{2j-1}\wedge\omega^{2j}-\omega^{(2j-1)*}\wedge\omega^{(2j)*}),
\end{aligned} \tag{4.37}$$

where θ is the slant angle of M. Hence, Θ es closed if and only if:

$$c = \frac{s(m-3)}{m+1}.$$

Proof By using (2.8) for an adapted slant frame, we get:

$$d\Theta = -\sum_{i,j=1}^{m}(\omega_j^{i*}\wedge\omega_i^j+\omega_{j*}^{i*}\wedge\omega_i^{j*})-\sum_{\alpha=1}^{s}\sum_{i=1}^{m}\omega_{m+\alpha}^{i*}\wedge\omega_i^{m+\alpha}+\sum_{i=1}^{m}\Omega_i^{i*}, \tag{4.38}$$

where we are denoting $e_{m+\alpha} = \xi_\alpha$, $\alpha = 1, \ldots, s$. Now, from (4.32)–(4.34) and following the same steps as in the proof of Theorem 3.1 of [10], we obtain:

$$\sum_{i,j=1}^{m}(\omega_j^{i*}\wedge\omega_i^j+\omega_{j*}^{i*}\wedge\omega_i^{j*}) = 0. \tag{4.39}$$

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Moreover, since from (2.7), (3.12) and (3.13),

$$\omega_{m+\alpha}^{i*}(X) = g(\widetilde{\nabla}_X \xi_\alpha, e_{i*}) = -g(fX, e_{i*}) = g(X, fe_{i*})$$

and

$$\omega_i^{m+\alpha}(X) = g(\widetilde{\nabla}_X e_i, \xi_\alpha) = -g(e_i, \widetilde{\nabla}_X \xi_\alpha) = g(fX, e_i) = -g(X, fe_i),$$

for any vector field X in $\widetilde{M}$, any $i = 1, \ldots, m$ and any $\alpha = 1, \ldots, s$, we have that, from (4.28)–(4.30):

$$\omega_{m+\alpha}^{(2j-1)*} = -\sin\theta\omega^{2j-1} - \cos\theta\omega^{(2j)*}; \quad \omega_{m+\alpha}^{(2j)*} = -\sin\theta\omega^{2j} + \cos\theta\omega^{(2j-1)*}, \tag{4.40}$$

$$\omega_{2j-1}^{m+\alpha} = -\cos\theta\omega^{2j} - \sin\theta\omega^{(2j-1)*}; \quad \omega_{2j}^{m+\alpha} = cos\theta\omega^{2j-1} - \sin\theta\omega^{(2j)*}, \tag{4.41}$$

for any $j = 1, \ldots, k$ and any $\alpha = 1, \ldots, s$. Consequently, taking into account (4.40) and (4.41), we compute

$$\begin{aligned} -\sum_{i=1}^{m} \omega_{m+\alpha}^{i*} \wedge \omega_i^{m+\alpha} &= 2\sin\theta\cos\theta \sum_{j=1}^{k} (-\omega^{2j-1} \wedge \omega^{2j} + \omega^{(2j-1)*} \wedge \omega^{(2j)*}) \\ &\quad + (\cos^2\theta - \sin^2\theta) \sum_{j=1}^{k} (-\omega^{2j-1} \wedge \omega^{(2j-1)*} + \omega^{2j} \wedge \omega^{(2j)*}), \end{aligned} \tag{4.42}$$

for any $\alpha = 1, \ldots, s$. Finally, from (4.35), (4.36), (4.38), (4.39) and (4.42), we deduce (4.37). The remaining part of the proof follows directly from (4.37). □

Observe that the above lemma does not give a closed 1-form on M if the constant f-sectional curvature of the ambient S-space-form is equal to $-3s$ (for instance, $\mathbf{R}^{2m+s}$ with the S-structure given in [13]). However, we can define another 1-form on M:

$$\omega = \Theta + m\sin\theta \sum_{\alpha=1}^{s} \eta_\alpha. \tag{4.43}$$

For this form, we can prove:

Lemma 2 *Let M^{m+s} be an $(m+s)$-dimensional $(m = 2k)$ proper slant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$. Then, the 1-form ω satisfies*

$$\begin{aligned} d\omega &= -\sin\theta\cos\theta \left(\frac{(m+1)(c+3s)}{2}\right) \sum_{j=1}^{k} (\omega^{2j-1} \wedge \omega^{2j} - \omega^{(2j-1)*} \wedge \omega^{(2j)*}) \\ &\quad - \sin^2\theta \left(\frac{(m+1)(c+3s)}{2}\right) \sum_{j=1}^{k} (\omega^{2j-1} \wedge \omega^{(2j-1)*} + \omega^{2j} \wedge \omega^{(2j)*}), \end{aligned} \tag{4.44}$$

where θ denotes the slant angle of M. Hence, ω is closed if and only if $c = -3s$.

Proof Since from (2.7), (3.12) and (3.13),

$$\omega_{i*}^{m+\alpha}(X) = g(\widetilde{\nabla}_X e_{i*}, \xi_\alpha) = -g(e_{i*}, \widetilde{\nabla}_X \xi_\alpha) = g(fX, e_{i*}) = -g(X, fe_{i*}),$$

and, from (3.10),

$$\omega_\beta^\alpha(X) = g(\widetilde{\nabla}_X \xi_\beta, \xi_a) = -g(fX, \xi_\alpha) = 0,$$

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for any vector field X tangent to M, any $i = 1, \ldots, m$ and any $\alpha, \beta = 1, \ldots, s$, we have that, from (4.28)-(4.30),

$$\omega^{m+\alpha}_{(2j-1)*} = \sin\theta\omega^{2j-1} + \cos\theta\omega^{(2j)*}; \quad \omega^{m+\alpha}_{(2j)*} = \sin\theta\omega^{2j} - \cos\theta\omega^{(2j-1)*}, \tag{4.45}$$

for any $j = 1, \ldots, k$ and any $\alpha = 1, \ldots, s$. Thus, from (2.8) and taking into account (4.45), we compute

$$\begin{aligned} d\eta_\alpha &= -\sum_{i=1}^{m}(\omega_i^{m+\alpha} \wedge \omega^i + \omega_{i*}^{m+\alpha} \wedge \omega^{i*}) \\ &= -2\sum_{j=1}^{k}\Big(\cos\theta(\omega^{2j-1} \wedge \omega^{2j} - \omega^{(2j-1)*} \wedge \omega^{(2j)*}) \\ &\quad + \sin\theta(\omega^{2j-1} \wedge \omega^{(2j-1)*} + \omega^{2j} \wedge \omega^{(2j)*})\Big), \end{aligned} \tag{4.46}$$

for any $\alpha = 1, \ldots, s$. Next, since $\widetilde{M}$ is an S-manifold, we know that $F = d\eta_\alpha$, for any α too. Consequently, by using (4.37), (4.43) and (4.46), we obtain (4.44). The rest of the proof is immediate. □

For S-slant submanifolds, the Maslov form ω_H and ω are related by the following theorem.

Theorem 4.1 *Let M^{m+s} be an $(m+s)$-dimensional S-slant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$. Then,*

$$\omega_H = -\frac{\sin\theta}{m+s}\omega, \tag{4.47}$$

where θ is the slant angle. Consequently, the Maslov form is closed if and only if $c = -3s$.

Proof We consider an adapted slant frame $\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1*}, \ldots, e_{m*}\}$ of M in $\widetilde{M}$. Then, from (2.4), (3.19), (3.21) and the definition of H, we have, for any $i = 1, \ldots, m$:

$$\omega_H(e_i) = -\frac{1}{m+s}\sum_{j=1}^{m} g(A_{Ne_i}e_j, e_j). \tag{4.48}$$

But, from (3.22), (3.26) and the symmetry of σ, we get

$$\begin{aligned} g(A_{Ne_i}e_j, e_j) &= g((\nabla_{e_j}T)e_i, e_j) - g(t\sigma(e_i, e_j), e_j) \\ &= g((\nabla_{e_i}T)e_j, e_j) - g(t\sigma(e_i, e_j), e_j) = g(A_{Ne_j}e_i, e_j), \end{aligned} \tag{4.49}$$

for any $i, j = 1, \ldots, m$. Thus, replacing (4.49) into (4.48) and taking into account (2.7) and (4.29),

$$\begin{aligned} \omega_H(e_i) &= -\frac{1}{m+s}\sum_{j=1}^{m} g(A_{Ne_j}e_i, e_j) = -\frac{1}{m+s}\sin\theta\sum_{j=1}^{m} g(\sigma(e_i, e_j), e_{j*}) \\ &= -\frac{1}{m+s}\sin\theta\sum_{j=1}^{m} g(\widetilde{\nabla}_{e_i}e_j, e_{j*}) = -\frac{1}{m+s}\sin\theta\sum_{j=1}^{m}\sum_{A=1}^{2m+s}\omega_j^A(e_i)g(e_A, e_{j*}) \\ &= -\frac{1}{m+s}\sin\theta\sum_{j=1}^{m}\omega_j^{j*}(e_i) = -\frac{1}{m+s}\sin\theta\Theta(e_i), \end{aligned} \tag{4.50}$$

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for any $i = 1, \ldots, m$, where we have denoted $e_{2m+\alpha} = \xi_\alpha$, $\alpha = 1, \ldots, s$ and used (4.31).

On the hand, by using (2.3), (2.7), (3.20), (4.29) and (4.31), we compute

$$\Theta(e_{2m+\alpha}) = \sum_{i=1}^{m} g(\widetilde{\nabla}_{e_{2m+\alpha}} e_i, e_{i*}) = \sum_{i=1}^{m} g(\sigma(e_{2m+\alpha}, e_i), e_{i*})$$
$$= -\sum_{i=1}^{m} g(Ne_i, e_{i*}) = -m \sin\theta,$$

for any $\alpha = 1, \ldots, s$. Consequently,

$$\sum_{i=1}^{m} \Theta(e_i)\omega^i = \Theta + m \sin\theta \sum_{\alpha=1}^{s} \eta_a = \omega. \tag{4.51}$$

Finally, since from (3.10) we easily get that $\omega_H(e_{2m+\alpha}) = 0$, for any $\alpha = 1, \ldots, s$, (4.50) and (4.51) give

$$\omega_H = -\frac{\sin\theta}{m+s} \sum_{i=1} \Theta(e_i)\omega^i = -\frac{\sin\theta}{m+s}\omega$$

and we complete the proof. □

Now, we are going to consider an $(m+s)$-dimensional anti-invariant submanifold M^{m+s} of an S-space-form $\widetilde{M}^{2m+s}(c)$, tangent to the structure vector fields $\xi_1, \ldots, \xi_s$, which is a particular case of slant immersion. In this situation, $tH = fH$. Moreover, we cannot use an adapted slant frame, but, if we take any local orthonormal frame

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s\}$$

on M, it is easy to show that

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\},$$

where $e_{i^*} = fe_i$, for any $i = 1, \ldots, m$, is a local orthonormal frame on $\widetilde{M}$, called an *adapted anti-invariant frame* of M in $\widetilde{M}$. Observe that adapted anti-invariant frames are not particular cases of adapted slant frames. So, it is necessary to make all the computations again, although we can use the same line of reasoning. So, we do not explicit them as detailed as above. First, we get that, for an adapted anti-invariant frame,

$$\omega_i^{j^*} = \omega_j^{i^*}, \quad \omega_{i^*}^{j^*} = -\omega_i^j, \tag{4.52}$$

for any $i, j = 1, \ldots, m$. Furthermore, from (3.15), the curvature forms are given by

$$\Omega_i^{i^*} = -\frac{c+s}{2}\omega^i \wedge \omega^{i^*} - \frac{c-s}{2}\sum_{j=1}^{n} \omega^j \wedge \omega^{j^*}, \tag{4.53}$$

for any $i = 1, \ldots, n$. Therefore, for the 1-form Θ, defined as in (4.31), by using (2.8), (4.52) and (4.53), we have:

Lemma 3 *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$, tangent to the structure vector fields. Then, the 1-form Θ satisfies:*

$$d\Theta = -\frac{(m+1)c - s(m-1) - 2s}{2}\sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}. \tag{4.54}$$

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Consequently, Θ is closed if and only if:

$$c = \frac{s(m-3)}{m+1}.$$

The above lemma implies that the 1-form Θ is not closed if $c = -3s$. Then, we consider the 1-form

$$\omega = \Theta + m\sum_{\alpha=1}^{s}\eta_\alpha$$

and we can prove:

Lemma 4 *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$, tangent to the structure vector fields. Then, the 1-form ω satisfies:*

$$d\omega = -\frac{(m+1)(c+3s)}{2}\sum_{i=1}^{m}\omega^i \wedge \omega^{i^*}. \tag{4.55}$$

Consequently, ω is closed if and only if $c = -3s$.

The relationship between ω and the Maslov form is given in the following theorem.

Theorem 4.2 *Let M^{m+s} be an $(m+s)$ an anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(c)$ of dimension $2m+s$, tangent to the structure vector fields. Then:*

$$\omega_H = -\frac{1}{m+s}\omega. \tag{4.56}$$

Consequently, ω_H is closed if and only if $c = -3s$.

Proof Considering an adapted anti-invariant frame

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1*}, \ldots, e_{m*}\}$$

of M in $\widetilde{M}$ and taking into account the Gauss formula, (2.4) and (3.21), we have

$$\omega_H(e_i) = -\frac{1}{m+s}\sum_{j=1}^{m} g(A_{e_{i*}}e_j, e_j), \tag{4.57}$$

for any $i = 1, \ldots, n$.. Now, from (3.22), since $T = 0$ and σ is symmetric:

$$g(A_{e_{i*}}e_j, e_j) = g(A_{e_{j*}}e_i, e_j). \tag{4.58}$$

Then, replacing (4.58) into (4.57) and from (2.3), we get

$$\omega_H(e_i) = -\frac{1}{m+s}\sum_{j=1}^{m} g(\widetilde{\nabla}_{e_i}e_j, e_{j*}) = -\frac{1}{m+s}\sum_{j=1}^{m}\omega_j^{j*}(e_i) = -\frac{1}{m+1}\Theta(e_i), \tag{4.59}$$

for any $i = 1, \ldots, n$. On the other hand, by using (2.7), (3.20) and that M is an anti-invariant submanifold:

$$\begin{aligned}\Theta(\xi_\alpha) &= \sum_{i=1}^{m} g(\widetilde{\nabla}_{\xi_\alpha}e_i, e_{i*}) = \sum_{i=1}^{m} g(\sigma(e_i, \xi_\alpha), e_{i^*}) \\ &= -\sum_{i=1}^{m} g(e_{i*}, e_{i*}) = -m.\end{aligned} \tag{4.60}$$

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Finally, by using (4.59) and (4.60) and since it is easy to show that $\omega_H(\xi_\alpha) = 0$, for any $\alpha = 1, \ldots, s$, we get:

$$\omega_H = \sum_{i=1}^{m} \omega_H(e_i)\omega^i = -\frac{1}{m+s}\sum_{i=1}^{m} \Theta(e_i)\omega^i$$
$$= -\frac{1}{m+s}\left(\Theta + m\sum_{\alpha=1}^{s}\eta_\alpha\right) = -\frac{1}{m+s}\omega.$$

□

From Theorems 4.1 and 4.2, we can prove the following topological obstruction to S-slant immersions as well as to anti-invariant immersions tangent to the structure vector fields into an S-space-form of constant f-sectional curvature $c = -3s$:

Theorem 4.3 *Let M^{m+s} be a compact simply connected $(m+s)$-dimensional differentiable manifold. Then, M cannot be immersed in any $(2m+s)$-dimensional S-space-form $\widetilde{M}^{2m+s}(-3s)$ as an anti-invariant submanifold tangent to the structure vector fields with no minimal points. Moreover, if m is even, M cannot be immersed in such a S-space-form as an S-slant submanifold with no minimal points either. In particular, if $m = 2$, M cannot be immersed in $\widetilde{M}(-3s)$ as a non-invariant slant submanifold with no minimal points.*

Proof Let us suppose that M is an anti-invariant submanifold of $\widetilde{M}(-3s)$, tangent to the structure vector fields, with no minimal points. Then, H is nowhere zero and, consequently, the Maslov form ω_H is also nowhere zero because M has codimension m (to check this, it is enough to consider an adapted anti-invariant frame). From Theorem 4.2, ω_H is closed and so, it represents a cohomology class $[\omega_H] \in H^1(M, \mathbf{R})$. Since M is compact, ω_H cannot be exact. Therefore, $[\omega_H]$ is a non-trivial cohomology class and then, the first cohomology group $H^1(M, \mathbf{R})$ is non-trivial. Hence, M is not simply connected, which is a contradiction.

In the case of being m even, the second part of the proof follows analogously from Theorem 4.1.

5 Conformal Maslov form

In this section, we want to study whether the Maslov form of an $(m+s)$-dimensional $(s \geq 1)$ anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(-3s)$ of dimension $2m+s$, tangent to the structure vector fields, can be conformal in M. First, we have a more general result:

Theorem 5.4 *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-manifold $\widetilde{M}^{2m+s}$ of dimension $2m+s$, tangent to the structure vector fields and such that its Maslov form is closed. Then, this Maslov form is conformal in M if and only if the mean curvature vector is parallel.*

Proof From (2.1) and (3.20), if Y is a tangent vector field to M, we get,

$$g(\nabla_{\xi_\alpha} fH, Y) = g(\nabla_Y fH, \xi_\alpha) = -g(fH, \nabla_Y \xi_\alpha) = g(fH, TY) = 0$$

and so:

$$\nabla_{\xi_\alpha} fH = 0.$$

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Consequently, by using (2.2), we have that ω_H is conformal in M if and only if

$$\nabla_X fH = 0,$$

for any vector field X tangent to M. But, since from (3.14), $(\widetilde{\nabla}_X f)H = 0$, taking into account the tangent component of this formula, we obtain $\nabla_X fH - fD_X H = 0$, that is, ω_H is conformal in M if and only if $D_X H = 0$ and the proof is complete. □

Due to the above theorem, it is necessary to introduce a more restrictive notion, and we say that the Maslov form is $\mathcal{L}$-*conformal* if $\nabla_Y fH = hY$, for any $Y \in \mathcal{L}$, being h a differentiable function. Then, we can prove the following theorem.

Theorem 5.5 *Let M^{m+s} be an $(m+s)$-dimensional anti-invariant submanifold of an S-space-form $\widetilde{M}^{2m+s}(-3s)$ of dimension $2m+s$, tangent to the structure vector fields. If*

$$\begin{aligned}\sigma(X,Y) = \frac{m+s}{m+s+1}\Bigg\{& g(fX,fY)H - \Big(\omega_H(X) + \frac{m+s+1}{m+s}\sum_{\alpha=1}^{s}\eta_\alpha(X)\Big)fY \\ &-\Big(\omega_H(Y) + \frac{m+s+1}{m+s}\sum_{\alpha=1}^{s}\eta_\alpha(Y)\Big)fX\Bigg\},\end{aligned} \tag{5.61}$$

for any tangent vector fields X, Y tangent to M, then the Maslov form of M is $\mathcal{L}$-conformal.

Proof Let $X, Y \in \mathcal{L}$ be two orthogonal vector fields such that $g(Y,Y) = 1$. Then, from (5.61):

$$\sigma(Y,Y) = \frac{m+s}{m+s+1}\{H + 2g(fY,H)fY\}. \tag{5.62}$$

Differentiating (5.62) with respect to X:

$$D_X\sigma(Y,Y) = \frac{m+s}{m+s+1}\{D_X H + 2Xg(fY,H)fY + 2g(fY,H)D_X fY\}. \tag{5.63}$$

Now, since from (3.14), we have that $(\widetilde{\nabla}_X f)Y = 0$, then, by using the Weingarten formula (2.3), we deduce $Xg(fY,H) = g(f\widetilde{\nabla}_X Y, H) + g(fY, D_X H)$ and, substituting into (5.63):

$$\begin{aligned}D_X\sigma(Y,Y) = \frac{m+s}{m+s+1}&\{D_X H + 2g(f\widetilde{\nabla}_X Y,H)fY \\ &+ 2g(fY,D_X H)fY + 2g(fY,H)D_X fY\}.\end{aligned} \tag{5.64}$$

On the other hand, from (5.61) and by using that $g(\nabla_X Y, Y) = 0 = \eta_\alpha(\nabla_X Y)$, we get:

$$\sigma(\nabla_X Y, Y) = \frac{m+s}{m+s+1}\{g(f\nabla_X Y,H)fY + g(fY,H)f\nabla_X Y\}. \tag{5.65}$$

Thus, from (2.6), (5.64) and (5.65):

$$\begin{aligned}(\widetilde{\nabla}_X\sigma)(Y,Y) = \frac{m+s}{m+s+1}&\{D_X H + 2g(f\widetilde{\nabla}_X Y,H)fY + 2g(fY,D_X H)fY \\ &+ 2g(fY,H)D_X fY - 2g(f\nabla_X Y,H)fY - 2g(fY,H)f\nabla_X Y\}.\end{aligned} \tag{5.66}$$

But, since fH is a tangent vector field to M and $(\widetilde{\nabla}_X f)Y = 0$, from (2.3) and (3.12), we obtain $g(f\widetilde{\nabla}_X Y, H) = g(f\nabla_X Y, H)$ and $D_X fY = f\nabla_X Y$, so (5.66) reduces to:

$$(\widetilde{\nabla}_X\sigma)(Y,Y) = \frac{m+s}{m+s+1}\{D_X H + 2g(D_X H, fY)fY\}. \tag{5.67}$$

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Next, from (2.6) again and by using a similar line of reasoning to above, a straightforward computation gives:

$$(\widetilde{\nabla}_Y\sigma)(X,Y)=\frac{m+s}{m+s+1}\{g(fX,D_YH)fY+g(fY,D_YH)fX\}. \tag{5.68}$$

Consequently, since from (3.15) we have that $(\widetilde{R}(X,Y)Y)^{\perp}=0$, then, from the Codazzi Eq. (2.5), we deduce that $(\widetilde{\nabla}_X\sigma)(Y,Y)-(\widetilde{\nabla}_Y\sigma)(X,Y)=0$ and subtracting (5.68) from (5.67):

$$D_XH=g(fX,D_YH)fY+g(fY,D_YH)fX-2g(D_XH,fY)fY. \tag{5.69}$$

Moreover, since from (3.14), $\widetilde{\nabla}_X fH=f\widetilde{\nabla}_X H$, taking into account (2.3) and (3.12), we get $g(D_XH,fY)=-g(\nabla_X fH,Y)$ and $g(D_YH,fX)=-g(\nabla_Y fH,X)$. But, from (2.1), since ω_H is closed, $g(\nabla_X fH,Y)=g(\nabla_Y fH,X)$, therefore (5.69) reduces to:

$$D_XH=g(D_YH,fY)fX-g(D_XH,fY)fY.$$

Thus, $g(D_XH,fY)=-g(D_XH,fY)$, because $g(fX,fY)=0$ and $g(fY,fY)=1$, that is, $g(D_XH,fY)=0$. Then, we obtain:

$$D_XH=g(D_YH,fY)fX. \tag{5.70}$$

Finally, from (2.3), (3.12), (3.14) and (5.70), we easily check that

$$g(\nabla_X fH,Z)=-g(D_YH,fY)g(X,Z),$$

for any vector field Z tangent to M. This implies that $\nabla_X fH=-g(D_YH,fY)X$, which completes the proof. □

Notice that, from Theorem 5.4, this result cannot be improved. However, it is interesting to ask about examples of submanifolds satisfying (5.61). Firstly, we have the totally f-geodesic submanifolds. It is easy to show that these submanifolds are minimal. Thus, (3.20) and (3.23) imply that any anti-invariant and totally f-geodesic submanifold satisfies (5.61).

On the other hand, in the case $s=1$, if we consider $\mathbf{R}^{2m+1}$ as the ambient Sasakian manifold, A. Carriazo and D. E. Blair proved in [2] that the condition (5.61) characterizes anti-invariant $(m+1)$-dimensional submanifolds satisfying the equality case of

$$\|H\|^2\geq\frac{2(m+2)}{(m+1)^2(m-1)}\tau,$$

where τ denotes the scalar curvature of the submanifold.

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GENERALIZED S-SPACE-FORMS

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Abstract. We introduce and study generalized S-space-forms. Moreover, we investigate generalized S-space-forms endowed with an additional structure and we obtain some obstructions for them to be S-manifolds.

1. Introduction

It is an interesting problem to analyze what kind of Riemannian manifolds may be determined by special pointwise expressions for their curvatures. For instance, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if (M, g) is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field R has the pointwise expression

$$R(X, Y)Z = \lambda\{g(X, Z)Y - g(Y, Z)X\},$$

where λ is a differentiable function on M, then M is a space of constant sectional curvature, that is, a real-space-form and λ is a constant function.

Further, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from the special form of the curvature tensor field for this structure too. Thus, an almost-Hermitian manifold (M, J, g) is said to be a *generalized complex-space-form* [**9**] if its curvature tensor satisfies

$$\begin{aligned} R(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\}, \end{aligned} \tag{1.1}$$

where f_1 and f_2 are differentiable functions on M. This name derives from the fact that, when M is a complex-space-form, that is, a Kaehlerian manifold of constant holomorphic curvature equal to c, the curvature tensor field of M satisfies (1.1) with $f_1 = f_2 = c/4$.

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Since Sasakian-spaces-forms play a similar role in contact metric geometry to that of complex-space-forms in complex geometry, Alegre, Blair and Carriazo have defined and studied *generalized Sasakian-space forms* [**1**] as those almost-contact metric manifolds (M,ϕ,ξ,η,g) whose curvature tensor field satisfies

$$\begin{aligned}R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + \\ + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned}$$

f_1, f_2, f_3 being differentiable functions on M. If M is actually a Sasakian-space-form, that is a Sasakian manifold with constant ϕ-sectional curvature equal to c, then $f_1 = \frac{1}{4}(c+3)$, $f_2 = f_3 = \frac{1}{4}(c-1)$.

More in general, Yano [**10**] introduced the notion of f-structure on a $(2n+s)$ -dimensional manifold as a tensor field f of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. In this context, Blair [**2**] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds in the almost contact geometry and he showed that the curvature of either S-manifolds or C-manifolds is completely determined by their f-sectional curvatures. Later, Kobayashi and Tsuchiya [**8**] got expressions of the curvature tensor field of S-manifolds and C-manifolds when their f-sectional curvature is constant depending on such a constant.

For these reasons, we consider that it is interesting to introduce a notion of generalized S-space-form on metric f-manifolds (see Section 2 for a precise definition of these manifolds). We observe that this work was made in [**5**] for metric f-manifolds with two structure vector fields, giving some interesting examples. Now, we present the definition for any number of structure vector fields. To this end, we have followed the same procedure as in almost complex and almost contact cases, that is, we have substituted the constants in the expression of the curvature tensor field of an S-space-form (an S-manifold of constant f-sectional curvature) obtained in [**8**] by certain differentiable functions on the manifold. So, S-space-forms are natural examples of generalized S-space-forms. Furthermore, we check that C-space-forms are also generalized S-space-forms.

We have organized the communication in the following way. In Section 2 we review definitions and formulas concerning metric f-manifolds which we shall use later. In Section 3 we define generalized S-space-forms and study the sectional curvatures of such manifolds. Moreover, we establish that the writing of the curvature tensor field is unique in terms of a family of differentiable functions on the manifold if and only if the dimension of the manifold is greater than $2+s$, s being the number of structure vector fields. In Section 4, we consider a different definition given by Falcitelli and Pastore in [**6**], comparing both definitions. Finally, in Section 5, we study generalized S-space-forms endowed with an additional structure and the relationships between the functions in such a case. Thus, we prove that any generalized S-space-form with a metric f-K-contact structure is actually an S-manifold and we deduce an obstruction for a generalized S-space-form to be an S-manifold, depending on the functions. The same result holds for a metric f-contact structure with some additional conditions on the functions. We also study generalized S-space-forms with an underlying C-structure and, more in general, with a K-structure.

2. Metric f-manifolds

A Riemannian manifold (M, g) of dimension $2n+s$ and endowed with an f-structure f (that is, a tensor field of type (1,1) and rank $2n$ satisfying $f^3+f=0$ [**10**]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1,\dots,\xi_s$ on M (called *structure vector fields*) such that, if $\eta_1,\dots,\eta_s$ are the dual 1-forms of $\xi_1,\dots,\xi_s$, then

$$f\xi_\alpha = 0;\ \ \eta_\alpha \circ f = 0;\ \ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X,Y) = g(fX,fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any $X, Y \in \mathcal{X}(M)$ and $\alpha = 1,\dots,s$. The distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, $TM = \mathcal{L} \oplus \mathcal{M}$. Moreover, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1,\dots,s$ and if $X \in \mathcal{M}$, then $fX = 0$.

Let F be the 2-form on M defined by $F(X,Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$. Since f is of rank $2n$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0$ and, particularly, M is orientable. A metric f-manifold is said to be a *metric f-contact manifold* if $F = \mathrm{d}\eta_\alpha$, for any $\alpha = 1,\dots,s$. On the other hand, a metric f-contact manifold is said to be a *metric f-K-contact manifold* if the structure vector fields are Killing vector fields. When $s = 1$, metric f-contact manifolds correspond to contact manifolds and metric f-K-contact manifolds to K-contact manifolds. Furthermore, in a metric f-K-contact manifold it easy to show that:

$$\nabla_X \xi_\alpha = -fX,\ X \in \mathcal{X}(M),\ \alpha = 1,\dots,s. \tag{2.1}$$

The f-structure f is said to be *normal* if $[f,f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$, where $[f,f]$ denotes the Nijenhuis tensor of f. Then, a metric f-manifold is said to be a K-manifold [**2**] if it is normal and $\mathrm{d}F = 0$. In a K-manifold M, the structure vector fields are Killing vector fields [**2**] and:

$$\nabla_{\xi_\alpha} \xi_\beta = 0,\ \alpha, \beta = 1,\dots,s. \tag{2.2}$$

A K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$, for any α (that is, if it is also a metric f-K-contact manifold) and a C-manifold if $\mathrm{d}\eta_\alpha = 0$, for any α. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geqslant 2$, non-trivial examples can be found in [**2, 3, 7**]. Moreover, a K-manifold M is an S-manifold if and only if

$$\nabla_X \xi_\alpha = -fX,\ X \in \mathcal{X}(M),\ \alpha = 1,\dots,s,$$

and it is a C-manifold if and only if $\nabla f = 0$ [**2**].

On the other hand, the curvature tensor field R of a K-manifold M satisfies

$$R(\xi_\alpha, X, \xi_\beta, Y) = -g(\nabla_X \xi_\beta, \nabla_Y \xi_\alpha), \tag{2.3}$$

for any $X, Y \in \mathcal{X}(M)$ and $\alpha, \beta = 1, \dots, s$ [**4**].

A plane section π on a metric f-manifold M is said to be an *f-section* if it is determined by a unit vector $X \in \mathcal{L}$ and fX. The sectional curvature $K(\pi)$ of π is called an *f-sectional curvature*. An S-manifold (resp., a C-manifold) is said to be an *S-space-form* (resp., a *C-space-form*) if it has a constant f-sectional curvature c and then, it is denoted by $M(c)$. In such cases, the curvature tensor field R of $M(c)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) = &\sum_{\alpha,\beta} \big(g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) \\ &\quad + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)\big) \\ &+ \frac{c+3s}{4}\big(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)\big) \\ &+ \frac{c-s}{4}\big(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) \\ &\qquad\qquad - 2F(X,Y)F(Z,W)\big), \end{aligned} \tag{2.4}$$

(resp.,

$$\begin{aligned} R(X,Y,Z,W) = &\frac{c}{4}(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)) \\ &+ F(X,W)F(Y,Z) - F(X,Z)F(Y,W) \\ &- 2F(X,Y)F(Z,W))), \end{aligned} \tag{2.5}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$ [**8**].

3. Generalized S-space-forms

A metric f-manifold $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$ is said to be a *generalized S-space-form* if there exists a family of differentiable functions on M,

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\},$$

such that the curvature tensor field R of M satisfies

$$R = F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}, \tag{3.1}$$

where

(3.2)

$$
\begin{aligned}
R_1(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W);\\
R_2(X,Y,Z,W) &= F(X,W)F(Y,Z) - F(X,Z)F(Y,W)\\
&\quad - 2F(X,Y)F(Z,W);\\
R_{\alpha\beta}(X,Y,Z,W) &= g(Y,W)\eta_\alpha(X)\eta_\beta(Z) - g(X,W)\eta_\alpha(Y)\eta_\beta(Z)\\
&\quad + g(X,Z)\eta_\alpha(Y)\eta_\beta(W) - g(Y,Z)\eta_\alpha(X)\eta_\beta(W);\\
\widetilde{R}_{\alpha\beta}(X,Y,Z,W) &= \eta_\alpha(X)\eta_\beta(Y)\eta_\beta(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\beta(Z)\eta_\alpha(W)\\
&\quad + \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\beta(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\beta(W);\\
R_{\alpha\beta\gamma}(X,Y,Z,W) &= \eta_\alpha(X)\eta_\beta(Y)\eta_\gamma(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\gamma(Z)\eta_\alpha(W)\\
&\quad + \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\gamma(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\gamma(W),
\end{aligned}
$$

for any $X,Y,Z,W \in \mathcal{X}(M)$.

This kind of manifold appears as a natural generalization of S-space-forms because a straightforward computation from (2.4) gives that any S-space-form $M(c)$ is a generalized S-space-form with functions

$$
\begin{gathered}
F_1 = \tfrac{1}{4}(c+3s);\quad F_2 = \tfrac{1}{4}(c-s);\quad F_{\alpha\alpha} = \tfrac{1}{4}(c+3s) - 1;\\
F_{\alpha\beta} = -1 \ (\alpha \neq \beta);\quad G_{\alpha\beta} = \tfrac{1}{4}(c+3s) - 2 \ (\alpha < \beta);\\
H_{\alpha\beta\gamma} = -1 \ (\alpha \neq \beta \neq \gamma \neq \alpha),
\end{gathered}
$$

where $\alpha,\beta,\gamma \in \{1,\dots,s\}$. Moreover, any C-space-form $M(c)$ is also a generalized S-space-form. In fact, from (2.5), we only have to take

$$
\begin{gathered}
F_1 = F_2 = F_{\alpha\alpha} = G_{\alpha\beta} = \tfrac{c}{4} \ (\alpha < \beta);\\
F_{\alpha\beta} = 0 \ (\alpha \neq \beta);\\
H_{\alpha\beta\gamma} = 0 \ (\alpha \neq \beta \neq \gamma \neq \alpha),
\end{gathered}
$$

where $\alpha,\beta,\gamma \in \{1,\dots,s\}$.

From (3.2) we easily deduce that $\widetilde{R}_{\alpha\alpha} = 0$; $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\beta\alpha}$; $R_{\alpha\beta\beta} = \widetilde{R}_{\alpha\beta}$; $R_{\alpha\alpha\alpha} = R_{\alpha\alpha\beta} = 0$, for any $\alpha,\beta = 1,\dots,s$. Furthermore, from (3.1) we get that

$$R(X,\xi_\alpha,X,\xi_\beta) = F_{\alpha\beta}, \tag{3.3}$$

$$R(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\alpha) = H_{\alpha\beta\gamma} - F_{\beta\gamma}, \tag{3.4}$$

for any unit vector field $X \in \mathcal{L}$ and any $\alpha,\beta,\gamma = 1,\dots,s$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Then, by using the symmetries of the curvature tensor field R, from (3.3) and (3.4) together, we obtain $F_{\alpha\beta} = F_{\beta\alpha}$ and $H_{\alpha\beta\gamma} = H_{\alpha\gamma\beta}$,$\alpha,\beta,\gamma = 1,\dots,s$, $\alpha \neq \beta \neq \gamma \neq \alpha$.

Now, we observe that, if $s = 2$, (3.1) agrees with (3.1) of [**5**]. In that paper, more examples of generalized S-space-forms with two structure vector fields were given and they can be generalized to any s. Thus, pseudo-umbilical, totally contact-umbilical, totally contact-geodesic, totally umbilical and totally geodesic hypersurfaces of a generalized S-space-form are also generalized S-space-forms and, moreover, the bundle space of a principal toroidal bundle over a Kaehlerian manifold and the warped product of $\mathbb{R}$ times a generalized S-space-form are generalized S-space-forms too.

Next, for the sectional curvatures of a generalized S-space form and by using (3.1) and (3.2), we can prove the following proposition.

PROPOSITION 3.1. *Let M be a generalized S-space-form with functions:*

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}.$$

Then, for any orthonormal vector fields $X, Y \in \mathcal{L}$ and $\alpha, \beta \in \{1, \dots, s\}$, we have

(i) $K(X,Y) = R(X,Y,Y,X) = F_1 + 3F_2 g(X, fY)^2$.
(ii) $H(X) = K(X, fX) = F_1 + 3F_2$.
(iii) $K(X, \xi_\alpha) = F_1 - F_{\alpha\alpha}$.
(iv) $K(\xi_\alpha, \xi_\beta) = F_1 - F_{\alpha\alpha} - F_{\beta\beta} + G_{\alpha\beta}$, $(\alpha < \beta)$.

We are going now to study if the writing of the curvature tensor field of a generalized S-space-form is unique. First, we can prove:

PROPOSITION 3.2. *Let M be a $(2n+s)$-dimensional generalized S-space-form. If $n \geqslant 2$, the writing of the curvature tensor field R of M in terms of a family of functions is unique.*

PROOF. Let us suppose that there exist two families of differentiable functions, $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and $\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}$, such that

$$\begin{aligned} R &= F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1 \leqslant \alpha < \beta \leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1, \\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma} \\ &= F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1 \leqslant \alpha < \beta \leqslant s} G_{\alpha\beta}^* \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1, \\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma}. \end{aligned} \tag{3.5}$$

Since $n \geqslant 2$, we can consider a pair of orthonormal vector fields $X, Y \in \mathcal{L}$ such that $g(X, fY) = 0$. From (3.5) we get that $R(X, Y, fX, fY) = F_2 = F_2^*$ and so, $R(X,Y,Y,X) = F_1 = F_1^*$. From (iii) and (iv) of Proposition 3.1 we deduce that $F_{\alpha\alpha} = F_{\alpha\alpha}^*$, for any $\alpha = 1, \dots, s$ and $G_{\alpha\beta} = G_{\alpha\beta}^*$, for any $\alpha, \beta = 1, \dots, s$, $\alpha < \beta$.

Finally, if $X \in \mathcal{L}$ is a unit vector field and $\alpha, \beta = 1, \dots, s$, $\alpha \neq \beta$, from (3.5) again, we get that $R(X, \xi_\alpha, X, \xi_\beta) = F_{\alpha\beta} = F_{\alpha\beta}^*$ and, by using (3.4), $H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma}^*$, for any $\alpha, \beta, \gamma \in \{1, \dots, s\}$, $\alpha \neq \beta \neq \gamma \neq \alpha$. □

Next, what about $(2+s)$-dimensional generalized S-space-forms? In this case, the writing of the curvature tensor field is not unique. Actually, if M is a generalized S-space-form of dimension $2+s$ such that its curvature tensor field R can be simultaneously written as

$$R = F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1 \leqslant \alpha < \beta \leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1, \\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}$$

and

$$R = F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}^* \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma},$$

then, given a unit vector field $X \in \mathcal{L}$ and $\alpha, \beta, \gamma \in \{1, \ldots, s\}$, from (3.3), (3.4) and Proposition 3.1, we obtain the system

$$\begin{aligned} F_1 - F_1^* &= 3(F_2^* - F_2); && \\ F_1 - F_1^* &= F_{\alpha\alpha} - F_{\alpha\alpha}^*; && \\ F_{\alpha\beta} - F_{\alpha\beta}^* &= 0; && (\alpha \neq \beta) \\ F_{\alpha\alpha} - F_{\alpha\alpha}^* &= G_{\alpha\beta} - G_{\alpha\beta}^*; && (\alpha < \beta) \\ F_{\beta\beta} - F_{\beta\beta}^* &= G_{\alpha\beta} - G_{\alpha\beta}^*; && (\alpha < \beta) \\ H_{\alpha\beta\gamma} - H_{\alpha\beta\gamma}^* &= 0, && (\alpha \neq \beta \neq \gamma \neq \alpha) \end{aligned}$$

whose general solution is given by

$$\begin{aligned} F_1^* &= F_1 + h, \qquad & F_2^* &= F_2 - \tfrac{1}{3}h, \qquad & F_{\alpha\alpha}^* &= F_{\alpha\alpha} + h, \\ G_{\alpha\beta}^* &= G_{\alpha\beta} + h, & F_{\alpha\beta}^* &= F_{\alpha\beta}, & H_{\alpha\beta\gamma}^* &= H_{\alpha\beta\gamma}, \end{aligned} \tag{3.6}$$

where h is a differentiable function on M. Consequently, if $h \neq 0$, the writing of R in not unique and the functions of two different writings are related by (3.6).

On the other hand, if M is a $(2+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and we define the functions

$$\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}$$

as in (3.6), for any differentiable function h on M, then we deduce:

$$\begin{aligned} R &= F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}, \\ &= F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}^* \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma} \\ &\qquad\qquad -hR_1 + \frac{h}{3} R_2 - h \sum_{\alpha=1}^{s} R_{\alpha\alpha} - h \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta}. \end{aligned}$$

But it is straightforward to check that

$$hR_1 - \frac{h}{3} R_2 + h \sum_{\alpha=1}^{s} R_{\alpha\alpha} + h \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta} = 0$$

and, consequently, M is also a generalized S-space-form with functions

$$\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}.$$

4. A different definition

In [**6**], Falcitelli and Pastore defined a generalized $f.pk$-space-form as a metric $f.pk$-manifold M of dimension $2n+s$ (actually, a metric f-manifold) endowed with a family of differentiable functions $\{\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_{\alpha\beta},\ \alpha,\beta = 1,\dots,s\}$, such that $\widetilde{F}_{\alpha\beta} = \widetilde{F}_{\beta\alpha}$, for any $\alpha,\beta \in \{1,\dots,s\}$ and such that the curvature tensor field R of M can be written as

$$\begin{aligned} R(X,Y)Z = {} & \widetilde{F}_1\big\{g(fX,fZ)f^2Y - g(fY,fZ)f^2X\big\} \\ & + \widetilde{F}_2\big\{g(X,fZ)fY + g(Y,fZ)fX + 2g(X,fY)fZ\big\} \\ & + \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\beta}\big\{\eta_\alpha(X)\eta_b(Z)f^2Y - \eta_\alpha(Y)\eta_b(Z)f^2X \\ & \qquad + g(fY,fZ)\eta_\alpha(X)\xi_\beta - g(fX,fZ)\eta_\alpha(Y)\xi_\beta\big\}, \end{aligned} \tag{4.1}$$

for any $X,Y,Z \in \mathcal{X}(M)$. This definition is more restrictive than the one concerning generalized S-space-form. In fact, we observe that, from (4.1), $R(\xi_\alpha,\xi_\beta)\xi_\gamma = 0$, for any $\alpha,\beta,\gamma \in \{1,\dots,s\}$ (this means that the distribution $\mathcal{M}$ is flat), but some examples of generalized S-space-forms not satisfying this condition were presented in [**5**].

Moreover, if M is a generalized $f.pk$-space-form, a straightforward computation using (3.2) gives

$$\begin{aligned} R = {} & \widetilde{F}_1 R_1 + \widetilde{F}_2 R_2 + \widetilde{F}_1\Bigg\{\sum_{\alpha=1}^{s} R_{\alpha\alpha} - \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta}\Bigg\} \\ & - \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\beta} R_{\alpha\beta} - \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\alpha}\widetilde{R}_{\alpha\beta} - \sum_{\substack{\alpha,\beta=1,\\ \alpha\neq\beta}}^{s} \widetilde{F}_{\alpha\beta}\Bigg\{\sum_{\substack{\gamma=1,\\ \alpha\neq\gamma\neq\beta}}^{s} R_{\gamma\alpha\beta}\Bigg\}. \end{aligned}$$

Consequently, M is a generalized S-space form with functions

$$\begin{gathered} F_1 = \widetilde{F}_1;\ \ F_2 = \widetilde{F}_2;\ \ F_{\alpha\alpha} = \widetilde{F}_1 - \widetilde{F}_{\alpha\alpha};\ \ F_{\alpha\beta} = -\widetilde{F}_{\alpha\beta}\ \ (\alpha\neq\beta); \\ G_{\alpha\beta} = \widetilde{F}_1 - \widetilde{F}_{\alpha\alpha} - \widetilde{F}_{\beta\beta};\ \ H_{\alpha\beta\gamma} = -\widetilde{F}_{\beta\gamma}. \end{gathered}$$

Conversely, if M is a generalized S-space-form with functions

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$$

such that the distribution $\mathcal{M}$ is flat, then, from (3.4) we get that $H_{\alpha\beta\gamma} = F_{\beta\gamma}$, for any $\alpha,\beta,\gamma = 1,\dots,s$, $\alpha\neq\beta\neq\gamma\neq\alpha$ and from (v) of Proposition 3.1, $G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta} - F_1$, $1\leqslant\alpha<\beta\leqslant s$. Then, it is easy to check that M is a generalized $f.pk$-space-form with functions:

$$\widetilde{F}_1 = F_1;\ \ \widetilde{F}_2 = F_2;\ \ \widetilde{F}_{\alpha\alpha} = F_1 - F_{\alpha\alpha};\ \ \widetilde{F}_{\alpha\beta} = -F_{\alpha\beta}\ \ (\alpha\neq\beta).$$

5. Generalized S-space-forms with additional structures

Taking into account the results of the above section, if M is a generalized S-space-form such that the distribution $\mathcal{M}$ is flat (for instance, if M is either a metric f-K-contact manifold or a K-manifold), we can apply the results of [**6**] to it. Firstly, we can prove:

Theorem 5.1. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$, such that $\nabla\xi_\alpha = -f$, for any $\alpha = 1, \dots, s$. Then, M is an S-manifold and*

$$F_1 = \tfrac{1}{4}(c+3s);\quad F_2 = \tfrac{1}{4}(c-s);\quad F_{\alpha\alpha} = \tfrac{1}{4}(c+3s) - 1;$$
$$F_{\alpha\beta} = -1\ \ (\alpha \neq \beta);\quad G_{\alpha\beta} = \tfrac{1}{4}(c+3s) - 2\ \ (\alpha < \beta);$$
$$H_{\alpha\beta\gamma} = -1\ \ (\alpha \neq \beta \neq \gamma \neq \alpha),$$

where $\alpha, \beta, \gamma \in \{1, \dots, s\}$ and $c = F_1 + 3F_2$. In particular, any generalized S-space-form with a metric f-K-contact-structure is an S-manifold.

Proof. Since, the condition of the statement implies that the distribution $\mathcal{M}$ is flat, we deduce that M is a generalized $f.pk$-space-form and we apply Proposition 7 of [**6**]. For metric f-K-contact manifolds we only have to consider (2.1). □

We point out here that, if $n \geqslant 2$, c becomes constant (see, for example, [**7**]) and M is actually an S-space-form. Moreover, we deduce:

Corollary 5.1. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$. If M is an S-manifold, then $F_1 - F_2 = s$.*

For C-manifolds, we have:

Theorem 5.2. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and with an underlying C-structure. Then*

$$F_1 = F_2 = F_{\alpha\alpha} = G_{\alpha\beta} = c/4, \quad \alpha < \beta; \tag{5.1}$$

$$F_{\alpha\beta} = H_{\alpha\beta\gamma} = 0, \quad \alpha \neq \beta \neq \gamma \neq \alpha, \tag{5.2}$$

where $\alpha, \beta, \gamma \in \{1, \dots, s\}$ and $c = F_1 + 3F_2$. Moreover, if $n > 1$, M is a C-space-form.

Proof. Since M is a C-manifold and so, a K-manifold, from (2.2), the distribution $\mathcal{M}$ is flat and M is also a generalized $f.pk$-space.form. Furthermore, the structure vector fields are parallel and, by using Proposition 8 and Remark 2 of [**6**] and applying the relationships obtained in the above section we get the desired results. Finally, from (3.1), the Ricci tensor field S and the scalar curvature ρ of M are given by

$$S(X,Y) = \frac{(n+1)c}{2}\left(g(X,Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)\right)$$

and $\rho = n(n+1)c$. Now, from the second Bianchi identity,

$$\nabla_i \rho = 2\sum_j \nabla_j S_i^j,$$

where S_i^j denotes the components of the Ricci tensor of type (1,1). Consequently, $(n-1)dc = 0$ and hence, $dc = 0$ if $n > 1$. □

Next, we are going to study generalized S-space-forms with more general structures. First, we get

THEOREM 5.3. *Let M be a generalized S-space-form with functions*

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}.$$

If M is a K-manifold, then

$$F_1 + G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta}; \quad F_1 - F_{\alpha\alpha} \geqslant 0, \ \textit{with } 1 \leqslant \alpha < \beta \leqslant s;$$
$$H_{\alpha\beta\gamma} = F_{\beta\gamma}, \ \textit{for any } \alpha, \beta, \gamma = 1, \ldots, s \ \textit{such that } \alpha \neq \beta \neq \gamma \neq \alpha.$$

PROOF. Since M is a K-manifold, from (2.2) we get that the distribution $\mathcal{M}$ is flat. Thus, M is a generalized $f.pk$-space-form and by using the results of Section 4, we deduce that $G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta} - F_1$, $1 \leqslant \alpha < \beta \leqslant s$ and $H_{\alpha\beta\gamma} = F_{\beta\gamma}$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Now, from (2.3) together (*iii*) of Proposition 3.1, we complete the proof. □

Finally, for metric f-contact structures, we can prove the following theorem.

THEOREM 5.4. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$. If M is a metric f-contact manifold and*

$$F_1 - F_{\alpha\alpha} = F_{\beta\beta} - G_{\alpha\beta} = 1, \ \ 1 \leqslant \alpha < \beta \leqslant s;$$
$$F_{\alpha\alpha} = F_{\beta\beta}, \ \textit{for any } \alpha, \beta = 1, \ldots, s,$$

then M is an S-manifold.

PROOF. First, from (v) of Proposition 3.1 and the hypothesis, we deduce that $K(\xi_\alpha, \xi_\beta) = 0$. Moreover, a direct computation by using (3.1) shows that $S(\xi_\alpha, \xi_\alpha) = 2n(F_1 - F_{\alpha\alpha}) = 2n$, $\alpha = 1, \ldots, s$, where S is the Ricci curvature tensor of M. Then, by using Theorem 3.8 of [**4**], we obtain that the structure vector fields are Killing vector fields, that is, M is a metric f-K-contact manifold. Thus, from Theorem 5.1, it is an S-manifold. □

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On η-Einstein Para-S-manifolds

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Abstract We introduce para-S-manifolds and obtain some results concerning the curvature of these manifolds. In particular, we prove that there does not exist Einstein para-S-manifold, and consequently, we investigate η-Einstein para-S-manifolds and the conditions for them to be ξ-conformally flat.

Keywords Almost para-f-structure · Para-S-manifold · η-Einstein · ξ-Conformally flat

Mathematics Subject Classification 53C99 · 53C15

1 Introduction

The study of paracomplex structures is a subject which has many applications to different topics and it is related to some physical problems (the nice survey [9] can be consulted for more details). When, moreover, a compatible pseudo-Riemannian metric is considered, we have the para-Hermitian and para-Kaehler manifolds and their variants.

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On the other hand, (almost) paracontact manifolds are semi-Riemannian manifolds which can be viewed as the odd dimensional counterpart of (almost) paracomplex manifolds. They were introduced by Sato in [17] and Kaneyuki and Williams in [12]. Recently, there seems to be an increasing interest in paracontact geometry and, in particular, in para-Sasakian manifolds, due to its links to more consolidated theory of para-Kaehler manifolds and to their role in geometry and mathematical physics (see, for instance, [8,10,11]).

Actually, the notion of almost paracontact structure is an analog of that one of almost contact structure and is closely related to the almost product structure. In this context, Bucki and Miernowski defined in [5] the notion of an almost r-paracontact structure which generalizes almost paracontact structure in a similar way to f-structures of co-rank greater than one generalize almost contact structures. They also started the study of almost r-paracontact manifolds equipped with a Riemannian compatible metric [3,4,13].

So, it is interesting to study what happens if instead of a Riemannian metric we consider a pseudo-Riemannian metric. Zamkovoy in [21] has obtained a complete arrangement of all the theory in the case of paracontact manifolds, and recently, Brunetti and Pastore have done a similar work in the context of indefinite globally framed f-manifolds in [2]. For these reasons, we want to introduce in this paper the notion of para-S-manifold and begin the study of some of its properties. The name is chosen to point out that it is the analog of S-manifolds introduced by Blair [1] in the setting of f-structures. We also observe that para-S-manifolds generalize para-Sasakian manifolds.

Firstly and after some preliminaries on almost para-f-structures, we define para-S-manifolds and obtain some results concerning the curvature tensor field and the Ricci tensor field of them. Since we show that they are not Einstein para-S-manifolds when the co-rank of the structure is greater than one, we also define η-Einstein para-S-manifolds and we prove that if the foliation generated by the structure vector fields is regular, then they project onto Einstein para-Kaehler manifolds and, consequently, their study is justified. Finally, if we consider the conformal curvature tensor field of the metric, introduced by Weyl in [19,20], we analyze the existence of ξ-conformally flat η-Einstein para-S-manifolds.

2 Preliminaries on (Almost) Para-f-manifolds

From now on, given a smooth manifold M, we shall denote by TM the Lie algebra of its tangent vector fields.

A $(2n+s)$-dimensional smooth manifold M is said to have an *almost para-f-structure* $(f, \eta_1, \ldots, \eta_s, \xi_1, \ldots, \xi_s)$ and it is called an *almost para-f-manifold* if it admits a tensor field f of type $(1, 1)$, s global tangent vector fields $\xi_1, \ldots, \xi_s$, called the *structure vector fields* and s 1-forms $\eta_1, \ldots, \eta_s$, satisfying the following compatibility conditions:

- $f(\xi_\alpha) = 0, \eta_\alpha \circ f = 0, \alpha = 1, \ldots, s;$
- $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \alpha, \beta = 1, \ldots, s;$

– $f^2 = Id - \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha$ and the eigendistributions of f corresponding to the eigenvalues 1 and -1, denoted by $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, have the same dimension equal to n.

An immediate consequence of the above definition is that the endomorphism f has rank $2n$. In particular, any almost s-paracontact manifold in the sense defined by Bucki and Miernowski (see [5]) is an almost para-f-structure.

If an almost para-f-manifold M admits a pseudo-Riemannian metric g such that

$$g(fX, fY) + g(X, Y) = \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{1}$$

for any $X, Y \in TM$, we say that M is a *metric almost para-f-manifold* and g is called a *compatible metric*. Putting $Y = \xi_\alpha$ in (1), we have that $\eta_\alpha(X) = g(X, \xi_\alpha)$, for any $\alpha = 1, \dots, s$.

Any compatible metric with a given almost para-f-structure is of signature $(n + s, n)$. Moreover, any almost para-f-structure admits a compatible metric. In fact, given a metric G on M, if we put

$$\bar{G}(X, Y) = G(f^2X, f^2Y) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

then, the metric g defined as

$$g(X, Y) = \frac{1}{2}\left(\bar{G}(X, Y) - \bar{G}(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)\right)$$

is a compatible metric with the structure. Observe that $\bar{G}(X, \xi_\alpha) = \eta_\alpha(X)$, for any α.

We should like to mention here that A. Bucky and A. Miernoski defined an almost s-paracontact metric structure (see [5]) if it admits a Riemannian metric g such that

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any tangent vector fields X, Y. For our interest, in this paper, the metric will be always pseudo-Riemannian and it will satisfy (1).

For a manifold M endowed with a metric almost para-f-structure, we can construct a very useful local orthonormal basis. To this end, let U be a coordinate neighborhood on M and E_1 any unit vector field on U orthogonal to the structure vector fields. Then, fE_1 is a vector field orthogonal to E_1 and to the structure vector fields too. Moreover, $g(fE_1, fE_1) = -1$. Now, if it is possible, we choose a unit vector field E_2 orthogonal to the structure vector fields, to E_1 and to fE_1. Then, fE_2 is also a vector field orthogonal to the structure vector fields, to E_1, to fE_1 and to E_2 and

$g(fE_2, fE_2) = -1$. Proceeding in this way, we obtain a local orthonormal basis $\{E_i, fE_i, \xi_\alpha\}$, $i = 1, \ldots, n$ and $\alpha = 1, \ldots, s$, called an *f-basis*.

On a metric almost para-f-manifold, we define a 2-form by $F(X, Y) = g(X, fY)$, for any $X, Y \in TM$, and we consider the following tensor fields,

- $N^{(1)}(X, Y) = [f, f](X, Y) - 2\sum_{\alpha=1}^{s} \mathrm{d}\eta_\alpha(X, Y)\xi_\alpha$,
- $N^{(2)}_\alpha(X, Y) = (L_{fX}\eta_\alpha)Y - (L_{fY}\eta_\alpha)X$,
- $N^{(3)}_\alpha(X) = (L_{\xi_\alpha} f)X$,
- $N^{(4)}_{\alpha,\beta}(X) = (L_{\xi_\alpha}\eta_\beta)X$,

for any $X, Y \in TM$ and $\alpha = 1, \ldots, s$, where $[f, f]$ is denoting the Nijenhuis tensor of f and $\mathcal{L}_X$ the Lie derivative with respect to the tangent vector field X. Firstly, we can prove the following proposition.

Proposition 1 *Let M be a metric almost para-f-manifold. Then, the covariant derivative ∇f of f with respect to the Levi-Civita connection ∇ of g is given by*

$$\begin{aligned}2g((\nabla_X f)Y, Z) = &-3\mathrm{d}F(X, fZ) - 3\mathrm{d}F(X, fY, fZ) - g(N^{(1)}(Y, Z), fX)\\ &+\sum_{\alpha=1}^{s}\{N^{(2)}_\alpha(Y, Z)\eta_\alpha(X) - 2\mathrm{d}\eta_\alpha(fZ, X)\eta_\alpha(Y)\\ &+2\mathrm{d}\eta_\alpha(fY, X)\eta_\alpha(Z)\},\end{aligned} \tag{2}$$

for any $X, Y, Z \in TM$.

Proof We know that the Levi-Civita connection ∇ of g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) = &Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z)\\ &+g([Z, X], Y) - g([Y, Z], X),\end{aligned}$$

for any $X, Y, Z \in TM$. On the other hand, $\mathrm{d}F$ can be expressed by

$$\begin{aligned}3\mathrm{d}F(X, Y, Z) = &XF(Y, Z) + YF(Z, X) + ZF(X, Y) - F([X, Y], Z)\\ &-F([Z, X], Y) - F([Y, Z], X).\end{aligned}$$

These two equations imply (2). □

If $F = \mathrm{d}\eta_\alpha$, for any $\alpha = 1, \ldots, s$, then we say that M is a *para-f-manifold*, and the structure $(f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ is called a *para-f-structure*. In this case, we have

Proposition 2 *Let M be a para-f-manifold. Then, $N^{(2)}_\alpha$ vanishes, for any $\alpha =$ $1, \ldots, s$. Consequently, (2) simplifies to*

$$2g((\nabla_X f)Y, Z) = -g(N^{(1)}(Y, Z), fX) \\ -2\sum_{\alpha=1}^{s}\{F(fZ, X)\eta_\alpha(Y) - F(fY, X)\eta_\alpha(Z)\}.$$

Proof Since the structure is a para-f-structure, $N_\alpha^{(2)}$ can be written as

$$\begin{aligned} N_\alpha^{(2)}(X, Y) &= 2\mathrm{d}\eta_\alpha(fX, Y) + 2\mathrm{d}\eta_\alpha(X, fY) = 2F(fX, Y) + 2F(X, fY) \\ &= 2g(fX, fY) + 2g(X, f^2Y) \\ &= 2g(fX, fY) + 2g(X, Y) - 2\sum_{\beta=1}^{s}\eta_\beta(X)\eta_\beta(Y) = 0, \end{aligned}$$

for any $X, Y \in TM$, where we have used (1). □

An almost para-f-structure is said to be *normal* if $N^{(1)}$ vanishes.

Proposition 3 [5] *Let M be a normal almost para-f-manifold. Then, $N_\alpha^{(2)}$, $N_\alpha^{(3)}$, and $N_{\alpha,\beta}^{(4)}$ vanish too, for any $\alpha, \beta = 1, \ldots, s$.*

3 Para-S-manifolds

In this section, we are going to introduce para-S-manifolds and study some basic properties of them.

Definition 1 A para-K-manifold is a normal almost para-f-manifold such that $\mathrm{d}F = 0$. A para-S-manifold is a normal para-f-manifold. In these cases, the structures are called para-K-structure and para-S-structure, respectively.

Observe that, if $s = 1$, a para-S-manifold is a para-Sasakian manifold (see, for instance, [21]). In general, it is clear that any para-S-manifold is a para-K-manifold. To find a necessary and sufficient condition for the converse, we have first to prove the following proposition.

Proposition 4 *Let M be a para-K-manifold and denote the para-K-structure by $(f, \eta_1, \ldots, \eta_s, \xi_1, \ldots, \xi_s, g)$. Then, we have*

(i) $[\xi_\alpha, \xi_\beta] = 0$, *for any* $\alpha, \beta = 1, \ldots, s$.
(ii) The structure vector fields $\xi_1, \ldots, \xi_s$ are Killing vector fields with respect to the metric g.
(iii) $\mathrm{d}\eta_\alpha(fX, Y) + \mathrm{d}\eta_\alpha(X, fY) = 0$, *for any $X, Y \in TM$ and any $\alpha = 1, \ldots, s$.*

Proof For (i), by using the normality of the structure, we obtain

$$\begin{aligned} 0 = N^{(1)}(X, \xi_\alpha) &= f^2[X, \xi_\alpha] - f[fX, \xi_\alpha] - 2\sum_{\gamma=1}^{s}\mathrm{d}\eta_\gamma(X, \xi_\alpha)\xi_\gamma, \\ &= [X, \xi_\alpha] - f[fX, \xi_\alpha] + \sum_{\gamma=1}^{s}(\xi_\alpha\eta_\gamma(X))\xi_\gamma, \end{aligned} \tag{3}$$

for any $X \in TM$ and any $\alpha = 1, \ldots, s$. Consequently, putting ξ_β in place of X, we deduce (i) from (3).

Now, since $\mathrm{d}F = 0$ and $F(X, \xi_\alpha) = 0$, for any $X \in TM$ and $\alpha = 1, \ldots, s$, from the well-known formula $L_{\xi_\alpha} F = \mathrm{d}i_{\xi_\alpha} F + i_{\xi_\alpha} \mathrm{d}F$, we get that $L_{\xi_a} F = 0$. Moreover, given any $X, Y \in TM$, it is easy to show that

$$0 = (L_{\xi_\alpha} F)(X, Y) = (L_{\xi_\alpha} g)(X, fY) + g(X, (L_{\xi_\alpha} f)Y).$$

But, since the structure is normal, by using Proposition 3, $(L_{\xi_\alpha} f)Y = 0$ and so, $(L_{\xi_\alpha} g)(X, fY) = 0$. Putting fY in place of Y and applying the definition of almost para-f-structure, we have

$$(L_{\xi_\alpha} g)(X, Y) = \sum_{\beta=1}^{s} \eta_\beta(Y)(L_{\xi_\alpha} g)(X, \xi_\beta).$$

But, a direct expansion gives that

$$(L_{\xi_\alpha} g)(X, \xi_\beta) = (L_{\xi_\alpha} \eta_\beta)X - g(X, [\xi_\alpha, \xi_\beta])$$

and we get (ii) from (i) and Proposition 3.

Finally, (iii) is a direct consequence from the fact that $N_\alpha^{(2)} = 0$, for any $\alpha = 1, \ldots, s$. □

Next, from (2) we have:

Proposition 5 *Let M be a para-S-manifold. Then,*

$$(\nabla_X f)Y = \sum_{\alpha=1}^{s} \left\{ g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X \right\}, \tag{4}$$

for any $X, Y \in TM$.

In this context, we can proof the desired characterization theorem.

Theorem 1 *A para-K-manifold M is a para-S-manifold if and only if (4) holds for any $X, Y \in TM$.*

Proof Let us denote by $(f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ the para-K-structure. We only have to prove that $F = \mathrm{d}\eta_\alpha$, for any $\alpha = 1, \ldots, s$. But, from (4), we get that $(\nabla_{fX} f)\xi_\alpha = fX$, for any $\alpha = 1, \ldots, s$ and $X \in TM$. On the other hand, from (2), we obtain that $g((\nabla_{fX} f)\xi_\alpha, Y) = -\mathrm{d}\eta_\alpha(fY, fX)$, for any $Y \in TM$. Therefore, $F(X, Y) = -\mathrm{d}\eta_\alpha(fX, fY)$.

Now, from (iii) of Proposition 4,

$$\mathrm{d}\eta_\alpha(fX, fY) = -\mathrm{d}\eta_\alpha(X, f^2 Y) = -\mathrm{d}\eta_\alpha(X, Y) + \sum_{\beta=1}^{s} \eta_\beta(Y) \mathrm{d}\eta_\alpha(X, \xi_\beta),$$

but, by using (3), $2\mathrm{d}\eta_\alpha(X, \xi_\beta) = -\xi_\beta \eta_\alpha(X) - \eta_\alpha([X, \xi_\beta]) = 0$ and this completes the proof. □

Example 1 Let $\mathbf{R}^{2n+s}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s$. We define

$$\eta_\alpha = \frac{1}{2}\left(\mathrm{d}z_\alpha - \sum_{i=1}^{n} y_i \mathrm{d}x_i\right),\ \xi_\alpha = 2\frac{\partial}{\partial z_\alpha};\ \alpha = 1, \ldots, s,$$
$$g = \sum_{\alpha=1}^{s}(\eta_\alpha \otimes \eta_\alpha) + \sum_{i=1}^{n}(\mathrm{d}x_i \otimes \mathrm{d}y_i + \mathrm{d}y_i \otimes \mathrm{d}x_i).$$

Then, we consider the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi_1, \ldots, \xi_s\}$, where

$$X_i = \frac{\partial}{\partial y_i};\ Y_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{s} y_i \frac{\partial}{\partial z_\alpha}$$

and we define

$$fX_i = X_i,\ fY_i = -Y_i,\ f\xi_\alpha = 0,\ 1 \le i \le n,\ 1 \le \alpha \le s.$$

Observe that we deduce

$$\begin{aligned} g(X_i, X_j) =&\ g(Y_i, Y_j) = 0,\ g(X_i, Y_j) = \delta_{ij},\ 1 \le i, j \le n; \\ g(\xi_\alpha, \xi_\beta) =&\ \delta_{\alpha\beta},\ 1 \le \alpha, \beta \le s, \\ g(X_i, \xi_\alpha) =&\ g(Y_i, \xi_\alpha) = 0,\ 1 \le i \le n,\ 1 \le \alpha \le s \end{aligned}$$

and $\eta_\alpha = g(., \xi_\alpha)$, for any $\alpha = 1, \ldots, s$.

In this context, it is straightforward to compute that $(f, \xi_\alpha, \eta_\alpha, g)$ is a para-S-structure on $\mathbf{R}^{2n+s}$.

From now on, M will always be a $(2n+s)$-dimensional para-S-manifold with para-S-structure given by $(f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$. If we denote by R the curvature tensor field associated with the Levi-Civita connection ∇ of M, from (4), a direct expansion gives that

$$\begin{aligned} & R(X, Y, fZ, W) + R(X, Y, Z, fW) \\ &= \sum_{\alpha=1}^{s}\{Xg(fY, fZ)\eta_\alpha(W) - g(fY, fZ)g(fX, W) \\ &\quad - X\eta_\alpha(Z)g(fY, fW) + \eta_\alpha(Z)g(\nabla_X f^2 Y, W) \\ &\quad + g(fX, f\nabla_Y Z)\eta_\alpha(W) - \eta_\alpha(\nabla_Y Z)g(fX, fW) \\ &\quad - Yg(fX, fZ)\eta_\alpha(W) + g(fX, fZ)g(fY, W) \\ &\quad + Y\eta_\alpha(Z)g(fX, fW) - \eta_\alpha(Z)g(\nabla_Y f^2 X, W) \\ &\quad - g(fY, f\nabla_X Z)\eta_\alpha(W) + \eta_\alpha(\nabla_X Z)g(fY, fW) \\ &\quad + g(f\nabla_X Y, fZ)\eta_\alpha(W) - g(f\nabla_Y X, fZ)\eta_\alpha(W) \\ &\quad - \eta_\alpha(Z)g(f\nabla_X Y, fW) + \eta_\alpha(Z)g(f\nabla_Y X, fW)\}, \end{aligned} \tag{5}$$

for any $X, Y, Z, W \in TM$. Therefore, if we choose these four vector fields orthogonal to the structure vector fields, from (5) we easily prove the following proposition.

Proposition 6 *Let M be a para-S-manifold. Then,*

$$\begin{aligned} &R(X, Y, Z, W) + R(X, Y, fZ, fW) \\ &\quad = s\{g(X, Z)g(Y, fW) - g(X, W)g(Y, fZ) \\ &\qquad + g(X, fZ)g(Y, W) - g(X, fW)g(Y, Z)\}, \end{aligned} \tag{6}$$

for any vector fields X, Y, Z, W orthogonal to the structure vector fields.

On the other hand, by using the fact of being the structure vector fields of M Killing vector fields, from (4) again, we deduce that

$$\nabla_X \xi_\alpha = -fX, \tag{7}$$

for any $X \in TM$ and any $\alpha = 1, \ldots, s$. Therefore, $R(X, Y)\xi_\alpha = (\nabla_Y f)X - (\nabla_X f)Y$, for any $X, Y \in TM$ and any $\alpha = 1, \ldots, s$. Thus, from (4), we get

$$R(X, Y)\xi_\alpha = \sum_{\beta=1}^{s} \left\{ \eta_\beta(X) f^2 Y - \eta_\beta(Y) f^2 X \right\}. \tag{8}$$

If we consider now an f-basis $\{E_i, fE_i, \xi_\alpha\}_{\{1 \le i \le n, 1 \le \alpha \le s\}}$ of TM, the Ricci tensor field of M is defined by

$$\begin{aligned} Ric(X, Y) = &\sum_{i=1}^{n} \{R(E_i, X, Y, E_i) - R(fE_i, X, Y, fE_i)\} \\ &+ \sum_{\alpha=1}^{s} R(\xi_\alpha, X, Y, \xi_a), \end{aligned} \tag{9}$$

for any $X, Y \in TM$. So, by using (8) we obtain

$$Ric(X, \xi_\alpha) = -2n \sum_{\beta=1}^{s} \eta_\beta(X), \tag{10}$$

for any $X \in TM$ and $\alpha = 1, \ldots, s$. Moreover, taking into account (6) and (8) again and by using the symmetries of the curvature tensor field, we have

$$Ric(E_i, E_i) = R(E_i, fE_i, E_j, fE_j) - s(2n - 1) \tag{11}$$

and

$$Ric(fE_i, fE_i) = -Ric(E_i, E_i), \tag{12}$$

for any $i = 1, \dots, n$. Denoting by r the scalar curvature of M and, following [14], r^* by

$$\begin{aligned} r^* &= \sum_{i=1,j}^{n} \{R(E_j, E_i, fE_i, fE_j) - R(fE_j, E_i, fE_i, f^2 E_j)\} \\ &\quad - \sum_{i,j=1}^{n} \{R(E_j, fE_i, f^2 E_i, fE_j) - R(fE_j, fE_i, f^2 E_i, f^2 E_j)\} \\ &= 2 \sum_{i,j=1}^{n} \{R(E_j, E_i, fE_i, fE_j) - R(fE_j, E_i, fE_i, E_j)\}, \end{aligned}$$

we show that $r + r^* + 4ns^2 = 0$. This formula was proved in [21] for para-Sasakian manifolds (case $s = 1$).

4 η-Einstein Para-S-manifolds

From (10), we easily deduce that $Ric(\xi_\alpha, \xi_\beta) = -2n$, for any $\alpha, \beta = 1, \dots, s$. Consequently,

Theorem 2 *For $s \geq 2$, there are not Einstein para-S-manifolds.*

This motivates, as in the case of Sasakian geometry, to introduce the notion of η-Einstein para-S-manifold.

Definition 2 A para-S-manifold M is said to be an η-Einstein manifold if its Ricci tensor field satisfies

$$Ric = ag + b \sum_{\alpha=1}^{s} \eta_\alpha \otimes \eta_\alpha + (a + b) \sum_{\alpha \neq \beta}^{s} \eta_\alpha \otimes \eta_\beta, \tag{13}$$

where a and b are differentiable functions on M.

From this definition, it follows that $Ric(\xi_\alpha, \xi_\beta) = a + b$, for any $\alpha, \beta = 1, \dots, s$. In particular, from (10), we have that $a + b = -2n$. Moreover, by using (13) and an f-basis, we deduce that the scalar curvature of M is given by $r = tr(Ric) = (2n + s)a + sb$. On the other hand, observe that, when $s = 1$, that is, if M is a para-Sasakian manifold, (13) reduces to $g(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, which was introduced in [21].

Example 2 It is straightforward to show that $\mathbf{R}^{2n+s}$ with the para-S-structure given in Example 1 is an η-Einstein manifold with functions $a = 0$ and $b = -2n$. To this end, we put

$$E_i = \frac{1}{\sqrt{2}}(X_i + Y_i),\ 1 \leq i \leq n,$$

and we have that $\{E_1, \dots, E_n, fE_1, \dots, fE_n, \xi_1, \dots, \xi_s\}$ is an f-basis.

Now, we can prove

Proposition 7 *Let M be a $(2n+s)$-dimensional η-Einstein para-S-manifold. If $n > 1$, then the functions a, b, and r are constant functions.*

Proof Given $X \in TM$, since $a + b = -2n$, we deduce that $Xa = -Xb$. Moreover, $Xr = (2n + s)Xa + sXb = -2nXb$. On the other hand, from Corollary 54 in [15], we know that $\mathrm{d}r = 2\mathrm{div}Ric$. By using an f-basis, the definition of the divergence of the Ricci tensor field (see [15], p. 86) and (13), we obtain that

$$\frac{1}{2}Xr = (\mathrm{div}Ric)X = Xa + \sum_{\alpha=1}^{s}(\xi_\alpha b)\eta_\alpha(X),$$

where we have used that $a + b$ is a constant function. Consequently,

$$(n-1)Xb = -\sum_{\alpha=1}^{s}(\xi_\alpha b)\eta_\alpha(X).$$

Putting $X = \xi_\beta$, $\beta \in \{1, \ldots, s\}$, we get from the latter formula that $\xi_\beta b = 0$, for any β and so, if $n > 1$, $Xb = 0$, that is, b is a constant function. Therefore, a and r are constant functions too and we complete the proof. □

The Definition 2 is motivated by the following theorem.

Theorem 3 *Let M be a $(2n + s)$-dimensional η-Einstein para-S-manifold. If we assume that the foliation generated by the structure vector fields is regular, then M projects onto an Einstein para-Kaehler manifold.*

Proof First, notice that $\xi_1, \ldots, \xi_s$ span an s-dimensional foliation ν on M. Indeed, from (i) of Proposition 4, we have that $[\xi_\alpha, \xi_\beta] = 0$, and hence, that foliation is an integrable one. The assumption that such foliation is regular ensures that the leaf space is a $2n$-dimensional manifold. Let us denote by π the global submersion $\pi : M \longrightarrow N$.

Since each ξ_α is a Killing vector field, we get that ν is a Riemannian foliation. Thus, the semi- Riemannian metric g projects onto a semi-Riemannian metric G on N and $\pi : (M, g) \longrightarrow (N, G)$ is a semi-Riemannian submersion, that is

$$g(X, Y) = G(\pi_* X, \pi_* Y) \circ \pi, \tag{14}$$

for any X, Y basic vector fields on M (following the terminology of [16]). Since $L_{\xi_\alpha} f = 0$, for any $\alpha = 1, \ldots, s$, f is also a projectable tensor, so that it projects onto a well-defined tensor field J on N such that

$$\pi_* \circ f = J \circ \pi_*. \tag{15}$$

Using (14) and (15), it is easy to check that $J^2 = Id$, $J \neq Id$ and that $G(J\widetilde{X}, J\widetilde{Y}) + G(\widetilde{X}, \widetilde{Y}) = 0$, for any $\widetilde{X}, \widetilde{Y} \in$ TN. Next, if X is a vector field on M such that X is orthogonal to ν and $fX = X$, we deduce that

$$[\xi_\alpha, X] = [\xi_\alpha, fX] = f[\xi_\alpha, X]$$

and this proves that $[\xi_\alpha, D^+] \subset D^+$, where D^+ denotes the eigendistribution of f corresponding to the eigenvalue 1. Similarly, we obtain that $[\xi_\alpha, D^-] \subset D^-$ for each $\alpha = 1, \ldots, s$. Thus, the distributions D^+ and D^- project onto two distributions $D^{'+}$ and $D^{'-}$ on N which coincide with the eigendistributions of J corresponding to the eigenvalues +1 and -1, respectively. It follows that $\dim D^{'+} = \dim D^+ = n$. Therefore, (N, J, G) is an almost para-Hermitian manifold.

Moreover, let $\widetilde{X}, \widetilde{Y} \in$ TN and let X, Y be basic vector fields on M such that $\pi_* X = \widetilde{X}$ and $\pi_* Y = \widetilde{Y}$. Then, denoting by $\widetilde{\nabla}$ the Levi-Civita connection associated with G,

$$(\widetilde{\nabla}_{\widetilde{X}} J)\widetilde{Y} = \pi_*((\nabla_X f)Y)^h = 0$$

and (N, G, J) is a para-Kaehler manifold, where ^{h} is denoting the horizontal component.

Now, we recall the O'Neill equation relating the Ricci tensor field Ric of the total space and that one $\widetilde{Ric}$ of the base space of a semi-Riemannian submersion (see [7,16] for more details). This equation is

$$\begin{aligned} Ric(X, Y) = \widetilde{Ric}(\pi_* X, \pi_* Y) \circ \pi &+ \frac{1}{2}(g(\nabla_X N, Y) + g(\nabla_Y N, X)) \\ &- 2\sum_{i=1}^{n} g(A_X E_i, A_Y E_i) + 2\sum_{i=1}^{n} g(A_X f E_i, A_Y f E_i) \\ &- \sum_{\alpha=1}^{s} g(T_{\xi_\alpha} X, T_{\xi_\alpha} Y), \end{aligned} \tag{16}$$

where

- $\{E_i, fE_i, \xi_\alpha\}_{\{i=1,\ldots,n,\, \alpha=1,\ldots,s\}}$ is an f-basis;
- A and T are the O'Neill tensor fields (see [16])
- $N = \sum_{\alpha=1}^{s} T_{\xi_\alpha} \xi_a$.

However, since the leaves of ν are totally geodesic (because $\nabla_{\xi_a} \xi_\beta = 0$, for any α, β), in this case, we have $T \equiv 0$ and hence, $N \equiv 0$. In addition, for any X, Y orthogonal to ν, we have [16]:

$$A_X Y = -A_Y X = \frac{1}{2}\sum_{\alpha=1}^{s} \eta_\alpha([X, Y])\xi_\alpha = -\sum_{\alpha=1}^{s} \mathrm{d}\eta_\alpha(X, Y)\xi_\alpha.$$

Thus, putting $\widetilde{X} = \pi_* X$ and $\widetilde{Y} = \pi_* Y$, (16) becomes

$$Ric(X, Y) = \widetilde{Ric}(\widetilde{X}, \widetilde{Y}) \circ \pi + 2sg(X, Y).$$

From the fact of being M an η-Einstein manifold, $Ric(X,Y) = ag(X,Y)$ (observe that X and Y are orthogonal to each ξ_α). Then, the above relation implies that

$$\begin{aligned}\widetilde{Ric}(\widetilde{X},\widetilde{Y})\circ\pi &= Ric(X,Y) - 2sg(X,Y)\\ &= (a-2s)g(X,Y) = (a-2s)G(\widetilde{X},\widetilde{Y})\circ\pi\end{aligned}$$

and, therefore, (N, G) is an Einstein manifold. This completes the proof. $\square$

For a m-dimensional Riemannian manifold (M, g), Weyl [19,20] introduced a generalized curvature tensor field which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. For this reason, he called it the *conformal curvature tensor* of the metric. Schouten [18] showed that for $m > 3$, the converse is true. If $m \geq 3$, the Weyl conformal curvature tensor is defined as a map $C : TM \times TM \times TM \longrightarrow TM$ such that

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z&\\ &-\frac{1}{m-2}\{Ric(Y,Z)X + g(Y,Z)QX\\ &-Ric(X,Z)Y - g(X,Z)QY\}\\ &+\frac{r}{(m-1)(m-2)}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{17}$$

where Q is denoting the Ricci operator defined by $g(QX,Y) = Ric(X,Y)$, for any $X, Y \in TM$.

In this context, a $(2n+s)$-dimensional para-S-manifold M is said to be *ξ-conformally flat* if the linear operator $C(X,Y)$ is an endomorphism of fTM, that is, if

$$C(X,Y)fTM \subseteq fTM.$$

Equivalently, ξ-conformally flatness means that the projection of the operator $C(X,Y)fTM$ onto the distribution spanned by the structure vector fields is zero. That is, if $C(X,Y)\xi_\alpha = 0$, for any $X, Y \in TM$ and any $\alpha = 1,\ldots,s$.

Now, we can prove the following theorem.

Theorem 4 *Let M be a $(2n+s)$-dimensional η-Einstein para-S-manifold with $n \geq 1$. Then*

(i) If $s = 1$, that is, is M is a para-Sasakian manifold, M is ξ-conformally flat.
(ii) If $s = 2$, M is ξ-conformally flat if and only if $a = -4n$.
(iii) If $s > 2$, M cannot be ξ-conformally flat.

Proof Since M is η-Einstein, we know that $a + b = -2n$ and $r = (2n+s)a + sb = 2n(a-s)$. Furthermore, from (13),

$$QX = aX + b\sum_{\alpha=1}^{s}\eta_\alpha(X)\xi_\alpha + (a+b)\sum_{\alpha\neq\beta}\eta_\alpha(X)\xi_\beta, \tag{18}$$

for any $X \in TM$.

Firstly, if $s = 1$, the above formula reduces to $QX = aX + b\eta(X)\xi$. Moreover, from (8), we deduce that $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$. Then, a direct expansion gives that $C(X, Y)\xi = 0$, for any $X, Y \in TM$ and M is ξ-conformally flat.

If $s = 2$, from (8) and (18), since $r = 2n(a - 2)$, we compute that

$$\begin{aligned} C(X, Y)\xi_1 = \frac{a + 2b}{2n} \Big\{ & \frac{1}{2n + 1}(\eta_1(Y)X - \eta_1(X)Y) \\ & + \eta_1(X)\eta_2(Y)\xi_2 - \eta_2(X)\eta_1(Y)\xi_2 \Big\}, \end{aligned}$$

and

$$\begin{aligned} C(X, Y)\xi_2 = \frac{a + 2b}{2n} \Big\{ & \frac{1}{2n + 1}(\eta_2(Y)X - \eta_2(X)Y) \\ & + \eta_2(X)\eta_1(Y)\xi_1 - \eta_1(X)\eta_2(Y)\xi_1 \Big\}, \end{aligned}$$

for any $X, Y \in TM$. Consequently, M is ξ-conformally flat if and only if $a = -4n$ (and $b = 2n$).

Finally, if $s > 2$, a long straightforward computation gives that, using (8) and (18) again,

$$\begin{aligned} & C(X, Y)\xi_1 \\ & = \left(1 + \frac{2a + b}{2n + s - 2} + \frac{r}{(2n + s - 2)(2n + s - 1)}\right)(\eta_1(X)Y - \eta_1(Y)X) \\ & \quad + \frac{s - 2}{2n + s - 2}\sum_{\alpha=2}^{s}(\eta_a(X)Y - \eta_\alpha(Y)X) \\ & \quad + \frac{2 - s + a + 2b}{2n + s - 2}\sum_{\alpha=2}^{s}(\eta_1(X)\eta_\alpha(Y)\xi_\alpha - \eta_a(X)\eta_1(Y)\xi_\alpha) \\ & \quad + \frac{2 - s}{2n + s - 2}\left(\sum_{\alpha=2}^{s}(\eta_a(X)\eta_1(Y) - \eta_1(X)\eta_\alpha(Y)\right)\xi_1 \\ & \quad - \sum_{\substack{\beta,\gamma=2 \\ \beta\neq\gamma}}^{s}(\eta_\beta(X)\eta_\gamma(Y)\xi_\gamma - \eta_\beta(X)\eta_\gamma(Y)\xi_\beta) \\ & \quad - \frac{a + b}{2n + s - 2}\sum_{\substack{\beta,\gamma=2 \\ \beta\neq\gamma}}^{s}(\eta_\beta(X)\eta_1(Y)\xi_\gamma - \eta_1(X)\eta_\beta(Y)\xi_\gamma), \end{aligned}$$

for any $X, Y \in TM$. Consequently, M cannot be ξ-conformally flat. $\square$

In Sasakian geometry, it is known that ξ-conformally flatness is equivalent to be η-Einstein (see [6]). Now, we can prove the same result for para-Sasakian manifolds.

Theorem 5 *A para-Sasakian manifold M is ξ-conformally flat if and only if an η-Einstein manifold.*

Proof We only have to prove that ξ-conformally flatness implies to be η-Einstein. But, from (8), (10), (17) and since $s = 1$, we deduce

$$QX = \left(1 + \frac{r}{2n}\right) X + \left(-1 - 2n - \frac{r}{2n}\right) \eta(X)\xi,$$

for any $X \in TM$. Thus, M is an η-Einstein manifold. □

Finally, for $s = 2$ we have

Theorem 6 *Let M be a ξ-conformally flat para-S-manifold with two structure vector fields. Then, M is an η-Einstein manifold with $a = -4n$.*

Proof Firstly, from (8), (10), (17) and since $C(\xi_\alpha, \xi_\beta)\xi_\beta = 0$, we deduce that $r = -4n(2n+1)$. Now, aplying the same formulas, $C(X, \xi_1)\xi_1 = 0$ implies that

$$\begin{aligned} QX = &-4nX + 2n(\eta_1(X)\xi_1 + \eta_2(X)\xi_2) \\ &-2n(\eta_1(X)\xi_2 + \eta_2(X)\xi_1), \end{aligned}$$

for any $X \in TM$. This completes the proof. □

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