A LOWER BOUND FOR THE EQUILATERAL NUMBER OF NORMED SPACES

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Abstract. We show that if the Banach-Mazur distance between an n-dimensional normed space X and ℓ_∞^n is at most $3/2$, then there exist $n+1$ equidistant points in X. By a well-known result of Alon and Milman, this implies that an arbitrary n-dimensional normed space admits at least $e^{c\sqrt{\log n}}$ equidistant points, where $c>0$ is an absolute constant. We also show that there exist n equidistant points in spaces sufficiently close to ℓ_p^n, $1<p<\infty$.

1. Notation

Throughout the paper we use the same symbol c for different absolute positive constants. Let X denote a normed space of finite dimension $\dim X = n$. Let $e(X)$ denote the largest size of an equilateral set in X. As usual, the space ℓ_p^n, $1 \le p < \infty$, is defined as $\mathbb{R}^n$ with the norm $\|(x_1,x_2,\dots,x_n)\|_p = (\sum_{i+1^n}|x_i|^p)^{1/p}$, and ℓ_∞^n is $\mathbb{R}^n$ with the norm $\|(x_1,x_2,\dots,x_n)\|_\infty = \max_i|x_i|$. The *Banach-Mazur distance* between two n-dimensional normed spaces is defined as $d(X,Y) = \inf\|T\|\,\|T^{-1}\|$, where the infimum is taken over all linear, invertible operators $T : X \to Y$. We say that X is a $(1+\varepsilon)$-*copy* of Y if $d(X,Y) \le 1+\varepsilon$.

2. The main theorems

It is conjectured [4, 9, 11, 12, 16] that $e(X) \ge n+1$ for all n-dimensional normed spaces X. This is known for $n \le 3$ [12] but open for $n \ge 4$. It is true for spaces sufficiently close to Euclidean:

Theorem 1 (Brass [3] & Dekster [6]). *Let X be an n-dimensional normed space with Banach-Mazur distance $d(X,\ell_2^n) \le 1+\frac{1}{n}$. Then an equilateral set in X of at most n points can be extended to one of $n+1$ points. In particular, $e(X) \ge n+1$.*

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Combining this theorem with Gordon's estimate [8] in the Dvoretzky theorem [7, §4], the following general lower bound follows:

$$e(X) \geq c(\log n)^{1/3}.$$

We improve this to the following:

Theorem A. *For any n-dimensional normed space X we have $e(X) \geq e^{c\sqrt{\log n}}$, where $c > 0$ is an absolute constant.*

The proof works by looking for large subspaces close to either ℓ_2^k or ℓ_∞^k. To this end we use the following theorem:

Theorem 2 (Alon-Milman [1])**.** *Let X be an n-dimensional normed space. Then for each $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that X either contains a $(1+\varepsilon)$-isomorphic copy of ℓ_2^m for some m satisfying $\log\log m \geq \frac{1}{2}\log\log n$ or contains a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^k for some k satisfying $\log\log k > \frac{1}{2}\log\log n - c$.*

If the second case occurs in the above theorem, i.e., if we find a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^k in X, then we need a result similar to Theorem 1 for spaces near ℓ_∞^n. This we provide as follows:

Theorem B. *Let X be an n-dimensional normed space with Banach-Mazur distance $d(X, \ell_\infty^n) \leq 3/2$. Then $e(X) \geq n+1$.*

We can then choose $\varepsilon = 1/2$ in Theorem 2 to obtain $e(X) \geq e^{c\sqrt{\log n}}$ in this case. On the other hand, if we find a $(1+\frac{1}{2})$-isomorphic copy of ℓ_2^m where $m > e^{\sqrt{\log n}}$, we cannot yet apply Theorem 1. We first have to find a k-dimensional subspace of the m-dimensional space that is $(1+\frac{1}{k})$-isomorphic to ℓ_2^k. This is provided by the following:

Theorem 3 (Milman [10])**.** *Let X be an m-dimensional normed space and $0 < \varepsilon < 1$. Then X contains a k-dimensional subspace Y where $k \geq c\varepsilon^2 m / d^2(X, \ell_2^m)$ and $d(Y, \ell_2^k) \leq 1+\varepsilon$.*

See also [7, Corollary 4.2.2]. Putting $\varepsilon = 1/k$ and X the m-dimensional subspace that is $3/2$-isomorphic to ℓ_2^m into the above theorem, we obtain $k > cm^{1/3}$, and Theorem 1 then gives $e(X) > cm^{1/3} > ce^{(1/3)\sqrt{\log n}}$. To complete the proof of Theorem A, it only remains to prove Theorem B.

Proof of Theorem B. We use the Brouwer fixed point theorem [5, §14.3], as in Brass' proof of Theorem 1. Without loss of generality we may assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \leq \|x\|_\infty \leq \frac{3}{2}\|x\| \quad \text{for all } x \in X.$$

Let $I = \{(i,j) : 1 \leq i < j \leq n+1\}$, with $|I| = n(n+1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j)\in I} \in [0, \frac{1}{2}]^N$, let

$$\begin{aligned} p_1(\varepsilon) &= (-1, 0, \ldots 0), \\ p_j(\varepsilon) &= (\varepsilon_{1,j}, \ldots \varepsilon_{j-1,j}, -1, 0, \ldots 0), \qquad 2 \leq j \leq n-1, \\ p_n(\varepsilon) &= (\varepsilon_{1,n}, \ldots \varepsilon_{n-1,n}, -1), \\ p_{n+1}(\varepsilon) &= (\varepsilon_{1,n+1}, \ldots \varepsilon_{n,n+1}). \end{aligned}$$

For $1 \leq i < j \leq n$ we have $\|p_i(\varepsilon) - p_j(\varepsilon)\|_\infty = 1 + \varepsilon_{i,j}$. Define $\varphi : [0,1/2]^N \to [0,1/2]^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - \|p_i(\varepsilon) - p_j(\varepsilon)\|$, $1 \leq i < j \leq n$. Note that

$$\varphi_{i,j}(\varepsilon) \geq 1 + \varepsilon_{i,j} - \|p_i - p_j\|_\infty = 0$$

and

$$\varphi_{i,j}(\varepsilon) \leq 1 + \varepsilon_{i,j} - \frac{2}{3}\|p_i - p_j\|_\infty = \frac{1}{3}(1 + \varepsilon_{i,j}) \leq \frac{1}{2},$$

so φ is well-defined. Brouwer now gives the existence of a point $\varepsilon' = (\varepsilon'_{i,j}) \in [0,1/2]^N$ with $\varphi(\varepsilon') = \varepsilon'$, which implies that $\|p_i(\varepsilon') - p_j(\varepsilon')\| = 1$ for all $1 \leq i < j \leq m$. We have obtained $n+1$ equilateral points. □

3. A generalization to ℓ_p^n

The following theorem partially generalizes Theorem B to all ℓ_p^n spaces with $1 < p < \infty$.

Theorem C. *For each $n > 2$ and $p \in (1, \infty)$ there exists $R(p,n) > 1$ such that for any n-dimensional normed space X with Banach-Mazur distance $d(X, \ell_p^n) \leq R(p,n)$ we have $e(X) \geq n$. In fact,*

$$\begin{aligned} R(p,n) &= \max_{\theta > 0} \left(\frac{1 + (1+\theta)^p}{2 + (n-2)\theta^p} \right)^{1/p} \\ &\sim 1 + \frac{p-1}{2p} n^{-\frac{1}{p-1}} \quad \textit{as } n \to \infty \textit{ with } p \textit{ fixed.} \end{aligned}$$

Proof. We follow the proof of Theorem B. Assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \leq \|x\|_p \leq R\|x\| \quad \text{ for all } x \in X.$$

Fix $\beta, \gamma > 0$. Let $I = \{(i,j) : 1 \leq i < j \leq n\}$, with $|I| = n(n-1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j)\in I} \in [0, \beta]^N$, let

$$\begin{aligned} p_1(\varepsilon) &= (-\gamma, 0, \ldots 0), \\ p_j(\varepsilon) &= (\varepsilon_{1,j}, \ldots \varepsilon_{j-1,j}, -\gamma, 0, \ldots 0), \qquad 2 \leq j \leq n-1, \\ p_n(\varepsilon) &= (\varepsilon_{1,n}, \ldots \varepsilon_{n-1,n}, -\gamma). \end{aligned}$$

For $1 \leq i < j \leq n$ we have

$$\|p_j - p_i\|_p^p = \sum_{k=1}^{i-1} |\varepsilon_{k,j} - \varepsilon_{k,i}|^p + (\varepsilon_{i,j} + \gamma)^p + \sum_{k=i+1}^{j} \varepsilon_{k,j}^p + \gamma^p.$$

Define $\varphi : [0,\beta]^N \to [0,\beta]^N$ by $\varphi_{i,j}(\varepsilon) = 1+\varepsilon_{i,j} - \|p_i - p_j\|$ for $1 \le i < j \le n$. On the one hand,

$$\begin{aligned}\varphi_{i,j}(\varepsilon) &\le 1 + \varepsilon_{i,j} - R^{-1}\|p_i - p_j\|_p \\ &\le 1 + \varepsilon_{i,j} - R^{-1}\left[(\gamma + \varepsilon_{i,j})^p + \gamma^p\right]^{1/q}.\end{aligned}$$

Taking into account that the latter is increasing with respect to $\varepsilon_{i,j}$, the inequality $\varepsilon_{i,j} \le \beta$ implies

$$\varphi_{i,j}(\varepsilon) \le 1 + \beta - R^{-1}\left[(\gamma + \beta)^p + \gamma^p\right]^{1/q}.$$

Therefore, if $(\gamma+\beta)^p + \gamma^p \ge R^p$ then $\varphi_{i,j}(\varepsilon) \le \beta$. On the other hand

$$\begin{aligned}\varphi_{i,j}(\varepsilon) &\ge 1 + \varepsilon_{i,j} - \|p_i - p_j\|_p \\ &\ge 1 + \varepsilon_{i,j} - \left[(n-2)\beta^p + (\gamma + \varepsilon_{i,j})^p + \gamma^p\right]^{1/q}.\end{aligned}$$

Again the latter is increasing with respect to $\varepsilon_{i,j}$, so using $\varepsilon_{i,j} \ge 0$ we have

$$\varphi_{i,j}(\varepsilon) \ge 1 - \left[(n-2)\beta^p + 2\gamma^p\right]^{1/q}.$$

Then $\varphi_{i,j}(\varepsilon_1, \dots \varepsilon_m) \ge 0$ would follow if $(n-2)\beta^p + 2\gamma^p \le 1$. Subsequently, if

$$(*) \qquad (\gamma+\beta)^p + \gamma^p \ge R^p \qquad \text{and} \qquad (n-2)\beta^p + 2\gamma^p \le 1,$$

then φ is well defined. Brouwer now gives a point $\varepsilon' = (\varepsilon'_{i,j}) \in [0,\beta]^N$ such that $\varphi(\varepsilon') = \varepsilon'$, implying that the points $p_1(\varepsilon'), \dots p_n(\varepsilon')$ are equilateral.

Finally, to take the best choice for the parameters in $(*)$ we have to maximize the expression $(\gamma+\beta)^p+\gamma^p$ under the constraints $(n-2)\beta^p+2\gamma^p \le 1$ and $\beta, \gamma \ge 0$. Setting $\theta = \beta/\gamma$, we obtain

$$R^p = \max_{\theta > 0} \frac{1 + (1+\theta)^p}{2 + (n-2)\theta^p}.$$

It is not difficult to see that for θ close to $n^{-1/(p-1)}$ the right-hand side is > 1 and $R - 1 \sim \frac{p-1}{2p} n^{-\frac{1}{p-1}}$. □

4. Concluding remarks

For $p = 2$ the estimate in the above theorem is $d(X, \ell_2^n) \lesssim 1 + \frac{1}{4n}$, slightly worse than Theorem 1. However, we don't know how to obtain $n+1$ equidistant points as in Theorem B. It would also be interesting to know whether arbitrary equilateral sets of at most n points in spaces near ℓ_p^n can be extended as in Theorem 1. A different idea will be needed to extend the above theorem to the case $p = 1$. See [15] for a survey on equilateral sets, as well as [2, 13, 14] for further results on equilateral sets in ℓ_p^n.

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