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ON A UNIFIED STUDY OF RELATIVE CHEBYSHEV RADII AND HAUSDORFF MEASURES OF NONCOMPACTNESS

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Abstract. The paper is concerned with the notion of the Lifschitz modulus and its relationship with both relative Chebyshev radii and Hausdorff measures of noncompactness.

1. Introduction

Throughout the paper we shall use the following notation: X will denote a real normed space, A a nonempty, bounded subset of X and G a nonempty subset of X. As usual, $B(x,r)$ will stand for the closed ball centered at $x \in X$ with radius $r > 0$, diam A for the diameter of A, and $\operatorname{dist}(x, A)$ for the distance from x to A. We shall also write

$$B(A,r) = \{x \in X : \operatorname{dist}(x,A) \le r\},$$
$$\operatorname{dist}(A,G) = \inf\{\operatorname{dist}(x,G) : x \in A\},$$
$$d(A,G) = \inf\{r > 0 : A \subset B(G,r)\}$$

and

$$d(\mathcal{H},G) = \inf\{d(F,G) : F \in \mathcal{H}\}$$

if $\mathcal{H}$ is a nonempty family of nonempty bounded subsets of X.

Let us recall that the relative Chebyshev radius $r_G(A)$ is given by

$$r_G(A) = \inf\{r > 0 : A \subset B(y,r) \ \text{ for some } y \in G\}$$

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and the relative Hausdorff measure of noncompactness $\chi_G(A)$ by

$$\chi_G(A) = \inf\{r > 0 : A \subset B(F, r) \text{ for some finite set } F \subset G\}.$$

If $G = X$ we shall abbreviate $r(A)$ and $\chi(A)$ respectively. Finally, for $\epsilon \geq 0$,

$$E^{\varepsilon}(A) = \{y \in X : A \subset B(y, r(A) + \varepsilon)\},$$

$$\mathcal{H}^{\varepsilon}(A) = \{F \subset X : A \subset B(F, \chi(A) + \varepsilon) \text{ and } F \text{ is finite}\}.$$

Note that $\mathcal{H}^{\varepsilon}(A) \neq \emptyset$ and $E^{\varepsilon}(A) \neq \emptyset$ for every bounded $A \subset X$ and $\varepsilon > 0$.

The central theme of this paper is the concept of the Lifschitz modulus, defined in [16] as follows:

Definition 1.1. The *Lifschitz modulus* of a normed space X is the function $\tilde{\kappa}_X(\cdot)$ defined on $[0, +\infty)$ by

$$\tilde{\kappa}_X(d) = \sup\{k > 0 : \exists \alpha \in (0,1)\, \forall x, y \in X\, \forall r > 0\, \exists z \in X\ (\|z - y\| \leq \alpha d r \wedge B(x, r) \cap B(y, kr) \subset B(z, r))\}.$$

The origins of this notion come from [13], where slightly similar ideas were used to prove a certain fixed point theorem for uniformly Lipschitzian mappings. Although the definition of the Lifschitz modulus seems to be a little artificial, it is based on very natural geometric intuition (see Section 2 for more details). With the help of this notion we managed [16] to extend the formula

$$r_G(A) = r(A) + \operatorname{dist}\left(E^0(A), G\right)$$

(proved in the case $X = C(K)$ by Smith and Ward in [15], see also [7]), for a wider class of spaces including c_0 as well as some subspaces of $C(K)$. Moreover, the concept of the Lifschitz modulus let us treat the notions of relative Chebyshev radii and relative Hausdorff measures of noncompactness in a unified way, since Theorem 1.2, stated below, implies that if $\tilde{\kappa}_X(d) = 1 + d$ for all $d \geq 0$, then

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^{\varepsilon}(A), G)$$

and

$$\chi_G(A) = \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^{\varepsilon}(A), G).$$

The following theorem is a minor modification of the result given in [16]:

Theorem 1.2. *Let A and G be nonempty subsets of a normed space X with A nonsingleton and bounded. Then*

$$r_G(A) \geq r(A)\, \tilde{\kappa}_X\left(\tfrac{d_r(A,G)}{r(A)}\right),$$

where

$$d_r(A,G) = \begin{cases} \operatorname{dist}\left(E^0(A),G\right) & if\ E^0(A) \neq \emptyset, \\ \lim\limits_{\varepsilon\to 0^+} \operatorname{dist}\left(E^\varepsilon(A),G\right) & if\ E^0(A) = \emptyset. \end{cases}$$

If $\chi(A) \neq 0$, then

$$\chi_G(A) \geq \chi(A)\,\tilde{\kappa}_X\left(\tfrac{d_\chi(A,G)}{\chi(A)}\right),$$

where

$$d_\chi(A,G) = \begin{cases} d\left(\mathcal{H}^0(A),G\right) & if\ \mathcal{H}^0(A) \neq \emptyset, \\ \lim\limits_{\varepsilon\to 0^+} d\left(\mathcal{H}^\varepsilon(A),G\right) & if\ \mathcal{H}^0(A) = \emptyset. \end{cases}$$

In [6] we put these results in a more general framework and gave the following characterization of $C(K)$ and $C_0(\Omega)$ spaces:

Theorem 1.3. *Let A and G be nonempty subsets of a Banach space X with A bounded. The following assertions are equivalent:*

(1) $r_G(A) = r(A) + \operatorname{dist}\left(E^0(A),G\right)$ for every A and G, as above.

(2) $r_G(A) = r(A) + \lim\limits_{\varepsilon\to 0^+} \operatorname{dist}\left(E^\varepsilon(A),G\right)$ for every A and G, as above.

(3) X is isometric to $C(K)$ for some compact Hausdorff topological space K or to $C_0(\Omega)$ for some locally compact Hausdorff space Ω.

(4) $\tilde{\kappa}_X(d) = 1 + d$ for all $d \geq 0$.

The present paper deals with the following two questions:

1) How “good” are the estimations obtained in Theorem 1.2?

2) Does the characterization of $C(K)$ and $C_0(\Omega)$ spaces given in Theorem 1.3 still hold if we replace the notion of relative Chebyshev radii by relative Hausdorff measures of noncompactness?

In Section 2 we discuss some properties of the Lifschitz modulus. With the use of these properties we show in Section 3 that the estimation of $r_G(A)$ given in Theorem 1.2 is, in a sense, optimal. We recall Theorem 1.3 in Section 4 in order to study its counterpart for the relative Hausdorff measure of noncompactness. It clearly follows that if X is isometric to $C(K)$ or $C_0(\Omega)$, then

$$(\star) \qquad \chi_G(A) = \chi(A) + \lim_{\varepsilon\to 0^+} d\left(\mathcal{H}^\varepsilon(A),G\right).$$

However, far from stating an equivalent characterizing result we prove that formula $(\star)$ does hold in the space ℓ^1 which is well-known to be nonisometric to the spaces $C(K)$ and $C_0(\Omega)$. Thus, we discover the rather surprising fact that the notions of relative Chebyshev radii and Hausdorff measures of noncompactness behave in a different way in this problem.

Section 5 contains some remarks concerning another problem on relative Chebyshev radii and relative Hausdorff measures of noncompactness.

2. Properties of the Lifschitz Modulus

It is easily seen that in any normed space X

$$\max\{1, d-1\} \leq \tilde{\kappa}_X(d) \leq d+1$$

for all $d \geq 0$. The next two propositions show that we may treat the parameter d in the definition of the modulus of Lifschitz as the distance between x and y.

Proposition 2.1. *For all $d \geq 0$*

$$\tilde{\kappa}_X(d) = \sup\{k > 0 : \exists \alpha \in (0,1)\, \forall x, y \in X \ (\|x-y\| \leq d \Rightarrow$$
$$\exists z \in X \ (\|z-y\| \leq \alpha d \ \wedge \ B(x,1) \cap B(y,k) \subset B(z,1)))\}.$$

Proof. Let us first notice that we can fix $r = 1$ in the definition of $\tilde{\kappa}_X$. Put $d > 0$ and take $k \geq 1$ and $\alpha \in (0,1)$ such that for every $x, y \in X$, we obtain

$$\|x-y\| \leq d \Rightarrow \exists z \in X \ (\|z-y\| \leq \alpha d \ \wedge \ B(x,1) \cap B(y,k) \subset B(z,1))\,. \tag{2.1}$$

It is sufficient to prove that $k \leq \tilde{\kappa}_X(d)$. Let $x, y \in X$ with $\|x-y\| > d$. We need to construct $z \in X$ such that $\|z-y\| \leq \alpha d \wedge B(x,1) \cap B(y,k) \subset B(z,1)$. Let y_1 be the element of the segment $[x,y]$ satisfying the equality $\|x-y_1\| = d$. Therefore there exists $z_1 \in X$ with the property that $\|z_1 - y_1\| \leq \alpha d$ and $B(x,1) \cap B(y_1,k) \subset B(z_1,1)$. We now consider two cases.

Assume first that $\|y_1 - y\| \leq (1-\alpha)d$. Therefore $\|z_1 - y\| \leq \|z_1 - y_1\| + \|y_1 - y\| \leq \alpha d + (1-\alpha)d = d$ and hence using (2.1), we obtain $B(z_1,1) \cap B(y,k) \subset B(z,1)$ for some z with $\|z-y\| \leq \alpha d$. Thus, since $k \geq 1$,

$$B(x,1) \cap B(y,k) = B(x,1) \cap B(y_1,k) \cap B(y,k) \subset B(z_1,1) \cap B(y,k) \subset B(z,1)$$

and we have the desired result.

Assume now that $\|y_1 - y\| > (1-\alpha)d$. Therefore $\|y_1 - y\| \leq n(1-\alpha)d$ for some $n \in \mathbb{N}$ and we can find the elements $y_2, y_3, ..., y_n$ of the segment $[y_1, y]$ such that $\|y_1 - y_2\| = (1-\alpha)d$, $\|y_1 - y_3\| = 2(1-\alpha)d, ..., \|y_1 - y_n\| = (n-1)(1-\alpha)d$ and $\|y_n - y\| \leq (1-\alpha)d$. Using similar arguments as before we find the elements $z_2, z_3, ..., z_n$ such that

$$B(x,1) \cap B(y_2,k) \subset B(z_2,1),\ \|z_2 - y_2\| \leq \alpha d,$$
$$B(x,1) \cap B(y_3,k) \subset B(z_3,1),\ \|z_3 - y_3\| \leq \alpha d,$$
$$......$$
$$B(x,1) \cap B(y_n,k) \subset B(z_n,1),\ \|z_n - y_n\| \leq \alpha d.$$

Finally, $B(x,1) \cap B(y,k) \subset B(z,1)$ for some z satisfying $\|z-y\| \leq \alpha d$ and the proof is complete. □

Proposition 2.1 leads to the aforementioned desired result.

Proposition 2.2. *For all* $d \geq 0$

$$\tilde{\kappa}_X(d) = \sup\{k > 0 : \exists \alpha \in (0,1)\, \forall x, y \in X \ (\|x-y\| = d \Rightarrow \\ \exists z \in X \ (\|z-y\| \leq \alpha d \ \wedge \ B(x,1) \cap B(y,k) \subset B(z,1)))\}.$$

Proof. Fixing $d > 0$ and taking $k > 0$ and $\alpha_0 \in (0,1)$ such that for every $x, y \in X$, we obtain

$$\|x-y\| = d \Rightarrow \exists z \in X \ (\|z-y\| \leq \alpha_0 d \ \wedge \ B(x,1) \cap B(y,k) \subset B(z,1)).$$

It is sufficient to prove that $k \leq \tilde{\kappa}_X(d)$. Choose an arbitrary $\alpha \in (\alpha_0, 1)$ and $x, y \in X$ with $\|x-y\| < d$. Let y_1 be the element of the half-line xy such that $\|x - y_1\| = d$. Therefore there exists $z \in X$ with the property that $\|z - y_1\| \leq \alpha_0 d$ and $B(x,1) \cap B(y_1,k) \subset B(z,1)$.

If $\|x-y\| \leq \alpha d$ we have $B(x,1) \cap B(y,k) \subset B(x,1)$ and x itself is the required center.

If $\alpha d < \|x-y\| < d$, then $\|y - y_1\| < (1-\alpha)d$ and

$$B(x,1) \cap B(y, k-(1-\alpha)d) \subset B(x,1) \cap B(y_1,k) \subset B(z,1).$$

Moreover $\|z-y\| \leq (1+\alpha_0-\alpha)d$.

Combining these two cases and bearing Proposition 2.1 in mind we conclude that $k-(1-\alpha)d \leq \tilde{\kappa}_X(d)$. Taking α close to 1 we complete the proof. □

From Proposition 2.1 we also obtain the following proposition.

Proposition 2.3. *The function* $\tilde{\kappa}_X$ *is nondecreasing on* $[0,\infty)$.

Nonexpansivity follows in a less trivial way.

Proposition 2.4. *The function* $\tilde{\kappa}_X$ *is nonexpansive, i.e.*

$$\tilde{\kappa}_X(d_2) - \tilde{\kappa}_X(d_1) \leq d_2 - d_1$$

for every $0 \leq d_1 \leq d_2$.

Proof. Fix $0 < d_1 < d_2$ and choose $k < \tilde{\kappa}_X(d_2)$. Therefore there exists $\alpha \in (0,1)$ such that

$$\forall x, y \in X\, \exists z \in X \ (\|z-y\| \leq \alpha d_2 \ \wedge \ B(x,1) \cap B(y,k) \subset B(z,1)).$$

We will now prove that $\tilde{\kappa}_X(d_1) \geq k-(d_2-d_1)$. By Proposition 2.2 it is sufficient to consider $x, y_1 \in X$ with $\|x-y_1\| = d_1$. Let y_2 be the element on the half-line xy_1 such that $\|x-y_2\| = d_2$. Hence

$$B(x,1) \cap B(y_1, k-(d_2-d_1)) \subset B(x,1) \cap B(y_2,k) \subset B(z,1)$$

for some $z \in X$ with $\|z - y_2\| \leq \alpha d_2$. We notice that $y_1 = \frac{d_2 - d_1}{d_2} x + \frac{d_1}{d_2} y_2$ and set $p = \frac{d_2 - d_1}{d_2} x + \frac{d_1}{d_2} z$. Therefore $\|p - y_1\| = \frac{d_1}{d_2} \|z - y_2\| \leq \alpha d_1$. Moreover it is not difficult to see that

$$B(x, 1) \cap B(y_1, k - (d_2 - d_1)) \subset B(p, 1).$$

Thus $\tilde{\kappa}_X(d_1) \geq k - (d_2 - d_1)$ and, taking supremum over k, we can deduce $\tilde{\kappa}_X(d_1) \geq \tilde{\kappa}_X(d_2) - (d_2 - d_1)$. □

We shall use Propositions 2.2, 2.3 and 2.4 in the next section.

3. The Optimality Theorem

In this section we show that the evaluation of $r_G(A)$ given in Theorem 1.2 is in a sense optimal.

Theorem 3.1. *Let X be a normed space. Then for every $d \geq 0$ and $\varepsilon > 0$ there exist sets $A, G \subset X$ such that*

$$| \tfrac{d_r(A,G)}{r(A)} - d | < \varepsilon \quad \textit{and} \quad \frac{r_G(A)}{r(A)} < \tilde{\kappa}_X \left(\tfrac{d_r(A,G)}{r(A)} \right) + \varepsilon. \tag{3.1}$$

Proof. Fix $d \geq 0, \varepsilon > 0$, and take k satisfying the inequality $\tilde{\kappa}_X(d) < k < \tilde{\kappa}_X(d) + \frac{\varepsilon}{2}$ and choose $\alpha \in (0, 1)$ (notice that $k > 1$). Then there exist $x, y \in X$ with $\|x - y\| = d$ such that

$$\forall z \in X \ \ (B(x, 1) \cap B(y, k) \subset B(z, 1) \Rightarrow \|z - y\| > \alpha d). \tag{3.2}$$

We set $A = B(x, 1) \cap B(y, k),\ G = \{y\}$. It is clear that $r_G(A) = k, r(A) \leq 1$ and $d_r(A, G) \leq d$. We claim that

$$d_r(A, G) \geq \alpha d. \tag{3.3}$$

Indeed, if $r(A) = 1$, then $x \in E^0(A)$ and $E^0(A)$ is nonempty. Using this fact and (3.2), we can deduce $d_r(A, G) = \operatorname{dist}\left(E^0(A), y\right) \geq \alpha d$. If $r(A) < 1$, then we set $0 < \delta < 1 - r(A)$ and, again, $d_r(A, G) \geq \operatorname{dist}\left(E^\delta(A), y\right) \geq \alpha d$ which proves (3.3). Moreover

$$r(A) \geq 1 - (1 - \alpha) d. \tag{3.4}$$

Assume conversely, that there exist $z \in X$ and $r_0 < 1 - (1 - \alpha)d$ such that $A \subset B(z, r_0)$. In the case $0 \leq d \leq 1$ we may use the fact that $\|z - y\| \geq \alpha d$ and that $B(y, 1 - d) \subseteq A$. If we take the point $u \in A$ in the line zy which is the farthest from z in $B(y, 1 - d)$, it can be seen that $\|u - z\| \geq 1 - d + \alpha d = 1 - (1 - \alpha)d$, which is a contradiction with $u \in A$. In the case $d > 1$, let $p \in A$ be the element of the half-line xy such that $\|x - p\| = 1$. Then $\|y - p\| = d - 1$ and it follows from (3.2) that $\|z-p\| \geq \|z-y\| - \|y-p\| > 1-(1-\alpha)d$ which contradicts $p \in A$. Thus (3.4) is proved.

Summarizing we have

$$1-(1-\alpha)d \leq r(A) \leq 1 \text{ and } \alpha d \leq d_r(A,G) \leq d. \tag{3.5}$$

From this fact and Propositions 2.3 and 2.4

$$r_G(A) = k \leq \tilde{\kappa}_X(d) + \frac{\varepsilon}{2} \leq \tilde{\kappa}_X\left(\tfrac{d_r(A,G)}{\alpha r(A)}\right) + \frac{\varepsilon}{2} \leq$$
$$\tilde{\kappa}_X\left(\tfrac{d_r(A,G)}{r(A)}\right) + \tfrac{d_r(A,G)}{r(A)}\left(\frac{1}{\alpha}-1\right) + \frac{\varepsilon}{2}.$$

Finally

$$\frac{r_G(A)}{r(A)} \leq \frac{\tilde{\kappa}_X(d)+\frac{\varepsilon}{2}}{1-(1-\alpha)d} \leq \frac{\tilde{\kappa}_X\left(\frac{d_r(A,G)}{r(A)}\right) + \frac{d_r(A,G)}{r(A)}\left(\frac{1}{\alpha}-1\right)+\frac{\varepsilon}{2}}{1-(1-\alpha)d}$$

and from (3.5)

$$\alpha d \leq \frac{d_r(A,G)}{r(A)} \leq \frac{d}{1-(1-\alpha)d}. \tag{3.6}$$

Taking α sufficiently close to 1 the formula (3.1) holds. □

Theorem 1.2 also states

$$\tilde{\kappa}_X(d) \leq \inf\{r_G(A) : A, G \subset X,\ r(A) = 1 \text{ and } d = d_r(A,G)\}.$$

Using slightly more subtle arguments than in the proof of Theorem 3.1 we may improve the previous formulae. We shall need the following observation, which is easy to check.

Lemma 3.2. *Let E be a convex subset of a normed space X, $y \in X$ and $dist(y,E) = d > 0$. Then for every $\varepsilon > 0$ there exist $y_1, y_2 \in B(y,\varepsilon)$ such that $dist(y_1,E) \leq \max\{d-\frac{\varepsilon}{2}, 0\}$, $dist(y_2,E) \geq d+\frac{\varepsilon}{2}$.*

Theorem 3.3. *In any normed space X*

$$\tilde{\kappa}_X(d) = \inf\{r_G(A) : A, G \subset X,\ r(A) = 1 \text{ and } d = d_r(A,G)\}.$$

Proof. Fix $d, \varepsilon > 0$. As in the proof of Theorem 3.1, for sufficiently large $\alpha < 1$, there exist $A \subset X$ and $y \in X$ such that $\tilde{\kappa}_X(d) \geq \frac{r_{\{y\}}(A)}{r(A)} - \varepsilon$ and $\alpha d \leq \frac{d_r(A,\{y\})}{r(A)} \leq \frac{d}{1-(1-\alpha)d}$ (see (3.6)). Moreover, it follows from Lemma 3.3 that we can fix α close to 1 for which there exist $y_1, y_2 \in B(y,\varepsilon)$ with $d_r(A,\{y_1\}) < dr(A)$ and $d_r(A,\{y_2\}) > dr(A)$. Since the function $x \to d_r(A,\{x\})$ is continuous, there exists $z \in B(y,\varepsilon)$ with $d_r(A,\{z\}) = dr(A)$. Put $\bar{A} = \frac{A}{r(A)}$ and $\bar{z} = \frac{z}{r(A)}$. Then

$d_r\left(\bar{A},\{\bar{z}\}\right)=d$, $r\left(\bar{A}\right)=1$ and $\tilde{\kappa}_X\left(d\right)\geq r_{\{\bar{z}\}}\left(\bar{A}\right)-2\varepsilon$. This completes the proof. □

Notice that Theorem 3.3 gives a geometric description of the Lifschitz modulus.

4. The Case of Relative Hausdorff Measures of Noncompactness

It is not difficult to see that in any normed space X,

$$\chi_G\left(A\right)\leq\chi\left(A\right)+\lim_{\epsilon\to0^+}d\left(\mathcal{H}^\epsilon\left(A\right),G\right).$$

If X is isometric to $C(K)$ or $C_0(\Omega)$, then by Theorem 1.3 $\tilde{\kappa}_X(d)=1+d$ and by Theorem 1.2

$$\chi_G\left(A\right)=\chi\left(A\right)+\lim_{\varepsilon\to0^+}d\left(\mathcal{H}^\varepsilon\left(A\right),G\right).$$

In this section we extend this result to the space ℓ^1. Let us first recall the notion of minimal sets introduced by Domínguez Benavides in [5].

Definition 4.1. Let M be a metric space and φ a measure of noncompactness. A bounded, infinite set $A\subset M$ is said to be *minimal* for the measure φ (or, in short, φ*-minimal*) if $\varphi(A)=\varphi(B)$ for every infinite subset B of A.

Definition 4.2. A measure of noncompactness φ is said to be *strictly minimalizing* for a metric space M if for every bounded $A\subset M$ there exists a φ - minimal set $B\subset A$ such that $\varphi(B)=\varphi(A)$.

It is known [3] that the Hausdorff measure of noncompactness χ is strictly minimalizing for a wide class of spaces including separable as well as reflexive Banach spaces. In particular, χ is strictly minimalizing in ℓ^1 space. Thus, for any bounded set $A\subset\ell^1$, there exists a χ-minimal sequence for A, that is, a sequence $\{x_n\}$ in A such that $\chi\left(A\right)=\chi\left(\{x_1,x_2,...\}\right)$.

We shall also need the following result concerning ℓ^1 space. We say that a bounded sequence $\{x_n\}$ of points of ℓ^1 is coordinate-wise convergent to $x\in\ell^1$ if for any $i=1,2,...$, $\lim_{n\to\infty}x_n^i=x^i$. For a fixed sequence $\{x_n\}$ in ℓ^1 and arbitrary $y\in\ell^1$, we set

$$r_a(y,\{x_n\})=\limsup_{n\to\infty}\|x_n-y\|.$$

Proposition 4.3. *([9]). If $\{x_n\}$ is a bounded sequence in ℓ^1 converging coordinate-wise to $x\in\ell^1$, then for any $y\in\ell^1$*

$$r_a(y,\{x_n\})=r_a(x,\{x_n\})+\|x-y\|.$$

To establish the main result of this section we shall use the following lemmas.

Lemma 4.4. *Fix $x \in \ell^1$, $r > 0$,$\varepsilon \geq 0$ and let $\{x_n\}$ be a sequence of points of the ball $B(x,r)$, which is coordinate-wise convergent to x_0. If $\chi(\{x_1,x_2,x_3,...\}) \geq r-\varepsilon$, then $\|x-x_0\| \leq \varepsilon$.*

Proof. Notice that $r_a(x_0,\{x_n\}) \geq r-\varepsilon$ and it follows from Proposition 4.3 that

$$\|x-x_0\| = r_a(x,\{x_n\}) - r_a(x_0,\{x_n\}) \leq r-(r-\varepsilon) = \varepsilon.$$

□

Lemma 4.5. *Let A be a bounded subset of a Banach space X and $\varepsilon > 0$. Then there exist $x_1,...,x_n \in X$, $r_1,....,r_n > 0$ such that $A \subset \bigcup_{i=1}^n B(x_i,r_i)$ and $\chi(A \cap B(x_i,r_i)) \geq r_i - \varepsilon$, $i=1,...,n$.*

Proof. Let $A \subset \bigcup_{i=1}^m B(y_i,a_i)$, where $y_1,...,y_m \in X$ and $a_1,...,a_m > 0$. If $\chi(A \cap B(y_j,a_j)) < a_j - \varepsilon$ for some $j \in \{1,...,m\}$, there exist $z_1,...,z_l \in X$, $b_1,....,b_l < a_j - \varepsilon$ such that $A \cap B(y_j,a_j) \subset \bigcup_{i=1}^l B(z_i,b_i)$. If, again,

$$\chi(A \cap B(y_j,a_j) \cap B(z_k,b_k)) < b_k - \varepsilon$$

for some $k \in \{1,...,l\}$ we have $A \cap B(y_j,a_j) \cap B(z_k,b_k) \subset \bigcup_{i=1}^p B(u_i,c_i)$, where $u_1,...,u_p \in X$ and $c_1,...,c_p < b_k - \varepsilon < a_j - 2\varepsilon$. After a finite number of steps we obtain the desired cover. □

Lemma 4.6. *Let $x,y \in \ell^1$, $0 < r \leq k$ and $\varepsilon > 0$. Therefore there exists a finite set $F \subset \ell^1$ with the properties $B(x,r) \cap B(y,k) \subset B(F,r)$ and $F \subset B(y,k-r+\varepsilon)$.*

Proof. It follows from Lemma 4.5 that there exist $f_1,...,f_m \in \ell^1$ and $r_1,....,r_m \leq r$ such that $B(x,r) \cap B(y,k) \subset \bigcup_{i=1}^m B(f_i,r_i)$ and

$$\chi(B(x,r) \cap B(y,k) \cap B(f_i,r_i)) \geq r_i - \frac{\varepsilon}{2},\ i=1,...,m.$$

For each set $U_i = B(x,r) \cap B(y,k) \cap B(f_i,r_i)$ we select a χ - minimal sequence $\{x_1^i, x_2^i, ...\} \subset U_i$. We can assume, by taking subsequences if necessary, that the sequences are coordinate wise convergent to some $w_1,...,w_m \in \ell^1$, respectively. Since $\chi(\{x_1^i,x_2^i,...\}) \geq r_i - \frac{\varepsilon}{2}$, it follows from Lemma 4.4 that $\|f_i - w_i\| \leq \frac{\varepsilon}{2}$. Moreover

$$\|w_i - y\| = r_a(y,\{x_n^i\}) - r_a(w_i,\{x_n^i\}) \leq k - r_i + \frac{\varepsilon}{2}$$

and hence

$$\|f_i - y\| \leq \|f_i - w_i\| + \|w_i - y\| \leq k - r_i + \varepsilon,\ i=1,...,m.$$

Let us construct the set $F = \{\bar{f}_1,...,\bar{f}_m\}$ in the following way: For $i=1,...,m$, if $\|f_i - y\| \geq r - r_i$, $\bar{f}_i$ is denoted as the point of the segment $[f_i,y]$ satisfying the equality $\|f_i - \bar{f}_i\| = r - r_i$, $r - r_i \geq 0$. If $\|f_i - y\| < r - r_i$ we set $\bar{f}_i = y$. It is not difficult to see that $F \subset B(y, k-r+\varepsilon)$. Moreover, since every $z \in B(x,r) \cap B(y,k)$ is in some $B(f_i, r_i)$, we have $\|z - \bar{f}_i\| \leq \|z - f_i\| + \|f_i - \bar{f}_i\| \leq r$ and therefore $B(x,r) \cap B(y,k) \subset B(F,r)$. □

We can now prove

Theorem 4.7. *In ℓ^1 space*

$$\chi_G(A) = \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G)$$

for every $A, G \subset \ell^1$ with A bounded.

Proof. *Write* $\chi_G(A) = k$, $\chi(A) = r$, $\lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G) = d$ and assume that $k < r + d$. Choose $\varepsilon > 0$ which satisfies $k - r < d - 2\varepsilon$ and $\delta > 0$ which satisfies

$$d\left(\mathcal{H}^\delta(A), G\right) > d - \varepsilon. \tag{4.1}$$

Now we can select $F_0 = \{f_1, ..., f_n\}$ and $G_0 = \{g_1, ..., g_m\} \subset G$ such that $A \subset \bigcup_{i=1}^n \bigcup_{j=1}^m B(f_i, r+\delta) \cap B(g_j, k+\delta)$. It follows from Lemma 4.6 that there exist finite sets F_{ij} with $F_{ij} \subset B(g_j, k-r+\varepsilon)$, $1 \leq i \leq n, 1 \leq j \leq m$, and $A \subset \bigcup_{i=1}^n \bigcup_{j=1}^m B(F_{ij}, r+\delta)$. Assuming $F = \bigcup_{i=1}^n \bigcup_{j=1}^m F_{ij}$ we obtain $F \in \mathcal{H}^\delta(A)$. Therefore

$$d\left(\mathcal{H}^\delta(A), G\right) \leq \sup_{f \in F} \inf_{g \in G} d(f,g) \leq k - r + \epsilon < d - \varepsilon,$$

which contradicts (4.1). □

It is well known that ℓ^1 is not isometric to $C(K)$ or $C_0(\Omega)$. Therefore, the class of spaces satisfying the equality $\chi_G(A) = \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G)$ is strictly larger than the other class with $r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G)$. Even more can be concluded. In [6] we show that, for $d \geq 3$, $\tilde{\kappa}_{l^1}(d) = d - 1$, which is the smallest possible value for the modulus!

5. Variations on a Certain Characterization of Hilbert Spaces

In this section we study another problem in which the notions of relative Chebyshev radii and Hausdorff measures of noncompactness behave in a different way.

Let G be a linear subspace of a Banach space X and put

$$C_G(X) = \sup\left\{\frac{r_G(A)}{r_X(A)} : A \subset G \text{ is bounded and } r_X(A) \neq 0\right\},$$
$$C(X) = \sup\{C_G(X) : G \text{ is a linear subspace of } X\}.$$

It is easy to see that $1 \leq C(X) \leq 2$ in any Banach space X. The following theorem follows from the well known results of Klee and Garkavi (see [11], [12]).

Theorem 5.1. *Let X be a Banach space. Then*

$$C(X) = 1 \Leftrightarrow X \text{ is a Hilbert space or } \dim X \leq 2.$$

The natural question arises of whether a similar statement is true if we use relative Hausdorff measures of noncompactness instead of Chebyshev radii. As before, we set

$$H_G(X) = \sup \left\{ \frac{\chi_G(A)}{\chi_X(A)} : A \subset G \text{ is bounded and } \chi_X(A) \neq 0 \right\}$$

and

$$H(X) = \sup \{ H_G(X) : G \text{ is a linear subspace of } X \}.$$

Obviously $1 \leq H(X) \leq 2$ in any Banach space X and $H(Y) = 1$ in a Hilbert space Y. However, it turns out that the equality $H(X) = 1$ is characteristic for a larger class of spaces. This follows from the work of Ayerbe and Domínguez Benavides [2]. They proved, using the notion of χ - minimal sets, that $H(X) = 1$ in every reflexive Banach space X satisfying the (nonstrict) Opial condition, that is,

$$\liminf_{n\to\infty} \| x_n - x_0 \| \leq \liminf_{n\to\infty} \| x_n - x \|$$

for every sequence $\{x_n\}$ converging weakly to zero and every $x \in X$. In particular, $H(\ell^p) = 1$ for $p \in (1, \infty)$.

We point out the difference between $C(X)$ and $H(X)$ in the following example.

Example 5.2. Let X be a Banach space $\ell^2 \times \mathbb{R}$ with the norm

$$\| (x,t) \|_X = \max \{ \| x \|, | t | \}.$$

It is clear that X is not a Hilbert space. We show that $H(X) = 1$.

Let G be a linear subspace of $\ell^2 \times \mathbb{R}$ and put $G_1 = \left\{ x \in \ell^2 : \exists t \in \mathbb{R} \ (x,t) \in G \right\}$. Then $G = G_1 \times \mathbb{R}$ or G is a subspace of $G_1 \times \mathbb{R}$ of codimension one. Therefore it is sufficient to consider subspaces of the form

$$G = \{ (x,t) \in X : \langle x, x_0 \rangle = at \},$$

where $x_0 \in \ell^2$, $\| x_0 \| = 1$ and $a \neq 0$. Assuming that $A \subset G$, we show that $\chi_G(A) \leq \chi_X(A)$.

Let

$$A \subset \bigcup_{i=1}^{n} B((x_i, t_i), r)$$

for some $(x_i, t_i) \in X$, $i = 1, ..., n$ and $r > 0$. We set $(x,t) \in A$ and choose (x_i, t_i) such that

$$\| (x,t) - (x_i, t_i) \|_X \leq r.$$

Notice that $(x_i, s_i) \in G$ for $s_i = \frac{\langle x_i, x_0 \rangle}{a}$. Moreover

$$\| (x,t) - (x_i, s_i) - s \left(x_0, \frac{1}{a} \right) \|_X = \| x - x_i - s x_0 \|,$$

if we assume that s satisfies the condition $t - s_i - \frac{s}{a} = 0$. This means that

$$s = ta - s_i a = \langle x, x_0 \rangle - \langle x_i, x_0 \rangle.$$

Hence

$$\| (x,t) - (x_i, s_i) - s \left(x_0, \frac{1}{a} \right) \|_X^2 = \| x - x_i - s x_0 \|^2$$

$$= \| x - x_i \|^2 + s^2 \| x_0 \|^2 - 2s \langle x - x_i, x_0 \rangle = \| x - x_i \|^2 + s^2 - 2s \left(\langle x, x_0 \rangle - \langle x_i, x_0 \rangle \right)$$

$$= \| x - x_i \|^2 + s^2 - 2s^2 \leq \| x - x_i \|^2 \leq \| (x,t) - (x_i, t_i) \|_X^2 \leq r^2.$$

Since A is bounded, there exists a constant $c > 0$ such that $A \subset B(F, r)$, where

$$F = \left\{ (x_i, s_i) + s \left(x_0, \frac{1}{a} \right) : | s | \leq c \right\}.$$

From the compactness of F, we conclude that $\chi_G(A) \leq \chi_X(A)$, and the proof is complete.

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