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A GEOMETRICAL COEFFICIENT IMPLYING THE FIXED POINT PROPERTY AND STABILITY RESULTS

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ABSTRACT. In this paper we define a new geometric constant $M(X)$ in Banach spaces such that X has the fixed point property for nonexpansive mappings if $M(X) > 1$. We prove that $M(X) \geq WCS(X)$, the inequality being strict in many important classes of Banach spaces and we obtain lower bounds for $M(X)$ based upon either the modulus of near uniform smoothness or the modulus of the Opial property of the conjugated space. We show that this new constant gives us stability results for the fixed point property with respect to ℓ_p-spaces which improve all previous results.

Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in M$. A Banach space X is said to have the fixed point property (f.p.p.) for nonexpansive mappings if for every convex and weakly compact subset C of X, every nonexpansive mapping $T : C \to C$ has a fixed point. In 1965 Browder [B] and Kirk [K], respectively, proved that every uniformly convex Banach space and any Banach space with normal structure has the f.p.p. In 1981 Alspach [A] showed that L_1 fails to have the f.p.p. Over the last 30 years many papers have appeared studying geometric properties of the Banach spaces (uniform

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convexity, uniform smoothness, near uniform convexity, unconditional basis, etc) which assure either normal structure or the f.p.p. (see, for instance, [Ma, GK]). A method to assure the f.p.p. for a Banach space X is to use the "proximity" of X to another Banach space Y which "strongly" satisfies this property. To use this method we need a quantification of the f.p.p. The first results in this direction were obtained by Bynum [By1] defining certain normal structure coefficients. In this paper and in latter papers [Pr3, DL] several lower bounds for the normal structure coefficients were obtained based upon the value of certain geometric coefficients (Clarkson modulus of uniform convexity, modulus of uniform smoothness, modulus of near uniform convexity, etc). These bounds can be understood as stability results for the f.p.p. Recently García-Falset [Ga1] defined a new geometric coefficient $R(X)$ which assures the f.p.p. (in particular he proved that near uniformly smooth spaces have the f.p.p. in spite this spaces can fail to have normal structure) and he obtained stability results using this coefficient. In this paper, following the idea in [Ga1], we define a new coefficient $M(X)$ and we prove that X has the f.p.p. if $M(X) > 1$. This coefficient is, in general, equal or greater than Bynum's weakly convergent sequence coefficient $WCS(X)$, and strictly bigger than $WCS(X)$ in many special spaces (see Theorem 4.1 and remark after Theorem 2.5.). So we can improve a classic result in metric fixed point theory: Every Banach space with weak uniform normal structure has the f.p.p. Obviously, all lower bounds for $WCS(X)$ also hold for $M(X)$ and, in addition, we show lower bounds for $M(X)$ using either the modulus of near uniform smoothness, defined in [Do], (recall that $WCS(X)$ can be equal to 1 in near uniformly smooth spaces) or the Opial modulus (see [LTX]) of the dual space. In the case of ℓ_p-spaces we can directly obtain the value of $M(X)$. This value gives us stability results, which are strictely better than all previous stability results in these spaces [JL, Kh, Pr2].

1. Notations and preliminaries.

In the following, X will be a Banach space, B_X the closed unit ball, of X, and S_X the unit sphere. We shall often use Bynum's weakly convergent sequence coefficient $WCS(X)$. Before introducing it, we recall some definitions.

The asymptotic diameter and radius of a sequence $\{x_n\}$ in a Banach space X will be defined by:

$$\operatorname{diam}_a(\{x_n\}) = \limsup_k \sup\{\|x_n - x_m\| : n, m \geq k\},$$
$$r_a\{x_n\} = \inf\{\limsup_n \|x_n - y\| : y \in \{x_n\}\},$$

The weakly convergent sequence coefficient of a Banach space X is defined by

$$WCS(X) = \inf\{\frac{\operatorname{diam}_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence which is not norm convergent}\}.$$

It is known [By1] that X has weak normal structure, that is, every weakly compact convex subset of X with more than one member is not diametral, when $WCS(X) > 1$.

The following result shows how the coefficient $WCS(X)$ can be useful to prove the stability of the fixed point property.

Theorem 1.1 [By1]. *Let X and Y be isomorphic Banach spaces, then*

$$WCS(X) \leq d(X, Y) WCS(Y).$$

Several improvements of this result can be found in [Pr2]. The following form of $WCS(X)$ [DL, DLX1] will be very important in this paper

Theorem 1.2. *Let X be a Banach space without the Schur property. Then:*

$$WCS(X) = \inf\{\frac{\lim_{n,m\,;n\neq m}\|x_n - x_m\|}{\limsup \|x_n\|} : \{x_n\} \text{ converges weakly to zero and } \lim_{n,m\,;n\neq m} \|x_n - x_m\| \text{ exists}\}$$

We recall that the mapping $\rho_X(t)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1\,,\ \|y\| \leq t\right\}$$

is called modulus of uniform smoothness of X. A Banach space X is said to be uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

A more general concept is the *near uniform smoothness*, the dual notion of the near uniform convexity (see [Pr 1]). A Banach space X is said to be near uniformly smooth if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for each t, $0 < t < \eta$, and for each basic sequence $\{x_n\}$ in B_X there exists $k > 1$ such that

$$\|x_1 + tx_k\| < 1 + \varepsilon t.$$

In [Do] a modulus of near uniform smoothness is defined in reflexive Banach spaces by

$$\Gamma(t) = \sup\left\{\inf\left\{\frac{\|x_1 + tx_n\| + \|x_1 - tx_n\|}{2} - 1 : n > 1\right\} : \{x_n\} \text{weakly null in } B_X\right\}.$$

It is easy to check that $0 \leq \Gamma(t) \leq t$ for every $t \geq 0$.

A Banach space X is said to satisfy the *Opial condition* [Op] if

$$\liminf \|x_n - x\| < \liminf \|x_n - y\|$$

for every sequence $\{x_n\}$ in X weakly convergent to x and every point $y \neq x$. We say that X satisfies the *uniform Opial condition* [Pr4] if for every $c > 0$, there exists an $r = r(c) > 0$ such that

$$1 + r \leq \liminf \|x + x_n\|$$

for all $x \in X$ with $\|x\| \geq c$ and all weakly null sequences $\{x_n\}$ in X such that $\liminf_{n\to\infty} \|x_n\| \geq 1$.

In [LTX] the following modulus associated to the Opial condition has been defined:

Definition 1.3. Let X be a Banach space. The *modulus of Opial of* X is defined as

$$r_X(c) := \inf\{\liminf_{n\to\infty} \|x + x_n\| - 1\}, \quad c \geq 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all weakly null sequences $\{x_n\}$ in X with $\liminf \|x_n\| \geq 1$.

It is easily seen that the uniform Opial condition implies the Opial condition and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$.

The following constant for a Banach space X is defined in [Ga2]:

$$R(X) = \sup\{\liminf \|x + x_n\|\}$$

where the supremum is taken over all weakly null sequences in B_X and over all vectors x in B_X. In [Ga1] the following result of existence of fixed points and stability of the f.p.p. is proved

Theorem 1.4. *Let X and Y isomorphic Banach spaces. If $d(X,Y)R(X) < 2$ then Y has the f.p.p.*

Finally, for a Banach space X, $[X]$ will denote, as usual, the quotient space $\ell_\infty(X)/c_0(X)$ endowed with the norm $\|[z_n]\| = \limsup \|z_n\|$, where $[z_n]$ denotes the equivalent class of $\{z_n\} \in \ell_\infty(X)$. By identifying $x \in X$ with the class $[x, x, ...]$ we can consider X as a subset of $[X]$. If K is a subset of X we can consider the set $[K] = \{[z_n] \in [X] : z_n \in K$ for every $n \in \mathbb{N}\}$. If T is a mapping from K into K we define $[T] : [K] \to [K]$ by $[T]([x_n]) = [Tx_n]$.

The following lemma is a basic tool in this paper:

Lin's lemma 1.5 [L]. *Let X be a Banach space and K be a minimal weakly compact convex subset of X which is invariant under a nonexpansive mapping T. If $[W]$ is a nonempty closed convex subset of $[K]$ which is invariant under $[T]$ then*

$$\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = \ diam(K)$$

for every $x \in K$.

2. The coefficient $M(X)$ and the f.p.p.

In this section we are going to introduce a new coefficient in Banach spaces which yields a new fixed point theorem. As we shall see, this theorem enables us to prove the existence of a fixed point in Banach spaces without normal structure. Previously we need to define a uniparameter family of coefficients.

Definition 2.1. Let X be a Banach space. For any nonnegative number a we define the coefficient

$$R(a, X) = \sup\{\liminf \|x_n + x\|\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences in B_X such that $\lim_{n,m\,;n\neq m} \|x_n - x_m\| \leq 1$.

Theorem 2.2. *Let X be a Banach space and assume that for some $a \geq 0$ we have $R(a, X) < 1 + a$. Then X has the fixed point property.*

Proof. We follow an argument similar to that in [Ga1]. Assume that X fails to have the f.p.p. Then we can find a weakly compact and convex subset K of X such that diam $(K) = 1$ and K is minimal invariant for a nonexpansive mapping T which has no fixed point and we can also find a weakly null approximated fixed point sequence $\{x_n\}$ of T in K. We consider the set

$$[W] = \{[z_n] \in [K] : \|[z_n] - [x_n]\| \leq 1 - t \text{ and } \limsup_n \limsup_m \|z_n - z_m\| \leq t\}$$

where $t = 1/(1 + a)$. It is easy to check that $[W]$ is a closed,convex and $[T]$-invariant set. Furthermore $[W]$ is non-empty because it contains $[tx_n]$. Therefore, from Lemma 1.5 we know that

$$\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = 1$$

for every $x \in K$. We take $[z_n] \in [W]$ and choose a weakly convergent subsequence $\{y_n\}$ of $\{z_n\}$ such that $\limsup \|z_n\| = \lim \|y_n\|$ and $\lim_{n,m\,;n \neq m} \|y_n - y_m\|$ exists. In this way we have

$$\lim_{n,m\,;n \neq m} \|y_n - y_m\| = \limsup_n \limsup_m \|y_n - y_m\| \leq \limsup_n \limsup_m \|z_n - z_m\| \leq t.$$

We denote the weak limit of $\{y_n\}$ as y. For every $n \in \mathbb{N}$ we have $\|y_n - y\| \leq \liminf_m \|y_n - y_m\|$. Hence

$$\limsup_n \|y_n - y\| = \limsup_n \limsup_m \|y_n - y_m\| \leq t.$$

A positive η can be chosen such that $\eta R(a, X) < 1 - R(a, X)/(1 + a)$. For a large enough n we have $\|y_n - y\| \leq t + \eta$. Furthermore $\|y\| \leq \liminf \|y_n - x_n\| \leq 1 - t$. Hence

$$\left\| \frac{y_n}{t + \eta} \right\| = \left\| \frac{y_n - y}{t + \eta} + \frac{y}{t + \eta} \right\| \leq R\left(\frac{1 - t}{t}, X \right) = R(a, X).$$

Thus $\limsup \|z_n\| = \lim \|y_n\| \leq R(a, X)(t + \eta) < 1$ which is contradiction with Lemma 1.5. □

The following stability result, similar to those in Theorems 1.1 and 1.4, can be proved by a straightforward argument:

Theorem 2.3. *Let X and Y be isomorphic Banach spaces. Then*

$$R(a, Y) \leq d(X, Y) R(a, X)$$

for every nonnegative number a.

Definition 2.4. Let X be a Banach space. We define the coefficient $M(X)$ as

$$\sup\left\{\frac{1+a}{R(a, X)} : a \geq 0\right\}.$$

The following result is a direct consequence of Theorems 2.2 and 2.3:

Theorem 2.5. *Let X be a Banach space. If $M(X) > 1$ then X has the f.p.p. If Y is another Banach space which is isomorphic to X and $d(X, Y) < M(X)$ then Y has the f.p.p.*

Remarks. (a) From Theorem 1.2 it is clear that $R(0, X) = 1/WCS(X)$. Thus $M(X) \geq WCS(X)$. This inequality can be strict. For instance, we consider Bynum's space $X = \ell_{2,\infty}$, that is, X is ℓ_2 with the norm $|x| = \max\{\|x^+\|, \|x^-\|\}$ where $x^+(n) = \max\{x(n), 0\}$ and $x^-(n) = \max\{-x(n), 0\}$ are respectively the positive and the negative part of x, and $\|\cdot\|$ is the euclidean norm. Since $\ell_{2,\infty}$ fails to have normal structure [By2] we know that $WCS(\ell_{2,\infty}) = 1$. However we shall prove in Section 4 that $M(\ell_{2,\infty}) = \sqrt{2}$.

(b) Theorem 2.4 is also a strict improvement of the result in [Ga1]. Indeed, consider $X = \ell_{2,1}$, that is, ℓ_2 with the norm

$$|x| = \|x^+\| + \|x^-\|.$$

This space has normal structure [By2, DLX], so $M(X) \geq WCS(X) = \sqrt{2}$. However, considering the vector $x = e_1$ and the sequence $x_n = -e_{n+1}$ it is clear that $R(\ell_{2,1}) = 2$.

3. Lower bounds for $M(X)$.

Since $M(X) \geq WCS(X)$, all lower bounds for $WCS(X)$ based upon the Clarkson modulus of convexity, the modulus of near uniform convexity and the modulus of uniform smoothness (see [By1, Pr3, DL]) also hold for $M(X)$. We shall give in this section several new bounds which do not longer hold for $WCS(X)$.

Theorem 3.1. *Let X be a reflexive Banach space and denote*

$$\Gamma = \inf\left\{1 + \Gamma(s) - \frac{s}{2} : s \in [0,1]\right\}.$$

Then $R(a,X) \leq 1 + a\Gamma$ if $a \leq 2$ and $R(a,X) \leq a + 2\Gamma - 1$ if $a \geq 2$. In particular, $M(X) \geq 3/(1+2\Gamma)$ and $M(X) > 1$ if $\Gamma'(0) < 1/2$.

Proof. The statement is obvious if $a = 0$. Assume $2 \geq a > 0$. Let $\{x_n\}$ be a weakly null sequence in B_X and $x \in X$ be a vector such that $\|x\| = r \leq a$. Taking subsequences we can assume that $\lim \|x_n + x\|$ exists. For an arbitrary positive number η, a number $t \in [0, r/2]$ can be chosen such that

$$1 - \frac{t}{r} + \Gamma\left(\frac{2t}{r}\right) < \Gamma + \eta.$$

With these assumptions we have

$$\|x + x_n\| = r\left\|\frac{x}{r} + \frac{x_n}{r}\right\| \leq r\left\|\frac{x}{r} + \frac{t}{r}x_n\right\| + (1-t).$$

If $\{x_n\} \to 0$ it is clear that

$$\liminf \|x + x_n\| \leq r\left(1 + \Gamma\left(\frac{2t}{r}\right)\right) + (1-t) \leq r\eta + r\Gamma + 1 \leq 1 + a\Gamma + a\eta.$$

(Note that $\Gamma(t) \geq 0$ implies $\Gamma \geq 1/2 > 0$ and thus $r\Gamma \leq a\Gamma$). If $\{x_n\}$ does not converge to zero, we can assume that the sequence $\{y_n\}$ defined by $y_1 = x$, $y_n = x_{n-1}$ for $n > 1$ is a basic sequence with arbitrary basic constant $c > 1$ (see [LT, page 5]). Hence, we have

$$\|x + tx_n\| = \frac{1}{2}\|2x + 2tx_n\| \leq \frac{1}{2}(\|x\| + \|x + 2tx_n\|) \leq \frac{1}{2}(c\|x - 2tx_n\| + \|x + 2tx_n\|).$$

Taking again subsequences we can assume

$$\frac{1}{2}\left(\left\|\frac{x}{r} + \frac{2t}{r}x_n\right\| + \left\|\frac{x}{r} - \frac{2t}{r}x_n\right\|\right) - 1 \leq \Gamma\left(\frac{2t}{r}\right) + \eta.$$

Thus

$$\|x + x_n\| \le \frac{rc}{2}\left(\left\|\frac{x}{r} - \frac{2t}{r}x_n\right\| + \left\|\frac{x}{r} + \frac{2t}{r}x_n\right\|\right) + (1-t) \le$$

$$rc\left[1 + \Gamma\left(\frac{2t}{r}\right) + \eta\right] + (1-t) \le c\left[r\left(1 + \Gamma\left(\frac{2t}{r}\right) + \eta - \frac{t}{r}\right) + 1\right] + (c-1)t \le$$

$$c(1 + r\Gamma + 2r\eta) + (c-1)a \le c(1 + a\Gamma + 2a\eta) + (c-1)a.$$

Hence

$$R(a, X) \le c(1 + a\Gamma + 2a\eta) + (c-1)a.$$

Since $c > 1$ and $\eta > 0$ are arbitrary we obtain $R(a, X) \le 1 + a\Gamma$. If $a \ge 2$ we have

$$\|x + x_n\| \le (r-2) + \left\|\frac{2x}{r} + x_n\right\|.$$

Applying the above result for the sequence $2x/r + x_n$ we have $R(a, X) \le (a-2) + 1 + 2\Gamma = a - 1 + 2\Gamma$. Taking $a = 2$ we obtain $M(X) \ge 3/(1+2\Gamma)$. Finally, if $\Gamma'(0) < 1/2$ it is clear that $\Gamma < 1$. □

We have not used in the proof of Theorem 3.1 the condition $\lim \|x_n - x_m\| \le 1$. This condition lets us improve the result:

Theorem 3.2. *Let X be a reflexive Banach space and denote*

$$\Gamma' = \inf\left\{1 + \Gamma(s) - \frac{sWCS(X)}{2} : s \in [0,1]\right\}.$$

Then $M(X) \ge 3/(1+2\Gamma')$. In particular, $M(X) > 1$ if $\Gamma'(0) < WCS(X)/2$.

Proof. We use the same arguments as those in the proof of Theorem 3.1, noting that the condition $\lim_{n,m\,;n\neq m} \|x_n - x_m\| \le 1$ lets us assume $\lim \|x_n\| \le 1/WCS(X)$. □

It is an open question for us if $1/\Gamma$ is a lower bound for $M(X)$ in a similar way as the lower bound for $WCS(X)$ obtained in [Pr3], using the modulus of uniform smoothness.

Theorem 3.3. *Let X be a reflexive Banach space. If $c_0 \in (0,1)$ satisfies $r_{X^\star}(c_0) > 0$ then*

$$R(a,X) \le \max\left\{1 + ac_0, a + \frac{1}{1+r_{X^\star}(c_0)}\right\}.$$

In particular $R(a,X) < 1+a$ and $M(X) > 1$ if $r_{X^\star}(1) > 0$.

Proof. We assume that $\{x_n\}$ is a weakly null sequence in B_X and that $x \in X$ satisfies $\|x\| \le a$. We choose $z_n^\star \in S_{X^\star}$ such that $z_n^\star(x+x_n) = \|x+x_n\|$. Taking subsequences we can assume that $\{z_n^\star\}$ is weakly convergent to a point, say $z^\star$, and that $\lim \|z_n^\star - z^\star\| = d$ exists. Let $\|z^\star\| = c$. If $c \le c_0$ we have

$$\liminf \|x+x_n\| = \liminf z_n^\star(x+x_n) \le z^\star(x) + \liminf z_n^\star(x_n) \le ca+1 \le 1+ac_0.$$

If $c > c_0$ we claim that $d \le 1/(1+r_{X^\star}(c))$. Indeed, if $d > 1/(1+r_{X^\star}(c))$ we can choose $\alpha > 1$ satisfying $1/d < \alpha < 1+r_{X^\star}(c)$. Since

$$\|\alpha(z_n^\star - z^\star)\| > \left\|\frac{z_n^\star - z^\star}{d}\right\|$$

we have $\lim \|\alpha(z_n^\star - z^\star)\| \ge 1$. Thus

$$\alpha = \|\alpha z_n^\star\| = \|\alpha(z_n^\star - z^\star) + \alpha z^\star\| = \lim \|\alpha(z_n^\star - z^\star) + \alpha z^\star\| \ge 1 + r_{X^\star}(\alpha c) \ge 1 + r_{X^\star}(c)$$

which is a contradiction. So we have $d \le 1/(1+r_{X^\star}(c))$ and

$$\liminf \|x+x_n\| = \lim z^\star(x+x_n) + \liminf (z_n^\star - z^\star)(x+x_n) =$$

$$z^\star(x) + \liminf (z_n^\star - z^\star)x_n \le a + \frac{1}{1+r_{X^\star}(c)} \le a + \frac{1}{1+r_{X^\star}(c_0)}.$$

Finally, if $r_{X^\star}(1) < 0$ the continuity of the Opial modulus (see[LTX]) implies that there exists $c_0 < 1$ such that $r_{X^\star}(c_0) < 1$ which implies $R(a,X) < 1$ and $M(X) > 1$. □

Remark. The space $\ell_{2,\infty}$, as the dual space of $\ell_{2,1}$, again provides us an example where Theorem 3.3 assures $M(\ell_{2,\infty}) > 1$. Indeed, let $\{x_n\}$ be a weakly null sequence in $\ell_{2,1}$ such that $\lim \|x_n\| \geq 1$ and $x \in \ell_{2,1}$ be a vector satisfying $\|x\| \geq c$. By standard arguments we suppose $\operatorname{supp} x \cap \operatorname{supp} x_n = \emptyset$. Thus

$$|x+x_n| = \|x^+ + x_n^+\| + \|x^- + x_n^-\| = \sqrt{\|x^+\|^2 + \|x_n^+\|^2} + \sqrt{\|x^-\|^2 + \|x_n^-\|^2} \geq$$

$$\sqrt{(\|x^+\| + \|x^-\|)^2 + (\|x_n^+\| + \|x_n^-\|)^2} = \sqrt{|x|^2 + |x_n|^2} \geq \sqrt{c^2 + |x_n|^2}$$

where we have used the inequality

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a+c)^2 + (b+d)^2}$$

which holds for every nonnegative numbers a, b, c, d. Thus

$$r_{\ell_{2,1}}(c) = \sqrt{1 + c^2} - 1.$$

4. Stability in ℓ_p-spaces.

In this section we are going to compute $M(X)$ when X is either the space ℓ_p or Bynum's space $\ell_{p,\infty}$. We will obtain stability results which are strict improvements of those which appear in [JL, Pr2, Kh, BS]. We recall the definition of Bynum's space $\ell_{p,\infty}$. Assume $1 \leq p < \infty$. The space $\ell_{p,\infty}$ is ℓ_p with the norm

$$|x| = \max\{\|x^+\|, \|x^-\|\}$$

where $\|\cdot\|$ is the usual norm in ℓ_p.

Theorem 4.1. *Let p be a number in $(1, \infty)$. Then*

(a) $$R(a, \ell_p) = \left(a^p + \tfrac{1}{2}\right)^{\frac{1}{p}}$$

for every nonnegative a and

(b) $$M(\ell_p) = \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.$$

$$R(a, \ell_{p,\infty}) = (1 + a^p)^{\frac{1}{p}}$$

for every nonnegative a and $M(\ell_{p,\infty}) = 2^{\frac{p-1}{p}}$.

Proof. Let X be the space ℓ_p. We consider a weakly null sequence $\{x_n\}$ in B_X such that $\lim_{n,m\,;n\neq m}\|x_n - x_m\| \leq 1$ and a vector $x \in X$ such that $\|x\| \leq a$. From Theorem 1.2 we know

$$\liminf \|x_n\| \leq \frac{1}{WCS(\ell_p)} = 2^{\frac{-1}{p}}.$$

Let ε be an arbitrary positive number. Using standard arguments we can find a subsequence of $\{x_n\}$, denoted again, $\{x_n\}$, a sequence $\{y_n\} \in B_X$ and a vector $y \in B_X$ such that $|\|x\| - \|y\|| < \varepsilon$, $\lim\|x_n - y_n\| = 0$ and $\operatorname{supp} y_n \cap \operatorname{supp} y = \emptyset$ for every $n \in \mathbb{N}$. Therefore

$$\lim\|y + y_n\|^p = \|y\|^p + \lim\|y_n\|^p \leq (a+\varepsilon)^p + \tfrac{1}{2}.$$

Taking limits

$$\liminf\|x + x_n\| \leq \liminf\|y + y_n\| + \varepsilon \leq \left((a+\varepsilon)^p + \tfrac{1}{2}\right)^{\frac{1}{p}} + \varepsilon.$$

Since ε is arbitrary we obtain

$$R(a, X) \leq \left(a^p + \tfrac{1}{2}\right)^{\frac{1}{p}}.$$

Considering the vector ae_1 and the sequence $x_n = 2^{-1/p}e_{n+1}$, it is easy to check that $\|x_n - x_m\| = 1$ and $\|x_n + x\| = (a^p + \frac{1}{2})^{1/p}$.

By elementary calculus we can see that the function $(1+a)/(a^p + 2^{-1})^{1/p}$ attains its maximum at $a = 2^{1/(1-p)}$ where the corresponding value is

$$M(X) = \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.$$

Finally we consider $X = \ell_{p,\infty}$. We can use similar arguments as above. Therefore we obtain $\|y^+\| \leq a + \varepsilon$, $\|y^-\| \leq a + \varepsilon$, $\lim\|y_n^+\| \leq 1$ and $\|y_n^-\| \leq 1$. Thus

$$\lim\|(y + y_n)^+\| = \lim\|y^+ + y_n^+\| \leq (1 + (a+\varepsilon)^p)^{\frac{1}{p}}$$

and the same inequality for $\lim\|(y + y_n)^-\|$. Hence $R(a, X) \leq (1 + a^p)^{1/p}$. The sequence $x_n = e_{n+1}$ and the point $x = e_1$ show that $R(a, X) = (1 + a^p)^{1/p}$ which implies $M(X) = 2^{(p-1)/p}$. □

Remark. The above method can be applied to compute $M(c_0) = 2$. But this value can also be obtained noting that $M(X) \geq 2/R(X)$ and $R(c_0) = 1$ (see [Ga2]).

Corollary 4.2. *Let X be a Banach space isomorphic to ℓ_2. If $d(X,\ell_2) < \sqrt{3}$, then X has the f.p.p.*

Remark. The best bound, until we know, for a stability result in ℓ_2 was $\sqrt{(3+\sqrt{5})/2}$ [JL] that is obviously smaller than $\sqrt{3}$.

Corollary 4.3. *Let X be a Banach space isomorphic to ℓ_p, $1 < p < \infty$. If*

$$d(X,\ell_p) < \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}$$

then X has the f.p.p.

Remark. Note that

$$\left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \geq \max\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}.$$

Thus the result in Corollary 4.3 is better than the stability result in [BS]. Furthermore, this bound is greater than that obtained in [Kh,Pr2] for the stability of the f.p.p. with respect to ℓ_p-spaces. Indeed, in these papers it is proved that X has the f.p.p. if $d(X,\ell_p) < c_p$ where c_p^p is the maximum value of the function $(1+x^p)/((1-x)^p + x^p)$ in [0,1]. We shall prove that c_p is strictly smaller than the constant obtained in Corollary 4.3. To this end, denote $a = (1+2^{1/(p-1)})^{p-1}$. Since $1 + 2^{1/(p-1)} > 2^{1/(p-1)}$ we have $a > 2$ and this inequality implies $a/(a-1) < 2$. We claim that $(1+x^p)/((1-x)^p + x^p) < a$ for every $x \in [0,1]$ and this inequality will prove the result because it implies $c_p^p < a$ and so

$$c_p < \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.$$

To prove this claim we only need elemental calculus. The wanted inequality is equivalent to

$$1 < (1-x)^p a + (a-1)x^p =: f(x)$$

and it is not difficult to compute the minimal value of f in [0,1]. Indeed, f' only vanishes if

$$x = x_0 = \frac{\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}}{1 + \left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}}.$$

Since $f''(x) > 0$ for every $x \in [0,1]$, the function f attains a minimum at x_0. The value of $f(x_0)$ is

$$\frac{a + \left(\frac{a}{a-1}\right)^{\frac{p}{p-1}}(a-1)}{\left(1+\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}\right)^p} = \frac{a\left(1+\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}\right)}{\left(1+\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}\right)^p} = \frac{a}{\left(1+\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}}\right)^{p-1}}.$$

Since

$$\left(\frac{a}{a-1}\right)^{\frac{1}{p-1}} < 2^{\frac{1}{p-1}}$$

we have

$$f(x_0) > \frac{a}{\left(1+2^{\frac{1}{p-1}}\right)^{p-1}} = 1.$$

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