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ON UNIVERSALITY OF COMPOSITION OPERATORS IN SEVERAL VARIABLES

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Abstract. In this paper we characterize the universality of a sequence of composition operators generated by automorphisms of the N-dimensional unit polydisc $\mathbf{D}_N$, on Hardy spaces of $\mathbf{D}_N$. In addition, we provide suitable conditions for the universality of partial derivative-composition operators in certain spaces X of holomorphic functions in $\mathbf{D}_N$ and the N-dimensional unit ball. Our theorems improve or extend earlier ones due to Bourdon and Shapiro, Herzog, León, Bernal and the authors, among others.

1. Introduction

It is well known that the group $\text{Aut}(\mathbf{D})$ of automorphisms of the unit disc $\mathbf{D}$ is the set of Möbius transformations $\{\sigma_{a,k} : \ |a| < 1 = |k|\}$, where $\sigma_{a,k}(z) = k \cdot \frac{z-a}{1-\bar{a}z}$. In 1941 W. Seidel and J. L. Walsh [14] established the existence of a function $f \in H(\mathbf{D})$ such that, given a simply connected domain $G \subset \mathbf{D}$ and a function $g \in H(G)$, there is a sequence $\{a_n\}_1^\infty \subset \mathbf{D}$ depending on g such that $f \circ \sigma_{a_n,1} \to g$ $(n \to \infty)$ in $H(G)$. This result is in turn a non–Euclidean version of Birkhoff's theorem about density of translates of certain entire functions [4]. In 1995 L. Bernal-González and A. Montes–Rodríguez [3] extended Seidel–Walsh's theorem by showing that if $\{S_n = \sigma_{a_n,k_n} : \ n \in \mathbf{N}\} \subset \text{Aut}(\mathbf{D})$, then the set $\{f \in H(\mathbf{D}) : \ \{f \circ S_n\}$ is dense in $H(\mathbf{D})\}$ is not empty if and only if

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it is residual if and only if $\limsup_{n\to\infty}|a_n| = 1$ if and only if the action of $\{S_n\}_1^\infty$ is properly discontinuous on $\mathbf{D}$ (i.e., given a compact subset $K \subset \mathbf{D}$ there exists $m = m(K) \in \mathbf{N}$ such that $K \cap S_m(K) = \emptyset$; the same definition is kept for any domain $G \subset \mathbf{C}^N$). In 1995 G. Herzog [11] proved the following "Seidel–Walsh theorem for derivatives": If X is a Banach space of holomorphic functions on $\mathbf{D}$ with $A(\mathbf{D}) := C(\overline{\mathbf{D}}) \cap H(\mathbf{D}) \subset X$ such that convergence in X implies compact convergence on $\mathbf{D}$ and polynomials are dense in X, then for every sequence $\{a_n\}_n \subset \mathbf{D}$ with $|a_n| \to 1$ $(n \to \infty)$ the set $\{f \in X : \{f' \circ \sigma_{a_n,1} : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})\}$ is a residual subset of X. Trivially, the expression $f' \circ \sigma_{a_n,1}$ cannot be changed to $f \circ \sigma_{a_n,1}$ (just take $X = A(\mathbf{D})$).

In 1999 Bernal-González and the second author [2] extended Herzog's result to an operator of the form $\Phi(D)$, where D is the differentiation operator $(Df = f')$ and Φ is a non-constant polynomial, and in fact to a C–bounded sequence of polynomials. A sequence $\{\Phi_n(z) = \sum_{j=0}^N b_j^{(n)} z^j\}_1^\infty$ of polynomials of the same degree $N \in \mathbf{N}$ is *C–bounded* whenever each sequence $\{b_j^{(n)} : n \in \mathbf{N}\}$ $(j = 0, 1, 2, \dots, N)$ is bounded and there exists a positive constant α such that $|b_N^{(n)}| \geq \alpha$ for all $n \in \mathbf{N}$. In [2] the following is shown: If X is a Fréchet space (in fact, it holds for an F–space) of holomorphic functions in $\mathbf{D}$ with $A(\mathbf{D}) \subset X$ such that convergence in X implies compact convergence on $\mathbf{D}$ and polynomials are dense in X, and if $\{S_n\}_1^\infty \subset \mathrm{Aut}(\mathbf{D})$ and $\{\Phi_n\}_1^\infty$ is a C–bounded sequence of polynomials, then the set $\{f \in X : \{(\Phi_n(D)f) \circ S_n : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})\}$ is a residual subset of X if and only if it is not empty if and only if the action of $\{S_n\}_1^\infty$ is properly discontinuous on $\mathbf{D}$.

If $1 \leq p < \infty$, the Hardy space $H^p(\mathbf{D})$ is defined as $H^p(\mathbf{D}) = \{f \in H(\mathbf{D}) : \|f\|_p < \infty\}$, where $\|f\|_p = \sup\limits_{0<r<1} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$. It becomes a Banach space if it is endowed with this norm. $H^\infty(\mathbf{D})$ is the space of all $f \in H(\mathbf{D})$ which are bounded on $\mathbf{D}$. It becomes a Banach space when endowed with $\|\cdot\|_\infty$. For $N \in \mathbf{N}$, denote the higher order Hardy space $H_N^p(\mathbf{D}) = \{f \in H(\mathbf{D}) : f^{(N)} \in H^p(\mathbf{D})\}$, which becomes a Banach space whenever it is endowed the norm $\|f\| = \|f^{(N)}\|_p + \sum_{j=0}^{N-1} \|f^{(j)}\|_\infty$.

In 1997 Bourdon and Shapiro consider $\varphi \in \mathrm{Aut}(\mathbf{D})$ and $S_n = \varphi \circ \cdots \circ \varphi$ (n times), and prove that the set $\mathcal{U} = \{f \in H^p(\mathbf{D}) : \{f \circ S_n : n \in \mathbf{N}\}$ is dense in $H^p(\mathbf{D})\}$ is residual in $H^p(\mathbf{D})$ if and only if it is not empty if and only if φ has no fixed point in $\mathbf{D}$ ([5, Theorem 2.3 and Proposition 0.1], see also [15, Chap.7]). Recently, Bernal-González and the authors [1] extended this classical result in two ways: to the derivative operator of order N from $H_N^p(\mathbf{D})$ to $H^p(\mathbf{D})$, and to the unit ball $\mathbf{B}_N$ in $\mathbf{C}^N$, that is, $\mathbf{B}_N = \{z = (z_1, \ldots, z_N) \in \mathbf{C}^N : \sum_{j=1}^N |z_j|^2 < 1\}$. They also dropped the condition $A(\mathbf{D}) \subset X$ in Herzog's result to be $H_N^p(\mathbf{D}) \subset X$ whenever $\{\Phi_n\}_n$ is a C-bounded sequence of polynomials of degree N, and they study the problem when $X = H^\infty(\mathbf{D})$. We stress that here the polynomials are not dense.

In 1998 F. León-Saavedra [13] proves that if $\{S_n\}_n \in \mathrm{Aut}(\mathbf{B}_N)$, then there exists $f \in H(\mathbf{B}_N)$ such that $\{f \circ S_n : n \in \mathbf{N}\}$ is dense in $H(\mathbf{B}_N)$ if and only if the action of $\{S_n\}_n$ is properly discontinuous on $\mathbf{B}_N$ [13, Theorem 5] (see also [6, Theorem 3]). He also provides an analogous statement for the unit polydisc $\mathbf{D}_N = \{z = (z_1, \ldots, z_N) \in \mathbf{C}^N : |z_j| < 1 \; j = 1, \ldots, N)\}$ [13, Theorem 6].

All above results about density can be expressed in the language of universality. If X and Y are topological spaces, a sequence $T_n : X \to Y$ $(n \in \mathbf{N})$ of continuous mappings is called *universal* whenever the set $\mathcal{U}$ of elements $x \in X$ such that the orbit $\{T_n x\}_1^\infty$ is dense in Y is not empty. Each element of $\mathcal{U}$ is said to be *universal* for $\{T_n\}_1^\infty$. If X and Y are topological vector spaces and each T_n is linear then the words *hypercyclic* and *universal* are synonymous. See [10] for a good up-to-day survey about these topics.

In this paper, we extend the Herzog's result to several variables, changing $\mathbf{D}$ to $\mathbf{D}_N$ or $\mathbf{B}_N$. We also provide a generalization of Bourdon-Shapiro's theorem to $\mathbf{D}_N$. From the above statements we improve Theorem 5-6 of [13] and, simultaneously, extend the results which are mentioned along the Introduction.

2. Preliminary results

In this section we include all the auxiliary statements which provide us the desired research about universality in $\mathbf{D}_N$ and $\mathbf{B}_N$. Firstly, the sufficient condition about universality can be found in [9, Satz 1.2.2 and Satz 1.4.2] (see also [10, Proposition 6]).

Theorem 2.1. *Let X, Y be metrizable topological vector spaces with X complete and Y separable, and let $\Lambda = \{L_n\}_1^\infty$ be a sequence of continuous linear operators from X to Y. Then the following statements are equivalent:*

(a) *The set of Λ-universal elements is a residual subset of X.*
(b) *The set of Λ-universal elements is a dense subset of X.*
(c) *The set $\{(x, L_n(x)) : x \in X, n \in \mathbf{N}\}$ is dense in $X \times Y$.*

If, in addition, there is a dense subset C of X such that $\lim_{n\to\infty} L_n(x)$ exists for all $x \in C$, then (a), (b) and (c) are equivalent to

(d) *The set of Λ-universal elements is not empty.*

If π is a permutation of $\{1,\dots,N\}$ and $I_{\pi(j)}$ is the identity representation of the unit disc $\mathbf{D}_j$ onto $\mathbf{D}_{\pi(j)}$, we denote by I_π the automorphism of $\mathbf{D}_N$ given by $(I_{\pi(1)},\dots,I_{\pi(N)})$. It is known that the set of automorphisms of $\mathbf{D}_N$ is $\mathrm{Aut}(\mathbf{D}_N) := \{\varphi\circ I_\pi :\ \pi \text{ is a permutation of } \{1,\dots,N\} \text{ and each component of } \varphi \text{ is in } \mathrm{Aut}(\mathbf{D})\}$. From this the following lemma is obvious.

Lemma 2.2. *Let* $\{\varphi_n = (\varphi_{1,n},\dots,\varphi_{N,n})\circ I_{\pi_n}\}_n \subset Aut(\mathbf{D}_N)$. *Then* $\{\varphi_n\}_n$ *is properly discontinuous on* $\mathbf{D}_N$ *if and only if* $\{\varphi_{j,n}\}_n$ *is properly discontinuous on* $\mathbf{D}$ *for some* $j\in\{1,\dots,N\}$.

It is known that for any sequence $\{\sigma_n\}_n$ of automorphisms on $\mathbf{D}$ there exist a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ and a function $S\in H(\mathbf{D})$ such that $\sigma_{n_k}\to S$ $(k\to\infty)$ on $H(\mathbf{D})$. In particular, the case $S =$ a constant function $=\gamma$, with $\gamma\in\partial\mathbf{D}$ ($\equiv$ the boundary of $\mathbf{D}$) is possible if and only if $\{\sigma_n\}_n$ is properly discontinuous on $\mathbf{D}$. Then we have $\sigma_{n_k}(z)\to\gamma$ $(k\to\infty)$ pointwise in $\overline{\mathbf{D}}\setminus\{\gamma\}$. In the next lemma we provide an analogous result for the unit polydisc.

Lemma 2.3. *Let* $\{\varphi_n = (\varphi_{1,n},\dots,\varphi_{N,n})\circ I_{\pi_n}\}_n \subset Aut(\mathbf{D}_N)$. *Then there exist holomorphic functions* $\{S_j\}_{j=1}^N \subset H(\mathbf{D})$, *a permutation* π *of* $\{1,\dots,N\}$ *and a subsequence* $\{n_k\}_k$ *of* $\mathbf{N}$ *such that*

$$\varphi_{n_k}\to (S_1,\dots,S_N)\circ I_\pi \qquad \text{on } H(\mathbf{D}_N) \qquad (k\to\infty).$$

Proof. Let $j=1$. Since $\{\varphi_{1,n}\}_n$ is a sequence of automorphisms on $\mathbf{D}$, there exist a subsequence $\{m_{1,k}:\ k\in\mathbf{N}\}$ of positive integers and a holomorphic function S_1 in $\mathbf{D}$ such that

$$\varphi_{1,m_{1,k}}(z)\to S_1(z) \qquad \text{on } H(\mathbf{D}) \qquad (k\to\infty).$$

By taking a new subsequence, if it is necessary, there exists $i_1\in\{1,\dots,N\}$ such that

$$\pi_{m_{1,k}}(1)=i_1 \qquad \text{for all } k\in\mathbf{N}.$$

But the sequence $\{\varphi_{2,m_{1,k}}(z)\}_k$ is also a sequence of automorphisms in $\mathbf{D}$, so there exist S_2 an holomorphic function in $\mathbf{D}$ and a subsequence $\{m_{2,k}:\ k\in\mathbf{N}\}$ of $\{m_{1,k}:\ k\in\mathbf{N}\}$ such that

$$\varphi_{2,m_{2,k}}(z)\to S_2(z) \qquad \text{on } H(\mathbf{D}) \qquad (k\to\infty).$$

As before, consider a new subsequence if it is necessary, there exists $i_2\in\{1,\dots,N\}$ such that $\pi_{m_{2,k}}(2)=i_2$. And it is obvious that we have $i_1\neq i_2$.

Continuing this process gives after finitely many steps a sequence $\{n_k=m_{N,k}:\ k\in\mathbf{N}\}$ of positive integers, a finite sequence $\{S_j\}_{j=1}^N$ of holomorphic functions in $\mathbf{D}$ and $i_1, \dots i_N \in \{1, \dots, N\}$ different each to other them such that

$$\varphi_{j,n_k}(z) \to S_j(z) \quad \text{on } H(\mathbf{D}) \qquad (k \to \infty),$$

$$\pi_{n_k}(j) = i_j \qquad \text{for all } k \in \mathbf{N},$$

for any $j \in \{1, \dots N\}$. If we now define π as the permutation of $\{1, \dots, N\}$ such that $\pi(j) = i_j$, then the conclusion of the lemma is evident. □

We can give a related statement for the unit ball $\mathbf{B}_N$. Recall that the set of automorphisms of $\mathbf{B}_N$ is $\mathrm{Aut}(\mathbf{B}_N) = \{M \circ \varphi_a : M$ is a unitary transformation of $\mathbf{C}^N$ and $a \in \mathbf{B}_N\}$, where $\varphi_a(z) = \frac{a - P_a(z) - (1-|a|^2)^{1/2} Q_a(z)}{1 - <z,a>}$, with $P_a(z) = \frac{<z,a>}{|a|^2} \cdot a$ and $Q_a = I - P_a$ (see [7]). In the following $\mathbf{S}_N$ denote the unit sphere in $\mathbf{C}^N$.

Lemma 2.4. *Let $\{\varphi_n\}_n \subset Aut(\mathbf{B}_N)$. If $\{\varphi_n\}_n$ is properly discontinuous on $\mathbf{B}_N$ then there exist a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and a point $\gamma \in \mathbf{S}_N$ such that*

$$\varphi_{n_k} \to \gamma \quad \text{on } H(\mathbf{B}_N) \qquad (k \to \infty).$$

The proof is analogous to that one we can find as part of the proof of [1, Theorem 4.3]. In fact they provide two points $\gamma, \gamma_1 \in \mathbf{S}_N$ such that $\varphi_{n_k} \to \gamma$ pointwise in $\overline{\mathbf{B}_N} \setminus \{\gamma_1\}$ $(k \to \infty)$.

We conclude this section giving a left-inverse operator for the partial derivative operator ∂^α in $H(G)$, where G is a polydisc or a ball. We denote $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and, given $\alpha = (\alpha_1, \dots, \alpha_N)$ and $\beta = (\beta_1, \dots, \beta_N)$, we write $\beta \leq \alpha$ whenever $\beta_j \leq \alpha_j$ for any $j \in \{1, \dots, N\}$.

Lemma 2.5. *Let $G \subset \mathbf{C}^N$ be a polydisc or a ball with center $(a_1, \dots, a_N)$, $f \in H(G)$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}_0^N \setminus \{0\}$. Define, for any $z = (z_1, \dots, z_N) \in G$, $\partial^{-\alpha} f(z)$ as*

$$\partial^{-\alpha} f(z) = \int_{a_N}^{z_N} \cdots \int_{a_1}^{z_1} \frac{(z_1 - \xi_1)^{\alpha_1 - 1} \cdots (z_N - \xi_N)^{\alpha_N - 1}}{(\alpha_1 - 1)! \cdots (\alpha_N - 1)!} f(\xi_1, \dots, \xi_N) d\xi_1 \cdots d\xi_N,$$

if $\alpha \in \mathbf{N}^N$ or as the analogous expression without the corresponding term in z_j if $\alpha_j = 0$ for some j, where the integration is taken along the segments $[a_j, z_j]$. We set $\partial^0 f = f$. Then $\partial^{-\alpha}$ is well-defined for every α, $\partial^{-\alpha} f \in H(G)$ and

$$\partial^\beta(\partial^{-\alpha} f) = \partial^{-\alpha+\beta} f \qquad \text{for any } \beta \leq \alpha.$$

3. A Bourdon-Shapiro theorem for the unit polydisc

In the following, let $\mathbf{T}^N$ denote the distinguished boundary of $\mathbf{D}_N$ and τ_N the measure on $\mathbf{T}^N$ that is product of normalized Lebesgue measure on the circles $|z_j| = 1$.

For $0 < p < \infty$ the Hardy space on $\mathbf{D}_N$ is defined as

$$H^p(\mathbf{D}_N) := \{f \in H(\mathbf{D}_N) : \ \|f\|_p := \sup_{0<r<1} \left(\int_{\mathbf{T}^N} |f(r\xi)|^p d\tau_N(\xi) \right)^{1/p} < +\infty\}$$

(see [7]). These Hardy spaces are functional Banach spaces for $p \geq 1$. In [12, Proposition 1] it is proved that any automorphism φ of $\mathbf{D}_N$ generates a bounded composition operator $C_\varphi f = f \circ \varphi$ of $H^p(\mathbf{D}_N)$, for $1 \leq p < \infty$. If we take account it we can establish the next "Seidel-Walsh" theorem.

Theorem 3.1. *Let* $\{\varphi_n\}_1^\infty = \{(\varphi_{1,n}, \dots, \varphi_{N,n}) \circ I_{\pi_n}\}_{n=1}^\infty \subset Aut(\mathbf{D}_N)$ *and* $p \in [1, +\infty)$. *Then the set*

$$\mathcal{U} = \{f \in H^p(\mathbf{D}_N) : \ \{f \circ \varphi_n : \ n \in \mathbf{N}\} \text{ is dense in } H^p(\mathbf{D}_N)\}$$

is residual in $H^p(\mathbf{D}_N)$ *if and only if the action of* $\{\varphi_n\}_1^\infty$ *is properly discontinuous on* $\mathbf{D}_N$.

Proof. If $\mathcal{U}$ is residual in $H^p(\mathbf{D}_N)$ then there exists $f \in H(\mathbf{D}_N)$ such that its orbit $\{f \circ \varphi_n : \ n \in \mathbf{N}\}$ is dense in $H(\mathbf{D}_N)$, because convergence in $H^p(\mathbf{D}_N)$ implies compact convergence. Now, we only need apply [13, Proposition 1].

As for the converse, we will use Theorem 2.1. Recall that the set of polynomials is dense in $H^p(\mathbf{D}_N)$. Fix two polynomials $p(z)$, $q(z)$ and a number $\varepsilon > 0$. We should find a function $g \in H^p(\mathbf{D}_N)$ and a positive integer n_0 such that $\|p-g\|_p < \varepsilon$ and $\|q - (g \circ \varphi_{n_0})\|_p < \varepsilon$.

From the fact $\{\varphi_n\}_{n=1}^\infty$ is properly discontinuous on $\mathbf{D}_N$ and by Lemma 2.2, we can suppose for the sake of a simpler notation, without loss of generality, that there is a point $\gamma_1 \in \partial\mathbf{D}$ such that $\varphi_{1,n} \to \gamma_1$ $(n \to \infty)$ pointwise in $\overline{\mathbf{D}} \setminus \{\gamma_1\}$, and that $\pi_n(1) = 1$ for all $n \in \mathbf{N}$.

Consider the "peak function" for γ_1 defined as

$$a_1(w) = \frac{1 + \overline{\gamma_1} w}{2} \qquad (w \in \mathbf{C}).$$

This is a peak–function at γ_1 for $\overline{\mathbf{D}}$ in the sense that $a_1(w)$ is continuous on $\overline{\mathbf{D}}$, holomorphic in $\mathbf{D}$, $a_1(\gamma_1) = 1$ and $|a_1(w)| < 1$ for all $w \in \overline{\mathbf{D}} \setminus \{\gamma_1\}$ (see [8, p. 189]).

Let $a(z_1, \dots, z_N) := a_1(z_1)$ and choose a positive integer m such that

$$\|a^m\|_p < \frac{\varepsilon}{\|p\|_\infty + \|q\|_\infty}, \tag{1}$$

which is possible because of the Lebesgue Bounded Convergence Theorem. Here, and from now on, we denote $\|\cdot\|_\infty := \|\cdot\|_{\overline{\mathbf{D}_N}}$.

Since $(a\circ\varphi_n)(z_1,\dots,z_N) = a_1(\varphi_{n,1}(z_{\pi_n(1)})) = a_1(\varphi_{n,1}(z_1))$, $a_1(w)$ is continuous on $\overline{\mathbf{D}}$ and $a_1(\gamma_1)=1$, we have that

$$1-[(a\circ\varphi_n)(z)]^m \to 0 \qquad (n\to\infty)$$

for τ_N-almost every point of $\mathbf{T}^N$, because the measure of the set $\{(\gamma_1,z_2,\dots,z_N)\in\mathbf{T}^N\}$ is zero. Again by the Lebesgue Bounded Convergence Theorem, we derive that $\|1-[a(\varphi_n(\cdot))]^m\|_p \to 0$ $(n\to\infty)$, so there is $n_0\in\mathbf{N}$ with

$$\|1-[a(\varphi_{n_0}(\cdot))]^m\|_p < \frac{\varepsilon}{\|p\|_\infty+\|q\|_\infty}. \tag{2}$$

Finally, define

$$g(z) = p(z) + a(z)^m\cdot[q(\varphi_{n_0}^{-1}(z)) - p(z)].$$

Then g is continuous on $\overline{\mathbf{D}_N}$, so $g\in H^p(\mathbf{D}_N)$. We have, from (1) and (2), that

$$\|p-g\|_p = \|a(z)^m\cdot[q(\varphi_{n_0}^{-1}(z)) - p(z)]\|_p < \varepsilon$$

and

$$\begin{aligned}&\|q(z)-(g\circ\varphi_{n_0})(z)\|_p\\ &= \|q(z)-p(\varphi_{n_0}(z)) - a(\varphi_{n_0}(z))^m\cdot[q(z)-p(\varphi_{n_0}(z))]\|_p\\ &= \|(1-[a(\varphi_{n_0}(z))]^m)\cdot(q(z)-p(\varphi_{n_0}(z)))\|_p<\varepsilon.\end{aligned}$$

An application of Theorem 2.1 with $X=Y=H^p(\mathbf{D}_N)$ and $L_n = C_{\varphi_n}$ $(n\in\mathbf{N})$ yields the desired result. □

Remark. If, in the above theorem, we consider $N=1$ and S_n as the nth-iterate of a single $\varphi\in\mathrm{Aut}(\mathbf{D})$ we obtain the Bourdon-Shapiro theorem mentioned in the Introduction.

Remark. It is clear that we also obtain the following result [13, Theorem 6]: Let $\{\varphi_n\}_n \subset \mathrm{Aut}(\mathbf{D}_N)$. Then there exists a function $f\in H(\mathbf{D}_N)$ such that $\{f\circ\varphi_n:\ n\in\mathbf{N}\}$ is dense in $H(\mathbf{D}_N)$ if and only if $\{\varphi_n\}_n$ is properly discontinuous on $\mathbf{D}_N$.

To finish this section we should say that in [1, Theorem 4.3] it is provided an analogous statement to Theorem 3.1 for the unit ball $\mathbf{B}_N$.

4. A generalization of Herzog's theorem to several variables

In the following, given $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}_0^N$, we denote $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$.

Theorem 4.1. *Let X be a F-space of holomorphic functions on $\mathbf{D}_N$ having the following properties:*

(a) *Convergence in X implies compact convergence on $\mathbf{D}_N$.*
(b) *$A(\mathbf{D}_N) \subset X$.*
(c) *The polynomials are dense in X.*

Assume that $\{\varphi_n\}_1^\infty = \{(\varphi_{1,n}, \dots, \varphi_{N,n}) \circ I_{\pi_n}\}_{n=1}^\infty$ is a sequence of automorphisms of $\mathbf{D}_N$ and that $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \geq 1$. Consider the set

$$\mathcal{U} = \{f \in X : \{(\partial^\alpha f) \circ \varphi_n : n \in \mathbf{N}\} \text{ is dense in } H(\mathbf{D}_N)\}.$$

If $j_0 \in \{1, \dots, N\}$ is such that $\alpha_{j_0} \neq 0$ and the action of the j_0-component sequence $\{\varphi_{j_0,n}\}_{n=1}^\infty \subset Aut(\mathbf{D})$ of $\{\varphi_n\}_n$ is properly discontinuous on $\mathbf{D}$, then $\mathcal{U}$ is a residual set of X.

Proof. We may suppose without loss of generality that $j_0 = 1$ and that $J \in \{1, \dots, N\}$ is such that $\alpha_j \neq 0$ for $j = 1, \dots, J$ and $\alpha_j = 0$ for $j = J+1, \dots, N$.

Define the mappings

$$L_n : X \to H(\mathbf{D}_N) \qquad (n \in \mathbf{N})$$

by $L_n f = (\partial^\alpha f) \circ \varphi_n$. Each L_n is linear, and continuous because X satisfies (a). If we prove that the set

$$\mathcal{G} = \{(f, L_n f) : \ f \in X, \ n \in \mathbf{N}\}$$

is dense in $X \times H(\mathbf{D}_N)$, an application of Theorem 2.1 would yield the conclusion.

Let us prove it first in the case $X = A(\mathbf{D}_N)$. Since the polynomials are dense in $A(\mathbf{D}_N)$ and in $H(\mathbf{D}_N)$, it is sufficient to prove that given two polynomials p, q and $\varepsilon, r \in (0,1)$, there exist $g \in A(\mathbf{D}_N)$ and $n_0 \in \mathbf{N}$ such that

$$\|p - g\|_\infty < \varepsilon$$

and

$$\|q - L_{n_0} g\|_{r\overline{\mathbf{D}} \times \cdots \times r\overline{\mathbf{D}}} < \varepsilon.$$

Let $\delta = 1 + \|q\|_\infty + \|\partial^\alpha p\|_\infty$ and choose $m \in \mathbf{N}$ such that

$$m > \frac{2\delta}{\varepsilon}.$$

From the fact $\{\varphi_{1,n}\}_{n \in \mathbf{N}}$ is properly discontinuous on $\mathbf{D}$ we can suppose without loss of generality, by taking a subsequence if necessary, that there is a point

$\gamma_1 \in \partial\mathbf{D}$ such that $\varphi_{1,n} \to \gamma_1$ $(n \to \infty)$ uniformly on compact subsets of $\mathbf{D}$. Hence there exists $n_0 \in \mathbf{N}$ satisfying

$$\|1 - \gamma_1^{-m} \varphi_{1,n_0}(z)^m\|_{r\overline{\mathbf{D}}} \leq \frac{\varepsilon}{2\delta}. \tag{1}$$

Take the function

$$F(z) = z_1^m (q(\varphi_{n_0}^{-1}(z)) - (\partial^\alpha p)(z)) \qquad (z = (z_1, \ldots, z_N)),$$

which is holomorphic in the polydisc $G := |\varphi_{1,n_0}(0)|^{-1}\mathbf{D} \times \cdots \times |\varphi_{N,n_0}(0)|^{-1}\mathbf{D}$. With the notation of Lemma 2.5, define the function

$$h = \frac{1}{\gamma_1^m} \partial^{-\alpha} F$$

on the domain G. Then $h \in H(G)$, so $h \in A(\mathbf{D}_N)$, and for any $z \in \overline{\mathbf{D}} \times \cdots \times \overline{\mathbf{D}}$,

$$|h(z)| = \left| \frac{1}{\gamma_1^m} \partial^{-\alpha} F(z_1, \ldots, z_N) \right| =$$

$$\left| \int_0^{z_J} \cdots \int_0^{z_1} \frac{(z_1 - \xi_1)^{\alpha_1 - 1} \ldots (z_J - \xi_J)^{\alpha_J - 1}}{(\alpha_1 - 1)! \cdots (\alpha_J - 1)!} \xi_1^m \cdot \right.$$

$$\left. (q(\varphi_{n_0}^{-1}(\xi_1, \ldots, \xi_J, z_{J+1}, \ldots, z_N)) - (\partial^\alpha p)(\xi_1, \ldots, \xi_J, z_{J+1}, \ldots, z_N)) d\xi_1 \cdots d\xi_J \right| =$$

$$\left| \int_0^1 \cdots \int_0^1 \frac{z_1^{\alpha_1 - 1}(1 - t_1)^{\alpha_1 - 1} \ldots z_J^{\alpha_J - 1}(1 - t_J)^{\alpha_J - 1}}{(\alpha_1 - 1)! \cdots (\alpha_J - 1)!} (z_1 \cdots z_J) z_1^m t_1^m \cdot \right.$$

$$\left. (q(\varphi_{n_0}^{-1}(z_1 t_1, \ldots, z_J t_J, z_{J+1}, \ldots, z_N)) - (\partial^\alpha p)(z_1 t_1, \ldots, z_J t_J, z_{J+1}, \ldots, z_N)) dt_1 \cdots dt_J \right|.$$

Since, trivially, we have

$$\left| \frac{z_1^{\alpha_1 + m}(1 - t_1)^{\alpha_1 - 1}}{(\alpha_1 - 1)!} \right| \leq 1 \qquad (z_1 \in \overline{\mathbf{D}},\ t_1 \in [0, 1]),$$

$$\left| \frac{z_j^{\alpha_j}(1 - t_j)^{\alpha_j - 1}}{(\alpha_j - 1)!} \right| \leq 1 \qquad (z_j \in \overline{\mathbf{D}},\ t_j \in [0, 1], j \in \{2, \ldots, J\}),$$

we may obtain the inequality

$$|h(z)| \leq \int_0^1 t_1^m \delta \, dt_1 = \frac{\delta}{m + 1} < \frac{\varepsilon}{2}.$$

In particular if we define $g = p + h$, then one gets

$$g \in A(\mathbf{D}_N)$$

and

$$\|g - p\|_\infty = \|h\|_\infty < \varepsilon.$$

Moreover,

$$q(z) - L_{n_0} g(z) = q(z) - (L_{n_0} p)(z) - (L_{n_0} h)(z) =$$

$$q(z) - (\partial^\alpha p)(\varphi_{n_0}(z)) - \frac{1}{\gamma_1^m}\varphi_{1,n_0}^m(z_{\pi_{n_0}(1)})[q(z) - (\partial^\alpha p)(\varphi_{n_0}(z))] = (1 - \gamma_1^{-m}\varphi_{1,n_0}^m(z_{\pi_{n_0}(1)}))(q(z) - (\partial^\alpha p)(\varphi_{n_0}(z))),$$

for all $z = (z_1, \dots, z_N) \in G$.

Assume that $z = (z_1, \dots, z_N) \in r\overline{\mathbf{D}} \times \cdots \times r\overline{\mathbf{D}}$. Then $z_{\pi_{n_0}(1)} \in r\overline{\mathbf{D}}$ and by (1) we get

$$|q(z) - L_{n_0}g(z)| = |1 - \gamma_1^{-m}\varphi_{1,n_0}^m(z_{\pi_{n_0}(1)})| \cdot |q(z) - (\partial^\alpha p)(\varphi_{n_0}(z))| < \frac{\varepsilon}{2\delta} \cdot \delta = \frac{\varepsilon}{2}.$$

Thus the set $\mathcal{G}$ is dense in $A(\mathbf{D}_N) \times H(\mathbf{D}_N)$, as required.

Now, let us see the general case where X is an F-space as in the hypothesis. By Lemma 2.3 there exist holomorphic functions $S_1, \dots, S_N$ on $\mathbf{D}$, and a permutation π of $\{1, \dots, N\}$ such that for some subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ it holds that

$$\varphi_{n_k}(z_1, \dots, z_N) \to S \equiv (S_1(z_{\pi(1)}), \dots, S_N(z_{\pi(N)})) \quad \text{on } H(\mathbf{D}_N) \qquad (k \to \infty).$$

Then, for any polynomial P on $\mathbf{C}^N$ we have

$$(\partial^\alpha P) \circ \varphi_{n_k} \to (\partial^\alpha P) \circ S \quad \text{on } H(\mathbf{D}_N) \qquad (k \to \infty).$$

Thus, $\lim_{k\to\infty} L_{n_k}P$ exists in $H(\mathbf{D}_N)$ for every polynomial P. Since X satisfies (b), the set $\mathcal{U}$ is non-empty and by (c), Theorem 2.1 yields that $\mathcal{U}$ is a residual subset of X. The proof is finished. □

Remark. Although the above theorem only provides with a sufficient condition, this is optimal in the following sense. Consider $X = A(\mathbf{D}_N)$ and suppose that for all $j \in \{1, \dots, N\}$ such that $\alpha_j \geq 1$ the sequence $\{\varphi_{j,n}\}_{n=1}^\infty$ is not properly discontinuous on $\mathbf{D}$. Then for any $r \in (0,1)$ there exists a compact subset $L_j \subset \mathbf{D}$ with

$$\varphi_{j,n}(|z| \leq r) \subset L_j \qquad \text{for all } n \in \mathbf{N}.$$

If $\alpha_j = 0$ we set $L_j = \overline{\mathbf{D}}$. Hence

$$((\partial^\alpha f) \circ \varphi_n)(r\overline{\mathbf{D}} \times \cdots \times r\overline{\mathbf{D}}) \subset \partial^\alpha f(L_1 \times \cdots \times L_N),$$

for all $n \in \mathbf{N}$. In particular, if $f \in A(\mathbf{D}_N)$, we have that the set

$$\bigcup_{n\in\mathbf{N}} [((\partial^\alpha f) \circ \varphi_n)(r\overline{\mathbf{D}} \times \cdots \times r\overline{\mathbf{D}})],$$

is bounded. Thus $f \notin \mathcal{U}$ and $\mathcal{U}$ is empty.

Now, suppose that $X = H^p(\mathbf{D}_N)$ and there exists a unique $j \in \{1, \dots, N\}$ such that the "component" $\{\varphi_{j,n}\}_n$ of $\{\varphi_n\} \subset \mathrm{Aut}(\mathbf{D}_N)$ is properly discontinuous on $\mathbf{D}$. Then for any $\alpha \in \mathbf{N}_0^N$, $\mathcal{U}$ is residual. If $\alpha_j \neq 0$ it is a direct consequence of

Theorem 4.1, but if $\alpha_j = 0$ we can make a similar proof as in Theorem 3.1 by introducing a "peak" function. We left the details to the interested reader.

It is known that for every automorphism φ of the unit ball $\mathbf{B}_N$ we can find a concentric open ball B such that $\overline{\mathbf{B}_N} \subset B$ and φ^{-1} is holomorphic in B. With this and Lemma 2.4 in mind we can adapt the proof of Theorem 4.1 to get the statement contained in Theorem 4.3, see below. Although the details are left to the interested reader, we indicate that now we must work directly with the automorphisms $\{\varphi_n\}$ and not with their components, and take the function $F(z) = a(z)^m(q(\varphi_{n_0}^{-1}(z)) - (\partial^\alpha p)(z))$, where $a(z) = \frac{1+\langle z,\gamma\rangle}{2}$.

Theorem 4.2. *Assume that $\alpha \in \mathbf{N}_0^N \setminus \{0\}$. Let X be an F-space of holomorphic functions on $\mathbf{B}_N$ having the following properties:*

(a) *Convergence in X implies compact convergence on $\mathbf{B}_N$.*
(b) *$A(\mathbf{B}_N) \subset X$.*
(c) *The polynomials are dense in X.*

Assume that $\{S_n\}_n$ is a sequence of automorphisms of $\mathbf{B}_N$. Consider the set

$$\mathcal{U} = \{f \in X : \ \{(\partial^\alpha f) \circ S_n : \ n \in \mathbf{N}\} \ \text{ is dense in } H(\mathbf{B}_N)\}.$$

Then $\mathcal{U}$ is residual in X if and only if the action of $\{S_n\}_n$ is properly discontinuous on $\mathbf{B}_N$.

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