A GENERALIZATION OF THE CLASSICAL LAGUERRE POLYNOMIALS. [1]

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Abstract

We consider a modification of the gamma distribution by adding a discrete measure supported in the point $x = 0$. For large n we analyze the existence of orthogonal polynomials with respect to such a distribution. Finally we represent them as the hypergeometric function ${}_3F_3$.

§1 Introduction.

The study of orthogonal polynomials with respect to some modifications of a weight function via the addition of one or two delta Dirac measures started in a paper by H.L. Krall [10]. In fact, when the research of the polynomial solution of the fourth order differential equations such as

$$\sum_{i=0}^{4} a_i(x)Y^{(i)}(x) = \lambda_n Y(x),$$

is realized, where $a_i(x)$ are polynomials of degree, at most, i, three new classes of polynomials orthogonal with respect to a such kind of modifications appear:

1. Laguerre-type case: $e^{-x}\,dx + M\delta(x)$ $\quad M > 0, \quad x \in \mathbb{R}^+,$

2. Legendre-type case: $\dfrac{\alpha}{2}\,dx + \dfrac{\delta(x-1)}{2} + \dfrac{\delta(x+1)}{2}$ $\quad \alpha > 0, \quad x \in (-1, 1),$

3. Jacobi-type case: $(1-x)^{\alpha}\,dx + M\delta(x)$ $\quad \alpha > -1, \quad M > 0, \quad x \in (0, 1).$

Some authors have considered more general situations and they have studied some properties of the new classes of orthogonal polynomials; the algebraic properties of the new orthogonal polynomials when a mass point outside the support of the measure is added (Chihara [4]), asymptotic properties in terms of the location of the mass point (Nevai [14]), distribution of their zeros (Buendía et al. [3]), differential equations (Marcellán and Maroni [12], Marcellán and Ronveaux [13]) which they satisfy and so on. For two mass points, see the paper by Koornwinder [8] and Draïdi and Maroni [5].

More recently, S.Belmehdi and F. Marcellán [1] have considered, in the framework of linear functionals, a modification using the first derivative of the delta Dirac measures. This question

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can be considered as a limit case of two masses located in two near points.

In an other way, the denominators Q_n of the main diagonal sequence for Padé approximants of Stieltjes type meromorphic functions,

$$\int \frac{d\mu(x)}{z-x} + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}\frac{i!}{(z-c_j)^{i+1}} \qquad\qquad A_{N_j,j} \neq 0,$$

satisfy orthogonal relations such as

$$\int p(x)Q_n(x)d\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}(p(z)Q_n(z))^{(i)}_{z=c_j} = 0,$$

where $p(x)$ is a polynomial of degree at most $n-1$, i.e., they are orthogonal with respect to a modification of the measure μ through $\sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}\delta^{(i)}(x-c_j)$.

Their study has known an increasing interest during the past years since their applications in approximation theory [11] and numerical integration [2] among other domains.

In our paper, we consider a particular case ($m = 1$ and $N_1 = 1$) and we analyze the corresponding polynomials when μ is the Laguerre measure. More precisely, we find explicitly such polynomials and identify them as a hypergeometric function. In other direction Koekoek and Meijer [6] (see also [7]) have considered some special inner products of Sobolev-type as

$$< p, q >= \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \sum_{k=0}^{N} M_k p^{(k)}(0)q^{(k)}(0).$$

Our case is very different with respect to this one. In fact, if $\{M_k\}_{k=0}^N$ are non-negative, then the above bilinear form is positive-definite. If we define the bilinear form associated with the functional $\mathcal{U}$ (see formula (13) from below),

$$(p, q) = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \sum_{k=0}^{N} M_k(p(z)q(z))^{(k)}_{z=0},$$

this bilinear form is not positive definite and, even, in general is not quasi-definite. This means that the monic orthogonal polynomial sequence does not exists for all values of M_k.

The structure of the paper is as follows. In Section 2 we provide the required background of the Laguerre polynomials. In Section 3, we deduce an expression of the generalized Laguerre polynomials $L_n^{\alpha,M_0,M_1}(x)$ in terms of the nth Laguerre polynomial and their first and second derivative. Finally, in Section 4, we obtain its representation as a hypergeometric function ${}_3F_3$.

§2 Some Preliminar Results.

In this section we have enclosed some formulas for the classical Laguerre polynomials which are useful to obtain the generalized polynomials orthogonal with respect to the linear functional (13). All the formulas as well as some special properties for the classical Laguerre polynomials can be found in a lot of books, see for instance the classical monograph *Orthogonal Polynomials* by G. Szegö [17], Chapter 5.

In this work we will use monic polynomials, i.e., the polynomials with leading coefficient equal to 1, ($P_n(x) = x^n+$ lower order terms).

The classical Laguerre polynomials are the polynomial solution of the second order linear differential equation of hypergeometric type

$$x(L_n^\alpha(x))'' + (\alpha+1-x)(L_n^\alpha(x))' + nL_n^\alpha(x) = 0. \tag{1}$$

They are orthogonal with respect to the linear functional $\mathcal{L}$ on the linear space of polynomials with real coefficients defined by

$$< \mathcal{L}, P > \; = \int_0^\infty P(x) x^\alpha e^{-x} dx \quad , \qquad \alpha > -1. \tag{2}$$

The orthogonality relation is

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-x} dx = \delta_{nm} \Gamma(n+\alpha+1) n! = \delta_{nm} d_n^2. \tag{3}$$

They satisfy the differentiation formula

$$(L_n^\alpha(x))^{(\nu)} = \frac{n!}{(n-\nu)!} L_{n-\nu}^{\alpha+\nu}(x) \qquad \nu = 0, 1, 2, , ..., \quad n = 0, 1, 2, ..., \tag{4}$$

where $(L_n^\alpha(x))^{(\nu)}$ denotes the ν times derivative of the function. Also the following *structure relation*

$$x(L_n^\alpha(x))' = n L_n^\alpha(x) + (\alpha+n) n L_{n-1}^\alpha(x). \tag{5}$$

holds.

The Christoffel-Darboux formula is, in this situation,

$$\sum_{m=0}^{n-1} \frac{L_m^\alpha(x) L_m^\alpha(y)}{\Gamma(m+\alpha+1) m!} = \frac{1}{x-y} \frac{L_n^\alpha(x) L_{n-1}^\alpha(y) - L_{n-1}^\alpha(x) L_n^\alpha(y)}{\Gamma(n+\alpha)(n-1)!} \qquad n = 1, 2, 3..., \tag{6}$$

The classical Laguerre polynomials are represented as the hypergeometric series

$$L_n^\alpha(x) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \; {}_1F_1\left({-n \atop \alpha+1} \middle| x\right), \tag{7}$$

where

$${}_pF_q\left({b_1, b_2, \ldots, b_q \atop a_1, a_2, \ldots, a_p} \middle| x\right) = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1), \qquad k = 1, 2, 3, ...$$

A consequence of this representation is

$$L_n^\alpha(0) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{8}$$

Now we will obtain an useful property of the Laguerre kernels (6). Taking derivates in this formula with respect to y then

$$\begin{aligned} &\sum_{m=0}^{n-1} \frac{L_m^\alpha(x) (L_m^\alpha)'(y)}{\Gamma(m+\alpha+1) m!} = \\ &= \frac{1}{(x-y)} \sum_{m=0}^{n-1} \frac{L_m^\alpha(x) L_m^\alpha(y)}{\Gamma(m+\alpha+1) m!} + \frac{1}{x-y} \frac{L_n^\alpha(x) (L_{n-1}^\alpha)'(y) - L_{n-1}^\alpha(x) (L_n^\alpha)'(y)}{\Gamma(n+\alpha)(n-1)!}. \end{aligned} \tag{9}$$

If we evaluate (6) and (8) in $y = 0$ and using (5), we obtain

$$\sum_{m=0}^{n-1} \frac{L_m^\alpha(x) L_m^\alpha(0)}{\Gamma(m+\alpha+1) m!} = \frac{(-1)^{n-1}}{\Gamma(\alpha+1) n!} (L_n^\alpha)'(x), \tag{10}$$

$$\sum_{m=0}^{n-1} \frac{L_m^\alpha(x)(L_m^\alpha)'(0)}{\Gamma(m+\alpha+1)m!} =$$

$$= \frac{(-1)^{n-1}}{x\Gamma(\alpha+1)n!}(L_n^\alpha)'(x) + \frac{(-1)^n[(n-1)L_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x)}{x\Gamma(\alpha+2)(n-1)!} = \tag{11}$$

$$= \frac{(-1)^{n-1}}{x\Gamma(\alpha+2)n!}[(\alpha+1)(L_n^\alpha)'(x) - n(n-1)L_n^\alpha(x) - n^2(n+\alpha)L_{n-1}^\alpha(x)].$$

If we use the second order differential equation (1) and the structure relation (5) we can obtain finally

$$\sum_{m=0}^{n-1} \frac{L_m^\alpha(x)(L_m^\alpha)'(0)}{\Gamma(m+\alpha+1)m!} = \frac{(-1)^n}{\Gamma(\alpha+2)n!}[(L_n^\alpha)''(x) + (n-1)(L_n^\alpha)'(x)]. \tag{12}$$

Formulas (10) and (12) will be used to obtain the expression (18).

§3 The definition and orthogonal relation.

Consider the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{U}, P > \;= \int_0^\infty P(x) \;\; x^\alpha e^{-x} dx + M_0 P(0) + M_1 P'(0) \;\;\; , \alpha > -1. \tag{13}$$

Notice that $< \mathcal{U}, P > = < \mathcal{L}, P > + M_0 P(0) + M_1 P'(0)$ where $\mathcal{L}$ is the Laguerre's functional (2).

For large n we will determine the monic polynomial $L_n^{\alpha,M_0,M_1}(x)$ which is orthogonal with respect to the functional (13). The reason for this assumption is to guarantee the existence of the polynomials for all values of the masses M_0 and M_1.

To obtain this we may write the generalized polynomials like a Fourier series

$$L_n^{\alpha,M_0,M_1}(x) = L_n^\alpha(x) + \sum_{k=0}^{n-1} a_{n,k} L_k^\alpha(x), \tag{14}$$

where $L_n^\alpha(x)$ denotes the classical Laguerre monic polynomial of degree n.

To find the unknowns coefficients $a_{n,k}$ we can use the orthogonality of the polynomials $L_n^{\alpha,M_0,M_1}(x)$ with respect to $\mathcal{U}$, i.e.,

$$< \mathcal{U}, L_n^{\alpha,M_0,M_1}(x) L_k^\alpha(x) > = 0 \qquad 0 \le k < n.$$

Putting (14) in (13) we find:

$$\begin{aligned} &< \mathcal{U}, L_n^{\alpha,M_0,M_1}(x) L_k^\alpha(x) > = \\ &= < \mathcal{L}, L_n^{\alpha,M_0,M_1}(x) L_k^\alpha(x) > \;+\; M_0 L_n^{\alpha,M_0,M_1}(0) L_k^\alpha(0) \;+ \\ &+ \;\; M_1 (L_n^{\alpha,M_0,M_1})'(0) L_k^\alpha(0) \;+\; M_1 L_n^{\alpha,M_0,M_1}(0)(L_k^\alpha)'(0), \end{aligned} \tag{15}$$

where $(L_n^{\alpha,M_0,M_1})'(0)$ and $(L_n^\alpha)'(0)$ denote the first derivatives of the generalized and the classical Laguerre polynomials, respectively, evaluated in $x = 0$. If we use the decomposition (14) and taking into account the orthogonality of the classical Laguerre polynomials with respect to the linear functional $\mathcal{L}$, hence the coefficients $a_{n,k}$ are given by:

$$a_{n,k} = -\frac{M_0 L_n^{\alpha,M_0,M_1}(0) L_k^{\alpha}(0) + M_1 (L_n^{\alpha,M_0,M_1})'(0) L_k^{\alpha}(0) + M_1 L_n^{\alpha,M_0,M_1}(0)(L_k^{\alpha})'(0)}{d_k^2}, \tag{16}$$

where by d_k^2 we will denote the norm of the classical Laguerre polynomials (3). Finally the equation (14) provides us the expression

$$\begin{aligned} L_n^{\alpha,M_0,M_1}(x) &= L_n^{\alpha}(x) - M_0 L_n^{\alpha,M_0,M_1}(0) \sum_{k=0}^{n-1} \frac{L_k^{\alpha}(0) L_k^{\alpha}(x)}{d_k^2} - \\ & - M_1 (L_n^{\alpha,M_0,M_1})'(0) \sum_{k=0}^{n-1} \frac{L_k^{\alpha}(0) L_k^{\alpha}(x)}{d_k^2} - \\ & - M_1 L_n^{\alpha,M_0,M_1}(0) \sum_{k=0}^{n-1} \frac{(L_k^{\alpha})'(0) L_k^{\alpha}(x)}{d_k^2}, \end{aligned} \tag{17}$$

In order to obtain an explicit expression for the polynomials we need some properties of the classical Laguerre polynomials. Doing some algebraic calculations in (17) and taking into account formulas (10) and (12), we obtain the following expression for the generalized Laguerre polynomials :

$$L_n^{\alpha,M_0,M_1}(x) = L_n^{\alpha}(x) + A_1 (L_n^{\alpha})'(x) + A_2 (L_n^{\alpha})''(x), \tag{18}$$

where

$$A_1 = \frac{(-1)^n [M_0 L_n^{\alpha,M_0,M_1}(0) + M_1 (L_n^{\alpha,M_0,M_1})'(0)]}{\Gamma(\alpha+1) n!} - \frac{(-1)^n M_1 (n-1) L_n^{\alpha,M_0,M_1}(0)}{\Gamma(\alpha+2) n!}, \tag{19}$$

$$A_2 = -\frac{(-1)^n M_1 L_n^{\alpha,M_0,M_1}(0)}{\Gamma(\alpha+2) n!}. \tag{20}$$

In the representation (18) appear the values of the polynomials $L_n^{\alpha,M_0,M_1}(x)$ and their first derivatives evaluated in $x = 0$. Then, to obtain the analytical expression of A_1 and A_2 is sufficient to evaluate (18) and its derivative in $x = 0$. The solution of these two equations yields

$$L_n^{\alpha,M_0,M_1}(0) = \frac{\begin{vmatrix} (-1)^n n! \binom{n+\alpha}{n} & \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} \\ -(-1)^n n! \binom{n+\alpha}{n-1} & 1 - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} & \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} \\ -\frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)} \frac{n(\alpha+2)-\alpha-1}{(n-2)} \binom{n+\alpha}{n-3} & 1 - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} \end{vmatrix}}, \tag{21}$$

and

$$(L_n^{\alpha,M_0,M_1})'(0) =$$

$$= \frac{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} & (-1)^n n! \binom{n+\alpha}{n} \\ -\frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)} \frac{n(\alpha+2)-\alpha-1}{(n-2)} \binom{n+\alpha}{n-3} & -(-1)^n n! \binom{n+\alpha}{n-1} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} & \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-1} \\ -\frac{M_0}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)} \frac{n(\alpha+2)-\alpha-1}{(n-2)} \binom{n+\alpha}{n-3} & 1 - \frac{M_1}{\Gamma(\alpha+1)} \binom{n+\alpha}{n-2} \end{vmatrix}}. \tag{22}$$

In formulas (21) and (22) the denominator take the form:

$$1+\frac{M_0(\alpha+1)_n}{(n-1)!\Gamma(\alpha+2)}-\frac{M_1(\alpha+1)_n}{(n-2)!\Gamma(\alpha+3)}-\frac{M_1^2(\alpha+1)_n^2}{(n-2)!^2\Gamma(\alpha+3)^2}\frac{n+\alpha+1}{(n-1)(\alpha+3)(\alpha+1)}$$

where $(\alpha+1)_n$ is the *shifted factorial* or Pochhamer symbol defined by

$$(a)_0 := 1, (a)_k := a(a+1)\cdots(a+k-1).$$

Now if we use the asymptotic formula for the Gamma function [15] (formula 8.16 page 88)

$$\Gamma(x) \sim e^{-x}x^x\sqrt{\frac{2\pi}{x}}, \qquad x >> 1, x \in \mathbb{R},$$

then for large n we obtain the following asymptotic relation for the denominator in (21) and (22):

$$1+\frac{M_0 n^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+2)}-\frac{2M_1(n-1)n^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+3)}-\frac{M_1^2(n-1)(n+\alpha+1)n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(\alpha+4)}$$

This means, that for any fixed M_0 and M_1 and for sufficiently large n the denominator could be taken as $O(-n^{2\alpha+4})$, i.e., for large n we can guarantee the existence of the polynomials for all values of the masses M_0 and M_1.

A very simple consequence of (18) is an operational Viskov-type formula for the generalized polynomials. From a result by Viskov (see [18] and [16]) a new representation for the Laguerre polynomials via a second order differential operator is

$$L_n^\alpha(x) = e^x[xD^2+(\alpha+1)D]^n(e^{-x}), \tag{23}$$

where $D := \frac{d}{dx}$ is the usual differential operator. Using this formula as well as (18), we obtain the following operational representation for the polynomial $L_n^{\alpha,M_0,M_1}(x)$

$$L_n^{\alpha,M_0,M_1}(x) = e^x[1+A_1+A_2+(A_1+2A_2)D+A_2D^2][xD^2+(\alpha+1)D]^n(e^{-x}),$$

or

$$L_n^{\alpha,M_0,M_1}(x) = e^x[1+L_2(A_1,A_2)][xD^2+(\alpha+1)D]^n(e^{-x}),$$

where $L_2(A_1,A_2) = A_1+A_2+(A_1+2A_2)D+A_2D^2$ is a second order differential operator. When $M_0 = M_1 = 0$ it is straighforward that the last formula becomes into the classical representation. For this reason it could be considered as a perturbation of the formula by Viskov (23).

§4 Representation as hypergeometric series.

In this section we will prove the following proposition:

Proposition 1 *The orthogonal polynomial $L_n^{\alpha,M_0,M_1}(x)$ is, up to a constant factor, a generalized hypergeometric serie. More precisely*

$$L_n^{\alpha,M_0,M_1}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)(1+A_1+A_2)\beta_0\beta_1}{\Gamma(\alpha+3)} \;{}_3F_3\left(\begin{matrix}-n,\beta_0+1,\beta_1+1\\ \alpha+3,\ \beta_0,\ \ \beta_1\end{matrix}\Big|x\right).$$

Proof: From (18) and the hypergeometric representation of the Laguerre polynomials (7) we can write

$$L_n^{\alpha,M_0,M_1}(x) =$$

$$=\frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\sum_{m=0}^{\infty}\left[\frac{(-n)_m}{(\alpha+1)_m}+\frac{A_1(-n)_{m+1}}{(\alpha+1)_{m+1}}+\frac{A_2(-n)_{m+2}}{(\alpha+1)_{m+2}}\right]\frac{x^m}{m!}$$

or, equivalently,

$$L_n^{\alpha,M_0,M_1}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{x^m(-n)_m}{m!}\times$$

$$\frac{(m+\alpha+1)(m+\alpha+2)+(m+\alpha+2)(m-n)A_1+(m-n)(m-n+1)A_2}{(\alpha+1)_{m+2}}.$$

Taking into account that the expression $1+A_1+A_2 \neq 0$ as well as the expression inside the cuadratic brackets is a polynomial in m of degree 2, we can write

$$L_n^{\alpha,M_0,M_1}(x) = \\ = \frac{(-1)^n\Gamma(n+\alpha+1)(1+A_1+A_2)}{\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{(-n)_m(m+\beta_0)(m+\beta_1)}{(\alpha+1)_{m+2}} \quad \frac{x^m}{m!}, \tag{24}$$

where $\beta_i = \beta_i(n,\alpha,A_1,A_2)$. Since

$$(m+\beta_i) = \frac{\beta_i(\beta_i+1)_m}{(\beta_i)_m}, \qquad i=0,1 \tag{25}$$

then (24) becomes

$$L_n^{\alpha,M_0,M_1}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)(1+A_1+A_2)\beta_0\beta_1}{\Gamma(\alpha+3)}\times \\ \times \sum_{m=0}^{\infty} \frac{(-n)_m(\beta_0+1)_m(\beta_1+1)_m}{(\beta_0)_m(\beta_1)_m(\alpha+3)_m} \quad \frac{x^m}{m!}, \tag{26}$$

or in terms of the hypergeometric series

$$L_n^{\alpha,M_0,M_1}(x) = \\ = \frac{(-1)^n\Gamma(n+\alpha+1)(1+A_1+A_2)\beta_0\beta_1}{\Gamma(\alpha+3)} \quad {}_3F_3\left({}^{-n,\beta_0+1,\beta_1+1}_{\alpha+3,\ \beta_0,\ \ \beta_1} \Big| x \right). \tag{27}$$

Here the coefficients β_0 and β_1 are the solutions of a second order equation at m (See formula (24)) and they are, in general, complex numbers. In the case when for some k, β_k is a nonpositive integer we need to take the analytic continuation of the hypergeometric series (27). Solving the second order equation in m we find

$$\beta_{0,1} = \frac{2\alpha+3+(\alpha+2-n)A_1-(2n-1)A_2}{(1+A_1+A_2)} \quad [1 \pm \sqrt{D(n,A_1,A_2)}], \tag{28}$$

where

$$D(n,A_1,A_2) = \\ = 1 - \frac{4(1+A_1+A_2)(n(n-1)A_2+n(\alpha-2)A_1+(\alpha+1)(\alpha+2))}{[2\alpha+3+(\alpha+2-n)A_1-(2n-1)A_2)]^2}. \tag{29}$$

Equation (27) can be rewritten in the form

$$L_n^{\alpha,M_0,M_1}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+3)}\times \\ \times\frac{(1-D(n,A_1,A_2)[2\alpha+3+(\alpha+2-n)A_1-(2n-1)A_2]^2}{(1+A_1+A_2)} \quad {}_3F_3\left({}^{-n,\beta_0+1,\beta_1+1}_{\alpha+3,\ \beta_0,\ \ \beta_1} \Big| x \right). \tag{30}$$

It is straighforward to show that for $M_0 = M_1 = 0$ the hypergeometric function (27) becomes

$$L_n^{\alpha,0,0}(x) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \; {}_3F_3\left({-n,\alpha+2,\alpha+3 \atop \alpha+3,\alpha+1,\alpha+2}\Big| x\right) =$$
$$= \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \; {}_1F_1\left({-n \atop \alpha+1}\Big| x\right) = L_n^{\alpha}(x). \tag{31}$$

So (27) can be considered as a generalization of the representation as hypergeometric series of the classical Laguerre polynomials $L_n^\alpha(x)$. ■

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