RECURRENCE RELATIONS FOR CONNECTION COEFFICIENTS BETWEEN Q-ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLES IN THE NON-UNIFORM LATTICE $x(s) = q^{2s}$. [1]

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Abstract

We obtain the structure relations for q-orthogonal polynomials in the exponential lattice q^{2s} and from that we construct the recurrence relation for the connection coefficients between two families of polynomials belonging to the classical class of discrete q-orthogonal polynomials. An explicit example is also given.

1 Introduction.

Given two families of Polynomials, denoted by $P_n(x)$ and $Q_m(x)$, of degree exactly equal to respectively n and m, the *Connection Problem* asks to compute the so-called *Connection Coefficients* $C_m(n)$ defined by the relation:

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x).$$

When both families are *orthogonal* with respect to two different measures the Connections Coefficients satisfy a relative simple recurrence relation, but mixing in the (m, n) table three adjacent m and three adjacent n *crossing* at (m, n).

The first survey on this topic was given by Askey 20 years ago [1]-[2], giving in some cases explicit expression for the Coefficients and discussing also the positivity properties of these Coefficients.

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It was noticed only recently that an additional assumption on the orthogonality measure gives for $C_m(n)$ a recurrence only in m, n being fixed. This Orthogonality class is called *semi-classical* and is very large [11], [7] . The classical (continuous) family: *Jacobi, Bessel, Laguerre, Hermite* (see for instance [12]) and the classical(discrete) family: *Hahn , Kravchuk, Meixner, Charlier* (see for instance [13]) are of course included in the semi-classical class. When the orthogonality measure is defined by a weight $\rho(x)$, the semi-classical class covers all weights solution of a linear first order differential (or difference) equation with polynomial coefficients.

The key property inside the semi-classical class, in order to obtain a one index (m) recurrence relation for $C_m(n)$, comes from the existence of a so called Structure Relation, linking linearly the derivative (or difference) of $P_n(x)$ times a polynomial, to a fixed combination of $P_k(x)$.

An algorithm has been given recently building for both discrete and continuous classical families (see [3], [15] and [16]) the explicit recurrences for $C_m(n)$, solving in many cases theses recurrences with the help of *Mathematica* [20].

Looking for the situation for which a structure relation is known explicitly, we realize that, from the data of Orthogonal Polynomial on the exponential lattice $x(s) = q^{2s}$ (a small subset of the q-world). Here we need to point out that exits two different point of view in the study of the q-polynomials. The first one, in the framework of the q-basic hypergeometric series [6], [8], [9] and the second, in the framework of the theory of difference equations developed by Nikiforov et al. [12], [13], [14]. In this work we will use the second one because it gives us the possibility to provide an uniform treatment of several classes of orthogonal polynomials and, probably, it is the best way to find further applications.

This paper shows how to apply the technique to a particular (simple) case: the exponential lattice, building first the corresponding Structure Relations.

2 Structure relations for q-orthogonal polynomials on the exponential lattice $x(s) = q^{2s}$.

Let us to start with the study of some general properties of orthogonal polynomials of a discrete variable in non-uniform lattices. Let be

$$\tilde{\sigma}(x(s))\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\triangledown Y(s)}{\triangledown x(s)} + \frac{\tilde{\tau}(x(s))}{2}\left[\frac{\triangle Y(s)}{\triangle x(s)} + \frac{\triangledown Y(s)}{\triangledown x(s)}\right] + \lambda Y(s) = 0, \tag{1}$$

$$\triangledown f(s) = f(s) - f(s-1), \triangle f(s) = f(s+1) - f(s)\,,$$

the *second order difference equation of hypergeometric type* for some lattice function $x(s)$, where $\triangledown f(s) = f(s) - f(s-1)$ and $\triangle f(s) = f(s+1) - f(s)$ denote the backward and forward finite difference quotients, respectively. Here $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant.

The previous equation (1) can be obtained from the classical *hypergeometric equation*

$$\tilde{\sigma}(x)y''(x) + \tilde{\tau}(x)y'(x) + \lambda y(s) = 0,$$

via the discretization of the first and second derivatives y' and y'' in an apropiate lattice [12], [13]. It is better to rewrite (1) in the equivalent form (see [13] and [14])

$$\begin{gathered}\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \tau(s)\frac{\triangle Y(s)}{\triangle x(s)} + \lambda Y(s) = 0, \\ \sigma(s) = \tilde{\sigma}(x(s)) - \tfrac{1}{2}\tilde{\tau}(x(s))\triangle x(s-\tfrac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)).\end{gathered} \tag{2}$$

The *q-orthogonal polynomials* $P_n(x(s))_q \equiv P_n(s)_q$ on the exponential lattice $x(s) = q^{2s}$ are, for given functions $\sigma(s)$ and $\tau(s)$, the polynomial (in powers of $x(s) = q^{2s}$) solution of the second order difference equation (2).

The *k-order difference derivative* of the polynomials $P_n(x(s))_q$, defined by

$$v_{kn}(s) = \frac{\triangle}{\triangle x_{k-1}(s)}\frac{\triangle}{\triangle x_{k-2}(s)}\cdots\frac{\triangle}{\triangle x(s)}[P_n(x(s))_q] \equiv \triangle^{(k)}[P_n(x(s))_q],$$

and

$$x_m(s) = x(s + \tfrac{m}{2}),$$

also satisfy the difference equation of hypergeometric type of the form

$$\sigma(s)\frac{\triangle}{\triangle x_k(s-\frac{1}{2})}\left[\frac{\nabla v_{kn}(s)}{\nabla x_k(s)}\right] + \tau_k(s)\frac{\triangle v_{kn}(s)}{\triangle x_k(s)} + \mu_k v_{kn}(s) = 0, \tag{3}$$

where (see [13], page 62, Equation (3.1.29))

$$\tau_k(s) = \frac{\sigma(s+k) - \sigma(s) + \tau(s+k)\triangle x(s+k-\frac{1}{2})}{\triangle x_{k-1}(s)},$$

and

$$\mu_k = \lambda_n + \sum_{m=0}^{k-1}\frac{\triangle\tau_m(s)}{\triangle x_m(s)}.$$

These polynomial solutions denoted by $P_n(x(s))_q \equiv P_n(s)_q$ satisfy the orthogonality property

$$\sum_{s_i=a}^{b-1} P_n(x(s_i))_q P_m(x(s_i))_q \rho(s_i)\triangle x(s_i - \tfrac{1}{2}) = \delta_{nm}d_n^2, \tag{4}$$

where $\rho(x)$ is some non-negative function (weight-function), i.e.,

$$\rho(s_i)\triangle x(s_i - \tfrac{1}{2}) > 0 \quad (a \le s_i \le b-1),$$

supported in a countable subset of the real line $[a, b]$ (a, b can be $\pm\infty$). The functions $\rho(s)$ and $\rho_k(s)$ are the solutions of the Pearson-type difference equations ([13], Eq.(3.2.9) and (3.2.10) page 64)

$$\frac{\triangle}{\triangle x(s-\frac{1}{2})}[\sigma(s)\rho(s)] = \tau(s)\rho(s), \tag{5}$$

and

$$\frac{\triangle}{\triangle x_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)] = \tau_k(s)\rho_k(s) \tag{6}$$

and $\rho(s)$ satisfy the condition [14]:

$$\sigma(s)\rho(s)x^k(s-\tfrac{1}{2})|_{s=a,b} = 0, \quad \forall k,l \in \mathbb{N} \quad (\mathbb{N} = \{0,1,2,...\}).$$

In (4) d_n^2 denotes the square of the norm of the corresponding orthogonal polynomials.

The q-orthogonal polynomials satisfy a three term recurrence relations (TTRR) of the form

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \tag{7}$$

with the initial conditions

$$P_{-1}(s)_q = 0, \quad P_0(s)_q = 1.$$

It is well known [13]-[14], that the polynomial solutions of equation (2), denoted by $P_n(x(s))_q$, are uniquely determined, up to a normalizing factor B_n, by the difference analog of the Rodrigues formula (see [13] page 66 Eq. (3.2.19)):

$$P_n(s)_q = \frac{B_n}{\rho(s)} \bigtriangledown_n^{(n)} [\rho_n(s)] \quad \bigtriangledown_n^{(n)} = \frac{\bigtriangledown}{\bigtriangledown x_1(s)} \frac{\bigtriangledown}{\bigtriangledown x_2(s)} \cdots \frac{\bigtriangledown}{\bigtriangledown x_n(s)} [\rho_n(s)], \tag{8}$$

where $\rho_n(s) = \rho(n+s)\prod_{k=1}^{n}\sigma(s+k)$. These solutions correspond to some values of λ_n - the eigenvalues of equation (2), which is computed from (see [13], page 104 and [14])

$$\lambda_n = -\frac{1}{2}[n]_q\{(q^{n-1}+q^{-n+1})\tilde{\tau}' + [n-1]_q\tilde{\sigma}''\}, \tag{9}$$

where $\tilde{\sigma}(s) = \sigma(s) + \frac{1}{2}\tilde{\tau}(s) \triangle x(s-\frac{1}{2})$ and $\tilde{\tau}(s) = \tau(s)$ (see Eq. (2)).

Here $[n]_q$ denotes the so called *q-numbers*

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh(hn)}{\sinh(h)}, \quad q = e^h.$$

2.1 The first structure relation for the q-polynomials in the lattice $x(s) = q^{2s}$.

Let us now try to obtain a structure relation for the q-polynomials in the exponential lattice $x(s) = q^{2s}$. (For the linear lattice see [13] Eq.(2.2.10) page 24.)

First of all, we rewrite the Rodrigues equation (8) in another form. We will use the linearity of the operator $\bigtriangledown_n^{(n)}$, as well as the identity

$$\bigtriangledown x_k(s) = q^k \bigtriangledown x(s).$$

Then, a straightforward calculation gives us

$$P_n(s)_q = \frac{q^{-\frac{n(n+1)}{2}} B_n}{\rho(s)} \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n [\rho_n(s)], \quad \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n = \overbrace{\frac{\bigtriangledown}{\bigtriangledown x(s)} \ldots \frac{\bigtriangledown}{\bigtriangledown x(s)}}^{\text{n-times}}. \tag{10}$$

Now, from formulas (5) and (10) we find

$$\frac{\bigtriangledown \rho_{n+1}(s)}{\bigtriangledown x_{n+1}(s)} = \frac{\bigtriangledown[\rho_n(s+1)\sigma(s+1)]}{\bigtriangledown x_n(s+\frac{1}{2})} = \frac{\bigtriangleup[\sigma(s)\rho_n(s)]}{\bigtriangleup x_n(s-\frac{1}{2})} = \tau_n(s)\rho_n(s).$$

Then by using the Rodrigues formula (8) we obtain

$$\begin{aligned} P_{n+1}(s)_q &= \frac{B_{n+1}}{\rho(s)} \bigtriangledown_{n+1}^{(n+1)} [\rho_n(s)] = \frac{B_{n+1}}{\rho(s)} \bigtriangledown_n^{(n)} \frac{\bigtriangledown \rho_{n+1}(s)}{\bigtriangledown x_{n+1}(s)} = \\ &= \frac{B_{n+1}}{\rho(s)} \bigtriangledown_n^{(n)} [\tau_n(s)\rho_n(s)] = q^{-\frac{n(n+1)}{2}} \frac{B_{n+1}}{\rho(s)} \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n [\tau_n(s)\rho_n(s)]. \end{aligned} \tag{11}$$

In order to obtain an expression for $\left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n [\tau_n(s)\rho_n(s)]$ we successively apply the formula $\bigtriangledown f(s)g(s) = f(s) \bigtriangledown g(s) + g(s-1) \bigtriangledown f(s)$, as well as formulas

$$\frac{\bigtriangleup \tau_n(s)}{\bigtriangleup x(s)} = q^n \tau_n', \quad \left[\frac{\bigtriangledown}{\bigtriangledown x(s-1)}\right]^n = q^{2n} \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n.$$

Then, Eq. (11) gives us the following

$$\begin{aligned} P_{n+1}(s)_q = \quad & \frac{q^{-\frac{n(n+1)}{2}} B_{n+1}}{\rho(s)} \times \\ & \times \left(\tau_n(s) \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^n [\rho_n(s)] + q^{2n-1} [n]_q \tau_n' \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^{n-1} [\rho_n(s-1)] \right). \end{aligned} \tag{12}$$

Using the Rodrigues formula for the difference derivative of the polynomial ([13], Eq. (3.2.18) page 66) we find (notice that $\bigtriangleup x(s-1) = q^{-2} \bigtriangleup x(s)$):

$$\begin{aligned} \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)} &= \frac{\bigtriangleup P_n(s-1)_q}{\bigtriangleup x(s-1)} = \frac{-q^{-\frac{(n-1)(n+2)}{2}} \lambda_n B_n}{\sigma(s)\rho(s)} \left[\frac{\bigtriangledown}{\bigtriangledown x(s-1)}\right]^{n-1} [\rho_n(s-1)] = \\ &= \frac{-q^{-\frac{(n-1)(n-2)}{2}} \lambda_n B_n}{\sigma(s)\rho(s)} \left[\frac{\bigtriangledown}{\bigtriangledown x(s)}\right]^{n-1} [\rho_n(s-1)]. \end{aligned}$$

Therefore, equation (12) can be rewritten in the form

$$P_{n+1}(s)_q = \frac{B_{n+1}\tau_n(s)}{B_n} P_n(s)_q - \frac{[n]_q B_{n+1} \tau_n' \sigma(s)}{\lambda_n B_n} \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)}$$

and then, the following *differentiation formula* holds

$$\sigma(s) \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)} = \frac{\lambda_n}{[n]_q \tau_n'} \left[\tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \tag{13}$$

If we now use the power expansion of $\tau_n(s)$, i.e., $\tau_n(s) = \tau_n' x_n(s) + \tau_n(0) = \tau_n' q^n x(s) + \tau_n(0)$ and the TTRR (7) we obtain the *first structure relation*

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \tilde{S}_n P_{n+1}(s)_q + \tilde{T}_n P_n(s)_q + \tilde{R}_n P_{n-1}(s)_q, \tag{14}$$

where

$$\begin{aligned} \tilde{S}_n &= \frac{\lambda_n}{[n]_q}\left[q^n\alpha_n - \frac{B_n}{\tau_n' B_{n+1}}\right], \\ \tilde{T}_n &= \frac{\lambda_n}{[n]_q}\left[q^n\beta_n - \frac{\tau_n(0)}{\tau_n'}\right], \\ \tilde{R}_n &= \frac{\lambda_n q^n \gamma_n}{[n]_q}. \end{aligned} \tag{15}$$

2.2 The second structure relation for the q-polynomials in the lattice $x(s) = q^{2s}$.

Let us try to obtain now the second structure relation. Firstly, we notice that

$$\triangle\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\triangle P_n(s)_q}{\triangle x(s)} - \frac{\nabla P_n(s)_q}{\nabla x(s)}.$$

Then, by using the difference equation (2)

$$\begin{aligned} \sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} &= \sigma(s)\frac{\triangle P_n(s)_q}{\triangle x(s)} - \sigma(s)\triangle\frac{\nabla P_n(s)_q}{\nabla x(s)} = \\ &= [\sigma(s) + \tau(s)\triangle x(s-\tfrac{1}{2})]\frac{\triangle P_n(s)_q}{\triangle x(s)} + \lambda_n \triangle x(s-\tfrac{1}{2}) P_n(s)_q. \end{aligned}$$

and (14) we find

$$\begin{aligned} [\sigma(s) + \tau(s)\triangle x(s-\tfrac{1}{2})]\frac{\triangle P_n(s)_q}{\triangle x(s)} &= \tilde{S}_n P_{n+1}(s)_q + \\ &+ (\tilde{T}_n - \lambda_n \triangle x(s-\tfrac{1}{2}))P_n(s)_q + \tilde{R}_n P_{n-1}(s)_q, \end{aligned} \tag{16}$$

Now, taking into account that $\triangle x(s-\frac{1}{2}) = (q-q^{-1})x(s)$, and using the TTRR (7) we finally obtain the *second structure relation*

$$[\sigma(s) + \tau(s)\triangle x(s-\frac{1}{2})]\frac{\triangle P_n(s)_q}{\triangle x(s)} = S_n P_{n+1}(s)_q + T_n P_n(s)_q + R_n P_{n-1}(s)_q, \tag{17}$$

where

$$\begin{aligned} S_n &= \tilde{S}_n - (q-q^{-1})\lambda_n\alpha_n, \\ T_n &= \tilde{T}_n - (q-q^{-1})\lambda_n\beta_n, \\ R_n &= \tilde{R}_n - (q-q^{-1})\lambda_n\gamma_n. \end{aligned} \tag{18}$$

3 Recurrence relations for connection coefficients.

Let us consider two families of q-polynomials $P_n(x)$ and $Q_n(x)$ belonging to the class of discrete orthogonal polynomials in the exponential lattice $x(s) = q^{2s}$. Each polynomial $P_n(x)$ can be represented as a linear combination of the polynomials $Q_n(x)$. In particular

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x). \tag{19}$$

For the family $P_n(x)$ we will use the notation

1. $\sigma(s)$, $\tau(s)$ and λ_n for the difference equation (2)
2. α_n, β_n and γ_n for the TTRR (7) coefficients
3. S_n, R_n and T_n for the second structure relation (17)

and for the $Q_n(x)$

1. $\bar{\sigma}(s)$, $\bar{\tau}(s)$ and $\bar{\lambda}_n$ for the difference equation (2)
2. $\bar{\alpha}_n$, $\bar{\beta}_n$ and $\bar{\gamma}_n$ for the TTRR (7) coefficients
3. $\bar{S}_n$, $\bar{R}_n$ and $\bar{T}_n$ for the second structure relation (17)

Since the polynomials of the family $P_n(x)$ are solutions of the second order difference equation (2) the action of the difference operator of second order $\hat{L}$, defined by

$$\hat{L} = \sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\left[\frac{\nabla}{\nabla x(s)}\right] + \tau(s)\frac{\triangle}{\triangle x(s)} + \lambda_n,$$

on Eq. (19) gives us

$$\sum_{m=0}^{n} C_m(n)\left[\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\left[\frac{\nabla Q_m(x)}{\nabla x(s)}\right] + \tau(s)\frac{\triangle Q_m(x)}{\triangle x(s)} + \lambda_n Q_m(x)\right] = 0. \tag{20}$$

Multiplying by $\bar{\sigma}(s)$ and using

$$\bar{\sigma}(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\left[\frac{\nabla Q_m(x)}{\nabla x(s)}\right] = -\bar{\tau}(s)\frac{\triangle Q_m(x)}{\triangle x(s)} - \bar{\lambda}_n Q_m(x),$$

we obtain the relation

$$\sum_{m=0}^{n} C_m(n)\left[(\tau(s)\bar{\sigma}(s) - \bar{\tau}(s)\sigma(s))\frac{\triangle Q_m(x)}{\triangle x(s)} + (\lambda_n\bar{\sigma}(s) - \sigma(s)\bar{\lambda}_m)Q_m(x)\right] = 0. \tag{21}$$

In order to eliminate $\frac{\triangle Q_m(x)}{\triangle x(s)}$, we multiply (21) by $\bar{\sigma}(s) + \bar{\tau}(s)\triangle x(s-\frac{1}{2})$ and use the second structure relation (17) for the $Q_m(x)$ family, obtaining

$$\begin{aligned}\sum_{m=0}^{n} C_m(n)\,[&(\tau(s)\bar{\sigma}(s) - \bar{\tau}(s)\sigma(s))(\bar{S}_m Q_{m+1}(x) + \bar{R}_m Q_{m-1}(x) + \bar{T}_m Q_m(x)) + \\ &+ (\bar{\sigma}(s) + \bar{\tau}(s)\triangle x(s-\tfrac{1}{2}))(\lambda_n\bar{\sigma}(s) - \sigma(s)\bar{\lambda}_m)Q_m(x)\big] = 0.\end{aligned} \tag{22}$$

The last step consists to expand the remaining terms of type $\bar{\sigma}^2(s)Q_m(x)$, $\bar{\sigma}(s)\sigma(s)Q_m(x)$, $\sigma(s)\bar{\tau}(s)Q_m(x)$ and $\bar{\sigma}(s)\tau(s)Q_m(x)$ in linear combination of $Q_m(x)$ by using the TTRR (7) repeatedly for the $Q_m(x)$ family.

After this process, (22) reduces to

$$\sum_{m=0}^{N} M_m\left[C_0(n), C_1(n), ..., C_n(n)\right] Q_m(x) \tag{23}$$

where $N = max\{n + deg\sigma + deg(\bar{\sigma}), n + 2deg(\bar{\sigma}), n + 1 + deg(\bar{\sigma}) + deg(\tau), n + 1 + deg(\bar{\tau}) + deg(\sigma), 1 + deg(\bar{\tau}) + deg(\bar{\sigma})\}$.

Taking into account the linear independence of the family $Q_m(x)$ we obtain the linear system

$$M_m\left[C_0(n), C_1(n), ..., C_n(n)\right] = 0. \tag{24}$$

These relations contain (linearly) several connection coefficients $C_i(n)$ depending essentially on the degrees of $\sigma(s)$ and $\bar{\sigma}(s)$. In the most general situation they are polynomials of second degree in $x(s) = q^{2s}$. In this case we obtain a relation of the following type the linear system we are looking for

$$M_m\left[C_{m+4}(n), ..., C_{m-4}(n)\right] = 0, \tag{25}$$

which is valid for n greater or equal than the number of initial conditions needed to start the recursion ($n \geq 8$). Notice that for ($n < 8$) the system also gives the solution, but not in a recurrent way.

Notice that for the q-Hahn, q-Meixner, q-Charlier and q-Kravchuk polynomials, as it is show in [13], table **3.3**, page 95, the $\sigma(s)$ is a polynomial of second degree in $x(s) = q^{2s}$. This implies that for such polynomials the recurrence relations for the connection coefficient all are of the form (25). Again we want to remark that we are follow the notation introduced by Nikiforov et al. [13].

4 Recurrence relations for connection coefficients: A simple example.

As we have noticed in the previous section the recurrence relation for connection coefficients for different classes of q-polynomials are too large (8-terms). Here we will analyze a more simple case. Firstly, notice that in the previous algorithm we have not used the orthogonality property of the polynomials P_n, and only that they satisfy a difference equation. On the other hand, for the polynomials Q_m we need to have structure relations as well as three term recurrence relations. Let us to show and example in which we will decompose a set of polynomials $P_n(s)$, satisfying a certain difference equation of first order in the lattice $x(s) = q^{2s}$, as a linear combination of the orthogonal q-polynomials defined in the same lattice, i.e., the q-Hahn, q-Meixner, q-Kravchuk and q-Charlier orthogonal polynomials (see [13], [4] and [17])

Let us define the quantities $(s)_q$ and $(s_n)_q$, defined by

$$(s)_q = \frac{q^{2s}-1}{q^2-1} = q^{s-1}[s]_q \tag{26}$$

and

$$(s_n)_q = (s)_q(s-1)_q \cdots (s-n+1)_q = \prod_{k=0}^{n-1} \frac{q^{2s+2k}-1}{q^2-1} \tag{27}$$

The quantities $(s_n)_q$ are closely related to the *q-Stirling numbers* $\tilde{S}_{q^2}(n,k)$, $s^*_{q^2}(n,k)$ [21] by formulas

$$(s)^n_q = \sum_{k=0}^{n} \tilde{S}_{q^2}(n,k)(s_k)_q, \qquad (s_n)_q = \sum_{k=0}^{n} s^*_{q^2}(n,k)(s)^k_q \tag{28}$$

and satisfy the following two difference equations (here, as before, $x(s) = q^{2s}$)

$$(q^{2s}-1)\frac{\nabla(s_n)_q}{\nabla x(s)} - q^{-n+1}[n]_q(s_n)_q = 0 \tag{29}$$

and

$$(q^{2s-2n+2}-1)\frac{\triangle(s_n)_q}{\triangle x(s)} - q^{-n+1}[n]_q(s_n)_q = 0. \tag{30}$$

Since $(s_n)_q$ is a polynomial in $x(s) = q^{2s}$, it can be represented as a linear combination of the polynomials $Q_m(x)$, the q-polynomials in the exponential lattice. In particular

$$(s_n)_q = \sum_{m=0}^{n} C_m(n)Q_m(x). \tag{31}$$

Let us to obtain the recurrence relation for the connection coefficients $C_m(n)$ between the $(s_n)_q$ and the q-Charlier, q-Meixner or q-Kravchuk. (For q-Hahn polynomials we will consider it separately). In order to do that we apply the operator

$$\tilde{\mathcal{L}} = (q^{2s}-1)\frac{\nabla}{\nabla x(s)} - q^{-n+1}[n]_q \tag{32}$$

to both sides of (31). Using formula (29) ($\tilde{\mathcal{L}}(s_n)_q = 0$) and multiplying by q^{2s} we obtain the following expression

$$0 = \sum_{m=0}^{n} C_m(n)\left\{q^{2s}(q^{2s}-1)\frac{\nabla Q_m(x)}{\nabla x(s)} - q^{-m+1}[m]_q q^{2s}Q_m(x)\right\}. \tag{33}$$

Taking into account that for q-Charlier, q-Meixner and q-Kravchuk the $\sigma(s)$ function in (2) coincide with $q^{2s}(q^{2s}-1)$ and applying the structure relation (14) and the TTRR (7) to the previous expression we find

$$0 = \sum_{m=0}^{n} C_m(n)\left\{A_m Q_{m+1}(x) + B_m Q_m(x) + \Gamma_m Q_{m-1}(x)\right\},$$

from where we obtain the following TTRR for the connection coefficients $C_m(n)$

$$A_{m-1}C_{m-1}(n) + B_m C_m(n) + \Gamma_{m+1}C_{m+1}(n) = 0, \tag{34}$$

where

$$
\begin{aligned}
A_{m-1} &= \tilde{S}_{m-1} - q^{-m+2}[m-1]_q\alpha_{m-1} = \\
&= \frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m-1}\alpha_{m-1} - \frac{B_{m-1}}{\tau'_{m-1}B_m}\right] - q^{-m+2}[m-1]_q\alpha_{m-1} \quad , \\
B_m &= \tilde{T}_m - q^{-m+1}[m]_q\beta_m = \\
&= \frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m}\beta_{m} - \frac{\tau_m(0)}{\tau'_m}\right] - q^{-m+1}[m]_q\beta_m \quad , \\
\Gamma_{m+1} &= \tilde{R}_{m+1} - q^{-m}\gamma_{m+1} = \frac{\lambda_{m+1}q^{m+1}\gamma_{m+1}}{[m+1]_q} - q^{-m}[m+1]_q\gamma_{m+1}.
\end{aligned}
\tag{35}
$$

In order to obtain the recurrence relation for connection coefficients in the q-Hahn case we apply the operator $\tilde{\mathcal{L}}$ (32) to both sides of (31). Taking into account that $\tilde{\mathcal{L}}(s_n)_q = 0$ and multiplying by $q^{2\alpha+2N} - q^{2s}$ we obtain the following expression

$$
0 = \sum_{m=0}^{n} C_m(n)\left\{(q^{2\alpha+2N} - q^{2s})(q^{2s}-1)\frac{\bigtriangledown Q_m(x)}{\bigtriangledown x(s)} - q^{-m+1}[m]_q q^{2s}Q_m(x)\right\}. \tag{36}
$$

Taking into account that for q-Hahn the $\sigma(s)$ function in (2) coincide with $(q^{2\alpha+2N} - q^{2s})(q^{2s}-1)$ (see [17]) and using the structure relation (14) and the TTRR (7) we obtain the same expression (34) as before for the TTRR for the connection coefficients $C_m(n)$, where now

$$
\begin{aligned}
A_{m-1} &= \tilde{S}_{m-1} + q^{-m+2}[m-1]_q\alpha_{m-1} = \\
&= \frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m-1}\alpha_{m-1} - \frac{B_{m-1}}{\tau'_{m-1}B_m}\right] + q^{-m+2}[m-1]_q\alpha_{m-1} \quad , \\
B_m &= \tilde{T}_m + q^{-m+1}[m]_q\beta_m - [m]_q q^{2N+2\alpha-m+1} = \\
&= \frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m}\beta_{m} - \frac{\tau_m(0)}{\tau'_m}\right] + q^{-m+1}[m]_q\beta_m - [m]_q q^{2N+2\alpha-m+1} \quad , \\
\Gamma_{m+1} &= \tilde{R}_{m+1} + q^{-m}\gamma_{m+1} = \frac{\lambda_{m+1}q^{m+1}\gamma_{m+1}}{[m+1]_q} + q^{-m}[m+1]_q\gamma_{m+1}.
\end{aligned}
\tag{37}
$$

4.1 The three term recurrence relation for connection coefficients of the q-powers $(s_n)_q$ and the q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$.

Here we will calculate the coefficients A_{m-1}, B_m and Γ_{m+1} of the three term recurrence relation for connection coefficients $C_m(n)$ (34) of the q-powers $(s_n)_q$ and the q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$, i.e.,

$$
(s_n)_q = \sum_{k=0}^{n} C_m(n) m_k^{\gamma,\mu}(s,q).
$$

The main data for the q-Meixner polynomials are provided in [4]. In our work we will use monic polynomials, i.e., the leading coefficient $a_n = 1$. In the Table I, at the end of this section, we provide the quantities needed for our calculations. (For more details see [4] and [13]). We want to point out that these monic q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$ [4] are connected with the monic little q-Jacobi polynomials $p_n(x;a,b|q)$ [6], [8] by the relation

$$m_n^{\gamma,\mu}(s,q) = p_n(q^{2s};\mu,q^{2\gamma-2}|q^2).$$

If we now apply formulas (15) and (14) we obtain for q-Meixner polynomials the structure relation

$$\sigma(s)\frac{\bigtriangledown m_n^{\gamma,\mu}(s,q)}{\bigtriangledown x(s)} = \tilde{S}_n m_{n+1}^{\gamma,\mu}(s,q) + \tilde{T}_n m_n^{\gamma,\mu}(s,q) + \tilde{R}_n m_{n-1}^{\gamma,\mu}(s,q), \tag{38}$$

where

$$\begin{aligned}
\tilde{S}_n = &\quad -q^{\gamma+\theta+1}\left(q^n[n+\gamma+\theta]_q + [2n+\gamma+\theta]_q\right) \quad , \\
\tilde{T}_n = &\quad -q^{2n+\theta+2}[n+\gamma+\theta]_q\left(\frac{[n+1]_q[n+\theta+1]_q}{[2n+\gamma+\theta+1]_q} - \frac{[n]_q[n+\theta]_q}{[2n+\gamma+\theta-1]_q}\right) - \\
&-\frac{q^{\theta+1}[n+\theta+1]_q[n+\gamma+\theta]_q}{[2n+\gamma+\theta+1]_q}, \\
\tilde{R}_n = &\quad -\frac{q^{-2\gamma+\theta-1}[n]_q[\gamma+n-1]_q[n+\gamma+\theta]_q[n+\gamma+\theta-1]_q[n+\theta]_q}{[2n+\gamma+\theta-2]_q[2n+\gamma+\theta-1]_q^2[2n+\gamma+\theta]_q}.
\end{aligned} \tag{39}$$

Then, by using (35) we finally find the coefficients A_{m-1}, B_m and Γ_{m+1}

$$A_{m-1} = -q^{\gamma+\theta+1}\left(q^{m-1}[m+\gamma+\theta-1]_q + [2m+\gamma+\theta-2]_q\right) - q^{-m+2}[m-1]_q, \tag{40}$$

$$\begin{aligned}
B_m = &\quad -\left(\frac{[m+1]_q[m+\theta+1]_q}{[2m+\gamma+\theta+1]_q} - \frac{[m]_q[m+\theta]_q}{[2m+\gamma+\theta-1]_q}\right)q^{\theta+1}[2m+\gamma+\theta]_q - \\
&-\frac{q^{\theta+1}[m+\gamma+\theta]_q[m+\theta+1]_q}{[2m+\gamma+\theta+1]_q},
\end{aligned} \tag{41}$$

$$\Gamma_{m+1} = \quad -\frac{q^{-m-2\gamma+\theta+2}[m+1]_q[\gamma+m]_q[m+\gamma+\theta]_q[m+\theta+1]_q}{[2m+\gamma+\theta]_q[2m+\gamma+\theta+1]_q[2m+\gamma+\theta+2]_q}. \tag{42}$$

Table I: The main data for q-Meixner Polynomials [4].

	$m_n^{\gamma,\mu}(s,q)$, $\mu = q^{2\theta}$
$\sigma(s)$	$q^{2s}(q^{2s}-1)$
$\tau(s)$	$q^{s+2\theta+\gamma+2}[s+\gamma]_q - q^s[s]_q$
λ_n	$-[n]_q q^{\gamma+\theta+1}[n+\gamma+\theta]_q$
τ_n'	$q^{\gamma+\theta+1}[2n+\gamma+\theta+1]_q$
$\tau_n(0)$	$-q^{\theta+1}[n+\theta+1]_q$
$\dfrac{B_n}{B_{n+1}}$	$-q^{\gamma+\theta+1}\dfrac{[2n+\gamma+\theta+1]_q[2n+\gamma+\theta]_q}{[n+\gamma+\theta]_q}$
α_n	1
β_n	$q^{-\gamma}\dfrac{[n+1]_q[n+\theta+1]_q}{[2n+\gamma+\theta+1]_q} - q^{-\gamma}\dfrac{[n]_q[n+\theta]_q}{[2n+\gamma+\theta-1]_q}$
γ_n	$\dfrac{q^{-n-3\gamma+2}[n]_q[\gamma+n-1]_q[n+\gamma+\theta-1]_q[n+\theta]_q}{[2n+\gamma+\theta-2]_q[2n+\gamma+\theta-1]_q^2[2n+\gamma+\theta]_q}$

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