ON THE PROPERTIES OF SPECIAL FUNCTIONS ON THE LINEAR-TYPE LATTICES

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Abstract. We present a general theory for studying the difference analogues of special functions of hypergeometric type on the linear-type lattices, i.e., the solutions of the second order linear difference equation of hypergeometric type on a special kind of lattices: the linear type lattices. In particular, using the integral representation of the solutions we obtain several difference-recurrence relations for such functions. Finally, applications to q-classical polynomials are given.

2000 Mathematics Subject Classification 33D15, 33D45

1. Introduction

The study of the so-called q-special functions has known an increasing interest in the last years due its connection with several problems in mathematics and mathematical-physics (see e.g. [3, 6, 8, 13, 17]). A systematic study starting from the second order linear difference equation that such functions satisfy was started by Nikiforov and Uvarov in 1983 and further developed by Atakishiyev and Suslov (for a very nice reviews see e.g. [7, 13, 16]). Of particular interest is the so-called q-classical polynomials (see e.g. [5]) introduced by Hahn in 1949 which are polynomials on the lattice q^s.

Our main aim in this paper is to present a constructive approach for generating recurrence relations and ladder-type operators for the difference analogues of special functions of hypergeometric type on the linear-type lattices. Here we will focus our attention on functions defined on the q-linear lattice (for the linear lattice $x(s) = s$ see [4] and references therein, and for the continuous case see e.g. [18]). Therefore we will complete the work started in [16] where few recurrence relations where obtained. In fact we will prove, by using the q-analoge of the technique introduced in [4] for the discrete case (uniform lattice), that the solutions (not only the polynomial ones) of the difference equation on the q-linear lattice $x(s) = c_1q^s + c_2$ satisfy a very general recurrent-difference relation from where several well known relations (such as the three-term recurrence relation and the ladder-type relations) follow.

The structure of the paper is as follows: In section 2 the needed results and notations from the q-special function theory are introduced. In sections 3 and 4 the general theorems for obtaining recurrences relations are presented. In section 5 the special case of classical q-polynomials are considered in details and some examples are worked out in details.

2. Some preliminar results

Here we collect the basic background [1, 13, 16] on q-hypergeometric functions needed in the rest of the work.

The hypergeometric functions on the non-uniform lattice $x(s)$ are the solutions of the second order linear difference equation of hypergeometric type on non-uniform lattices

$$\begin{aligned}&\sigma(s)\frac{\Delta}{\Delta x(s-\frac12)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0,\\ &\sigma(s)=\widetilde{\sigma}(x(s))-\tfrac12\widetilde{\tau}(x(s))\Delta x\left(s-\tfrac12\right),\quad \tau(s)=\widetilde{\tau}(x(s)),\end{aligned}\qquad(1)$$

Key words and phrases. q-hypergeometric functions, difference equations, recurrence relations, q-polynomials.

MAY 15, 2011

where $\Delta y(s) := y(s+1) - y(s)$, $\nabla y(s) := y(s) - y(s-1)$, are the forward and backward difference operators, respectively; $\widetilde{\sigma}(x(s))$ and $\widetilde{\tau}(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. Here we will deal with the linear and q-linear lattices, i.e., lattices of the form

$$x(s) = c_1 s + c_2 \quad \text{or} \quad x(s) = c_1(q) q^s + c_2(q), \tag{2}$$

respectively, with $c_1 \neq 0$ and $c_1(q) \neq 0$.

We will define the *k-order difference derivative* of a solution $y(s)$ of (1) by

$$y^{(k)}(s) := \Delta^{(k)}[y(s)] = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)}[y(s)],$$

where $x_\nu(s) = x(s + \frac{\nu}{2})$. It is known [13] that $y^{(k)}(s)$ also satisfy a difference equation of the same type. Moreover, for the solutions of the difference equation (1) the following theorem holds

Theorem 2.1. [12, 16] *The difference equation (1) has a particular solution of the form*

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \sum_{s=a}^{b-1} \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}}, \tag{3}$$

if the condition

$$\left. \frac{\sigma(s) \rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z+1)]^{(\nu+1)}} \right|_a^b = 0,$$

is satisfied, and of the form

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}} ds, \tag{4}$$

if the condition

$$\int_C \Delta_s \frac{\sigma(s) \rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z+1)]^{(\nu+1)}} = 0, \tag{5}$$

is satisfied. Here C is a contour in the complex plane, C_ν is a constant, $\rho(s)$ and $\rho_\nu(s)$ are the solution of the Pearson-type equations

$$\begin{aligned} \frac{\rho(s+1)}{\rho(s)} &= \frac{\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi(s)}{\sigma(s+1)}, \\ \frac{\rho_\nu(s+1)}{\rho_\nu(s)} &= \frac{\sigma(s) + \tau_\nu(s) \Delta x_\nu(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi_\nu(s)}{\sigma(s+1)}, \end{aligned} \tag{6}$$

where

$$\tau_\nu(s) = \frac{\sigma(s+\nu) - \sigma(s) + \tau(s+\nu) \Delta x(s + \nu - \frac{1}{2})}{\Delta x_{\nu-1}(s)}, \tag{7}$$

ν is the root of the equation

$$\lambda_\nu + [\nu]_q \left\{ \alpha_q(\nu-1) \widetilde{\tau}' + [\nu-1]_q \frac{\widetilde{\sigma}''}{2} \right\} = 0, \tag{8}$$

and $[\nu]_q$ and $\alpha_q(\nu)$ are the q-numbers

$$[\nu]_q = \frac{q^{\nu/2} - q^{-\nu/2}}{q^{1/2} - q^{-1/2}}, \qquad \alpha_q(\nu) = \frac{q^{\nu/2} + q^{-\nu/2}}{2}, \quad \forall \nu \in \mathbb{C}, \tag{9}$$

respectively. The generalized powers $[x_k(s) - x_k(z)]^{(\nu)}$ are defined by

$$[x_k(s) - x_k(z)]^{(\nu)} = (q-1)^\nu c_1^\nu q^{\nu(k-\nu+1)/2} q^{\nu z} \frac{\Gamma_q(s - z + \nu)}{\Gamma_q(s-z)}, \quad \nu \in \mathbb{R}, \tag{10}$$

for the q-linear (exponential) lattice $x(s) = c_1 q^s + c_2$ and

$$[x_k(s) - x_k(z)]^{(\nu)} = c_1^\nu \frac{\Gamma(s - z + \mu)}{\Gamma(s-z)}, \quad \nu \in \mathbb{R},$$

for the linear lattice $x(s)=c_1s+c_2$, respectively. For the definitions of the Gamma and the q-Gamma functions see, for instance, [6].

Remark 2.2. *For the special case when $\nu\in\mathbb{N}$, the generalized powers become*

$$\begin{aligned}[x_k(s)-x_k(z)]^{(n)} =&(-1)^n c_1^n q^{-n(n-1)/2} q^{n(z+k/2)}(q^{s-z};q)_n,\\ [x_k(s)-x_k(z)]^{(n)} =&c_1^n(s-z)_n,\end{aligned}$$

for q-linear and linear lattices, respectively.

We will need the following straightforward proposition which proof we omit here (see e.g. [1, 16])

Proposition 2.3. *Let μ and ν be complex numbers and m and k be positive integers with $m\geq k$. For the q-linear lattice $x(s)=c_1q^s+c_2$ we have*

(1) $\dfrac{[x_\mu(s)-x_\mu(z)]^{(m)}}{[x_\nu(s)-x_\nu(z)]^{(m)}}=q^{\frac{m(\mu-\nu)}{2}}$,

(2) $\dfrac{[x_\mu(s)-x_\mu(z)]^{(m)}}{[x_\mu(s)-x_\mu(z)]^{(k)}}=[x_\mu(s)-x_\mu(z-k)]^{(m-k)}$,

(3) $\dfrac{[x_\mu(s)-x_\mu(z)]^{(m)}}{[x_\nu(s)-x_\nu(z)]^{(k)}}=q^{\frac{k(\mu-\nu)}{2}}[x_\mu(s)-x_\mu(z-k)]^{(m-k)}$,

(4) $\dfrac{[x_\mu(s)-x_\mu(z)]^{(m+1)}}{[x_{\mu-1}(s+1)-x_{\mu-1}(z)]^{(m)}}=x_{\mu-m}(s)-x_{\mu-m}(z)$,

(5) $\dfrac{[x_\mu(s)-x_\mu(z)]^{(m+1)}}{[x_{\mu-1}(s)-x_{\mu-1}(z)]^{(m)}}=x_{\mu-m}(s+m)-x_{\mu-m}(z)$.

To obtain the result for the linear lattice one only has to put in the above formulas $q=1$.

3. The general recurrence relation in the linear-type lattices

In this section we will obtain several recurrence relations for the solutions (3) and (4) of the difference equation (1) in the linear-type lattices (2). Since the equation (1) is linear we can restrict ourselves to the canonical cases $x(s)=q^s$ and $x(s)=s$.

Let us define the functions[1]

$$\Phi_{\nu,\mu}(z)=\sum_{s=a}^{b-1}\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+1)}} \tag{11}$$

and

$$\Phi_{\nu,\mu}(z)=\int_C\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\nu+1)}}ds. \tag{12}$$

Notice that the functions y_ν and the functions $\Phi_{\nu,\mu}$ are related by the formula

$$y_\nu(z)=\frac{C_\nu}{\rho(z)}\Phi_{\nu,\nu}(z). \tag{13}$$

Lemma 3.1. *For the functions $\Phi_{\nu,\mu}(z)$ the following relation holds*

$$\nabla_z\,\Phi_{\nu,\mu}(z)=[\mu+1]_q\nabla x_{\nu-\mu}(z)\Phi_{\nu,\mu+1}(z), \tag{14}$$

where $[t]_q$ denotes the symmetric q-numbers (9).

[1]Obviously the functions (3) correspond to the functions (11), whereas the functions y_ν given by (4) correspond to those of (12).

Proof. We will prove it for the functions (11). The other case is analogous. Using (10), one gets

$$\begin{aligned}
\nabla_z \Phi_{\nu,\mu}(z) =& \sum_{s=a}^{b-1} \nabla_z \left(\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+1)}} \right)\\
=& \sum_{s=a}^{b-1} \left(\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+1)}} - \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z-1)]^{(\mu+1)}} \right)\\
=& \sum_{s=a}^{b-1} \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z-1)]^{(\mu)}} \left(\frac{1}{x_\nu(s)-x_\nu(z)} - \frac{1}{x_\nu(s)-x_\nu(z-1-\mu)} \right)\\
=& \sum_{s=a}^{b-1} \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z-1)]^{(\mu)}} \frac{x_\nu(z)-x_\nu(z-1-\mu)}{(x_\nu(s)-x_\nu(z))(x_\nu(s)-x_\nu(z-1-\mu))}\\
=& \sum_{s=a}^{b-1} \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+2)}} \left(x_\nu(z)-x_\nu(z-1-\mu)\right)
\end{aligned}$$

Since $x(s)-x(s-t)=[t]_q\nabla x\left(s-\frac{t-1}{2}\right)$ we then have

$$\begin{aligned}
\nabla_z \Phi_{\nu,\mu}(z) =& \sum_{s=a}^{b-1} \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+2)}} [\mu+1]_q \nabla x_\nu\left(z-\frac{\mu}{2}\right)\\
=& [\mu+1]_q \nabla x_{\nu-\mu}(z)\Phi_{\nu,\mu+1}(z)
\end{aligned}$$

which is (14). □

From (14) follows that

$$\Delta_z \Phi_{\nu,\mu}(z) = [\mu+1]_q \Delta x_{\nu-\mu}(z)\Phi_{\nu,\mu+1}(z+1).$$

Next we prove the following lemma that is the discrete analog of the Lemma in [14, page 14].

Lemma 3.2. . *Let $x(z)$ be $x(z)=q^z$ or $x(z)=z$. Then, any three functions $\Phi_{\nu_i,\mu_i}(z)$, $i=1,2,3$, are connected by a linear relation*

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i,\mu_i}(z)=0, \tag{15}$$

with non-zero at the same time polynomial coefficients on $x(z)$, $A_i(z)$, provided that the differences $\nu_i-\nu_j$ and $\mu_i-\mu_j$, $i,j=1,2,3$, are integers and that the following condition holds[2]

$$\left.\frac{x^k(s)\sigma(s)\rho_{\nu_0}(s)}{[x_{\nu_0-1}(s)-x_{\nu_0-1}(z)]^{(\mu_0)}}\right|_{s=a}^{s=b}=0, \quad k=0,1,2,\ldots, \tag{16}$$

when the functions Φ_{ν_i,μ_i} are given by (11) and

$$\int_C \Delta_s \frac{x^k(s)\sigma(s)\rho_{\nu_0}(s)\,ds}{[x_{\nu_0-1}(s)-x_{\nu_0-1}(z)]^{(\mu_0)}}=0, \quad k=0,1,2,\ldots, \tag{17}$$

when Φ_{ν_i,μ_i} are given by (12). Here ν_0 is the ν_i, $i=1,2,3$, with the smallest real part and μ_0 is the μ_i, $i=1,2,3$, with the largest real part.

Proof. Since in [4] we have proved the case when $x(s)=s$ (the uniform lattice) we will restrict here to the case of the q-linear lattice $x(s)=c_1q^s+c_2$). Moreover, we will give the proof for the case of functions of the form (11), the other case is completely equivalent. Using the identity

$$\nabla x_{\nu_i+1}(s)=q^{\frac{\nu_i-\nu_0}{2}}\nabla x_{\nu_0+1}(s),$$

[2] In some cases this condition is equivalent to the condition $x(s)^k\sigma(s)\rho_{\nu_0}(s)|_{s=a}^{s=b}=0$, $k=0,1,2,\ldots$.

as well as (3) of Proposition 2.3, we have

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i,\mu_i}(z) = \sum_{i=1}^{3} A_i(z) \sum_{s=a}^{b-1} \frac{\rho_{\nu_i}(s)\nabla x_{\nu_i+1}(s)}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i+1)}}$$
$$= \sum_{s=a}^{b-1}\sum_{i=1}^{3} A_i(z) \frac{\rho_{\nu_i}(s)\nabla x_{\nu_i+1}(s)}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i+1)}} = \sum_{s=a}^{b-1} \frac{1}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \times$$
$$\left(\sum_{i=1}^{3} A_i(z) q^{\frac{(\mu_i+1)(\nu_0-\nu_i)}{2}} [x_{\nu_0}(s) - x_{\nu_0}(z-\mu_i-1)]^{(\mu_0-\mu_i)} \rho_{\nu_i}(s)\nabla x_{\nu_i+1}(s) \right)$$
$$= \sum_{s=a}^{b-1} \frac{\rho_{\nu_0}(s)\nabla x_{\nu_0+1}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \times$$
$$\left(\sum_{i=1}^{3} A_i(z) q^{\frac{\mu_i(\nu_0-\nu_i)}{2}} [x_{\nu_0}(s) - x_{\nu_0}(z-\mu_i-1)]^{(\mu_0-\mu_i)} \frac{\rho_{\nu_i}(s)}{\rho_{\nu_0}(s)} \right).$$

Using the Pearson-type equation (6) we obtain

$$\rho_{\nu_i}(s) = \phi(s+\nu_0)\phi(s+\nu_0+1)\dots\phi(s+\nu_i-1)\rho_{\nu_0}(s), \tag{18}$$

so

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i,\mu_i}(z) = \sum_{s=a}^{b-1} \frac{\rho_{\nu_0}(s)\nabla x_{\nu_0+1}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \Pi(s)$$

where

$$\begin{aligned}\Pi(s) = \sum_{i=1}^{3} A_i(z) q^{\frac{\mu_i(\nu_0-\nu_i)}{2}} [x_{\nu_0}(s) - x_{\nu_0}(z-\mu_i-1)]^{(\mu_0-\mu_i)} \times \\ \phi(s+\nu_0)\phi(s+\nu_0+1)\cdots\phi(s+\nu_i-1)\,.\end{aligned} \tag{19}$$

Let us show that there exists a polynomial $Q(s)$ in $x(s)$ (in general, $Q \equiv Q(z,s)$ is a function of z and s) such that

$$\begin{aligned}\frac{\rho_{\nu_0}(s)\nabla x_{\nu_0+1}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}}\Pi(s) =&\Delta \left[\frac{\rho_{\nu_0}(s-1)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) \right] \\ =&\Delta \left[\frac{\sigma(s)\rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) \right] .\end{aligned} \tag{20}$$

If such polynomial exists, then, taking the sum in s from $s = a$ to $b-1$ and using the boundary conditions (16) we obtain (15).

To prove the existence of the polynomial $Q(s)$ in the variable $x(s)$ in (20) we write

$$\frac{\sigma(s+1)\rho_{\nu_0}(s+1)}{[x_{\nu_0-1}(s+1) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s+1) - \frac{\sigma(s)\rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) =$$
$$\frac{\rho_{\nu_0}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \left[\sigma(s+1)\frac{\rho_{\nu_0}(s+1)}{\rho_{\nu_0}(s)} \frac{[x_{\nu_0}(s)-x_{\nu_0}(z)]^{(\mu_0+1)}}{[x_{\nu_0-1}(s+1)-x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s+1) - \right.$$
$$\left. \sigma(s) \frac{[x_{\nu_0}(s)-x_{\nu_0}(z)]^{(\mu_0+1)}}{[x_{\nu_0-1}(s)-x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) \right] .$$

From (4) and (5) of Proposition 2.3, and using (6), the above expression becomes

$$\frac{\rho_{\nu_0}(s)}{[x_{\nu_0}(s)-x_{\nu_0}(z)]^{(\mu_0+1)}} \left\{ \phi_{\nu_0}(s) \left[x_{\nu_0-\mu_0}(s) - x_{\nu_0-\mu_0}(z) \right] Q(s+1) - \right.$$
$$\left. \sigma(s) \left[x_{\nu_0-\mu_0}(s+\mu_0) - x_{\nu_0-\mu_0}(z) \right] Q(s) \right\} .$$

Thus

$$\begin{aligned}(\sigma(s)+\tau_{\nu_0}(s)\nabla x_{\nu_0+1}(s))\,[x_{\nu_0-\mu_0}(s)-x_{\nu_0-\mu_0}(z)]\,Q(s+1)-\\ \sigma(s)\,[x_{\nu_0-\mu_0}(s+\mu_0)-x_{\nu_0-\mu_0}(z)]\,Q(s)=\nabla x_{\nu_0+1}(s)\Pi(s).\end{aligned} \tag{21}$$

Since $\nabla x_{\nu_0+1}(s)$ is a polynomial of degree one in $x(s)$, $x_k(s)$ and $\tau_{\nu_0}(s)$ are polynomials of degree at most one in $x(s)$, and $\sigma(s)$ is a polynomial of degree at most two in $x(s)$, we conclude that the degree of $Q(s)$ is, at least, two less than the degree of $\Pi(s)$, i.e., $\deg Q \geq \deg \Pi - 2$. Moreover, equating the coefficients of the powers of $x(s)=q^s$ on the two sides of the above equation (21), we find a system of linear equations in the coefficients of $Q(s)$ and the coefficients $A_i(z)$ which have at least one unknown more then the number of equations. Notice that the coefficients of the unknowns are polynomials in q^z, so that after one coefficient is selected the remaining coefficients are rational functions of q^z, therefore after multiplying by the common denominator of the $A_i(z)$ we obtain the linear relation with polynomial coefficients on $x\equiv x(z)=q^z$. This completes the proof. □

The above Lemma when $q\to 1$ and $x(s)=s$ leads to the corresponding result on the uniform lattice $x(s)$ [4].

3.1. **Some representative examples.** In the following examples, and for the sake of simplicity, we will use the notation

$$\sigma(s)=aq^{2s}+bq^s+c,\ \tau(s)=dq^s+e,\ \phi_\nu(s)=\sigma(s)+\tau_{\nu-1}(s)\nabla x_\nu(s)=fq^{2s}+gq^s+h. \tag{22}$$

Example 3.3. *The following relation holds*

$$A_1(z)\Phi_{\nu,\nu-1}(z)+A_2(z)\Phi_{\nu,\nu}+A_3(z)\Phi_{\nu+1,\nu}(z)=0,$$

where the coefficients A_1, A_2 *and* A_3, *are polynomials in* $x\equiv x(z)=q^z$, *given by*

$$A_1(z)=-\,eq^{\frac{\nu}{2}}+\frac{b+e\left(q^{\frac12}-q^{-\frac12}\right)}{a+d\left(q^{\frac12}-q^{-\frac12}\right)}\left(dq^{\frac{\nu}{2}}+a[\nu]_q\right)+\left(dq^\nu+a[2\nu]_q\right)q^{\frac{\nu}{2}+z},$$

$$A_2(z)=\frac{c\,(dq^\nu+a[2\nu]_q)}{a+d\left(q^{\frac12}-q^{-\frac12}\right)}+\frac{b+e\left(q^{\frac12}-q^{-\frac12}\right)}{q^{\frac12}-q^{-\frac12}}\left(q^\nu+\frac{a}{q^\nu\left(a+d\left(q^{\frac12}-q^{-\frac12}\right)\right)}\right)q^z+\left(dq^\nu+a[2\nu]_q\right)q^{2z},$$

$$A_3(z)=-\,\frac{dq^{\frac{\nu}{2}}+a[\nu]_q}{a+d\left(q^{\frac12}-q^{-\frac12}\right)},$$

where a, b, c, d, and e, are the coefficients of σ *and* τ *(22).*

Proof. Using the notations of Lemma 3.2 we have $\nu_1=\nu$, $\nu_2=\nu$, $\nu_3=\nu+1$, $\mu_1=\nu-1$, $\mu_2=\nu$ and $\mu_3=\nu$, thus $\nu_0=\nu$ and $\mu_0=\nu$. By (19)

$$\Pi(s)=A_1\left(q^{s+\frac{\nu}{2}}-q^{z-\frac{\nu}{2}}\right)+A_2+A_3q^{-\frac{\nu}{2}}\left[\left(a+d\left(q^{\frac12}-q^{-\frac12}\right)\right)q^{2\nu+2s}+\left(b+e\left(q^{\frac12}-q^{-\frac12}\right)\right)q^{\nu+s}+c\right]. \tag{23}$$

On the other hand, from (21) and because $Q(s)=k$ is a constant –notice that $\deg(\Pi)=2$– we have

$$\begin{aligned}\nabla x_{\nu_0+1}(s)\Pi(s)=k\Big\{\Big[\left(a+d\left(q^{\frac12}-q^{-\frac12}\right)\right)q^{2\nu+2s}+\left(b+e\left(q^{\frac12}-q^{-\frac12}\right)\right)q^{\nu+s}+c\Big]\,(q^s-q^z)-\\ \left(aq^{2s}+bq^s+c\right)\left(q^{\nu+s}-q^z\right)\Big\}\end{aligned} \tag{24}$$

where k is an arbitrary constant. Introducing (23) in (24), using the identity

$$\nabla x_{\nu_0+1}(s)=q^{\frac{\nu}{2}}\left(q^{\frac12}-q^{-\frac12}\right)q^s$$

and comparing the coefficients of the powers of $x(s)=q^s$ we get a linear system of three equations with four variables A_1, A_2, A_3 and k. Choosing $k=1$ and solving the corresponding system we get, after some simplifications, the coefficients A_1, A_2 and A_3. □

In the next examples, since the technique is similar to the previous one we will omit the details.

Example 3.4. *The following relation holds*

$$A_1(z)\Phi_{\nu,\nu}(z)+A_2(z)\Phi_{\nu,\nu+1}(z)+A_3(z)\Phi_{\nu+1,\nu+1}(z)=0\,,$$

where the coefficients A_1 *,* A_2 *and* A_3*, are polynomials in* $x\equiv x(z)=q^z$ *, given by*

$$\begin{aligned}
A_1(z) &= f\left(a-f\,q^{2\,\nu}\right)q^z+a\,g\,q-f\,b\,q^{\nu+1}\,,\\
A_2(z) &= q^{-\frac{\nu}{2}-1}\,\left(a-f\,q^{2\,\nu}\right)\,\left(f\,q^{2\,z}+g\,q^{z+1}+h\,q^2\right)\,,\\
A_3(z) &= \sqrt{q}\,\left(a\,q-f\,q^{\nu}\right),
\end{aligned}$$

where a*,* b*,* c*,* f*,* g *and* h*, are the coefficients of* σ *and* ϕ_ν *(22).*

Example 3.5. *The following relation holds*

$$A_1(z)\Phi_{\nu-1,\nu-1}(z)+A_2(z)\Phi_{\nu,\nu-1}(z)+A_3(z)\Phi_{\nu,\nu}(z)=0\,,$$

where the coefficients A_1 *,* A_2 *and* A_3*, are polynomials in* $x\equiv x(z)=q^z$ *, given by*

$$\begin{aligned}
A_1(z) =& q^{-\frac{1}{2}-\nu}\Big\{\,fq^{2z}\Big[-a^2hq^4+agbq^{\nu+4}-q^{2\nu+2}\left(ag^2q-2fah+fb^2\right)\\
&\quad -fgbq^{3\nu+1}\left(q^2-q-1\right)+fq^{4\nu}\left(g^2\left(q-1\right)q-fh\right)\Big]+\\
&\quad gq^{z+1}\Big[-a^2hq^5+aq^{\nu+2}\left(gbq^3+fhq^2-fh\right)-q^{2\nu+2}\big(\left(fah+fgb+ag^2\right)q^2-\\
&\quad f\left(2ah-b^2+gb\right)q-fah\big)+fq^{3\nu}\left(q^2\left(g^2q-fh+gb-g^2\right)+fh\right)+f^2hq^{4\nu}\left(q^2-q-1\right)\Big]\\
&\quad -a^2h^2q^6+aghq^{\nu+5}\left(bq+gq-g\right)+fghq^{3\nu+4}\left(gq+b-g\right)-f^2\,h^2q^{4\nu+2}\\
&\quad -hq^{2\nu+3}\Big(ag^2q^3+fgbq^2+fg^2q^2-2fahq+fb^2q-2fg^2q-fgb+fg^2\Big)\Big\}\,,\\
A_2(z) =& \left(q^{-\frac{\nu}{2}}-q^{\frac{\nu}{2}}\right)\left(fq^{2z}+gq^{z+1}+hq^2\right)\Big(fq^z\left(fq^{2\nu}-aq^2\right)+fq^{\nu+1}(gq+b-g)-agq^3\Big),\\
A_3(z) =& f\left(fq^{\nu}-aq\right)\Big[\left(fq^{2z}+hq^2\right)\left(fq^{2\nu}-aq^2\right)+gq^{z+1}\Big(fq^{\nu}\left(q^{\nu}+q-1\right)-aq^3\Big)\Big],
\end{aligned}$$

where a*,* b*,* c*,* f*,* g *and* h*, are the coefficients of* σ *and* ϕ_ν *(22).*

Example 3.6. *The following relation holds*

$$A_1(z)\Phi_{\nu-1,\nu-1}(z)+A_2(z)\Phi_{\nu,\nu}(z)+A_3(z)\Phi_{\nu,\nu+1}(z)=0\,,$$

where the coefficients A_1 *,* A_2 *and* A_3*, are polynomials in* $x\equiv x(z)=q^z$ *, given by*

$$\begin{aligned}
A_1(z) &= a^2hq^4-agbq^{\nu+3}+q^{2\nu+2}\left(fb^2-2fah+ag^2\right)-fgbq^{3\nu+1}+f^2hq^{4\nu},\\
A_2(z) &= q^{-\frac{1}{2}}\left(fq^{\nu}-aq^2\right)\left(fq^{z+2\nu}-aq^{z+2}+gq^{2\nu+1}-bq^{\nu+2}\right),\\
A_3(z) &= -q^{\frac{v-3}{2}}\left(f\,q^{2v}-aq^2\right)\left(q^{v+1}-1\right)\left(gq^{z+1}+fq^{2z}+hq^2\right),
\end{aligned}$$

where a*,* b*,* c*,* f*,* g *and* h*, are the coefficients of* σ *and* ϕ_ν *(22).*

Example 3.7. *The relation*

$$A_1(z)\Phi_{\nu,\nu-1}(z)+A_2(z)\Phi_{\nu,\nu}(z)+A_3(z)\Phi_{\nu+1,\nu+1}(z)=0\,,$$

is verified when the polynomial coefficients A_1 *,* A_2 *and* A_3*, in the variable* $x\equiv x(z)=q^z$ *, are given by*

$$\begin{aligned}
A_1(z) &= q^{\frac{\nu+1}{2}}\Big(fq^{z+\nu}\left(fq^{2\nu}-gq^{\nu}+b-a\right)-f\left(h-b\right)q^{2\nu+1}-aq\left(gq^{\nu}-h\right)\Big)\,,\\
A_2(z) &= q^{-z+\nu+\frac{1}{2}}\Big(q^z\left(fq^{2\nu}-a\right)+q^{\nu}\left(gq^{\nu}-b\right)\Big)\left(fq^{2z}+gq^{z+1}+hq^2\right)\,,\\
A_3(z) &= q^{2z}\left(fq^{\nu}-aq\right)+q^{z+\nu}\Big(q\left(gq^{\nu}-aq-b\right)-fq^{\nu}\left(q^{\nu+1}-q-1\right)\Big)+\\
&\quad q^{\nu+1}\big(\left(h-b\right)q^{\nu+1}+gq^{\nu}-h\big),
\end{aligned}$$

where a*,* b*,* c*,* f*,* g *and* h*, are the coefficients of* σ *and* ϕ_ν *(22).*

4. Recurrences involving the solutions y_ν

In [16] the following relevant relation was established

$$\Delta^{(k)} y_\nu(s) = \frac{C_\nu^{(k)}}{\rho_k(s)} \Phi_{\nu\,,\nu-k}(s), \tag{25}$$

where

$$C_\nu^{(k)} = C_\nu \prod_{m=0}^{k-1} \left[\alpha_q(\nu+m-1)\widetilde{\tau}' + [\nu+m-1]_q \frac{\widetilde{\sigma}''}{2} \right].$$

This relation is valid for solutions of the form (3) and (4) of the difference equation (1). In the following, $y_n^{(k)}(s)$ denotes the k-th differences $\Delta^{(k)} y_n(s)$.

Theorem 4.1. *In the same conditions as in Lemma 3.2, any three functions* $y_{\nu_i}^{(k_i)}(s)$, $i = 1, 2, 3$, *are connected by a linear relation*

$$\sum_{i=1}^{3} B_i(s) y_{\nu_i}^{(k_i)}(s) = 0, \tag{26}$$

where the $B_i(s)$, $i = 1, 2, 3$, *are polynomials.*

Proof. From Lemma 3.2 we know that there exists three polynomials $A_i(s)$, $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} A_i(s) \Phi_{\nu_i, \nu_i - k_i}(s) = 0,$$

then, using the relation (25), we find

$$\sum_{i=1}^{3} A_i(s) (C_\nu^{(k)})^{-1} \rho_{k_i}(s) y_{\nu_i}^{(k_i)}(s) = 0.$$

Now, dividing the last expression by $\rho_{k_0}(s)$, where $k_0 = \min\{k_1, k_2, k_3\}$, and using (18) we obtain

$$\sum_{i=1}^{3} B_i(s) y_{\nu_i}^{(k_i)}(s) = 0, \quad B_i(s) = A_i(s)(C_\nu^{(k)})^{-1} \phi(s+k_0) \cdots \phi(s+k_i-1),$$

which completes the proof. ☐

Corollary 4.2. *In the same conditions as in Lemma 3.2, the following three-term recurrence relation holds*

$$A_1(s) y_\nu(s) + A_2(s) y_{\nu+1}(s) + A_3(s) y_{\nu-1}(s) = 0,$$

with polynomial coefficients $A_i(s)$, $i = 1, 2, 3$.

Proof. It is sufficient to put $k_1 = k_2 = k_3 = 0$, $\nu_1 = \nu$, $\nu_2 = \nu + 1$ and $\nu_3 = \nu - 1$ in (26). ☐

Corollary 4.3. *In the same conditions as in Lemma 3.2, the following* Δ*-ladder-type relation holds*

$$B_1(s) y_\nu(s) + B_2(s) \frac{\Delta y_\nu(s)}{\Delta x(s)} + B_3(s) y_{\nu+m}(s) = 0, \qquad m \in \mathbb{Z}, \tag{27}$$

with polynomial coefficients $B_i(s)$, $i = 1, 2, 3$.

Proof. It is sufficient to put $k_1 = k_3 = 0$, $k_2 = 1$, $\nu_1 = \nu_2 = \nu$ and $\nu_3 = \nu + m$ in (26). ☐

Notice that for the case $m = \pm 1$ (27) becomes

$$B_1(s)y_\nu(s) + B_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)} + B_3(s)y_{\nu+1}(s) = 0, \tag{28}$$

$$\widetilde{B}_1(s)y_\nu(s) + \widetilde{B}_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)} + \widetilde{B}_3(s)y_{\nu-1}(s) = 0, \tag{29}$$

with polynomial coefficients $B_i(s)$ and $\widetilde{B}_i(s)$, $i = 1, 2, 3$. The above relations are usually called raising and lowering operators, respectively, for the functions y_ν.

Let us now obtain a raising and lowering operators for the functions y_ν but associated to the $\nabla/\nabla x(s)$ operators.

We start applying the operator $\nabla/\nabla x(s)$ to (13)

$$\begin{aligned}\frac{\nabla}{\nabla x(s)}y_\nu(s) =& \frac{\nabla}{\nabla x(s)}\left[\frac{C_\nu}{\rho(s)}\Phi_{\nu,\nu}(s)\right]\\ =& \frac{1}{\nabla x(s)}\left[C_\nu \Phi_{\nu\nu}(s)\left(\frac{1}{\rho(s)} - \frac{1}{\rho(s-1)}\right) + \frac{C_\nu}{\rho(s-1)}\nabla\Phi_{\nu\nu}(s)\right],\end{aligned}$$

or, equivalently,

$$\frac{\nabla\Phi_{\nu\nu}}{\nabla x(s)} = \frac{\rho(s-1)}{C_\nu}\frac{\nabla y_\nu(s)}{\nabla x(s)} - \frac{\Phi_{\nu\nu}(s)}{\nabla x(s)}\left[\frac{\rho(s-1)}{\rho(s)} - 1\right].$$

By Lemma (3.2) with $\nu_1 = \mu_1 = \nu_2 = \nu$, $\mu_2 = \nu + 1$ and $\nu_3 = \mu_3 = \nu + m$, there exist polynomial coefficients on $x(s)$, $A_i(s)$, $i = 1, 2, 3$, such that

$$A_1(s)\Phi_{\nu,\nu}(s) + A_2(s)\Phi_{\nu,\nu+1}(s) + A_3(s)\Phi_{\nu+m,\nu+m}(s) = 0.$$

From (14)

$$\Phi_{\nu,\nu+1}(s) = \frac{1}{[\nu+1]_q}\frac{\nabla\Phi_{\nu,\nu}}{\nabla x(z)} = \frac{1}{[\nu+1]_q}\frac{\nabla\Phi_{\nu,\nu}}{\nabla x(z)}.$$

Therefore

$$\begin{aligned}A_1(s)\Phi_{\nu,\nu} + \frac{A_2(s)}{[\nu+1]_q}\left[\frac{\rho(s-1)}{C_\nu}\frac{\nabla y_\nu}{\nabla x(s)} - \frac{\Phi_{\nu\nu}(s)}{\nabla x(s)}\left(\frac{\rho(s-1)}{\rho(s)} - 1\right)\right]&\\ + A_3\Phi_{\nu+m,\nu+m} = 0.&\end{aligned}$$

Using now the Pearson equation (6) and dividing by $\rho(s)$ we get

$$\begin{aligned}A_1(s)y_\nu(s) + \frac{A_2(q)}{[\nu+1]_q}\left[\frac{\sigma(s)}{\phi(s-1)}\frac{\nabla y_\nu}{\nabla x(s)} - \frac{y_\nu(s)}{\nabla x(s)}\left(\frac{\sigma(s)}{\phi(s-1)} - 1\right)\right]&\\ + A_3\frac{C_\nu}{C_{\nu+m}}y_{\nu+m}(s) = 0\,.&\end{aligned}$$

Multiplying both sides by $[\nu+1]_q\phi(s-1)$,

$$\begin{aligned}&A_1(s)[\nu+1]_q\phi(s-1)y_\nu(s) + A_2(s)\sigma(s)\frac{\nabla y_\nu}{\nabla x(s)} -\\ &A_2(s)\frac{\sigma(s) - \phi(s-1)}{\nabla x(s)}y_\nu(s) + [\mu+1]_q C_\nu C_{\nu+m}^{-1}A_3\phi(s-1)y_{\nu+m}(s) = 0.\end{aligned}$$

Thus we have proven the following

Theorem 4.4. *In the same conditions as in Lemma 3.2, the following ∇-ladder-type relation holds*

$$C_1(s)y_\nu(s) + C_2(s)\frac{\nabla y_\nu(s)}{\nabla x(s)} + C_3(s)y_{\nu+m}(s) = 0, \qquad m \in \mathbb{Z}, \tag{30}$$

with polynomial coefficients $C_i(s)$, $i = 1, 2, 3$.

Notice that for the case $m = \pm 1$ (30) becomes

$$C_1(s)y_\nu(s) + C_2(s)\frac{\nabla y_\nu(s)}{\nabla x(s)} + C_3(s)y_{\nu+1}(s) = 0, \tag{31}$$

$$\widetilde{C}_1(s)y_\nu(s) + \widetilde{C}_2(s)\frac{\nabla y_\nu(s)}{\nabla x(s)}y_\nu(s) + \widetilde{C}_3(s)y_{\nu-1}(s) = 0, \tag{32}$$

with polynomial coefficients $C_i(s)$ and $\widetilde{C}_i(s)$, $i = 1, 2, 3$. The above relation are usually called raising and lowering operators, respectively, for the functions y_n. Eq. (31) was firstly obtained in [16, Eq. (3.4)].

To conclude this section let us point that from formula (25) and the examples 3.3, 3.5, and 3.7 follow the relations

$$\begin{aligned} B_1(s)y_\nu^{(1)}(s) + B_2(s)y_\nu(s) + B_3(s)y_{\nu+1}^{(1)}(s) = 0, \\ B_1(s)y_\nu^{(1)}(s) + B_2(s)y_{\nu-1}(s) + B_3(s)y_\nu(s) = 0, \\ B_1(s)y_\nu^{(1)}(s) + B_2(s)y_\nu(s) + B_3(s)y_{\nu+1}(s) = 0, \end{aligned} \tag{33}$$

respectively, being the last two expressions the lowering and raising operators for the functions y_ν. Moreover, combining the explicit values of A_1, A_2 and A_3 with formula (25), one can obtain the explicit expressions for the coefficients B_1, B_2 and B_3 in (33).

5. Applications to q-classical polynomials

In this section we will apply the previous results to the q-classical orthogonal polynomials [2, 10, 11] in order to show how the method works. We first notice that these polynomials are instances of the functions y_ν on the lattice $x(s) = q^s$ defined in (4). In fact we have [13, 16]

$$P_n(x(s)) = \frac{[n]_q! B_n}{\rho(s)\ 2\pi i}\int_C \frac{\rho_n(z)\nabla x_{n+1}(z)}{[x_n(z) - x_n(s)]^{(n+1)}}dz, \tag{34}$$

where B_n is a normalizing constant, C is a closed contour surrounding the points $x = s, s-1, \dots, s-n$ and it is assumed that $\rho_n(s) = \rho(s+n)\prod_{m=1}^{n}\sigma(s+m)$ and $\rho_n(s+1)$ are analytic inside C (ρ is the solution of the Pearson equation (6)), i.e., the condition (5) holds.

A detailed study of the q-classical polynomials, including several characterization theorems, was done in [2, 9, 11]. In particular, a comparative analysis of the q-Hahn tableau with the q-Askey tableau [9] and Nikiforov-Uvarov tableau [15] was done in [5]. In the following we use the standard notation for the q-calculus [8]. In particular by $(a;q)_k = \prod_{m=0}^{k-1}(1 - aq^m)$, we denote the q-analogue of the Pochhammer symbol.

Since the q-classical polynomials are defined by (34) where the contour C is closed and ν is a non-negative integer, then the condition (17) is automatically fulfilled, so Lemma 3.2 holds for all of them. Moreover, the Theorem 4.1 holds and there exist the non vanishing polynomials B_1, B_2 and B_3 of (26).

In the following we will assume that the three term recurrence relation is known, i.e.,

$$\begin{aligned} &x(s)P_n(x(s)) = \alpha_n P_{n+1}(x(s)) + \beta_n P_n(x(s)) + \gamma_n P_{n-1}(x(s)) = 0, \qquad n \ge 0 \\ &P_{-1}(x(s)) = 0\,,\ \ P_0(x(s)) = 1, \qquad x(s) = q^s. \end{aligned} \tag{35}$$

where the coefficients α_n , β_n and γ_n can be computed using the coefficients σ , τ and $\lambda \equiv \lambda_n$ of (1), being λ_n given by (8) and (9) with $\nu = n$. For more details see, e.g., [1, 11].

Since the TTRR and the differentiation formulas for the q-polynomials are very well known (see e.g. [9, 11, 16]) we will obtain here two recurrent-difference relations involving the q-differences of the polynomials and the polynomials themselves.

5.1. **The first difference-recurrece relation.** If we choose $\nu_1 = n-1\,,\ \nu_2 = n\,,\ \nu_3 = n+1\,,\ k_1 = 1\,,\ k_2 = 1$ and $k_3 = 0$, in Theorem 4.1 one gets

$$A_1(s)\Delta^{(1)}P_{n-1}(x(s)) + A_2(s)\Delta^{(1)}P_n(x(s)) + A_3(s)P_{n+1}(x(s)) = 0\,.$$

Using [1, Eq. (6.14), page 193]

$$[\sigma(s)+\tau(s)\Delta x(s-1/2)]\Delta^{(1)}P_n(x(s)) = \widehat{\alpha}_n P_{n+1}(x(s)) + \widehat{\beta}_n P_n(x(s)) + \widehat{\gamma}_n P_{n-1}(x(s)),$$

where

$$\widehat{\alpha}_n = \frac{\lambda_n}{[n]_q}\left[q^{-\frac{n}{2}}\alpha_n - \frac{B_n}{\tau_n' B_{n+1}}\right],\quad \widehat{\beta}_n = \frac{\lambda_n}{[n]_q}\left[q^{-\frac{n}{2}}\beta_n + \frac{\tau_n(0)}{\tau_n'} - c_3(q^{-\frac{n}{2}}-1)\right],$$

$$\widehat{\gamma}_n = \frac{\lambda_n q^{-\frac{n}{2}}\gamma_n}{[n]_q},$$

to compute $\Delta^{(1)}P_n(x(s)) = \frac{\Delta P_n(x(s))}{\Delta x(s)}$ we get

$$\begin{aligned}
&\left[A_2(s)\tfrac{\lambda_n}{[n]_q}\left(q^{-\frac{n}{2}}\alpha_n - \tfrac{B_n}{\tau_n' B_{n+1}}\right) + \left(\sigma(s)+\tau(s)\Delta x\left(s-\tfrac12\right)\right)A_3(s)\right]P_{n+1}+\\
&\left[A_1(s)\tfrac{\lambda_{n-1}}{[n-1]_q}\left(q^{-\frac{n-1}{2}}\alpha_{n-1} - \tfrac{B_{n-1}}{\tau_{n-1}' B_n}\right) + A_2(s)\tfrac{\lambda_n}{[n]_q}\left(q^{-\frac{n}{2}}\beta_n + \tfrac{\tau_n(0)}{\tau_n'}\right)\right]P_n+\\
&\left[A_1(s)\tfrac{\lambda_{n-1}}{[n-1]_q}\left(q^{-\frac{n-1}{2}}\beta_{n-1} + \tfrac{\tau_{n-1}(0)}{\tau_{n-1}'}\right) + A_2(s)\tfrac{\lambda_n q^{-\frac{n}{2}}\gamma_n}{[n]_q}\right]P_{n-1}+\\
&A_1(s)\tfrac{\lambda_{n-1}q^{-\frac{n-1}{2}}\gamma_{n-1}}{[n-1]_q}P_{n-2} = 0\,,
\end{aligned}$$

By (35) we may write

$$P_{n-2}(x(s)) = \frac{x(s)-\beta_{n-1}}{\gamma_{n-1}}P_{n-1}(x(s)) - \frac{\alpha_{n-1}}{\gamma_{n-1}}P_n(x(s))$$

so the above equality becomes

$$\begin{aligned}
&\left[\tfrac{\lambda_n}{[n]_q}\left(q^{-\frac{n}{2}}\alpha_n - \tfrac{B_n}{\tau_n' B_{n+1}}\right)A_2(s) + \left(\sigma(s)+\tau(s)\Delta x\left(s-\tfrac12\right)\right)A_3(s)\right]P_{n+1}(x(s))+\\
&\left[-\tfrac{\lambda_{n-1}}{[n-1]_q}\tfrac{B_{n-1}}{\tau_{n-1}' B_n}A_1(s) + \tfrac{\lambda_n}{[n]_q}\left(q^{-\frac{n}{2}}\beta_n + \tfrac{\tau_n(0)}{\tau_n'}\right)A_2(s)\right]P_n(x(s))+ \qquad (36)\\
&\left[\tfrac{\lambda_{n-1}}{[n-1]_q}\left(\tfrac{\tau_{n-1}(0)}{\tau_{n-1}'} + q^{-\frac{n-1}{2}}x\right)A_1(s) + \tfrac{\lambda_n}{[n]_q}q^{-\frac{n}{2}}\gamma_n A_2(s)\right]P_{n-1}(x(s)) = 0\,.
\end{aligned}$$

Comparing the above equation with the TTRR (35) one can obtain the explicit values of A_1, A_2, and A_3.

5.1.1. *Some examples.* Since we are working in the q-linear lattice $x(s) = q^s$, for the sake of simplicity, we will use the letter x to denote the variable of the polynomials [9, 11]. We will consider monic polynomials, i.e., those with the leading coefficient equal to 1. In the following we need the value of $\tau_n(x)$ for each family, which can be computed using (7).

Al-Salam-Carlitz I q-polynomials. For the Al-Salam-Carlitz I monic polynomials $U_n^{(a)}(x;q)$ we have (see [1, see table 6.5, p.208] or [11])

$$\sigma(x) = (1-x)(a-x)\,,\quad \tau_n(x) = \tfrac{q^{\frac{1-n}{2}}}{1-q}\big(x-(1+a)\big)\,,$$
$$\tau(x) = \tau_0(x)\,,\quad \lambda_n = -\tfrac{q^{\frac32-n}(1-q^n)}{(1-q)^2}\,,$$

and

$$\alpha_n = 1\,,\quad \beta_n = (1+a)q^n\,,\quad \gamma_n = -aq^{n-1}(1-q^n)\,.$$

The constant B_n is given by [1, Eq. (5.57), p. 147], $B_n = q^{\frac{1}{4}n(3n-5)}(1-q)^n$. Introducing these values into the equation (36) it becomes

$$\left[q\left(q^{-\frac{n}{2}}-1\right)A_2(x)+a(1-q)q^nA_3(x)\right]U_{n+1}^{(a)}(x;q)+$$
$$\left[q^{-\frac{n}{2}-\frac{5}{2}}A_1(x)+q^{1+\frac{n}{2}}(1+a)\left(1-q^{\frac{n}{2}}\right)A_2(x)\right]U_n^{(a)}(x;q)+$$
$$\left[\left(q^{\frac{n+3}{2}}(1+a)-q^{2-n}x\right)A_1(x)+aq^n\left(1-q^n\right)A_2(x)\right]U_{n-1}^{(a)}(x;q)=0.$$

Comparing with the TTRR (35) for the Al-Salam I polynomials we obtain a linear system for getting the unknown coefficients A_1, A_2 and A_3

$$q\left(q^{-\frac{n}{2}}-1\right)A_2(x)+a(1-q)q^nA_3(x)=1\,,$$
$$q^{-\frac{n}{2}-\frac{5}{2}}A_1(x)+q^{1+\frac{n}{2}}(1+a)\left(1-q^{\frac{n}{2}}\right)A_2(x)=(1+a)q^n-x\,,$$
$$\left(q^{\frac{n+3}{2}}(1+a)-q^{2-n}x\right)A_1(x)+aq^n\left(1-q^n\right)A_2(x)=aq^{n-1}\left(q^n-1\right)\,.$$

The solution of the above system is

$$\begin{aligned}
A_1(x)&=\frac{aq^n\left(1+q^{\frac{n}{2}}\right)\left((1+a)-q^{-\frac{n}{2}}x\right)}{aq^{-\frac{5}{2}}\left(1+q^{\frac{n}{2}}\right)-q(1+a)\left(q^{\frac{n+3}{2}}(1+a)-q^{2-n}x\right)}\,,\\
A_2(x)&=\frac{-aq^{-\frac{7}{2}}(1-q^n)-\left((1+a)q^n-x\right)\left(q^{\frac{3}{2}}(1+a)-q^{2-\frac{3n}{2}}x\right)}{\left(1-q^{\frac{n}{2}}\right)\left[aq^{-\frac{5}{2}}\left(1+q^{\frac{n}{2}}\right)-q(1+a)\left(q^{\frac{n+3}{2}}(1+a)-q^{2-n}x\right)\right]}\,,\\
A_3(x)&=\frac{a+q^{\frac{11}{2}-2n}x^2+q^{-\frac{n}{2}}\left(a-(1+a)q^5x\right)}{a(1-q)\left[aq^n+q^{\frac{3n}{2}}\left(a-(1+a)^2q^5\right)+(1+a)q^{\frac{11}{2}}x\right]}\,.
\end{aligned}\tag{37}$$

Then, the Al-Salam I q-polynomials satisfy the the following relation

$$A_1(x)\Delta^{(1)}U_{n-1}^{(a)}(x;q)+A_2(x)\Delta^{(1)}U_n^{(a)}(x;q)+A_3(x)U_{n+1}^{(a)}(x;q)=0\,,\tag{38}$$

where the coefficients A_1, A_2 and A_3 are given by (37).

Notice that the coefficients A_1, A_2 and A_3 are rational functions on x. Therefore, multiplying (38) by and appropriate factor it becomes a linear relation with polynomials coefficients.

Alternative q-Charlier polynomials. In this case (see [1, table 6.6, p.209])

$$\sigma(x)=q^{-1}x(1-x)\,,\quad \tau_n(x)=-\frac{q^{-\frac{n+1}{2}}}{1-q}\Big(\left(1+aq^{1+2n}\right)x-1\Big),$$
$$\tau(x)=\tau_0(x)\,,\quad \lambda_n=\frac{q^{\frac{1}{2}-n}(1-q^n)(1+aq^n)}{(1-q)^2}\,,$$

and, for the monic case, $\alpha_n=1$

$$\beta_n=\frac{q^n\left(1+aq^{n-1}+aq^n-aq^{2n}\right)}{(1+aq^{2n-1})\,(1+aq^{2n+1})}\,,\quad \gamma_n=\frac{aq^{3n-2}\left(1-q^n\right)\left(1+aq^{n-1}\right)}{(1+aq^{2n-2})\,(1+aq^{2n-1})^2\,(1+aq^{2n})}\,.$$

The corresponding normalizing constant B_n is given by

$$B_n=\frac{(-1)^nq^{\frac{1}{4}n(3n-1)}(1-q)^n}{(-aq^n;q)_n}\,.$$

Following the same procedure as before we obtain the following relation for the alternative Charlier q-polynomials:

$$A_1(x)\Delta^{(1)}K_{n-1}(x;a;q)+A_2(x)\Delta^{(1)}K_n(x;a;q)+A_3(x)K_{n+1}(x;a;q)=0\,,$$

with the coefficients

$$A_1(x) = \frac{a\left(1+aq^{\frac{n}{2}}\right)\left(\left(1+aq^{2n+1}\right)x-q^{-\frac{n}{2}}\right)x}{q^2\left(1+aq^{2n-2}\right)\left(1+aq^{2n-1}\right)\left(1+aq^{2n}\right)\left(1+aq^{2n+1}\right)},$$

$$A_2(x) = \frac{-q^{\frac{3n+1}{2}}\left(1+aq^n\right)x+\left(1+aq^{2n}\right)\left(q^{1+\frac{n}{2}}\left(1+aq^{2n+1}\right)+aq^{2n+\frac{1}{2}}(1+q)+q^{\frac{3n}{2}}\left(1-aq^{2n}\right)\right)x^2}{q^{3n}\left(1+aq^n\right)\left(1+aq^{2n}\right)\left(1+aq^{2n+1}\right)} - \frac{q^{\frac{3}{2}}\left(1+aq^{2n-1}\right)\left(1+aq^{2n+1}\right)x^3}{q^{3n}\left(1+aq^n\right)\left(1+aq^{2n}\right)\left(1+aq^{2n+1}\right)},$$

$$A_3(x) = \frac{q^{\frac{n+1}{2}}+aq^{2n}\left(q^{\frac{n}{2}}+1+q^{\frac{1}{2}}\right)-q^{\frac{3}{2}}\left(1-aq^{\frac{3n}{2}}\right)\left(1+aq^{2n-1}\right)x}{q^{\frac{9n}{2}}\left(1+aq^n\right)}.$$

Big q-Jacobi polynomials. In this case (see [1, see table 6.2, p.204] or [11])

$$\sigma(x) = q^{-1}(x-aq)(x-cq)\,,\lambda_n = -q^{\frac{1}{2}-n}\frac{\left(1-abq^{1+n}\right)\left(1-q^n\right)}{(1-q)^2}\,,$$

$$\tau_n(x) = \frac{q^{\frac{1-n}{2}}}{1-q}\left(\frac{1-abq^{2+2n}}{q}x+a(b+c)q^{1+n}-(a+c)\right)\,,\tau(x)=\tau_0(x)\,,$$

and, for the monic case $\alpha_n = 1$,

$$\beta_n = \frac{c+a^2bq^n\left((1+b+c)q^{1+n}-q-1\right)+a\left(1+b+c-q^n\left(b(1+q)+c\left(1+q+b+bq-bq^{1+n}\right)\right)\right)}{q^{-1-n}(1-abq^{2n})(1-abq^{2n+2})},$$

$$\gamma_n = -\frac{a\left(1-q^n\right)\left(1-aq^n\right)\left(1-bq^n\right)\left(1-cq^n\right)\left(c-abq^n\right)}{q^{-1-n}\left(1-abq^{2n-1}\right)\left(1-abq^{2n}\right)^2\left(1-abq^{2n+1}\right)}.$$

The corresponding normalizing constant is

$$B_n = \frac{(1-q)^n q^{\frac{1}{4}n(3n-1)}}{(abq^{1+n};q)_n}.$$

The big q-Jacobi polynomials satisfy the following relation

$$A_1(x)\Delta^{(1)}p_{n-1}(x;a,b,c;q) + A_2(x)\Delta^{(1)}p_n(x;a,b,c;q) + A_3(x)p_{n+1}(x;a,b,c;q) = 0\,,$$

with the coefficients A_1, A_2 and A_3 given by

$$A_1(x) = \frac{aq^{-\frac{1}{2}+n}\left(1-abq^{n+1}\right)(1-x)(c-bx)\left(c-(b+c)x+bx^2\right)}{1-abq^{2n-1}}\times$$
$$\Bigg\{(1-q)q^{\frac{n}{2}}\left(1-abq^{2n+2}\right)\left[\frac{c+a\left(1+b+c+b(c+a(1+b+c))q^{2n+1}-(c+b(1+a+c))q^n(1+q)\right)}{q^{-(n+1)}\left(1-abq^{2n}\right)\left(1-abq^{2n+2}\right)}-x\right]D(x)-$$
$$(1-q)q^n\left(1-abq^{2n}\right)\Big[\left(1-abq^{2n}\right)\left(-c+a\big(-1+(b+c)q^{n+1}\big)\right)+$$
$$q^{\frac{n}{2}}\left(c+a\Big(1+b+c+b\big(c+a(1+b+c)\big)q^{2n+1}-\big(c+b(1+a+c)\big)q^n(1+q)\Big)\right)N(x)\Big]\Bigg\},$$

$$A_2(x) = a(1-q)q^n\left(1-abq^{2n}\right)^2\left(1-abq^{2n+2}\right)(1-x)(c-bx)\left(c-(b+c)x+bx^2\right)N(x),$$

$$A_3(x) = \left(1-abq^{n+1}\right)\left(1-abq^{2n+2}\right)(1-x)(c-bx)D(x)+$$
$$q^{-1-\frac{n}{2}}\left(1-q^{\frac{n}{2}}\right)\left(1+abq^{1+\frac{3n}{2}}\right)\left(1-abq^{2n}\right)^2\left(1-abq^{2n+2}\right)\left(c-(b+c)x+bx^2\right)N(x),$$

where the polynomials $N(x)$ and $D(x)$ are given by

$$N(x) = \frac{aq^2(1-q^n)(1-aq^n)(1-bq^n)(1-cq^n)(c-abq^n)}{(1-abq^{2n})^2(1-aq^{2n+1})} - \left[\frac{q\left(-c+a(-1+(b+c)q^n)\right)}{1-abq^{2n}}+q^{\frac{1-n}{2}}x\right]\times$$
$$\left[\frac{c+a^2bq^n\left(-1-q+(1+b+c)q^{n+1}\right)+a\left(1-(b+c)\left(-1+q^n+q^{n+1}\right)-bcq^n\left(1+q-q^{n+1}\right)\right)}{q^{-n-1}(1-abq^{2n})(1-abq^{2n+2})}-x\right]$$

and

$$D(x)=\tfrac{aq(1-q^n)(1-aq^n)(1-bq^n)(1-cq^n)(c-abq^n)}{1-abq^{2n+1}}+\tfrac{1-q^{\frac{n}{2}}}{1-abq^{2n+2}}\times$$
$$\Big\{-c+a^2bq^{\frac{3n}{2}}\left(-1-q+(b+c)q^{n+1}-q^{1+\frac{n}{2}}\right)+a\Big[-1+(b+c)\big(q^{\frac{n}{2}}+q^n+q^{n+1}\big)-$$
$$bc\big(q^{\frac{3n}{2}}+q^{1+\frac{3n}{2}}+q^{2n+1}\big)\Big]\Big[(c+a)q^{1+\frac{n}{2}}-a(b+c)q^{1+\frac{3n}{2}}-q^{\frac12}\left(1-abq^{2n}\right)x\Big]\Big\},$$

respectively.

5.2. **The second difference-recurrece relation.** If we choose $\nu_1=n-1$, $\nu_2=n$, $\nu_3=n+1$, $k_1=0$, $k_2=0$ and $k_3=1$ in Theorem 4.1, and proceeding as in the previous case one gets

$$A_1(x)P_{n-1}(x;q)+A_2(x)P_n(x;q)+A_3(x)\Delta^{(1)}P_{n+1}(x;q)=0\,, \tag{39}$$

where the coefficients A_1, A_2 and A_3, satisfy the linear relation

$$A_3(x)\Big[\Big(q^{-\frac{n+1}{2}}-\tfrac{B_{n+1}}{\alpha_{n+1}\tau'_{n+1}B_{n+2}}\Big)(x-\beta_{n+1})+\Big(q^{-\frac{n+1}{2}}\beta_{n+1}+\tfrac{\tau_{n+1}(0)}{\tau'_{n+1}}\Big)\Big]P_{n+1}+$$
$$\Big[A_3(x)\tfrac{B_{n+1}}{\alpha_{n+1}\tau'_{n+1}B_{n+2}}\gamma_{n+1}+\left(\sigma(x)+\tau(x)\Delta x\left(s-\tfrac12\right)\right)\tfrac{[n+1]_q}{\lambda_{n+1}}A_2(x)\Big]\,P_n+$$
$$\left(\sigma(x)+\tau(x)\Delta x\left(s-\tfrac12\right)\right)\tfrac{[n+1]_q}{\lambda_{n+1}}A_1(x)P_{n-1}=0\,.$$

Comparing the above relation with the three-term recurrence relation (35) one can obtain the explicit expressions for the coefficients A_1, A_2 and A_3 in (39).

5.2.1. *Some examples.*

Al-Salam and Carlitz I polynomials. Using the main data for the Al-Salam and Carlitz I polynomials we obtain the relation

$$A_1(x)U_{n-1}^{(a)}(x;q)+A_2(x)U_n^{(a)}(x;q)+A_3(x)\Delta^{(1)}U_{n+1}^{(a)}(x;q)=0$$

where

$$A_1(x)=aq^{n-1}\left(1-q^n\right)x,\quad A_2(x)=\Big[a\left(1+q^{\frac{n+1}{2}}\right)q^n-\Big((1+a)q^n-x\Big)x\Big],$$
$$A_3(x)=-\,a\frac{1-q}{1-q^{\frac{n+1}{2}}}q^{\frac{3n+1}{2}}.$$

Alternative q-Charlier polynomials. In this case, one gets

$$A_1(x)K_{n-1}(x;a;q)+A_2(x)K_n(x;a;q)+A_3(x)\Delta^{(1)}K_{n+1}(x;a;q)=0\,,$$

$$A_1(x)=\frac{a\left(1-q^n\right)\left(1+aq^{n-1}\right)\Big\{aq^n\left(1-q^{n+1}\right)+q^{-\frac{n+1}{2}}\left(1+aq^{2n+1}\right)\Big[\left(1+aq^{n+1}\right)-q^{-\frac{n+1}{2}}\left(1+aq^{2n+2}\right)\Big]x\Big\}}{q^{2-3n}\left(1+aq^{2n-2}\right)\left(1+aq^{2n-1}\right)\left(1+aq^{2n}\right)},$$
$$A_2(x)=-x\left\{aq^n\left(1-q^{n+1}\right)+q^{-\frac{n+1}{2}}\left(1+aq^{2n+1}\right)\Big[\left(1+aq^{n+1}\right)-q^{-\frac{n+1}{2}}\left(1+aq^{2n+2}\right)\Big]x\right\}+$$
$$\frac{a^2q^{3n-1}\left(1-q^n\right)\left(1-q^{n+1}\right)+q^{\frac{n-1}{2}}\left(1+aq^{n-1}+aq^n-aq^{2n}\right)\left(1+aq^{2n+1}\right)\Big[\left(1+aq^{n+1}\right)-q^{-\frac{n+1}{2}}\left(1+aq^{2n+2}\right)\Big]x}{\left(1+aq^{2n-1}\right)\left(1+aq^{2n+1}\right)},$$
$$A_3(x)=a(1-q)q^{\frac{n+1}{2}}\left(1+aq^{2n+1}\right)x^2$$

Concluding remarks. In this paper we present a constructive approach for finding recurrence relations for the hypergeometric-type functions on the linear-type lattices, i.e., the solutions of the hypergeometric difference equation (1) on the linear-type lattices. Important instances of "discret" functions are the celebrated Askey-Wilson polynomials and q-Racah polynomials. Such functions are defined on the non-uniform lattice of the form $x(s)=c_1(q)q^s+c_2(q)q^{-s}+c_2(q)$ with $c_1c_2\neq0$, i.e., a non linear-type lattice and therefore they require a more detailed study (some preliminar general results can be found in [16]).

Acknowledgements. The authors thank J. S. Dehesa and J.C. Petronilho for interesting discussions. The authors were partially supported by DGES grants MTM2009-12740-C03; PAI grant FQM-0262 (RAN) and Junta de Andalucía grants P09-FQM-4643, Spain; and CM-UTAD from UTAD (JLC).

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