On the connection and linearization problem for discrete hypergeometric $q-$polynomials.

R. Álvarez-Nodarse

Departamento de Análisis Matemático. Universidad de Sevilla. Apdo. 1160,
E-41080 Sevilla, and Instituto Carlos I de Física Teórica y Computacional,
Universidad de Granada, E-18071 Granada, Spain
E-mail: ran@cica.es

J. Arvesú

Departamento de Matemáticas, Universidad Carlos III de Madrid. Ave.
Universidad 30, E-28911, Leganés, Madrid, Spain
E-mail: jarvesu@math.uc3m.es

and

R. J. Yáñez

Departamento de Matemática Aplicada, Universidad de Granada, E-18071
Granada and Instituto Carlos I de Física Teórica y Computacional, Universidad de
Granada, E-18071 Granada, Spain
E-mail: ryanez@ugr.es

In the present paper, starting from the second-order difference hypergeometric equation on the non-uniform lattice $x(s)$ satisfied by the set of *discrete* hypergeometric orthogonal $q-$polynomials $\{p_n\}$, we find analytical expressions of the expansion coefficients of any $q-$polynomial $r_m(x(s))$ on $x(s)$ and of the product $r_m(x(s))q_j(x(s))$ in series of the set $\{p_n\}$. These coefficients are given in terms of the polynomial coefficients of the second-order difference equations satisfied by the involved discrete hypergeometric $q-$polynomials.
Key words and phrases: q-polynomials, connection and linearization problems.
AMS (MOS, 2000) subject classification: **33D45**

1. INTRODUCTION.

The expansion of any arbitrary discrete polynomial $q_m(x)$ in series of a general (albeit fixed) set of discrete hypergeometric polynomial $\{p_n(x)\}$ is a matter of great interest, solved only for some particular classical cases (for

a review see [8, 9, 17] up to the middle of seventies and [5, 40], since then up to now). This is particularly true for the deeper problem of linearization of a product of any two polynomials. Usually, the determination of the expansion coefficients in these particular cases required a deep knowledge of special functions and, at times, ingenious induction arguments based in the three-term recurrence relation of the involved orthogonal polynomials [9, 14, 16, 17, 22, 25, 34, 42, 43, 44]. Only recently, general and widely applicable strategies begin to appear [5, 6, 7, 13, 19, 24, 26, 27, 29, 30, 31, 39, 41].

One of the reasons for this increasing interest is the applications of such kind of problems in several branches of Mathematics and Physics. For example, Gasper in his paper [17], motivated the connection and linearization problem in the framework of the *positivity*. Nine years after, one of the most famous conjecture: *The Bieberbach conjecture* $(|a_n| \leq n)$ for analytic and univalent functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $|z| < 1$, has been solved by Louis de Branges using an inequality proved by Askey and Gasper in 1976 [10] in the framework of the positivity. In fact they proved that for $0 \leq t < 1$, $\alpha > -2$,

$$\sum_{k=0}^{n} P_k^{\alpha,0}(t) = \frac{(\alpha+2)_n}{n!} {}_3\mathrm{F}_2 \left(\begin{array}{c} -n, n+\alpha+2, \frac{\alpha+2}{2} \\ \frac{\alpha+3}{2}, \alpha+1 \end{array} \middle| t \right) \geq 0, \qquad (1)$$

where $(a)_n$ is the Pochhammer symbol and $P_n^{\alpha,\beta}(x)$ denotes the classical Jacobi polynomials. Here in this work we will study the problem of finding the the connection c_{mn} and linearization c_{jmn} coefficients, i.e., the coefficients on the expansions [9]

$$q_m(x) = \sum_{n=0}^{m} c_{mn} p_n(x), \qquad (2)$$

$$q_m(x) r_j(x) = \sum_{n=0}^{m+j} c_{jmn} p_n(x), \qquad (3)$$

respectively, where $q_m(x)$ and $r_j(x)$ are any mth-degree and jth-degree polynomials, and $\{p_n\}$ denotes an arbitrary set of polynomials.

Notice that, since the involved hypergeometric series in (1) is terminating, i.e., has a finite number of terms, the above problem can be considered as a connection problem between two families of polynomials where all the connection coefficients are positive (and equal to 1 in this example). So the Gasper's words about the importance in applications of the connection and linearization problems, and the positivity of the corresponding coefficients, become very actual and of interest.

The first who considered the linearization problem for discrete polynomials (notice that in the de Branges's proof the "continuous" Jacobi polynomials have been used) was Eagleson in 1969 for Kravchuk polynomials [14]. Later on, Gasper [17] studied the connection problem for the Hahn $h^{\alpha,\beta}(x, N)$ polynomials

$$h_j^{\gamma,\mu}(x, M) = \sum_{n=0}^{j} c_{jn}\, h_n^{\alpha,\beta}(x, N), \qquad j \leq \min\{N-1, M-1\},$$

and completely solved it (the particular case $N = M$, of interest because $c_{jn} \geq 0$, he solved one year earlier in [16]), from where, by limiting process it is possible to obtain the connection coefficients for Jacobi polynomials as well as for other continuous and discrete families (see [16, 17] for further information on this). Some years later, Askey and Gasper [11] considered the linearization problem when the involved polynomials were the discrete polynomials of Hahn, Meixner Kravchuk and Charlier (for a review on discrete polynomials see [32]) but only in the special case when all r_m, q_j and p_n belong to the same family with the same parameters (in [17] some preliminary results regarding to the positivity of such coefficients were discussed).

In all these cases, continuous and discrete, the proofs were based on very specific characteristic of the involved families, particularly their hypergeometric representation and generating functions have been exploited for finding the corresponding solution.

It is important to remark that, even in the case when it is possible to compute explicitly the connection or the linearization coefficients, not always is easy to show that they are nonnegative which were important as we already pointed out. This led to a recurrent method, i.e., to find a difference equation for the coefficients c_{mn} and c_{jmn}, respectively, and from it to deduce their non negativity. The first who did it was Hylleraas [22] in 1962 for a product of two Jacobi polynomials. In fact Hylleraas was able to solve the obtained recurrence relation for some special cases and prove the non negativity of the coefficients in some of these cases. Later, this method has been used by Askey and Gasper (see e.g. [11]) to prove the non negativity of the linearization coefficients for certain families of orthogonal polynomials.

More recently, Ronveaux, Zarzo, Area and Godoy [6, 19, 39], developed a recurrent method, called NAVIMA algorithm, for solving the connection problem (2) for all families of classical polynomials, as well as some special kind of linearization problem and used it for solving different problems related with the associated, Sobolev-type polynomials, etc [20, 21]. Although, they use it only for solving a very special linearization problem, it can be easily extended for solving the general problem (3) [13]. Let us

point out that there is a very similar algorithm for finding the recurrence relation for both, connection and linearization coefficients due to Lewanowicz [26, 27, 29]. The most important tool in the both aforesaid algorithms was the *structure relations* (or Al-Salam & Chihara characterization) that the polynomials p_n in (2) and (3) satisfy.

Both problems, connection and linearization, are of great interest also in Physics. For example, for the 2^l-pole transitions in hydrogen-like atoms (and other related systems) the radial part of the probability is proportional to integrals of the form

$$T_l^{1\,2} = \int_0^\infty [L_{n_1}^{2l_1+1}(\alpha_1 r)L_{n_2}^{2l_2+1}(\alpha_2 r)]r^m e^{-r} dr,$$

where L_n^l are the Laguerre polynomials. This kind of integrals also appears in the theory of Morse oscillators as well as in transitions for spherical-symmetric systems [34]. Furthermore, for spherical-symmetric the Wigner-Ekkart theorem [15] allows to write the matrix elements of certain irreducible operators in terms of products of two (or more) 3j symbols (Hahn and dual Hahn polynomials [32]), $6j$ symbols (Racah polynomials [32]), etc as well as their $q-$analogues.

To conclude this introduction we need to say that in the world of $q-$polynomials (discrete case) there are not so many results concerning to these problems. One of the first who was interested on this was Rogers [37, 38] who used a $q-$analogue of the connection formula for Jacobi polynomials $P_n^{\gamma,\gamma}(x) = \sum_{j=0}^{[n/2]} c_{j,n} P_{n-2j}^{\beta,\beta}(x)$, $c_{n,j} \geq 0$, for the $q-$ultraspherical polynomials to prove some Rogers-Ramanujan identities (see also [36]). Also, very recently, this problem has been considered in [3, 4, 28] for $q-$polynomials in the exponential lattice $x(s) = q^s$ [2, 33, 32], where the authors obtained recurrence relations for the coefficients in (2) and (3). Again, in these works the use of the structure relations plays a fundamental role. But not for any arbitrary family of $q-$polynomials there exist such relations. In [2] it is proven that all families of $q-$polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$ satisfy such a relation, but for the general lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ [12, 32] the problem is still open. Then, the following question naturally arises: What to do in case when we do not have structure relations? This question was solved for the continuous case in [7, 41] and for the discrete case in [5].

The main goal of the present paper is give and alternative method for finding the connection and linearization problem for q-hypergeometric polynomials obtaining explicit expressions for the coefficients c_{mn} and c_{jmn} in (2)-(3) in terms of the coefficients of the second order difference equation of hypergeometric-type on the general non-uniform lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ that such polynomials satisfy. The resulting expansion

coefficients are given in a compact form in terms of the polynomial coefficients of the corresponding second-order difference equations. Notice that the above lattice contains, as a particular case, the exponential lattice $x(s) = q^s$ which has been firstly considered in [4, 28]. The advantage of the present approach is that it only requires the knowledge of the second order difference equation satisfied by the involved hypergeometric q-polynomials as well as their hypergeometricity, i.e., the Rodrigues-type formula. Then, contrary to the algorithm presented in [4, 3, 28], we do not require information about any kind of recurrence relation of the involved discrete hypergeometric q-polynomials nor we need to solve any *"high"* order recurrence relation for the connection coefficients themselves.

The structure of the paper is as follows. In Section 2, we collect the basic background [32] used in the rest of the work; namely, the second-order hypergeometric difference equation on the uniform lattice $x(s)$ and its polynomial solutions (called as hypergeometric q-polynomials). In Section 3, we present the main results of the paper, namely, the expressions for the connection and linearization coefficients c_{mn} and c_{jmn}, respectively. In particular, we show how the main formulas and theorems given in [5] for the linear lattice $x(s) = s$, as well as the ones given in [7, 41] hold as particular cases. Finally, in Section 4, some examples are developed.

2. SOME BASIC PROPERTIES OF THE Q–POLYNOMIALS.

Here we will summarize some of the properties of the q-polynomials [32] useful for the rest of the work.

2.1.   The hypergeometric-type difference equation.

Let us consider the *second order difference equation of hypergeometric type*

$$\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)} + \tau(s)\frac{\triangle y(s)}{\triangle x(s)} + \lambda y(s) = 0,$$

$$\sigma(s) = \bar{\sigma}(x(s)) - \tfrac{1}{2}\bar{\tau}(x(s))\triangle x(s-\tfrac{1}{2}), \quad \tau(s) = \bar{\tau}(x(s)), \tag{4}$$

$$\nabla f(s) = f(s) - f(s-1), \triangle f(s) = f(s+1) - f(s),$$

where $\nabla f(s)$ and $\triangle f(s)$, denote the backward and forward finite difference derivatives, respectively, $\bar{\sigma}(x)$ and $\bar{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. It is important to notice that the above difference equation has polynomial solutions of hypergeometric type iff $x(s)$ is a function of the form

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q), \tag{5}$$

where c_1, c_2, c_3 and $q^\mu = \frac{c_1}{c_2}$ are constants which, in general, depend on q [12, 32, 33].

The polynomial solutions of (4) is determined by the $q-$analogue of the Rodrigues Formula on the non-uniform lattices [32, page 66, Eq. (3.2.19)]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \bigtriangledown^{(n)} [\rho_n(s)], \quad \bigtriangledown^{(n)} \equiv \frac{\bigtriangledown}{\bigtriangledown x_1(s)} \frac{\bigtriangledown}{\bigtriangledown x_2(s)} \cdots \frac{\bigtriangledown}{\bigtriangledown x_n(s)}, \quad (6)$$

where the function $\rho_n(s)$ is given by

$$\rho_n(s) = \rho(s + n) \prod_{i=1}^{n} \sigma(s + i), \quad \text{and} \quad x_m(s) = x(s + \tfrac{m}{2}), \quad (7)$$

and $\rho(s)$ is the solution of the Pearson-type difference equation

$$\frac{\bigtriangleup}{\bigtriangleup x(s - \frac{1}{2})} [\sigma(s)\rho(s)] = \tau(s)\rho(s). \quad (8)$$

Throughout the paper $[n]_q$ denotes the so called q-numbers and $[n]_q!$ are the q-factorial $[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$, $[n]_q! = [1]_q[2]_q \cdots [n]_q$.

Also the difference derivatives $y_{kn}(s)_q$ of the polynomial solution $P_n(s)_q$, defined by

$$y_{kn}(s)_q = \frac{\bigtriangleup}{\bigtriangleup x_{k-1}(s)} \frac{\bigtriangleup}{\bigtriangleup x_{k-2}(s)} \cdots \frac{\bigtriangleup}{\bigtriangleup x(s)} [P_n(s)_q] \equiv \bigtriangleup^{(k)} [P_n(s)_q], \quad (9)$$

satisfy a Rodrigues-type formula

$$y_{kn}(s)_q = \bigtriangleup^{(k)} P_n(s)_q = \frac{A_{nk} B_n}{\rho_k(s)} \bigtriangledown_k^{(n)} [\rho_n(s)], \quad (10)$$

where $\bigtriangledown_k^{(n)} f(s) = \frac{\bigtriangledown}{\bigtriangledown x_{k+1}(s)} \frac{\bigtriangledown}{\bigtriangledown x_{k+2}(s)} \cdots \frac{\bigtriangledown}{\bigtriangledown x_n(s)} [f(s)]$, is given by (see appendix A)

$$\bigtriangledown_k^{(n)} f(s) = \sum_{l=0}^{n-k} (-1)^l \begin{bmatrix} n-k \\ l \end{bmatrix}_q \frac{\bigtriangledown x_n(s - l + \frac{1}{2})}{\prod_{m=0}^{n-k} \bigtriangledown x_n(s - \frac{m+l-1}{2})} f(s - l), \quad (11)$$

and $A_{nk} = \frac{[n]_q!}{[n-k]_q!} \frac{a_k}{B_k}$, where a_n denotes the leading coefficient of the polynomial P_n, i.e., $P_n(x) = a_n x^n +$ lower order terms. Here, and through-

out the paper we will use the following notation for the symmetric q-binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}. \tag{12}$$

In this paper we will deal with discrete orthogonal q-polynomials, i.e., polynomials with a discrete orthogonality

$$\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s) \triangle x(s-\tfrac{1}{2}) = \delta_{nm} d_n^2, \quad s = a, a+1, \ldots, b-1, \tag{13}$$

where $\rho(x)$ is a solution of the Pearson-type equation (8), and it is a non-negative (not identically zero) weight function, i.e., $\rho(s) \triangle x(s-\tfrac{1}{2}) \geq 0$, $a \leq s \leq b-1$, supported on a countable subset of the real line $[a,b]$ (a, b can be $\pm\infty$). This condition follows from the difference equation of hypergeometric-type (4), providing that the following boundary condition

$$\sigma(s)\rho(s)x^k\left(s-\tfrac{1}{2}\right)\Big|_{s=a,b} = 0, \quad k = 0, 1, 2, \ldots, \tag{14}$$

holds [33]. Notice that the above boundary condition (14) is valid for $k = 0$. Moreover, if we assume that a is finite, then (14) is fulfilled at $s = a$ providing that $\sigma(a) = 0$ [32, §3.3, page 70]. In the following we will assume that this condition holds. The squared norm in (13) is given by [32, Chapter 3, Section **3.7.2**, pag. 104]

$$d_n^2 = (-1)^n A_{nn} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \triangle x_n(s-\tfrac{1}{2}). \tag{15}$$

In the most general case, the solution of the q-hypergeometric equation (4) corresponds to the case

$$\sigma(s) = A[s-s_1]_q[s-s_2]_q[s-s_3]_q[s-s_4]_q, \quad A = const \neq 0. \tag{16}$$

and has the form [33]

$$P_n(s)_q = D_n \, {}_4\varphi_3 \left(\begin{matrix} q^{-n}, q^{2\mu+n-1+\sum_{i=1}^4 s_i}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{matrix} \; ; q \, , q \right), \tag{17}$$

where, D_n is a normalizing constant, $\kappa_q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, and the basic hypergeometric series $_p\varphi_q$ are defined by [18]

$$
_r\varphi_p \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_p \end{matrix} \; ; \; q, \; z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1},
$$

and

$$
(a; q)_k = \prod_{m=0}^{k-1} (1 - aq^m), \tag{18}
$$

is a $q-$analogue of the Pochhammer symbol.

3. MAIN RESULTS.

Here we find the explicit expression of the coefficients c_{mn} in the expansion of an arbitrary q-polynomial $Q_m(s)_q$ on $x(s)$ in series of the orthogonal discrete hypergeometric set of q-polynomials $\{P_n\}$ in the same non-uniform lattice $x(s)$, i.e.

$$
Q_m(s)_q = \sum_{n=0}^{m} c_{mn} P_n(s)_q . \tag{19}
$$

THEOREM 3.1. *The explicit expression of the coefficients c_{mn} in the expansion (19) is*

$$
\begin{aligned}
c_{mn} &= \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \triangle^{(n)} \left[Q_m(s)_q \right] \rho_n(s) \triangle x_n(s - \tfrac{1}{2}) \\
&= (-1)^n \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \triangle^{(n)} [Q_m(s - n)] \rho_n(s - n) \triangle x(s - \tfrac{n+1}{2}).
\end{aligned} \tag{20}
$$

Proof. Multiplying both sides of Eq. (19) by $P_k(s)_q \rho(x) \triangle x(s - \tfrac{1}{2})$, and summing between a and $b - 1$, the orthogonality relation (13) immediately gives

$$
c_{mn} = \frac{1}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q P_n(s)_q \rho(s) \triangle x(s - \tfrac{1}{2}) . \tag{21}
$$

Next we use the Rodrigues formula (6) for $P_n(s)_q$. This yields

$$c_{mn} = \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q \, \nabla^{(n)} \left[\rho_n(s) \right] \triangle x(s - \tfrac{1}{2}) =$$
$$= \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q \, \nabla \left[\nabla_1^{(n)} [\rho_n(s)] \right].$$

Then, using the formula of summation by parts

$$\sum_{x_i=a}^{b-1} f(x_i) \, \nabla \, g(x_i) = f(x_i)g(x_i) \Big|_{a-1}^{b-1} - \sum_{x_i=a}^{b-1} g(x_i - 1) \, \nabla \, f(x_i),$$

we obtain for c_{mn} the expression

$$\frac{B_n}{d_n^2} Q_m(s)_q \, \nabla_1^{(n)} \left[\rho_n(s) \right] \Big|_{a-1}^{b-1} - \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \nabla Q_m(s)_q \, \nabla_1^{(n)} \left[\rho_n(t) \right] \Big|_{t=s-1}. \quad (22)$$

Notice that the first term is proportional to $\rho_1(s) = \sigma(s+1)\rho(s+1)$, so, since the condition (14), it vanishes. Now, making the change $s \to s-1$ in the second term, we find

$$c_{mn} = -\frac{B_n}{d_n^2} \sum_{s=a-1}^{b-2} \triangle Q_m(s)_q \, \nabla_1^{(n)} \left[\rho_n(s) \right].$$

But

$$\nabla_1^{(n)}[\rho_n(s)] = \frac{\nabla}{\nabla x_2(s)} \nabla_2^{(n)} \left[\rho_n(s) \right], \qquad \nabla x_2(s) = \nabla x(s+1) = \triangle x(s),$$

then, the last equation transforms

$$c_{mn} = -\frac{B_n}{d_n^2} \sum_{s=a-1}^{b-2} \frac{\triangle}{\triangle x(s)} [Q_m(s)_q] \, \nabla \left[\nabla_2^{(n)} [\rho_n(s)] \right].$$

Repeating this process k times, we obtain

$$c_{mn} = (-1)^k \frac{B_n}{d_n^2} \sum_{s=a-k}^{b-k-1} \frac{\triangle}{\triangle x_{k-1}(s)} \cdots \frac{\triangle}{\triangle x(s)} [Q_m(s)_q] \, \nabla \left[\nabla_{k+1}^{(n)} [\rho_n(s)] \right] =$$

$$= (-1)^k \frac{B_n}{d_n^2} \sum_{s=a-k}^{b-k-1} \triangle^{(k)} [Q_m(s)_q] \, \nabla_k^{(n)} \left[\rho_n(s) \right] \triangle x_k(s - \tfrac{1}{2}).$$

Putting $k = n$ and using $\nabla_n^{(n)}[\rho_n(s)] = \rho_n(s)$ as well as that $\rho_n(a - k) = 0$ for $k = 1, 2, ..., n$ (see Eq. (7)), we obtain the desired expression (20) for c_{mn}.

To obtain the second expression for c_{mn} we transform (20) using the identity

$$\nabla_1^{(n)}[\rho_n(s-1)] = \frac{\nabla}{\nabla x_2(s-1)} \, \nabla_2^{(n)}[\rho_n(s-1)] = \frac{\nabla}{\nabla x(s)} \, \nabla_2^{(n)}[\rho_n(s-1)].$$

Then Eq. (22) becomes

$$c_{mn} = -\frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s)}[Q_m(s)_q] \, \nabla \left[\nabla_2^{(n)}[\rho_n(s)] \right].$$

Applying $k-$times the same technique as before we find

$$c_{mn} = \; (-1)^k \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s - \frac{k-1}{2})} \frac{\nabla}{\nabla x(s - \frac{k}{2} + 1)} \cdots \frac{\nabla}{\nabla x(s)}[Q_m(s)_q] \times$$
$$\nabla_k^{(n)}[\rho_n(s-k)] \, \triangle x(s - \tfrac{k+1}{2}).$$

The change $k = n$ and the fact that

$$\triangle^{(n)} Q_m(s-n) = \frac{\nabla}{\nabla x(s - \frac{n-1}{2})} \cdots \frac{\nabla}{\nabla x(s)}[Q_m(s)_q],$$

lead us to the result. ∎

COROLLARY 3.1. *If Q_m is an hypergeometric polynomial satisfying an equation of the form (4) but with coefficients $\tilde{\sigma}$, $\tilde{\tau}$, and $\tilde{\lambda}_m$, then, the explicit expression of the coefficients c_{mn} in the expansion (19) is*

$$c_{mn} \; = \; \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{l=0}^{m-n} (-1)^l \begin{bmatrix} m-n \\ l \end{bmatrix}_q \times$$
$$\sum_{s=a}^{b-n-1} \frac{\tilde{\rho}_m(s-l)\rho_n(s)}{\tilde{\rho}_n(s)} \frac{\triangle x_m(s-l-\frac{1}{2})\,\triangle x_n(s-\frac{1}{2})}{\displaystyle\prod_{k=0}^{m-n} \triangle x_m(s - \frac{k+l+1}{2})}. \qquad (23)$$

Proof. The proof follows from the equations (20), (9) and (11). ∎

As a simple consequence of the theorem 3.1 we obtain the following result for the linearization problem, which consists of finding the expansion

coefficients c_{jmn} of the relation

$$R_j(s)_q Q_m(s)_q = \sum_{n=0}^{m+j} c_{jmn} P_n(s)_q \ , \qquad (24)$$

where $\{P_n\}$ is a discrete orthogonal set of hypergeometric q-polynomials which satisfy the difference equation (4) and Q_m and R_j are arbitrary q-polynomials on the same lattice $x(s)$.

THEOREM 3.2. *The explicit expression of the coefficients c_{jmn} in the expansion (24) is*

$$c_{jmn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \triangle^{(n)} \left[Q_m(s)_q R_j(s)_q \right] \rho_n(s) \triangle x_n(s - \tfrac{1}{2}) \quad (25)$$

$$= (-1)^n \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s - \frac{n-1}{2})} \cdots \frac{\nabla}{\nabla x(s)} [Q_m(s)_q R_j(s)_q] \rho_n(s-n) \triangle x(s - \tfrac{n+1}{2}).$$

THEOREM 3.3. *Let R_j be the $j-$degree q-hypergeometric polynomial solution of the second order difference equation on the non-uniform lattice $x(s)$*

$$\widetilde{\sigma}(s) \frac{\triangle}{\triangle x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \widetilde{\tau}(s) \frac{\triangle y(s)}{\triangle x(s)} + \widetilde{\lambda}_j y(s) = 0, \qquad (26)$$

Then, explicit expression of the coefficients c_{jmn} in the expansion (24) is given by

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \tilde{A}_{j\,k} \times$$

$$\sum_{s=a}^{b-n-1} \frac{\rho_n(s) \triangle x_n(s - \frac{1}{2})}{\tilde{\rho}_k(s+n-k)} [\triangle^{(n-k)} Q_m(s)_q] [\nabla_k^{(j)} \tilde{\rho}_j(s+n-k)] \ , \qquad (27)$$

or, equivalently,

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \tilde{A}_{j\,n-k} \times$$

$$\sum_{s=a}^{b-n-1} \frac{\rho_n(s) \triangle x_n(s - \frac{1}{2})}{\tilde{\rho}_{n-k}(s)} [\triangle^{(k)} Q_m(s+n-k)_q] [\nabla_{n-k}^{(j)} \tilde{\rho}_j(s)] \ . \qquad (28)$$

Proof. Using the $q-$analog of the Leibniz formula in the non-uniform lattice (5) [1]

$$\triangle^{(n)}[f(s)g(s)] = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \triangle^{(k)} f(s+n-k) \triangle^{(n-k)} g(s), \qquad (29)$$

for the expression $\triangle^{(n)} [Q_m(s)_q R_j(s)_q]$ in Eq. (25), as well as the Rodrigues-type formula (9) for $\triangle^{(k)} R_j(s+n-k)_q$

$$\triangle^{(k)} R_j(s+n-k)_q = \frac{\tilde{A}_{jk} \tilde{B}_j}{\tilde{\rho}_k(s+n-k)} \nabla_k^{(j)} [\tilde{\rho}_j(s+n-k)],$$

the desired result holds. The second formula can be obtained analogously but starting from the second equation of Theorem 3.2. ∎

COROLLARY 3.2. *The explicit expression of the coefficients c_{jmn} in the expansion (24) is given by*

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{A}_{jk} \sum_{l=0}^{j-k} (-1)^l \begin{bmatrix} j-k \\ l \end{bmatrix}_q \sum_{s=a}^{b-n-1} \frac{\tilde{\rho}_j(s+n-k-l)}{\tilde{\rho}_k(s+n-k)} \times$$

$$\frac{\triangle x_j(s+n-k-l-\frac{1}{2})}{\prod_{m=0}^{j-k} \triangle x_j(s+n-k-\frac{m+l+1}{2})} [\triangle^{(n-k)} Q_m(s)_q] \rho_n(s) \triangle x_n(s-\tfrac{1}{2}) .$$

Proof. To prove this it is sufficient to substitute the expression (11) in (27). ∎

Notice that the corollary 3.1 also follows from the above formula if we put $m = 0$ since $Q_0 \equiv 1$.

3.1. Special case: The classical discrete polynomials.

In the special case when $x(s)$ is the linear lattice, i.e, $x(s) = s$, from Theorem 3.1 and 3.2 we recover the main results in [5] for the connection and linearization problems, respectively

THEOREM 3.4. *Let be $x(s)$ the linear lattice $x(s) = s$. Then, the explicit expression of the coefficients c_{mn} in the expansion (19) is*

$$c_{mn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \triangle^n Q_m(s)\rho(s+n) \prod_{k=1}^{n} \sigma(s+k)$$

$$= \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-1} \nabla^n Q_m(s)\rho(s) \prod_{k=0}^{n-1} \sigma(s-k).$$

If Q_m is also an hypergeometric polynomial, then

$$c_{mn} = \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{s=a}^{b-n-1} \sum_{k=0}^{m-n} \frac{\rho_n(s)}{\tilde{\rho}_n(s)} \binom{m-n}{k} (-1)^k \tilde{\rho_m}(s-k)$$

$$= \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{s=a}^{b-1} \sum_{k=0}^{m-n} \frac{\rho_n(s-n)}{\tilde{\rho}_n(s-n)} \binom{m-n}{k} (-1)^k \tilde{\rho_m}(s-n-k).$$

THEOREM 3.5. *Let be $x(s)$ the linear lattice $x(s) = s$. Then, the explicit explicit expression of the coefficients c_{jmn} in the expansion (24) is given by*

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \sum_{s=a}^{b-n-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{\rho_n(s)}{\tilde{\rho}_k(s+n-k)} \times$$

$$\tilde{\rho}_j(s+n-k-l)[\nabla^{n-k} Q_m(s+n-k)] = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \times$$

$$= \sum_{s=a}^{b-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{\rho_n(s-n)}{\tilde{\rho}_k(s-k)} \tilde{\rho}_j(s-k-l)[\nabla^{n-k} Q_m(s-k)],$$

where $k_- = \max(0, n-m)$ and $k_+ = \min(n, j)$.

In all the above formulas $\binom{n}{m}$ denotes the binomial coefficients $\frac{n!}{m!(n-m)!}$.

3.2. Special case: The classical continuous case.

Finally, we will show how from Theorem 3.1 we can recover (formally) the general results for the continuous case [7, 41]. In order to do this we notice that, formally, if we make the change $x(s) = sh \to x$, then [32],

$$\frac{P_n(x(s+1)) - P_n(x(s))}{x_k(s+1) - x_k(s)} = \frac{P_n(sh+h) - P_n(sh)}{h} = \frac{P_n(x+h) - P_n(x)}{h}.$$

Here we have used the notation $P_n(x(s)) \equiv P_n(s)_q$. Thus, $\lim_{h \to 0} \frac{\triangle P_n(x(s))}{\triangle x_k(s)} = P'_n(x)$ and $\lim_{h \to 0} \triangle^{(k)} P_n(s)_q = \frac{d^k P_n(x)}{dx^k}$. Then, by similar limiting processes Eq. (4) transforms into the classical hypergeometric differential equation [32]

$$\sigma(x) P''_n(x) + \tau(x) P'_n(x) + \lambda_n P_n(x) = 0.$$

where $\sigma(x) = \lim_{h \to 0} \bar{\sigma}(x(s))$, $\tau(x) = \lim_{h \to 0} \bar{\tau}(x(s))$ being $x = sh$. Furthermore, the Pearson-type equation (8) becomes $[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$

and also [32] $\rho_n(s;h) \to \rho(x)\sigma^n(x)$. Finally, the Rodrigues-type formula (6) transforms into

$$\triangle^{(k)} P_n(s)_q = \frac{A_{nk} B_n(h)}{\rho_k(s)} \nabla_k^{(n)} [\rho_n(s)] \to \frac{d^k P_n(x)}{dx^k} = \frac{A_{nk} B_n}{\rho_k(x)} \frac{d^{n-k}}{dx^{n-k}} [\rho(x)\sigma^n(x)].$$

Now we put $x(s) = sh$ in (20)

$$c_{mn}(h) = \frac{(-1)^n B_n(h)}{d_n^2(h)} \sum_{x_i=ah}^{(b-1)h-nh} \triangle^{(n)} [Q_m(x_i)_q] \rho_n(x_i/h;h)h =$$

$$= \frac{(-1)^n B_n(h)}{d_n^2(h)} \sum_{x=A}^{B-nh} \triangle^{(n)} [Q_m(x_i)_q] \rho_n(x_i/h;h)h, \quad x_{i+1} = x_i + h.$$

Let us prove that the above sum transforms in the limit in a integral from which the main result in [41, Theorem 3.1, page 163] easily follows. More concretely,

$$\lim_{h\to 0} c_{mn}(h) = \frac{(-1)^n B_n}{d_n^2} \int_A^B \frac{d^k Q_m(x)}{dx^k} \rho(x)\sigma^n(x)\,dx,$$

where d_n^2 is the squared norm for the polynomials orthogonal with respect to $\rho(x)$ [32].

In order to do that, let us show that the quantity

$$I_n(Q_m, \rho_n) \equiv \left| \sum_{x=A}^{B-nh} \triangle^{(n)} [Q_m(sh)_q] \rho_n(x_i/h;h)h - \int_A^B Q_m^{(n)}(x)\rho(x)\sigma^n(x)\,dx \right|$$

can be small enough for h sufficiently small. So,

$$|I_n(Q_m, \rho_n)| \le \sum_{x_i=A}^{B-nh} \left| \triangle^{(n)} [Q_m(sh)_q] - Q_m^{(n)}(x_i) \right| \rho_n(x_i/h;h)h +$$

$$\sum_{x_i=A}^{B-nh} \left| Q_m^{(n)}(x_i) \{\rho_n(x_i/h;h) - \rho_n(x_i)\} \right| h +$$

$$\left| \sum_{x_i=A}^{B-nh} Q_m^{(n)}(x_i)\rho_n(x_i)h - \int_A^B Q_m^{(n)}(x)\rho_n(x)\,dx \right|,$$

where $Q_m^{(n)}$ denotes the n-th derivative of Q_m and $\rho_n(x) = \rho(x)\sigma^n(x)$. Let consider first the case when B is bounded. In this case the first integral

can be small enough (less that $\epsilon/3$) for h sufficient small providing that $\rho_n(x_i/h; h)$ is bounded. In the following we will suppose that the limit function $\rho_n(x)$, $n \geq 1$ is a continuous function in $[A, B]$. For the second sum we can do the same since Q_m is a polynomial and then it is bounded in any closed interval. Finally we will consider the last sum which can be rewritten in the form

$$\left| \sum_{x_i=A}^{B} Q_m^{(n)}(x_i)\rho_n(x_i)h - \int_A^B Q_m^{(n)}(x)\rho_n(x)\, dx \right| + \left| \sum_{x_i=B-hn}^{B} Q_m^{(n)}(x_i)\rho_n(x_i)h \right|.$$

Notice that the first sum can be less $\epsilon/6$ since it is a Riemann sum corresponding to the integral $\int_A^B Q_m^{(n)}(x)\rho_n(x)\, dx$, and the last sum obviously tends to zero so, for sufficiently small h, it is less than $\epsilon/6$. So, for any given $\epsilon > 0$, one can chose a sufficiently small h so that $|I_n(Q_m, \rho_n)| \leq \epsilon$.

Finally, to prove the result for the unbounded B we use the fact that, in this case, the functions $\rho_n(x_i/h; h)$ as well as $\rho_n(x_i)$ tend to zero faster than any polynomial tends to infinity when $x_i \to \infty$ (see the boundary conditions (14) for the polynomials on the lattice $x(s)$ as well as for the continuous case [32, Eq. (1.3.1) page 7]). Then,

$$|I_n(Q_m, \rho_n)| \leq \sum_{x_i=A}^{\infty} \left| \triangle^{(n)}[Q_m(sh)_q] - Q_m^{(n)}(x_i) \right| \rho_n(x_i/h; h)h +$$

$$\sum_{x_i=A}^{\infty} \left| Q_m^{(n)}(x_i)\{\rho_n(x_i/h; h) - \rho_n(x_i)\} \right| h +$$

$$+ \left| \sum_{x_i=A}^{\infty} Q_m^{(n)}(x_i)\rho_n(x_i)h - \int_A^{\infty} Q_m^{(n)}(x)\rho_n(x)\, dx \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

To conclude this Section let us point out that here we have taken the limit formally and have proved that our main result, i.e., formula (20), transforms into the corresponding one for the continuous case [41], but solving concrete examples one must to be very careful since, for instance, in the limit Hahn $\to$ Jacobi, the parameter $h = 1/N$ where N is the total number of points in the lattice and the Hahn polynomials explicitly depend on it. More information on how to take limits for concrete families can be found in [18, 23, 32, 33].

4. EXAMPLES.

4.1. Connection between $(q^s; q)_m$ and $c_n^\mu(x, q)$.

First of all we will apply theorem 3.1 for finding the connection coefficients c_{mn}^q in the expansion

$$(q^s; q)_m = \sum_{n=0}^{m} c_{m\,n}^q c_n^\mu(s, q), \tag{30}$$

where $(a; q)_k$ is defined in (18), and $c_n^\mu(s, q)$ denotes the q-Charlier polynomials on the exponential lattice $x(s) = \frac{q^s - 1}{q - 1}$ [2]

$$
\begin{aligned}
c_n^{(\mu)}(s, q) &= q^{\frac{n}{4}(n+5)} {}_2\varphi_0\left(\begin{array}{c} q^{-n}, q^{-s} \\ - \end{array}; q, -\frac{q^s}{(q-1)\mu}\right) = \\
&= q^{\frac{n}{4}(n+5)} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(q^s; q)^{[k]}}{\mu^k} \frac{1}{(1-q)^k}, \quad 1 < q < 1,\ 0 < \mu < 1.
\end{aligned}
\tag{31}
$$

where $(q^s; q)^{[k]} = \prod_{m=0}^{k-1}(1 - q^{s-m})$.

Obviously, these q-Charlier polynomials $c_n^{(\mu)}(s, q)$ are polynomials of degree n on any exponential lattice $x(s) = c_1 q^s + c_3$. For these polynomials we have [2]:

$$\rho(s) = \frac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad a_n = \frac{(-1)^n}{\mu^n} q^{-\frac{3n}{4}(n-1)+\frac{n}{2}}, \quad B_n = \frac{1}{\mu^n},$$

$$d_n^2 = \frac{e_q[(1-q)q^{n+1}\mu]}{e_q[(1-q)\mu]} \frac{[n]_q!}{q^{\frac{n}{4}(n-9)+\frac{1}{2}}\mu^n}, \quad \rho_n(s) = \frac{\mu^{s+n} q^{\frac{n}{2}(n+2s+1)}}{e_q[(1-q)\mu]\Gamma_q(s+1)},$$

where $e_q[z]$ denotes the q- exponential function [18] defined by

$$e_q[z] = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty}, \quad \Gamma_q(s) = (1-q)^{1-s} \frac{(q; q)_\infty}{(q^s; q)_\infty},$$

where $(z; q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k)$. Notice that all the characteristics of these $q-$Charlier polynomials, as well as the polynomials $c_n^{(\mu)}(s, q)$ themselves transform into the classical Charlier polynomials in the limit $q \to 1$. Notice also that the results presented here remain valid for the q-Charlier polynomials in the lattice $x(s) = q^s$ [1, 2] since their hypergeometric representation is given by the same basic hypergeometric series (31).

Now, using

$$\frac{\triangle(q^s; q)_n}{\triangle x(s)} = -q^{\frac{n-1}{2}}[n]_q c_1^{-1} (q^{s+1}; q)_{n-1}, \quad x(s) = c_1 q^s + c_3. \tag{32}$$

we get

$$\triangle^{(n)}\left[(q^s;q)_m\right] = q^{-\frac{n}{4}(n-1)}\left[\frac{\triangle}{\triangle x(s)}\right]^n (q^s;q)_m$$

$$= \frac{(1-q)^n[m]_q!q^{\frac{n}{2}(m-1)}}{[m-n]_q!}(q^{s+n};q)_{m-n}.$$

Therefore (20) gives

$$c_{m\,n}^q = \frac{q^{\frac{n}{4}(5n-7)}(q-1)^n\mu^n}{e_q[(1-q)\mu q^{n+1}]}\binom{m}{n}_q\sum_{s=0}^{\infty}\frac{(q^{s+n};q)_{m-n}\,[(1-q)\mu q^{n+1}]^s}{(q;q)_s},$$

where $\binom{m}{n}_q = \dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}$ are the classical q-binomial coefficients (do not confuse with the symmetric $\begin{bmatrix} m \\ n \end{bmatrix}_q$ q-binomial coefficients defined in (12)).

In order to take the sum in the above expression we will use the identity [18, Eq. (1.2.34) page 6] $(a\,q^s;q)_k = \frac{(a;q)_k(a\,q^k;q)_s}{(a;q)_s}$, as well as the expression [18, Eq. (1.5.2) page 11] $\frac{(q^m;q)_s}{(q^n;q)_s} = \sum_{k=0}^{s}\frac{(q^{-s};q)_k(q^{n-m};q)_k}{(q^n;q)_k}\frac{q^{m+s}}{(q;q)_k}$. Thus, denoting by $z = (1-q)\mu q^{n+1}$, we have

$$\sum_{s=0}^{\infty}\frac{(q^{s+n};q)_{m-n}\,z^s}{(q;q)_s} = \sum_{s=0}^{\infty}\frac{(q^n;q)_{m-n}(q^m;q)_s}{(q^n;q)_s(q;q)_s}z^s$$

$$= (q^n;q)_{m-n}\sum_{k=0}^{\infty}\frac{(q^{n-m};q)_kq^{mk}}{(q^n;q)_k(q;q)_k}\sum_{s=0}^{\infty}\frac{(q^{-s};q)_kq^{sk}}{(q;q)_s}z^s$$

$$= (q^n;q)_{m-n}\sum_{k=0}^{\infty}\frac{(q^{n-m};q)_kq^{mk}z^k}{(q^n;q)_k(q;q)_k}\left[(-1)^kq^{\frac{k}{2}(k-1)}\right]\sum_{s=k}^{\infty}\frac{z^{s-k}}{(q;q)_{s-k}}$$

$$= (q^n;q)_{m-n}e_q[(1-q)\mu q^{n+1}]_1\varphi_1\left(\begin{array}{c} q^{n-m} \\ q^n \end{array};q,\mu q^{n+m+1}(1-q)\right).$$

For the third equality we have used the identity [18, Eq. (1.2.32) page 6]

$$\frac{(q^{-s};q)_k}{(q;q)_s} = \frac{(-1)^kq^{\frac{k}{2}(k-1)-ks}}{(q;q)_{s-k}}. \tag{33}$$

Then, for the coefficients $c_{m\,n}^q$ we finally obtain

$$c_{m\,n}^q = (q^n;q)_{m-n}\mu^n(q-1)^nq^{\frac{n}{4}(5n-7)}\binom{m}{n}_q{}_1\varphi_1\left(\begin{array}{c} q^{n-m} \\ q^n \end{array};q,\mu q^{n+m+1}(1-q)\right).$$

Remark. Notice that, since $\frac{(q^s;q)_m}{(1-q)^m} = \sum_{n=0}^{m} \frac{c_{m\,n}^q}{(1-q)^m} c_n^\mu(x,q)$, and taking into account that $\lim_{q \to 1} \frac{(q^s;q)_m}{(1-q)^m} = (s)_m$, $\lim_{q \to 1} c_n^\mu(x,q) = c_n^\mu(s)$, we obtain taking the limit $q \to 1$

$$(s)_m = \sum_{n=0}^{m} c_{m\,n} c_n^\mu(s), \quad c_{m\,n} = \binom{m}{n} \frac{(m-1)!}{(n-1)!} (-\mu)^n {}_1F_1 \left(\begin{array}{c} n-m \\ n \end{array} \middle| -\mu \right),$$

where $c_n^\mu(s)$ denotes the classical (non monic) Charlier polynomials [32, 33]. Since for these polynomials the leading coefficients are given by $a_n = (-\mu)^{-n}$, the above result coincides with the classical result (see e.g. [5] and references therein).

4.2. Connection between $(q^s;q)^{[m]}$ and $c_n^\mu(x,q)$.

Now will apply theorem 3.1 for finding the connection coefficients c_{mn}^q in the expansion

$$(q^s;q)^{[m]} = \sum_{n=0}^{m} d_{m\,n}^q c_n^\mu(s,q), \tag{34}$$

where $(a;q)^{[k]} = \prod_{m=0}^{k-1} (1 - q^{s-m})$ and $c_n^\mu(s,q)$ is, as before, the q-Charlier polynomials on the lattice $x(s) = \frac{q^s-1}{q-1}$ (31). In this case, using

$$\frac{\triangle(q^s;q)^{[n]}}{\triangle x(s)} = -q^{-\frac{n-1}{2}} [n]_q c_1^{-1} \ (q^s;q)^{[n-1]}, \quad x(s) = c_1 q^s + c_3, \tag{35}$$

we find

$$\triangle^{(n)} \left[(q^s;q)^{[m]} \right] = q^{-\frac{n}{4}(n-1)} \left[\frac{\triangle}{\triangle x(s)} \right]^n (q^s;q)^{[m]}$$

$$= \frac{(1-q)^n [m]_q! q^{-\frac{n}{2}(m-1)}}{[m-n]_q!} (q^{s+n};q)_{m-n}.$$

Thus, using formula (20), the expression $\frac{(q^s;q)^{[m-n]}}{(q;q)_s} = \frac{1}{(q;q)_{s-m+n}}$, as well as

$$\sum_{s=0}^{\infty} \frac{(q^s;q)^{[m-n]} z^s}{(q;q)_s} = \sum_{s=m-n}^{\infty} \frac{(q^s;q)^{[m-n]} z^s}{(q;q)_s} = z^{m-n} \sum_{s=0}^{\infty} \frac{z^s}{(q;q)_s} = z^{m-n} e_q(z),$$

we obtain

$$d_{m\,n}^q = q^{m+\frac{n}{4}(n-7)} \binom{m}{n}_q (1-q)^m (-1)^n \mu^m. \tag{36}$$

The above formula is the q-analogue of the so-called *inversion formula* for hypergeometric polynomials (compare with the explicit expression of the q-Charlier polynomials (31).

Remark. If we rewrite (34) in the form

$$(s)_q^{[m]} = \sum_{n=0}^{m} \tilde{d}_{mn}^q c_n^\mu(s, q), \qquad \tilde{d}_{mn}^q = q^{m + \frac{n}{4}(n-7)} \binom{m}{n}_q (-1)^n \mu^m, \quad (37)$$

taking into account that $\lim_{q \to 1} \frac{(q^s; q)^{[m]}}{(1-q)^m} = (s)^{[m]}$, we obtain in the limit $q \to 1$

$$(s)^{[m]} = \sum_{n=0}^{m} d_{m\,n} c_n^\mu(s), \quad d_{m\,n} = \binom{m}{n} (-1)^n (\mu)^m.$$

Using again the fact that for the polynomials $c_n^\mu(s)$, the leading coefficients are given by $a_n = (-\mu)^{-n}$, the above result coincides with well know classical result (see e.g. [5] and references therein.)

4.3.   The $q-$Charlier polynomials in the exponential lattice.

Finally, we will solve now the connection problem

$$c_m^\gamma(s, q) = \sum_{n=0}^{m} c_{m\,n}^q c_n^\mu(s, q). \qquad (38)$$

Then, by using Eq. (23) of the corollary (3.1) where $Q_m(s)_q = c_m^\gamma(s, q)$ and $P_n(s)_q = c_n^\mu(s, q)$, respectively, we get for $c_{m\,n}^q$ the expression

$$\left(\frac{\mu}{\gamma}\right)^n \binom{m}{n}_q \frac{q^{\frac{1}{4}(m-n)(m-n+5)}}{e_q[(1-q)q^{n+1}\mu]} \sum_{l=0}^{m-n} \frac{(-1)^l q^{\frac{l}{2}(l-1)}}{q^{lm}(1-q)^l \gamma^l} \binom{m-n}{l}_q \sum_{s=l}^{\infty} \frac{[(1-q)\mu q^{n+1}]^{s-l}}{(q; q)_{s-l}}$$

$$= \left(\frac{\mu}{\gamma}\right)^n \binom{m}{n}_q q^{\frac{1}{4}(m-n)(m-n+5)} \sum_{l=0}^{m-n} (-1)^l \left(\frac{\mu}{\gamma} q^{n-m+1}\right)^l \binom{m-n}{l}_q q^{\frac{l(l-1)}{2}},$$

where we also use the fact that

$$\sum_{s=0}^{\infty} \frac{z^k}{\Gamma_q(s-k)} = \sum_{s=k}^{\infty} \frac{z^k}{\Gamma_q(s-k)} = \sum_{s=k}^{\infty} \frac{z^k (1-q)^{s-k}}{(q; q)_{s-k}}.$$

Now, applying the identity (33) to $(q; q)_{m-n-l}$ $(k = l)$, and using the q-binomial theorem [18, §1.3, Eq. (1.3.14) page 9],

$$\sum_{l=0}^{k} \frac{(q^{-k}; q)_l}{(q; q)_l} z^l = {}_1\varphi_0 \left(\begin{matrix} q^{-k} \\ - \end{matrix}; q, z\right) = (zq^{-k}; q)_k,$$

we obtain the following expression for the coefficient $c_{m\,n}^{q}$

$$c_{m\,n}^{q} = \left(\frac{\mu}{\gamma}\right)^{n} \binom{m}{n}_{q} q^{\frac{1}{4}(m-n)(m-n+5)} (q^{n-m+1}\,\mu\gamma^{-1}; q)_{m-n}. \qquad (39)$$

Notice the positivity of the coefficients (39) in the case when $\mu/\gamma < q^{m-1}$. In the case of the Charlier polynomials in the exponential lattice a similar results has been obtained in [28] by solving a recurrence relation for the coefficients (there the author does not give a closed formula for the connection coefficient).

Remark. A simple calculation shows that the equation (38) transforms in the limit $q \to 1$ into

$$c_{m}^{\gamma}(s) = \sum_{n=0}^{m} \binom{m}{n} \left(\frac{\mu}{\gamma}\right)^{n} \left(1 - \frac{\mu}{\gamma}\right)^{m-n} c_{n}^{\mu}(s),$$

for the (non monic) Charlier polynomials and this coincides with the classical results for monic polynomials (see e.g. [5]) since the leading coefficients for the Charlier polynomials $c_{n}^{\mu}(s)$ is equal to $(-\mu)^{-n}$.

4.4.   Further examples.

To conclude the paper we will show two more examples for polynomials on q−quadratic lattices, more exactly in the lattice $x(s) = [s]_{q}[s+1]_{q}$, i.e., $c_{1} = q^{\frac{1}{2}}\kappa_{q}^{-2}$ and $\mu = 1$. In this case there are not structure relations and then most of the aforesaid (in the introduction) methods can not be used.

In fact we will solve the following two examples:

$$(q^{-s}; q)_{m}(q^{s+1}; q)_{m} = \sum_{n=0}^{m} d_{m\,n} u_{n}^{\alpha,\beta}(x,0,b),$$

and

$$u_{n}^{\gamma,\delta}(x,0,d) = \sum_{n=0}^{m} c_{m\,n} u_{n}^{\alpha,\beta}(x,0,b),$$

where $u_{n}^{\alpha,\beta}(x,0,b)$ denotes the q−Racah polynomials introduced by Nikiforov and Uvarov in [33] (see also [1])

$$u_{n}^{\alpha,\beta}(x,0,b) = {}_{4}\varphi_{3}\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{-s}, q^{s+1} \\ q^{-b+1}, q^{\beta+1}, q^{b+\alpha+1} \end{array}; q\,,\,q\right). \qquad (40)$$

For these polynomials we have:

$$\rho(s) = \frac{q^{\frac{(b-\beta)(-1+b+\beta+2s)}{4}}\Gamma_q(s+\beta+1)\Gamma_q(s+\alpha+b+1)\Gamma_q(b+\alpha-s)}{\Gamma_q(s+b+1)\Gamma_q(s+-\beta+1)\Gamma_q(b-s)},$$

$$d_n^2 = q^{-\frac{\alpha(\alpha-1)}{2}-(\alpha+1)b+\frac{\beta}{2}+\alpha\beta+2n(\beta-\alpha-b)} \times$$

$$\frac{\Gamma_q(\alpha+n+1)\Gamma_q(\beta+n+1)\Gamma_q(b+\alpha-\beta+n+1)\Gamma_q(b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\Gamma_q(n+1)\Gamma_q(\alpha+\beta+n+1)\Gamma_q(b-n)\Gamma_q(b-\beta-n)}.$$

In the first case, using the identity

$$\frac{\Delta}{\Delta x(s)}(q^{s_1-s};q)_m(q^{s_1+s+\mu};q)_m$$
$$= -q^{s_1+\mu+\frac{-k+1}{2}}[k]_q c_1^{-1}(q^{s_1-s};q)_{m-1}(q^{s_1+s+\mu+1};q)_{m-1},$$

for the lattice $x(s) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q)$, and (20) we obtain

$$d_{mn} = \binom{m}{n}_q \frac{(-1)^n q^{\frac{n(n-1)}{2}}(q^{-b+1};q)_m(q^{\beta+1};q)_m(q^{b+\alpha+1};q)_m}{(q^{\alpha+\beta+n+1};q)_n(q^{\alpha+\beta+2+n+1};q)_n}.$$

Finally, using (23), after some straightforward but cumbersome calculations we find

$$c_{mn} = \frac{(-1)^n q^{\frac{n(n+1)}{2}}(q^{-m};q)_n(q^{\alpha+\beta+m+1};q)_n(q^{-d+1};q)_n(q^{\delta+1};q)_n(q^{d+\gamma+1};q)_n}{(q;q)_n(q^{-b+1};q)_n(q^{\beta+1};q)_n(q^{b+\alpha+1};q)_n(q^{\gamma+\delta+n+1};q)_n} \times$$

$$_5\varphi_4\left(\begin{array}{c} q^{n-m},q^{\alpha+\beta+n+m+1},q^{n-d+1},q^{\delta+n+1},q^{d+\gamma+n+1} \\ q^{\gamma+\delta+2n+1},q^{n-b+1},q^{n+\beta+1},q^{b+\alpha+n+1} \end{array}; q, q\right).$$

Notice that if we assume that $q \in (0,1)$ and take the limit $\gamma \to \infty$ we obtain the connection between $q-$Racah and $q-$Dual Hahn $W_n^{(\delta)}(x(s),0,d)_q = {}_3\varphi_2\left(\begin{array}{c} q^{-n},q^{-s},q^{s+1} \\ q^{-b+1},q^{\delta+1} \end{array}; q, q\right)$ introduced in [1]

$$c_{mn} = \frac{(-1)^n q^{\frac{n(n+1)}{2}}(q^{-m};q)_n(q^{\alpha+\beta+m+1};q)_n(q^{-d+1};q)_n(q^{\delta+1};q)_n}{(q;q)_n(q^{-b+1};q)_n(q^{\beta+1};q)_n(q^{b+\alpha+1};q)_n} \times$$

$$_4\varphi_3\left(\begin{array}{c} q^{n-m},q^{\alpha+\beta+n+m+1},q^{n-d+1},q^{\delta+n+1}, \\ q^{n-b+1},q^{n+\beta+1},q^{b+\alpha+n+1} \end{array}; q, q\right).$$

From the above equation, by taking the limits α, $\gamma \to \infty$, a formula for the connection coefficients for the $q-$Dual Hahn — $q-$Dual Hahn polynomials easily follows.

APPENDIX A

In this appendix we will prove the expression (11). In fact we will prove the following Lemma which is interesting in its own right:

Lemma. *Let $f(s)$ be an analytic function inside and on a curve C on the complex plane containing the points $z = s, s - 1, ..., s - n$, and $\nabla_k^{(n)}$ the operator*

$$\nabla_k^{(n)} \equiv \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)}.$$

Then

$$\nabla_k^{(n)} f(s) = \sum_{l=0}^{n-k} (-1)^l \begin{bmatrix} n-k \\ l \end{bmatrix}_q \frac{\nabla x_n(s - l + \frac{1}{2})}{\displaystyle\prod_{m=0}^{n-k} \nabla x_n(s - \frac{m+l-1}{2})} f(s-l). \quad \text{(A.1)}$$

Proof. First of all, notice that the function $x_m(z) = x(z + \frac{m}{2})$, where $x(s)$ is given by (5) satisfies

$$x(s) - x(s-t) = [t]_q \nabla x(s - \tfrac{t-1}{2}). \quad \text{(A.2)}$$

Then, by induction, one has

$$\nabla_k^{(n)} \left[\frac{1}{x_n(z) - x_n(s)} \right] = \frac{[n-k]_q!}{\displaystyle\prod_{m=0}^{k} [x_n(z) - x_n(s-m)]} = \frac{[n-k]_q!}{[x_n(z) - x_n(s)]^{(n-k+1)}},$$

where

$$[x_k(z) - x_k(s)]^{(m)} = \prod_{j=0}^{m-1} [x_k(z) - x_k(s-j)], \quad m = 0, 1, 2..., \quad \text{(A.3)}$$

denotes the generalized powers. Since f is analytic, then by using the Cauchy formula

$$f(s) = \frac{1}{2\pi i} \int_C \frac{f(z) x_n'(z)}{x_n(z) - x_n(s)} dz, \quad \text{(A.4)}$$

we have

$$f(s) = \frac{[n-k]_q!}{2\pi i} \int_C \frac{f(z) x_n'(z)}{[x_n(z) - x_n(s)]^{(n-k+1)}} dz. \quad \text{(A.5)}$$

If we now use the residue's theorem, and taking into account that the only singularities of the integrand are the simple poles located at $z = s - l$, $l = 0, 1, \cdots, n - k$, then

$$Res \left[\frac{f(z)x'_n(z)}{[x_n(z) - x_n(s)]^{(n-k+1)}} \right] = \frac{f(s-l)}{\displaystyle\prod_{\substack{m=0 \\ m \neq l}}^{n-k} [x_n(s-l) - x_n(s-m)]}.$$

Finally, using the property (A.2) the result follows. ▌

ACKNOWLEDGMENTS

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