On the Krall-type discrete polynomials

R. Álvarez-Nodarse[1] and J. Petronilho[2]

[1]Departamento de Análisis Matemático, Universidad de Sevilla.

Apdo. 1160, E-41080 Sevilla, Spain and

Instituto Carlos I de Física Teórica y Computacional,

Universidad de Granada. E-18071, Granada, Spain

E-mail: ran@us.es

[2]Dpto. de Matemática, FCTUC, Universidade de Coimbra,

Apdo. 3001-454 Coimbra, Portugal

E-mail:josep@mat.uc.pt

March 10, 2004

Abstract

In this paper we present a unified theory for studying the so called Krall-type discrete orthogonal polynomials. In particular, the three-term recurrence relation, lowering and raising operators as well as the second order linear difference equation that the sequences of monic orthogonal polynomials satisfy are established. Some relevant examples of q-Krall polynomials are considered in detail.

1

1 Introduction

Let $\mathfrak{u}$ be a quasi-definite linear functional in the vector space $\mathbb{P}$ of polynomials with complex coefficients. Then there exists a sequence of monic polynomials $(P_n)_n$ with $\deg P_n = n$, such that [14]

$$\langle \mathfrak{u}, P_n P_m \rangle = k_n \delta_{n,m}, \qquad k_n \neq 0, \quad n, m = 0, 1, 2, \ldots .$$

Special cases of quasi-definite linear functionals are the classical ones (those of Jacobi, Laguerre, Hermite, and Bessel). In the last years perturbations of the functional $\mathfrak{u}$ via the addition of Dirac delta functions —the so-called Krall-type orthogonal polynomials— have been extensively studied (see e.g. [6, 7, 16, 20, 21, 22, 26] and references therein), i.e., the linear functional

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{i=1}^{M} A_i \delta(x - a_i), \tag{1.1}$$

where $(A_i)_{i=1}^{M}$ are non-zero real numbers and $\delta(x - y)$ means the Dirac linear functional defined by $\langle \delta(x - y), p(x) \rangle = p(y)$, $\forall p \in \mathbb{P}$. In the very recent paper [2] we have considered the case of the more general functional $\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{i=1}^{M} A_i \delta(x - a_i) - \sum_{j=1}^{N} B_j \delta'(x - b_j)$, which also involves the case of derivatives of delta Dirac functionals defined by $\langle \delta'(x - a), p(x) \rangle = -p'(a)$. Moreover, in [2] a necessary and sufficient condition for the quasi-definiteness of the linear functional $\widetilde{\mathfrak{u}}$ was established and a detailed study when the original functional $\mathfrak{u}$ is a semiclassical functional was worked out in detail.

In the present paper we will suppose that the functional $\mathfrak{u}$ in (1.1) is a semiclassical discrete [31] or q-discrete [28] functional making an special emphasis in the case when $\mathfrak{u}$ is a classical discrete [17] or q-classical functional [29]. The interest of such modifications for the discrete case starts after the *Third International Symposium on Orthogonal Polynomials and their Applications* held in Erice (Italy) when R. Askey raised the question of identify and study the resulting polynomials of adding a delta Dirac measure to the classical Meixner linear functional. This problem was independently solved in [3] and [12] and it was extended to other families of classical polynomials (see [4] for the Hahn and Kravchuk cases and [5, 13] for the Charlier one, for a general framework see [19]). The case when $\mathfrak{u}$ is a q-classical linear functional is still open and only few results by Costas-Santos [15] are known. Another connected problem is related with the so called coherent pairs for measures [27, 30] that leads to similar linear discrete functionals [9, 10, 11].

Let us also point out that there are also the so-called discrete (see e.g. [8]) and q-discrete Sobolev type orthogonal polynomials associated with the classical discrete and q-classical functionals [23, 24]. In both cases the corresponding polynomials can be reduced to the Krall-type one (except for the q-case when the mass is added at zero where a more careful study is needed [23, 24]) since the differences $\Delta f(x) = f(x+1) - f(x)$ and $D_q f(x) = (f(qx) - f(x))/(qx - x)$.

The aim of this contribution is to present a simple and unified approach to the study of such perturbations of the semiclassical and q-semiclassical functionals.

The structure of the paper is as follows: In Section 2 some remarks on the general theory [2] are included as well as a detailed discussion when $\mathbf{u}$ is a semiclassical functional. In Section 3 the algebraic properties of the new family are obtained, and, finally, in Section 4 some examples are developed in details.

2 General theory

2.1 Representation formula

We follow [2]. If $\widetilde{\mathbf{u}}$ in (1.1) is quasi-definite then there exists a sequence of monic polynomials $(\widetilde{P}_n)_n$ orthogonal with respect to $\widetilde{\mathbf{u}}$ and therefore we can consider the Fourier expansion

$$\widetilde{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} \lambda_{n,k} P_k(x), \quad n = 0, 1, 2, \dots. \tag{2.1}$$

Then, for $0 \leq k \leq n - 1$,

$$\lambda_{n,k} = \frac{\langle \mathbf{u}, \widetilde{P}_n(x) P_k(x) \rangle}{\langle \mathbf{u}, P_k^2(x) \rangle} = -\sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \frac{P_k(a_i)}{\langle \mathbf{u}, P_k^2(x) \rangle}.$$

Thus, (2.1) becomes

$$\widetilde{P}_n(x) = P_n(x) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}(x, a_i) \tag{2.2}$$

where, as usual, $\mathrm{K}_n(x, y) = \sum_{l=0}^{n} \frac{P_l(x) P_l(y)}{\langle \mathbf{u}, P_l^2(x) \rangle}$ denotes the reproducing kernel associated with the linear functional $\mathbf{u}$. Therefore from (2.2) we get the following system of M linear equations in the M unknowns $(\widetilde{P}_n(a_k))_{k=1}^{M}$

$$\widetilde{P}_n(a_k) = P_n(a_k) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}(a_k, a_i), \quad k = 1, 2, \dots, M, \tag{2.3}$$

To simplify the above expressions we use the notations of [2] (A^T is the transpose of A):

$$\mathcal{P}_n(\vec{z}) = (P_n(z_1), P_n(z_2), \ldots, P_n(z_k))^T, \qquad \vec{z} = (z_1, z_2, \cdots, z_k)^T.$$

Also we introduce the matrices $\mathcal{K}_{n-1}(\vec{z}, \vec{y}) \in \mathbb{C}^{p \times q}$ whose (m, n) entry is $K_{n-1}(z_m, y_n)$. Here $\vec{z} = (z_1, z_2, \ldots, z_p)$ and $\vec{y} = (y_1, y_2, \ldots, y_q)$. Finally, we introduce the matrix associated with the mass points $D = \mathrm{diag}\,(A_1, A_2, \ldots, A_M)$. With this notation (2.3) can be rewritten as

$$\widetilde{\mathcal{P}}_n(\vec{a}) = \mathcal{P}_n(\vec{a}) - \mathbb{K}_{n-1} D \widetilde{\mathcal{P}}_n(\vec{a}), \qquad \mathbb{K}_{n-1} = \mathcal{K}_{n-1}(\vec{a}, \vec{a}), \tag{2.4}$$

where $\vec{a} = (a_1, a_2, \ldots, a_M)$. If the matrix $I + \mathbb{K}_{n-1}D$, where I is the identity matrix, is nonsingular, then we get the existence and uniqueness for the solution of (2.4) and therefore (2.2) becomes

$$\widetilde{P}_n(x) = P_n(x) - \mathcal{K}_{n-1}^T(x, \vec{a}) D (I + \mathbb{K}_{n-1}D)^{-1} \mathcal{P}_n(\vec{a}). \tag{2.5}$$

The above formula constitutes the first representation formula for the polynomials $(\widetilde{P}_n)_n$.

From the above expression and following [2] we obtain the following

Theorem 1 *The linear functional* $\widetilde{\mathfrak{u}}$ *defined in* (1.1) *is a quasi-definite linear functional if and only if*

(i) The matrix $I + \mathbb{K}_{n-1}D$ is nonsingular for every $n \in \mathbb{N}$.

(ii) $\langle \mathfrak{u}, P_n^2(x) \rangle + \mathcal{P}_n^T(\vec{a}) D (I + \mathbb{K}_{n-1}D)^{-1} \mathcal{P}_n(\vec{a}) \neq 0$, for every $n \in \mathbb{N}$.

In such a case the norm $\widetilde{d}_n^2 := \langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(x) \rangle$ is

$$\langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(x) \rangle = \langle \mathfrak{u}, P_n^2(x) \rangle + \mathcal{P}_n^T(\vec{a}) D (I + \mathbb{K}_{n-1}D)^{-1} \mathcal{P}_n(\vec{a}), \tag{2.6}$$

and the corresponding sequence $(\widetilde{P}_n)_n$ of monic orthogonal polynomials is given by (2.5).

Furthermore, taking into account $A_i \neq 0$, $i = 1, 2, \ldots, M$, then D is a nonsingular matrix. Thus $D(I + \mathbb{K}_{n-1}D)^{-1} = (D^{-1} + \mathbb{K}_{n-1})^{-1} := M_{n-1}$, so (ii) reads

$$1 + \varepsilon_n \widehat{\mathcal{P}}_n^T(\vec{a}) M_{n-1} \widehat{\mathcal{P}}_n(\vec{a}) \neq 0, \qquad \widehat{P}_n(a_i) = \frac{P_n(a_i)}{\sqrt{|\langle \mathfrak{u}, P_n^2(x) \rangle|}},$$

and $\varepsilon_n = e^{-i \arg \langle \mathfrak{u}, P_n^2 \rangle}$, where $\arg z$ means the principal argument of $z \in \mathbb{C}$.

Similarly to [2], if we multiply (2.5) by $\phi(x) = \prod_{i=1}^{M}(x - a_i)$, and use the Christoffel-Darboux formula

$$\mathrm{K}_{n-1}(x, y) = \frac{1}{k_n}\left[\frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x - y}\right], \qquad k_n = \langle \mathfrak{u}, P_n^2(x)\rangle, \qquad (2.7)$$

then we obtain the representation

$$\phi(x)\widetilde{P}_n(x) = A(x; n)P_n(x) + B(x; n)P_{n-1}(x), \qquad (2.8)$$

where $A(x; n)$ and $B(x; n)$ are polynomials of degree bounded by a number independent of n and at most M and $M - 1$, respectively. On the other hand, from the three-term recurrence relation that the sequence $(P_n)_n$ satisfies

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \qquad \gamma_n \neq 0, \quad \forall n \in \mathbb{N}, \qquad (2.9)$$

and taking into account (2.8) we get, for $n \geq 1$

$$\phi(x)\widetilde{P}_{n-1}(x) = C(x; n)P_n(x) + D(x; n)P_{n-1}(x),$$

$$\qquad (2.10)$$

$$C(x; n) = -\frac{B(x; n - 1)}{\gamma_{n-1}}, \quad D(x; n) = A(x; n - 1) + \frac{x - \beta_{n-1}}{\gamma_{n-1}}B(x; n - 1).$$

Let us point out that the above representations are valid for any family of polynomials orthogonal with respect to the linear functional (1.1).

Notice also that, as in the continuous case [2], an inverse process can be done in order to recover the linear functional $\mathfrak{u}$ in terms of $\widetilde{\mathfrak{u}}$ (it is sufficient to add to $\widetilde{\mathfrak{u}}$ the same masses but with opposite sign). Therefore, there exist two polynomials $\overline{A}(x; n)$ and $\overline{B}(x; n)$ with degrees bounded by a number independent of n such that

$$\phi(x)P_n(x) = \overline{A}(x; n)\widetilde{P}_n(x) + \overline{B}(x; n)\widetilde{P}_{n-1}(x). \qquad (2.11)$$

2.2 Representation formula in the semiclassical case

If $\mathfrak{u}$ is a semiclassical discrete linear functional, then there exist a polynomial $\psi(x)$ and two polynomials $M_1(x; n)$ and $N_1(x; n)$, with degree bounded by a number independent of n, such that [31]

$$\psi(x)\Delta P_n(x) = M_1(x; n)P_n(x) + N_1(x; n)P_{n-1}(x), \qquad (2.12)$$

where Δ is the forward difference operator $\Delta f(x) = f(x + 1) - f(x)$. Notice that using the TTRR (2.9) we get a similar expression but in terms of P_n and P_{n-1}

$$\psi(x)\Delta P_n(x) = M_2(x; n)P_n(x) + N_2(x; n)P_{n+1}(x). \qquad (2.13)$$

where again the degree of $M_2(x;n)$ and $N_2(x;n)$ are bounded by a number independent of n. Usually the formulas (2.12) and (2.13) are called the lowering and raising operators for the family $(P_n)_n$.

Similarly, for the q-semiclassical case a similar result is known [28], i.e., there exist a polynomial $\psi(x)$ as well as the polynomials $M_1(x;n)$, $N_1(x;n)$, $M_2(x;n)$, and $N_2(x;n)$, with degree bounded by a number independent of n, such that

$$\psi(x)D_qP_n(x) = M_1(x;n)P_n(x) + N_1(x;n)P_{n-1}(x), \qquad (2.14)$$

$$\psi(x)D_qP_n(x) = M_2(x;n)P_n(x) + N_2(x;n)P_{n+1}(x). \qquad (2.15)$$

where D_q is the q-Jackson derivative[1]

$$D_qP(x) = \frac{P(qx) - P(x)}{x(q-1)}, \qquad q \neq 0, \pm 1.$$

Using either (2.8) and (2.12) or (2.10) and (2.13) we obtain the following representation formula

$$\pi(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n(x+1), \qquad (2.16)$$

where a, b and π are polynomials of degree bounded by a number independent of n.

In the q-case the situation is the same. In fact using (2.8) and (2.14) or (2.10) and (2.15) we obtain the following representation formula

$$\pi(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n(qx), \qquad (2.17)$$

where a, b and π are polynomials of degree bounded by a number independent of n.

3 Algebraic properties of the polynomials $\widetilde{P}_n(x)$

3.1 The three-term recurrence relation for $(\widetilde{P}_n)_n$

In the sequel we assume that $\widetilde{\mathbf{u}}$ is quasi-definite. Then, the sequence $(\widetilde{P}_n)_n$ of monic polynomials orthogonal with respect to $\widetilde{\mathbf{u}}$ satisfies a three-term recurrence relation (TTRR)

$$x\widetilde{P}_n(x) = \widetilde{P}_{n+1}(x) + \widetilde{\beta}_n\widetilde{P}_n(x) + \widetilde{\gamma}_n\widetilde{P}_{n-1}(x), \qquad n \in \mathbb{N}, \qquad (3.1)$$

with the initial conditions $\widetilde{P}_{-1}(x) = 0$, $\widetilde{P}_0(x) = 1$. To obtain the coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ of the TTRR (3.1) for the polynomials $\widetilde{P}_n$ orthogonal with respect to $\widetilde{\mathbf{u}}$ we use

---

[1]Usually $q \in (0,1)$.

the standard formulas for orthogonal polynomials (see e.g. [14]). Thus, using (2.6) we find

$$\widetilde{\gamma}_n = \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x) \rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_{n-1}^2(x) \rangle} = \gamma_n \frac{1 + \varepsilon_n \widehat{\mathcal{P}}_n^T(\vec{a}) \mathrm{M}_{n-1} \widehat{\mathcal{P}}_n(\vec{a})}{1 + \varepsilon_{n-1} \widehat{\mathcal{P}}_{n-1}^T(\vec{a}) \mathrm{M}_{n-2} \widehat{\mathcal{P}}_{n-1}(\vec{a})}, \quad n > 1,$$

as well as, for $n = 1$

$$\widetilde{\gamma}_1 = \gamma_1 \frac{1 + \varepsilon_1 \widehat{\mathcal{P}}_n^T(\vec{a}) \mathrm{M}_0 \widehat{\mathcal{P}}_n(\vec{a})}{1 + \sum_{i=1}^M A_i / u_0},$$

where $u_0 = \langle \mathbf{u}, 1 \rangle$ is the first moment of the functional $\mathbf{u}$.

On the other hand, $\widetilde{\beta}_n = \widetilde{b}_n - \widetilde{b}_{n+1}$, where $\widetilde{b}_n$ denotes the coefficient of x^{n-1} for $\widetilde{P}_n$ and b_n is the corresponding coefficient of x^{n-1} for P_n. To compute $\widetilde{b}_n$ we use (2.5), so that

$$\widetilde{b}_n = b_n - \varepsilon_n \varepsilon_{n-1} |\gamma_n|^{1/2} \widehat{\mathcal{P}}_{n-1}^T(\vec{a}) \mathrm{M}_{n-1} \widehat{\mathcal{P}}_n^T(\vec{a})$$

and therefore

$$\widetilde{\beta}_n = \beta_n + \varepsilon_n \varepsilon_{n+1} |\gamma_{n+1}|^{1/2} \widehat{\mathcal{P}}_n^T(\vec{a}) \mathrm{M}_n \widehat{\mathcal{P}}_{n+1}^T(\vec{a}) - \varepsilon_n \varepsilon_{n-1} |\gamma_n|^{1/2} \widehat{\mathcal{P}}_{n-1}^T(\vec{a}) \mathrm{M}_{n-1} \widehat{\mathcal{P}}_n^T(\vec{a}).$$

Finally, for $n = 0$ we have

$$\widetilde{\beta}_0 = \frac{\langle \widetilde{\mathbf{u}}, x \rangle}{\langle \widetilde{\mathbf{u}}, 1 \rangle} = \frac{u_1 + \sum_{i=1}^M a_i A_i}{u_0 + \sum_{i=1}^M A_i}, \quad u_1 = \langle \mathbf{u}, x \rangle.$$

3.2 Second order difference equation for $(\widetilde{P}_n)_n$

In the following we assume that $\mathbf{u}$ is a semiclassical discrete or q-discrete functional.

From the representation formulas (2.8) and (2.16) and (2.17) follows that the polynomials $\widetilde{P}_n$ satisfy a second order difference equation. For the discrete case it is an immediate consequence of the Theorem 2.1 or Theorem 3.1 in [1]. In fact, we have

Theorem 2 *Suppose the polynomials $(\widetilde{P}_n)_n$ are defined by (2.16) where the polynomial P_n is a solution of a second order difference equation (SODE)*

$$\sigma(x; n) P_n(x-1) - \varphi(x; n) P_n(x) + \varsigma(x; n) P_n(x+1) = 0. \tag{3.2}$$

Then $\{\widetilde{P}_n\}$ satisfy the SODE

$$\widetilde{\sigma}(x; n) \Delta \nabla \widetilde{P}_n(x) + \widetilde{\tau}(x; n) \Delta \widetilde{P}_n(x) + \widetilde{\lambda}(x; n) \widetilde{P}_n(x) = 0, \tag{3.3}$$

where $\widetilde{\tau}(x; n) = \widetilde{\varsigma}(x; n) - \widetilde{\sigma}(x; n)$, $\widetilde{\lambda}(x; n) = \widetilde{\varsigma}(x; n) + \widetilde{\sigma}(x; n) + \widetilde{\varphi}(x; n)$, and $\widetilde{\sigma}$, $\widetilde{\varphi}$, and $\widetilde{\varsigma}$ are given explicitly in (3.8).

For the sake of completeness we present an sketch of the proof. We start with the representation formula (2.16)

$$\pi(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n(x+1), \qquad (3.4)$$

and evaluate it in $x \pm 1$ and then we use (3.2) to substitute the values $P_n(x-1)$ and $P_n(x+2)$. So, we obtain

$$r(x;n)\widetilde{P}_n(x+1) = c(x;n)P_n(x) + d(x;n)P_n(x+1),$$

$$r(x;n) = \varsigma(x+1;n)\pi(x+1;n), \quad c(x;n) = -\sigma(x+1;n)\,b(x+1;n), \qquad (3.5)$$

$$d(x;n) = a(x+1;n)\varsigma(x+1;n) + b(x+1;n)\varphi(x+1;n),$$

and

$$s(x;n)\widetilde{P}_n(x-1) = e(x;n)P_n(x) + f(x;n)P_n(x+1),$$

$$s(x;n) = \sigma(x;n)\pi(x-1;n), \quad e(x;n) = \sigma(x;n)b(x-1;n) + a(x-1;n)\varphi(x;n),$$

$$f(x;n) = -a(x-1;n)\varsigma(x;n).$$

$$(3.6)$$

Then, Eqs. (3.4–3.6) yield

$$\begin{vmatrix} \pi(x;n)\widetilde{P}_n(x) & a(x;n) & b(x;n) \\ r(x;n)\widetilde{P}_n(x+1) & c(x;n) & d(x;n) \\ s(x;n)\widetilde{P}_n(x-1) & e(x;n) & f(x;n) \end{vmatrix} = 0, \qquad (3.7)$$

where the functions π, a, and b are given by (2.16) as well as c, d, e, f, r, and s in (3.5) and (3.6). Expanding the determinant in (3.7) by the first column we get

$$\widetilde{\sigma}(x;n)\widetilde{P}_n(x-1) - \widetilde{\varphi}(x;n)\widetilde{P}_n(x) + \widetilde{\varsigma}(x;n)\widetilde{P}_n(x+1) = 0,$$

where

$$\widetilde{\sigma}(x;n) = s(x;n)[a(x;n)d(x;n) - c(x;n)b(x;n)],$$

$$\widetilde{\varphi}(x;n) = -\pi(x;n)[c(x;n)f(x;n) - e(x;n)d(x;n)], \qquad (3.8)$$

$$\widetilde{\varsigma}(x;n) = r(x;n)[e(x;n)b(x;n) - a(x;n)f(x;n)],$$

or, equivalently, (3.3). ∎

To conclude this section let notice that for the q-case a similar equation can be obtained using the same technique developed here. Nevertheless we can immediately obtain the result as follows.

8

Let us write $x = q^s$. Then $f(qx) = f(q^{s+1})$ and therefore (2.17) can be rewritten as follows

$$\pi(q^s; n)\widetilde{P}_n(q^s) = a(q^s; n)P_n(q^s) + b(q^s; n)P_n(q^{s+1}),$$

or, in terms of the s variable,

$$\pi(s; n)\widetilde{P}_n(s) = a(s; n)P_n(s) + b(s; n)P_n(s + 1),$$

i.e., they admit the same representation (2.16) changing x by q^s. But for the q-semiclassical polynomials the following second order q-difference equation is known (see e.g. [28])

$$\sigma(x; n)P_n(q^{-1}x) - \varphi(x; n)P_n(x) + \varsigma(x; n)P_n(qx) = 0, \tag{3.9}$$

which becomes into the equation (3.2) with the change $x \to q^s$. Thus the following result holds

Theorem 3 *Assume the polynomials $(\widetilde{P}_n)_n$ satisfy (2.17) where the polynomial P_n is a solution of a q-SODE (3.9). Then $(\widetilde{P}_n)_n$ satisfy the q-SODE*

$$\widetilde{\sigma}(x; n)\widetilde{P}_n(q^{-1}x) + \widetilde{\varphi}(x; n)\widetilde{P}_n(x) + \widetilde{\varsigma}(x; n)\widetilde{P}_n(qx) = 0, \tag{3.10}$$

where $\widetilde{\sigma}$, $\widetilde{\varphi}$, and $\widetilde{\varsigma}$ are given explicitly by (3.8) but now

$$r(x; n) = \varsigma(qx; n)\pi(qx; n), \qquad c(x; n) = -\sigma(qx; n)\, b(qx; n),$$

$$d(x; n) = a(qx; n)\varsigma(qx; n) + b(qx; n)\varphi(qx; n),$$

$$s(x; n) = \sigma(x; n)\pi(q^{-1}x; n), \qquad e(x; n) = \sigma(x; n)b(q^{-1}x; n) + a(q^{-1}x; n)\varphi(x; n),$$

$$f(x; n) = -a(q^{-1}x; n)\varsigma(x; n).$$

3.3   The lowering and raising operators

In this section we will prove that the polynomials $\widetilde{P}_n$ orthogonal with respect to the linear discrete functional $\widetilde{\mathfrak{u}}$, where $\mathfrak{u}$ is a semiclassical functional, have lowering and rising-type operators.

Proposition 4 *The lowering-type operator associated with the discrete linear functional $\widetilde{\mathfrak{u}}$ is given by the expression*

$$\alpha_l(x; n)\widetilde{P}_n(x) + \beta_l(x; n)\widetilde{P}_n(x + 1) = \gamma_l(x; n)\widetilde{P}_{n-1}(x), \tag{3.11}$$

where

$$\alpha_l(x; n) = \phi(x)d(x; n)\pi(x; n) - [a(x; n)d(x; n) - c(x; n)b(x; n)]\overline{A}(x; n),$$

$$\beta_l(x; n) = -\phi(x)b(x; n)r(x; n), \quad \gamma_l(x; n) = [a(x; n)d(x; n) - c(x; n)b(x; n)]\overline{B}(x; n).$$

Proof: Using formulas (3.4) and (3.5) we find

$$d(x; n)\pi(x; n)\widetilde{P}_n(x) - b(x; n)r(x; n)\widetilde{P}_n(x+1) = [a(x; n)d(x; n) - c(x; n)b(x; n)]P_n(x).$$

Multiplying the last formula by $\phi(x)$ and using (2.11) we obtain the result. ∎

Notice that from (3.11) and using the TTRR (3.1) we obtain the raising-type operator

$$\alpha_r(x; n)\widetilde{P}_n(x) + \beta_r(x; n)\widetilde{P}_n(x + 1) = \gamma_r(x; n)\widetilde{P}_{n+1}(x), \qquad (3.12)$$

where

$$\alpha_r(x; n) = \alpha_l(x; n) + \gamma_l(x; n)(\widetilde{\beta}_n - x)\widetilde{\gamma}_n^{-1}, \quad \beta_r(x; n) = \beta_l(x; n), \quad \gamma_r(x; n) = -\gamma_l(x; n)\widetilde{\gamma}_n^{-1}.$$

Notice that if instead of formula (3.5) we use (3.6) then we will find expressions similar to (3.11) and (3.12) but with the term $\widetilde{P}_n(x - 1)$ instead of $\widetilde{P}_n(x + 1)$.

In a complete analogous way but using (2.17) we have

Proposition 5 *The lowering operator associated with the q-linear functional $\widetilde{\mathfrak{u}}$ is*

$$\alpha_l(x; n)_q\widetilde{P}_n(x) + \beta_l(x; n)_q\widetilde{P}_n(qx) = \gamma_l(x; n)_q\widetilde{P}_{n-1}(x), \qquad (3.13)$$

where

$$\alpha_l(x; n)_q = \phi(x)d(x; n)\pi(x; n) - [a(x; n)d(x; n) - c(x; n)b(x; n)]\overline{A}(x; n),$$

$$\beta_l(x; n)_q = -\phi(x)b(x; n)r(x; n), \quad \gamma_l(x; n)_q = [a(x; n)d(x; n) - c(x; n)b(x; n)]\overline{B}(x; n).$$

The raising operator in this case is

$$\alpha_r(x; n)_q\widetilde{P}_n(x) + \beta_r(x; n)_q\widetilde{P}_n(qx) = \gamma_r(x; n)_q\widetilde{P}_{n+1}(x), \qquad (3.14)$$

where

$$\alpha_r(x; n)_q = \alpha_l(x; n)_q + \gamma_l(x; n)_q(\widetilde{\beta}_n - x)\widetilde{\gamma}_n^{-1}, \quad \beta_r(x; n)_q = \beta_l(x; n)_q,$$

$$\gamma_r(x; n)_q = -\gamma_l(x; n)_q\widetilde{\gamma}_n^{-1}.$$

As before, from the above equations similar expression involving the terms $\widetilde{P}_n(q^{-1}x)$ can be easily obtained.

10

4 Examples

Here we will consider some examples. Since the classical case with one or two extra delta Dirac measures (functionals) has been studied intensively (see e.g. [4, 5]) we will focus here our attention in the q-case. For the sake of simplicity we will choose the Al-Salam & Carlitz I polynomial as the starting family. The main data of such family can be found in [25, page 113].

The Al-Salam & Carlitz I polynomials are defined by

$$U_n^{(a)}(x) := U_n^{(a)}(x; q) = (-a)^n q^{\frac{1}{2}n(n-1)} {}_2\varphi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \Big| q; \frac{x\,q}{a} \right),$$

where the basic hypergeometric series ${}_r\varphi_p$ is defined by [18]

$${}_r\varphi_p \left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_p \end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k(k-1)}{2}} \right]^{p-r+1},$$

being $(a; q)_k = \prod_{m=0}^{k-1}(1 - aq^m)$ the q-shifted factorials. Also we will use the standard notation $(a_1, \ldots, a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k$ and $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$.

The polynomials $U_n^{(a)}(x)$ satisfy the following properties: a second order linear difference equation

$$aU_n^{(a)}(qx) - [a + q(1-x)(a-x)]U_n^{(a)}(x) + q(1-x)(a-x)U_n^{(a)}(q^{-1}x)$$
$$= q^{1-n}(1 - q^n)x^2 U_n^{(a)}(x), \tag{4.1}$$

i.e., an equation of the form (3.9) with

$$\sigma(x; n) = q(1-x)(a-x), \quad \varphi(x; n) = a + q(1-x)(a-x) + q^{1-n}(1-q^n)x^2, \quad \varsigma(x; n) = a,$$

the three-term recurrence relation

$$xU_n^{(a)}(x) = U_{n+1}^{(a)}(x) + (1+a)q^n U_n^{(a)}(x) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x), \quad n = 0, 1, 2, \ldots, \tag{4.2}$$

and the differentiation formula

$$U_n^{(a)}(x) - U_n^{(a)}(qx) = (1 - q^n)xU_{n-1}^{(a)}(x). \tag{4.3}$$

They satisfy the following orthogonality relation

$$\int_a^1 (qx; q)_\infty (qx/a; q)_\infty U_n^{(a)}(x)U_m^{(a)}(x)d_q x = d_n^2 \delta_{n,m}, \quad a < 0, \tag{4.4}$$

11

where
$$d_n^2 = (-a)^n (1-q)(q;q)_n (q;q)_\infty (a;q)_\infty (a^{-1}q;q)_\infty q^{\frac{1}{2}n(n-1)}.$$

Here $\int_a^b f(x)d_qx$ denotes the q-integral by Jackson (see e.g. [18, 25]).

From the above orthogonality relation we can define the positive definite linear functional $\mathfrak{u}$ as

$$\mathfrak{u}_a : \mathbb{P} \to \mathbb{C}, \quad \mathfrak{u}_a[P(x)] = \int_a^1 (qx;q)_\infty (qx/a;q)_\infty P(x)d_qx, \qquad a < 0. \qquad (4.5)$$

A particular case of this functional is $a = -1$ that leads to the discrete q-Hermite I polynomials, a q-analog of the Hermite polynomials.

4.1 Modification of the Al-Salam & Carlitz I polynomials

As an example we will consider the following *perturbed* functional $\widetilde{\mathfrak{u}}_a : \mathbb{P} \to \mathbb{C}$,

$$\widetilde{\mathfrak{u}}_a[P(x)] = \int_a^1 (qx;q)_\infty (qx/a;q)_\infty P(x)d_qx + AP(x_0), \quad a < 0 < A. \qquad (4.6)$$

The polynomials orthogonal with respect to the linear functional (4.6) will be denoted by $U_n^{(a),A}(x)$.

Using (2.3) and (2.7) (or (2.8)) we find

$$\begin{aligned}
(x - x_0)U_n^{(a),A}(x) = [x - x_0 - AU_n^{(a),A}(x_0)d_n^{-2}U_{n-1}^{(a)}(x_0)]U_n^{(a)}(x) \\
+ AU_n^{(a),A}(x_0)d_n^{-2}U_n^{(a)}(x_0)U_{n-1}^{(a)}(x),
\end{aligned} \qquad (4.7)$$

where

$$U_n^{(a),A}(x_0) = \frac{U_n^{(a)}(x_0)}{1 + A\sum_{k=0}^{n-1}(U_k^{(a)}(x_0))^2 d_k^{-2}} = \frac{U_n^{(a)}(x_0)}{1 + A\mathrm{K}_{n-1}(x_0, x_0)}.$$

Therefore, taking into account (4.3) and (4.7), we find that (2.17) holds with

$$\begin{aligned}
\pi(x;n) &= x(x - x_0), \\
a(x;n) &= x\left(x - x_0 - \frac{AU_n^{(a),A}(x_0)U_{n-1}^{(a)}(x_0)}{d_n^2}\right) + \frac{AU_n^{(a),A}(x_0)U_n^{(a)}(x_0)}{(1-q^n)d_n^2} \\
b(x;n) &= -\frac{AU_n^{(a),A}(x_0)U_n^{(a)}(x_0)}{(1-q^n)d_n^2}.
\end{aligned}$$

For these polynomials, by (2.6), we have

$$\widetilde{d_n}^2 = \langle \widetilde{\mathfrak{u}}, \left(U_n^{(a),A}\right)^2 \rangle = d_n^2 + A[U_n^{(a)}(x_0)]^2(1 + A\mathrm{K}_{n-1}(x_0, x_0))^{-1},$$

12

and therefore the coefficients of the TTRR are

$$\widetilde{\beta}_n = (1+a)q^n + A\left[\frac{U_{n+1}^{(a)}(x_0)U_n^{(a)}(x_0)}{d_{n+1}^2(1+A\mathrm{K}_n(x_0,x_0))} - \frac{U_n^{(a)}(x_0)U_{n-1}^{(a)}(x_0)}{d_n^2(1+A\mathrm{K}_{n-1}(x_0,x_0))}\right],$$

$$\widetilde{\gamma}_n = -aq^{n-1}(1-q^n)\frac{1+A[U_n^{(a)}(x_0)d_n^{-1}]^2(1+A\mathrm{K}_{n-1}(x_0,x_0))^{-1}}{1+A[U_{n-1}^{(a)}(x_0)d_{n-1}^{-1}]^2(1+A\mathrm{K}_{n-2}(x_0,x_0))^{-1}}.$$

(4.8)

Now, from the above explicit expressions of $\pi(x;n)$, $a(x;n)$, $b(x;n)$, $\sigma(x;n)$, $\varphi(x;n)$, and $\varsigma(x;n)$, we immediately obtain the second order difference equation (3.10). Finally, to deduce the lowering and raising operators we should obtain formula (2.11) that, for this case is

$$U_n^{(a)}(x) = U_n^{(a),A}(x) + AU_n^{(a)}(x_0)\widetilde{\mathrm{K}}_{n-1}(x,x_0),$$

or, equivalently

$$(x-x_0)U_n^{(a)}(x) = \overline{A}(x;n)U_n^{(a),A}(x) + \overline{B}(x;n)U_{n-1}^{(a),A}(x),$$

where

$$\overline{A}(x;n) = x - x_0 + \frac{AU_n^{(a)}(x_0)U_{n-1}^{(a),A}(x_0)}{\widetilde{d}_n^2}, \quad \overline{B}(x;n) = -\frac{AU_n^{(a)}(x_0)U_n^{(a),A}(x_0)}{\widetilde{d}_n^2}.$$

Therefore, (3.14) and (3.13) give the raising and lowering operators. For the sake of simplicity we will omit the explicit expressions of the q-SODE and the raising and lowering operators and we only present them for the special case of discrete q-Hermite I polynomials.

4.2   Modification of the discrete q-Hermite I polynomials

To conclude this work we will consider the discrete q-Hermite I polynomials, i.e. the polynomials $h_n(x;q) := U_n^{(-1)}(x;q)$, and let us study in detail the modification of these polynomials via the addition of a delta Dirac measure A at $x_0 = 0$, which will be denoted by $h_n^A(x;q)$. The main data for the q-Hermite I polynomials follow from the data of the Al-Salam & Carlitz I putting $a = -1$.

According to (4.7), in this case the representation formula (2.8) reads as

$$xh_n^A(x;q) = xh_n(x;q) + \Gamma_n h_{n-1}(x;q), \qquad n \geq 1,$$

(4.9)

where

$$\Gamma_n = \begin{cases} \dfrac{A[h_{2m}(0)]^2}{d_{2m}^2(1+A\mathrm{K}_{2m-1}(0,0))}, & n = 2m \\ \\ 0, & n = 2m-1, \end{cases}$$

13

$$d_n^2 = (1-q)(q;q)_n(q,-1,-q;q)_\infty q^{\binom{n}{2}},$$

$$h_n(0;q) = \begin{cases} q^{m(m-1)}(-1)^m(q;q^2)_m, & n = 2m \\ \\ 0, & n = 2m-1, \end{cases} \quad (m \in \mathbb{N}),$$

and $\mathrm{K}_{n-1}(0,0) = \sum_{k=0}^{n-1}[h_k(0;q)]^2 d_k^{-2}$. For the special case $n = 2m-1$ we have

$$\mathrm{K}_{2m-1}(0,0) = \frac{1}{(1-q)(q,-1,-q;q)_\infty}\sum_{k=0}^{m-1}\frac{q^{-k}(q;q^2)_k}{(q^2;q^2)_k}.$$

Notice that with the above notation

$$\Gamma_{2m} = \frac{1}{(1-q)(q,-1,-q;q)_\infty}\frac{A(q;q^2)_m q^{-m}}{(q^2;q^2)_m}\frac{1}{1+A\mathrm{K}_{2m-1}(0,0)}, \quad m \in \mathbb{N}.$$

If now we use (4.9) and the differentiation formula (4.3) with $a = -1$ we find

$$xh_n^A(x;q) = xh_n(x;q) + \frac{1-q}{1-q^n}\Gamma_n D_q h_n(x;q), \quad n \geq 1,$$

or, equivalently,

$$x^2 h_n^A(x;q) = (x^2 + \Lambda_n)h_n(x;q) - \Lambda_n h_n(qx;q), \quad n \geq 1, \tag{4.10}$$

where $\Lambda_n = \Gamma_n/(1-q^n)$.

Remark: Notice that since $\Gamma_{2m-1} = 0$ for all $m \in \mathbb{N}$ then, by (4.9), $h_{2m-1}^A(x;q) = h_{2m-1}(x;q)$, i.e., the odd degree polynomials are not affected with the addition of the Dirac measure.

Notice also that

$$xh_{2m}^A(x;q) = xh_{2m}(x;q) + \Gamma_{2m}h_{2m-1}(x;q) = xh_{2m}(x;q) + \Gamma_{2m}h_{2m-1}^A(x;q),$$

i.e., formula (2.11) takes the form

$$xh_n(x;q) = xh_n^A(x;q) - \Gamma_n h_{n-1}^A(x;q).$$

For this family the square of the norm is

$$\widetilde{d_n}^2 = (1-q)(q;q)_n(q,-1,-q;q)_\infty q^{\binom{n}{2}}(1+\Gamma_n).$$

14

Using the formulas in Section 3 (or (4.8) with $a = -1$ and $x_0 = 0$) we find

$$xh_n^A(x; q) = h_{n+1}^A(x; q) + \widetilde{\beta}_n h_n^A(x; q) + \widetilde{\gamma}_n h_{n-1}^A(x; q), \quad n \in \mathbb{N}, \tag{4.11}$$

where the coefficients of the TTRR are given by

$$\widetilde{\beta}_n = 0, \qquad \widetilde{\gamma}_n = q^{n-1}(1 - q^n) \frac{1 + \Gamma_n}{1 + \Gamma_{n-1}}, \quad n \in \mathbb{N}.$$

To compute the q-SODE we use Theorem 3 with the functions (see [25])

$$\sigma(x; n) = q(1 - x^2), \quad \varphi(x; n) = 1 + q - q^{1-n}x^2, \quad \varsigma(x; n) = 1$$

and (cf. (4.10))

$$\pi(x; n) = x^2, \quad a(x; n) = x^2 + \Lambda_n, \quad b(x; n) = -\Lambda_n.$$

Then we have

$$\widetilde{\sigma}(x; n)h_n^A(q^{-1}x; q) + \widetilde{\varphi}(x; n)h_n^A(x; q) + \widetilde{\varsigma}(x; n)h_n^A(qx; q) = 0,$$

where

$$\widetilde{\sigma}(x; n) = q^{-n}x^2 \left(-1 + x^2\right) \left(-q^2 \Lambda_n \left(x^2 + \Lambda_n\right) + q^n \left(\Lambda_n + q^2 \Lambda_n^2 - q \left(x^2 + \Lambda_n\right)\right)\right),$$

$$\widetilde{\varphi}(x; n) = q^{-1} \left(-1 + q^2 x^2\right) \Lambda_n \left(x^2 + q^2 \Lambda_n\right)$$
$$- q \left(q \left(-1 + x^2\right) \Lambda_n + \left(1 + q - q^{1-n}x^2\right) \left(q^{-2}x^2 + \Lambda_n\right)\right) \times$$
$$\left(-\Lambda_n + qx^2 \left(1 + q^{1-n}\Lambda_n\right)\right),$$

$$\widetilde{\varsigma}(x; n) = q^{-n}x^2 \left(q\Lambda_n \left(x^2 + q^2 \Lambda_n\right) + q^n \left(x^2 + q\Lambda_n \left(-1 + q - q^2 \Lambda_n\right)\right)\right).$$

We notice that these are the expressions in Theorem 3 up to the factor x^2.

For the lowering-type operator we have, from (3.13)

$$\alpha_l(x; n)_q h_n^A(x; q) + \beta_l(x; n)_q h_n^A(qx; q) = \gamma_l(x; n)_q h_{n-1}^A(x; q),$$

where (up to the factor $qx^2\Lambda_n$)

$$\alpha_l(x; n)_q = q^{1-n}x \left(-q\Lambda_n + q^n \left(-1 + q\Lambda_n\right)\right), \quad \beta_l(x; n)_q = qx,$$

$$\gamma_l(x; n)_q = (-1 + q^n) \left(-\Lambda_n + q \left(x^2 + \Lambda_n - q\Lambda_n^2 + q^{1-n}\Lambda_n \left(x^2 + \Lambda_n\right)\right)\right).$$

Combining the last expression with the TTRR (4.11) we obtain the raising-type operator.

15

To conclude this section let us show that the polynomials $h_n^A(x;q)$ can be expressed in terms of a basic series ${}_3\varphi_2$. For doing that we substitute the representation

$$h_n(x;q) = q^{\frac{1}{2}n(n-1)}{}_2\varphi_1\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array}\middle| q; -xq\right)$$

in (4.10). After some straightforward calculations, this leads to the expression

$$x^2 h_n^A(x;q) = q^{\frac{1}{2}n(n-1)}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{-1}x^{-1};q)_k(-qx)^k}{(q;q)_k}\frac{qx(x^2+\Lambda_n)}{(qx-1)}(1-\delta(x;n)q^k),$$

where $\delta(x;n) = (x/q+\Lambda_n)/(x^2+\Lambda_n)$. Finally, using the well-known identity $1-aq^k = (1-a)(aq;q)_k/(a;q)_k$ with $a = \delta(x;n)$, we obtain

$$h_n^A(x;q) = q^{\frac{1}{2}n(n-1)}{}_3\varphi_2\left(\begin{array}{c} q^{-n},\, q^{-1}x^{-1}, \delta(x;n)q \\ 0, \delta(x;n) \end{array}\middle| q;\, -qx\right),\quad \delta(x;n) = \frac{q^{-1}x+\Lambda_n}{x^2+\Lambda_n}.$$

References

[1] R. Álvarez-Nodarse: Second order difference equations for certain families of "discrete" polynomials. J. Comput. Appl. Math. **99** (1998), 135-156.

[2] R. Álvarez-Nodarse, J. Arvesú, and F. Marcellán: Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions. Indag. Mathem. N.S. (2003), in press.

[3] R. Álvarez-Nodarse and F. Marcellán: Difference equation for modification of Meixner polynomials. J. Math. Anal. Appl. **194** (1995), 250-258.

[4] R. Álvarez-Nodarse and F. Marcellán: The modification of classical Hahn polynomials of a discrete variable. Integral Transforms Spec. Funct. **4** (1995), 243-262.

[5] R. Álvarez-Nodarse, A.G. García, and F. Marcellán: On the properties for modification of classical orthogonal polynomials of discrete variables. J. Comput. Appl. Math. **65** (1995), 3-18.

[6] R. Álvarez-Nodarse and F. Marcellán: The limit relations between generalized orthogonal polynomials. Indag. Mathem.N.S. **8** (1997), 295-316.

[7] R. Álvarez-Nodarse, F. Marcellán and J. Petronilho: WKB approximation and Krall-type orthogonal polynomials. Acta Applicandae Mathematicae, **54** (1998), 27-58.

[8] M. Álvarez De Morales, T. Pérez, and M. Piñar: Orthogonal polynomials associated with a Δ-Sobolev inner product. J. Difference Eq. Appl. **8** (2002), 125–151.

[9] I. Area, E. Godoy and F. Marcellán: Classification of all Δ-coherent pairs. Integral Transforms Spec. Funct. **9** (2000), 1–18.

[10] I. Area, E. Godoy and F. Marcellán: q-Coherent pairs and q-orthogonal polynomials. Orthogonal systems and applications. Appl. Math. Comput. **128** (2002), 191–216.

[11] I. Area, E. Godoy, and F. Marcellán, Δ-coherent pairs and orthogonal polynomials of a discrete variable. Integral Transforms Spec. Funct. **14** (2003), 31–57.

[12] H. Bavinck and H. Haeringen: Difference Equations for Generalized Meixner Polynomials. J. Math. Anal. and Appl. **184** (1994), 453-463.

[13] H. Bavinck and R. Koekoek: On a difference equation for generalizations of Charlier polynomials. J. Approx. Theory **81** (1995), 195-206.

[14] T.S. Chihara: *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York, 1978.

[15] R. Costas-Santos: On q-Krall polynomials. In preparation.

[16] W. N. Everitt, K.H. Kwon, L. L. Littlejohn, and R. Wellman: Orthogonal polynomial solutions of linear ordinary differential equations. J. Comput. Appl. Math. **133** (2001), 85-109.

[17] A. G. García, F. Marcellán, and L. Salto: A distributional study of discrete classical orthogonal polynomials. J. Comput. Appl. Math. **57** (1995), 147-162.

[18] G. Gasper and M. Rahman: *Basic Hypergeometric Series.* Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990.

[19] E. Godoy, F. Marcellán, L. Salto, and A. Zarzo: Perturbations of discrete semiclassical functionals by Dirac masses. Integral Transforms and Special Functions **5** (1997), 19-46.

[20] F. A. Grünbaum and L. Haine: Orthogonal polynomials satisfying differential equations: the role of the Darboux transformations. *CRM Proceedings, Lect. Notes* **9**, Amer. Math. Soc. Providence, RI 1996, 143-154.

[21] F. A. Grünbaum and L. Haine: Bispectral Darboux transformation: An extension of the Krall polynomials. Internat. Math. Res. Not. **8** (1997), 359-392.

[22] F. A. Grünbaum, L. Haine, and E. Horozov: Some functions that generalize the Krall-Laguerre polynomials. J. Comput. Appl. Math. **106** (1999), 271-297.

[23] R. Koekoek: A generalization of Moak's q-Laguerre polynomials. Canad. J. Math. **42**(2) (1990), 280–303.

[24] R. Koekoek: Generalizations of a q-analogue of Laguerre polynomials. J. Approx. Theory **69** (1992), 55–83.

[25] R. Koekoek and R. F. Swarttouw: The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. *Reports of the Faculty of Technical Mathematics and Informatics* **No. 98-17**. Delft University of Technology, Delft, 1998.

[26] K.H. Kwon, G.J. Yoon, and L.L. Littlejohn: Bochner-Krall orthogonal polynomials. In *Special Functions* C. Dunkl, M. Ismail, and R. Wong Eds. World Scientific, Singapore, 2000, 181-193.

[27] F. Marcellán and J. Petronilho: Orthogonal polynomials and coherent pairs: the classical case. Indag. Mathem. N.S. **6**(3) (1995), 287-307.

[28] J. C. Medem: *Polinomios q-semiclásicos.* Tesis Doctoral. Universidad Politécnica. Madrid, 1996. In Spanish.

[29] J. C. Medem, R. Álvarez-Nodarse, and F. Marcellán: On the $q-$polynomials: A distributional study. J. Comput. Appl. Math. **135** (2001), 197-223.

[30] H. G. Meijer: Determination of all coherent pairs. J. Approx. Theory **89**(3) (1997), 321-343.

[31] L. Salto: *Polinomios D_w-semiclásicos.* Tesis Doctoral. Universidad de Alcalá de Henares. Madrid, 1995. In Spanish.