ON THE LIMIT OF NON-STANDARD q-RACAH POLYNOMIALS

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Abstract. The aim of this article is to study the limit transitions from non-standard q-Racah polynomials to big q-Jacobi, dual q-Hahn, and q-Hahn polynomials such that the orthogonality properties and the three-term recurrence relations remain valid.

1. Introduction

The Askey-Wilson polynomials and q-Racah polynomials are the most general classical orthogonal families from which all the other q-hypergeometric orthogonal polynomials can be obtained by (possibly successive) limit transitions. There are several ways of getting these limits, but must of them are not good enough by means of the orthogonality property or the three term recurrence relation as it was pointed out by Koornwinder in [7] for the case of q-Racah polynomials and big q-Jacobi polynomials. In fact, in [7] the author studied the limit relation from the standard q-Racah polynomials defined on the lattice [8, page 422] $x(s) = q^{-s} + \gamma\delta q^{s+1}$ to the big q-Jacobi polynomials such that the orthogonality properties remain valid.

In this article, we introduce some limit formulas from the non-standard q-Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)$ [3, 11] defined on the lattice $x(s) = [s]_q[s + 1]_q$ where $[s]_q$ are the symmetric q-numbers

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad s \in \mathbb{C},$$

to the big q-Jacobi polynomials. Let us point out that the lattice for the non-standard polynomials is more appropriate for numerical analysis since it does not depend on any parameters of the polynomials.

Furthermore, we consider the similar limit properties between the non-standard q-Racah-Krall-type polynomials [5] and the big q-Jacobi-Krall-type polynomials [4]. The Krall-type polynomials are polynomials which are orthogonal with respect to a linear functional $\widetilde{\mathbf{u}}$ obtained from a quasi-definite functional $\mathbf{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures, i.e., $\widetilde{\mathbf{u}}$ is the linear functional

$$\widetilde{\mathbf{u}} = \mathbf{u} + \sum_{k=1}^{N} A_k \delta_{x_k},$$

where $A_k \in \mathbb{R}$, $x_1, \ldots, x_k \in \mathbb{R}$ and δ_a is the delta Dirac functional at the point a, i.e., $\langle \delta_a, p \rangle = p(a)$, where $p \in \mathbb{P}$.

These kind of modifications firstly appeared as eigenfunctions of a fourth order linear differential operator with polynomial coefficients that do not depend on the degree of the polynomials (see [10] or the more recent reviews [2] and [9, chapter XV]).

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Our main aim in this note is to obtain the limit formulas between non-standard q-Racah polynomials and big q-Jacobi, dual q-Hahn, and q-Hahn polynomials, respectively, as well as between the corresponding Krall-type polynomials. In fact, the explicit limits from q-Racah polynomials to these other families are given in [8, (14.2.15), (14.2.19) and (14.2.16,17,18)], respectively. However, the formula [8, (14.2.15)], for example, is not valid for getting the orthogonality of the resulting polynomials as it is pointed out by Koornwinder in [7]. In [7] Koornwinder proposed a more convenient limit formula that allows him to show that the support of the measure for the q-Racah polynomials transforms into the measure of the big q-Jacobi polynomials. However, in [7] it is not shown how the orthogonality relation and the three-term recurrence relation (TTRR) of the q-Racah polynomials transform into the big q-Jacobi ones. We fill this gap in this paper but using the aforementioned non-standard q-Racah polynomials.

Following [7], we present an alternative limit formula (to the one suggested in [12]) from the non-standard q-Racah to big q-Jacobi polynomials and from the non-standard q-Racah-Krall-type polynomials to big q-Jacobi-Krall-type polynomials such that the orthogonality property remains present. In particular, we show that by taking a proper limit, not only the polynomials $u_n^{\alpha,\beta,A,B}(x(s),a,b)$ and $u_n^{\alpha,\beta}(x(s),a,b)$ become into $P_n^{\alpha,\beta,A,B}(x,\widetilde{a},\widetilde{b},\widetilde{c};q)$ and $P_n^{\alpha,\beta}(x,\widetilde{a},\widetilde{b},\widetilde{c};q)$, but also that the orthogonality relation and the three-term recurrence relation (TTRR) of the q-Racah polynomials become into the ones of big q-Jacobi polynomials.

The structure of the paper is as follows. We start by introducing some preliminary results on the q-Hahn polynomials and on the non-standard q-Racah polynomials and big q-Jacobi and Krall-type polynomials obtained via the addition of two mass points to the weight function of the these two polynomials in the forthcoming section 2. In section 3, we deal with the limit relation between the non-standard q-Racah polynomials and big q-Jacobi polynomials in detail. In section 4 we consider the limits from the non-standard q-Racah polynomials to dual q-Hahn and q-Hahn polynomials, respectively. Finally, in section 5 the limit between the corresponding Krall-type families are considered.

2. Preliminary

Here we include some properties of the non-standard q-Racah polynomials, non-standard q-Racah-Krall-type polynomials with two mass points, big q-Jacobi polynomials, and q-Hahn monic polynomials. In the following, and throughout the paper, we denote κ_q by the quantity

$$\kappa_q := q^{1/2} - q^{-1/2}.$$

The non-standard monic q-Racah polynomials are defined by the following hypergeometric representation [3, 11] (for the definition and properties of basic series see [6])

$$u_n^{\alpha,\beta}(s)_q := u_n^{\alpha,\beta}(\mu(s),a,b)_q = q^{-\frac{n}{2}(2a+1)}\frac{(q^{a-b+1},q^{\beta+1},q^{a+b+\alpha+1};q)_n}{\kappa_q^{2n}(q^{\alpha+\beta+n+1};q)_n}\times {}_4\varphi_3\left(\begin{matrix} q^{-n},q^{\alpha+\beta+n+1},q^{a-s},q^{a+s+1} \\ q^{a-b+1},q^{\beta+1},q^{a+b+\alpha+1}\end{matrix}\,\middle|\, q\,,\,q\right), \tag{1}$$

which are polynomials on the q-quadratic lattice of the form

$$\mu(s) = [s]_q[s+1]_q = c_1(q^s+q^{-s-1})+c_3, \quad c_1 = q^{1/2}\kappa_q^{-2},\ c_3 = -q^{-1/2}(1+q)\kappa_q^{-2}. \tag{2}$$

For $-\frac{1}{2} < a \le b-1$, $\alpha > -1$, $-1 < \beta < 2a+1$, they satisfy the orthogonality relation ($b-a = N \in \mathbb{N}$)

$$\sum_{s=a}^{b-1} u_n^{\alpha,\beta}(s)_q u_m^{\alpha,\beta}(s)_q \rho(s)\Delta\mu(s-\tfrac{1}{2}) = \delta_{n,m} d_n^2, \quad \Delta\mu(s-\tfrac{1}{2}) = [2s+1]_q, \tag{3}$$

where

$$d_n^2 = \frac{(q;q)_n (q^{\alpha+1};q)_n (q^{\beta+1};q)_n (q^{b-a+\alpha+\beta+1};q)_n (q^{a+b+\alpha+1};q)_n (q^{a-b+1};q)_n (q^{\beta-a-b+1};q)_n}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{4n} (q^{\alpha+\beta+2};q)_{2n} (q^{\alpha+\beta+n+1};q)_n},$$

and ρ is the weight function (see Table 1[1] in [5]) of the non-standard q-Racah polynomials

$$\begin{aligned} \rho(s) =& \frac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(\alpha+\beta+2)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(\alpha+1)\widetilde{\Gamma}_q(\beta+1)} \\ &\times \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(a+b-\beta)}{\widetilde{\Gamma}_q(b-a+\alpha+\beta+1)\widetilde{\Gamma}_q(a+b+\alpha+1)}, \end{aligned} \tag{4}$$

where $\widetilde{\Gamma}_q(x)$, introduced in [11, Eq. (3.2.24)], is related to the classical q-Gamma function, Γ_q, [8] by formula

$$\widetilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}} \Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}} (1-q)^{1-s} \frac{(q;q)_\infty}{(q^s;q)_\infty}, \quad 0 < q < 1.$$

The non-standard q-Racah polynomials satisfy the TTRR [3]

$$\mu(s) u_n^{\alpha,\beta}(s)_q = u_{n+1}^{\alpha,\beta}(s)_q + \beta_n u_n^{\alpha,\beta}(s)_q + \gamma_n u_{n-1}^{\alpha,\beta}(s)_q, \tag{5}$$

$$\begin{aligned} \beta_n =& [a]_q[a+1]_q - \frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q} \\ &- \frac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[-a-b+\beta+n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}, \\ \gamma_n =& \frac{[n]_q[\alpha+\beta+n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[-a-b+\beta+n]_q}{[\alpha+\beta+2n-1]_q([\alpha+\beta+2n]_q)^2[\alpha+\beta+2n+1]_q} \\ &\times [a-b+n]_q[a+b+\alpha+n]_q. \end{aligned}$$

The q-Racah-Krall-type polynomials with two mass points are orthogonal with respect to a linear functional $\widetilde{\mathbf{u}}$ obtained from a quasi-definite functional $\mathbf{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures, i.e., $\widetilde{\mathbf{u}}$ is the linear functional

$$\widetilde{\mathbf{u}} = \mathbf{u} + A\delta_a + B\delta_{b-1},$$

where $A, B \in \mathbb{R}$, δ_a and δ_{b-1} are the delta Dirac functionals at the point a and $b-1$, i.e., $\langle \delta_a, p\rangle = p(a)$ and $\langle \delta_{b-1}, p\rangle = p(b-1)$, where $p \in \mathbb{P}$ and the kernel

$$\mathrm{K}_n^{\alpha,\beta}(s_1,s_2) := \sum_{k=0}^{n} \frac{u_k^{\alpha,\beta}(s_1)_q u_k^{\alpha,\beta}(s_2)_q}{d_k^2} = \frac{\alpha_n}{d_n^2} \frac{u_{n+1}^{\alpha,\beta}(s_1)_q u_n^{\alpha,\beta}(s_2)_q - u_{n+1}^{\alpha,\beta}(s_2)_q u_n^{\alpha,\beta}(s_1)_q}{x(s_1) - x(s_2)}. \tag{6}$$

[1]We have chosen $\rho(s)$ in such a way that $\sum_{x=a}^{b-1} \rho(s)\mu(s-\frac{1}{2}) = 1$, i.e., to be a probability measure.

They satisfy the following orthogonality relation

$$\sum_{s=a}^{b-1} u_n^{\alpha,\beta,A,B}(s)_q u_m^{\alpha,\beta,A,B}(s)_q \rho(s)[2s+1]_q + A u_n^{\alpha,\beta,A,B}(a)_q u_m^{\alpha,\beta,A,B}(a)_q + B u_n^{\alpha,\beta,A,B}(b-1)_q u_m^{\alpha,\beta,A,B}(b-1)_q = \delta_{n,m}(d_n^{A,B})^2, \tag{7}$$

where ρ is the non-standard q-Racah weight function (see Table 1 in [5][2]). They can be written as [5]

$$u_n^{\alpha,\beta,A,B}(s)_q = u_n^{\alpha,\beta}(s)_q - A u_n^{\alpha,\beta,A,B}(a)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,a) - B u_n^{\alpha,\beta,A,B}(b-1)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,b-1). \tag{8}$$

Moreover, the following expressions for the the values at the points $s=a$ and $s=b-1$ and the norm $(d_n^{A,B})^2$ of the modified polynomials $u_n^{\alpha,\beta,A,B}(s)_q$, respectively, hold

$$\begin{aligned} u_n^{\alpha,\beta,A,B}(a)_q &= \frac{(1+B\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,b-1))u_n^{\alpha,\beta}(a)_q - B\mathrm{K}_{n-1}^{\alpha,\beta}(a,b-1)u_n^{\alpha,\beta}(b-1)_q}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \\ u_n^{\alpha,\beta,A,B}(b-1)_q &= \frac{-A\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,a)u_n^{\alpha,\beta}(a)_q + (1+A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a))u_n^{\alpha,\beta}(b-1)_q}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \end{aligned} \tag{9}$$

$$\begin{aligned} (d_n^{A,B})^2 = d_n^2 &+ \frac{A(u_n^{\alpha,\beta}(a)_q)^2\{1+B\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,b-1)\} + B(u_n^{\alpha,\beta}(b-1)_q)^2\{1+A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a)\}}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)} \\ &- \frac{2ABu_n^{\alpha,\beta}(a)_q u_n^{\alpha,\beta}(b-1)_q \mathrm{K}_{n-1}^{\alpha,\beta}(a,b-1)}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \end{aligned}$$

where

$$\begin{aligned} \kappa_m^{\alpha,\beta}(s,t) &= 1 + A\mathrm{K}_m^{\alpha,\beta}(s,s) + B\mathrm{K}_m^{\alpha,\beta}(t,t) \\ &\quad + AB\left\{\mathrm{K}_m^{\alpha,\beta}(s,s)\mathrm{K}_m^{\alpha,\beta}(t,t) - (\mathrm{K}_m^{\alpha,\beta}(s,t))^2\right\}, \end{aligned} \tag{10}$$

where $\mathrm{K}_m^{\alpha,\beta}(s,t)$ are the kernels defined by (6) and d_n^2 denotes the squared norm of the n-th non-standard q-Racah polynomials (see Table 1 in [5]).

The non-standard q-Racah-Krall-type polynomials satisfy the TTRR [5]

$$\mu(s)u_n^{\alpha,\beta,A,B}(s)_q = \alpha_n^{A,B}u_{n+1}^{\alpha,\beta,A,B}(s)_q + \beta_n^{A,B}u_n^{\alpha,\beta,A,B}(s)_q + \gamma_n^{A,B}u_{n-1}^{\alpha,\beta,A,B}(s)_q, \tag{11}$$

$$\begin{aligned} \alpha_n^{A,B} &= 1, \\ \beta_n^{A,B} &= \beta_n - A\left(\frac{u_n^{\alpha,\beta,A,B}(a)_q u_{n-1}^{\alpha,\beta}(a)_q}{d_{n-1}^2} - \frac{u_{n+1}^{\alpha,\beta,A,B}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2}\right) \\ &\quad - B\left(\frac{u_n^{\alpha,\beta,A,B}(b-1)_q u_{n-1}^{\alpha,\beta}(b-1)_q}{d_{n-1}^2} - \frac{u_{n+1}^{\alpha,\beta,A,B}(b-1)_q u_n^{\alpha,\beta}(b-1)_q}{d_n^2}\right), \\ \gamma_n^{A,B} &= \gamma_n \frac{1+\Delta_n^{A,B}}{1+\Delta_{n-1}^{A,B}}, \ \ \Delta_n^{A,B} = \frac{Au_n^{\alpha,\beta,A,B}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2} + \frac{Bu_n^{\alpha,\beta,A,B}(b-1)_q u_n^{\alpha,\beta}(b-1)_q}{d_n^2}. \end{aligned}$$

The monic big q-Jacobi polynomials are defined by the following basic series [8]

$$P_n^{\alpha,\beta}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) = (-\widetilde{a})^n q^{\frac{n(n+1)}{2}} \frac{(q\widetilde{b},q\widetilde{c};q)_n}{(\widetilde{a}\widetilde{b}q^{n+1};q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, \widetilde{a}\widetilde{b}q^{n+1}, q\widetilde{c}z^{-1} \\ q\widetilde{b}, q\widetilde{c} \end{matrix} \,\middle|\, q\,,\, \frac{z}{\widetilde{a}}\right). \tag{12}$$

[2]We have chosen $\rho(s)$ in such a way that $\sum_{s=a}^{b-1}\rho(s)[2s+1]_q = 1$, i.e., to be a probability measure.

They satisfy the orthogonality relation

$$
\begin{aligned}
&\int_{\widetilde{c}q}^{\widetilde{a}q} P_m(z;\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(z;\widetilde{a},\widetilde{b},\widetilde{c};q)\widetilde{\rho}(z)d_qz = \int_{\widetilde{c}q}^{0}[.]d_qz + \int_{0}^{\widetilde{a}q}[.]d_qz \\
&= (1-q)(-\widetilde{c})\sum_{s=0}^{\infty} P_n(\widetilde{c}q^{s+1})P_m(\widetilde{c}q^{s+1})\widetilde{\rho}(\widetilde{c}q^{s+1})q^{s+1} \\
&+ (1-q)\widetilde{a}\sum_{s=0}^{\infty} P_n(\widetilde{a}q^{s+1})P_m(\widetilde{a}q^{s+1})\widetilde{\rho}(\widetilde{a}q^{s+1})q^{s+1} = \widetilde{d}_n^2\delta_{mn}
\end{aligned}
\tag{13}
$$

for $0 < q\widetilde{a} < 1, 0 \leq q\widetilde{b} < 1, \widetilde{c} < 0$, where

$$
\begin{aligned}
\widetilde{\rho}(z) &= \frac{(\widetilde{a}^{-1}z, \widetilde{c}^{-1}z;q)_\infty}{(z,\widetilde{b}\widetilde{c}^{-1}z;q)_\infty}\frac{(\widetilde{a}q,\widetilde{b}q,\widetilde{c}q,\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q;q)_\infty}{\widetilde{a}q(1-q)(q,\widetilde{a}\widetilde{b}q^2,\widetilde{a}^{-1}\widetilde{c},\widetilde{a}\widetilde{c}^{-1}q;q)_\infty}, \\
\widetilde{d}_n^2 &= \frac{1-\widetilde{a}\widetilde{b}q}{1-\widetilde{a}\widetilde{b}q^{2n+1}}\frac{(q,\widetilde{b}q,\widetilde{a}q,\widetilde{c}q,\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q;q)_n}{(\widetilde{a}\widetilde{b}q,\widetilde{a}\widetilde{b}q^{n+1},\widetilde{a}\widetilde{b}q^{n+1};q)_n}(-\widetilde{a}\widetilde{c}q^2)^n q^{\binom{n}{2}},
\end{aligned}
\tag{14}
$$

and $\int_0^t f(z)d_qz$ is the q-Jackson integral [6].

The monic big q-Jacobi polynomials satisfy the following TTRR

$$
zP_n(z,\widetilde{a},\widetilde{b},\widetilde{c};q) = P_{n+1}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\beta}_nP_n(z,\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\gamma}_nP_{n-1}(z,\widetilde{a},\widetilde{b},\widetilde{c};q)
$$

where

$$
\begin{aligned}
\widetilde{\beta}_n &= 1 - \frac{(1-\widetilde{a}q^{n+1})(1-\widetilde{a}\widetilde{b}q^{n+1})(1-\widetilde{c}q^{n+1})}{(1-\widetilde{a}\widetilde{b}q^{2n+1})(1-\widetilde{a}\widetilde{b}q^{2n+2})} + \widetilde{a}\widetilde{c}q^{n+1}\frac{(1-q^n)(1-\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q^n)(1-\widetilde{b}q^n)}{(1-\widetilde{a}\widetilde{b}q^{2n})(1-\widetilde{a}\widetilde{b}q^{2n+1})}, \\
\widetilde{\gamma}_n &= -\widetilde{a}\widetilde{c}q^{n+1}\frac{(1-q^n)(1-\widetilde{a}q^n)(1-\widetilde{b}q^n)(1-\widetilde{c}q^n)(1-\widetilde{a}\widetilde{b}q^n)(1-\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q^n)}{(1-\widetilde{a}\widetilde{b}q^{2n-1})(1-\widetilde{a}\widetilde{b}q^{2n})^2(1-\widetilde{a}\widetilde{b}q^{2n+1})}.
\end{aligned}
\tag{15}
$$

The big q-Jacobi-Krall-type polynomials with two mass points satisfy the orthogonality relation

$$
\begin{aligned}
&\int_{\widetilde{c}q}^{\widetilde{a}q} P_m^{A,B}(z;\widetilde{a},\widetilde{b},\widetilde{c};q)P_n^{A,B}(z;\widetilde{a},\widetilde{b},\widetilde{c};q)\widetilde{\rho}(z)d_qz + AP_n^{A,B}(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q)P_m^{A,B}(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q) \\
&+ BP_n^{A,B}(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q)_qP_m^{A,B}(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q) = (1-q)(-\widetilde{c})\sum_{s=0}^{\infty} P_n^{A,B}(\widetilde{c}q^{s+1})P_m^{A,B}(\widetilde{c}q^{s+1})\widetilde{\rho}(\widetilde{c}q^{s+1})q^{s+1} \\
&+ (1-q)\widetilde{a}\sum_{s=0}^{\infty} P_n^{A,B}(\widetilde{a}q^{s+1})P_m^{A,B}(\widetilde{a}q^{s+1})\widetilde{\rho}(\widetilde{a}q^{s+1})q^{s+1} + AP_n^{A,B}(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q)P_m^{A,B}(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q) \\
&+ BP_n^{A,B}(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q)P_m^{A,B}(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q) = \delta_{n,m}(\widetilde{d}_n^{A,B})^2
\end{aligned}
\tag{16}
$$

where $\widetilde{\rho}(z)$ is the big q-Jacobi weight function (see Table 1 in [4][3]) and

$$
\begin{aligned}
P_n^{A,B}(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q) &= \frac{(1+B\mathrm{K}_{n-1}(\widetilde{a}q,\widetilde{a}q))P_n(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q) - B\mathrm{K}_{n-1}(\widetilde{c}q,\widetilde{a}q)P_n(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q)}{\kappa_{n-1}(\widetilde{c}q,\widetilde{a}q)}, \\
P_n^{A,B}(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q) &= \frac{-A\mathrm{K}_{n-1}(\widetilde{a}q,\widetilde{c}q)P_n(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q) + (1+A\mathrm{K}_{n-1}(\widetilde{c}q,\widetilde{c}q))P_n(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q)}{\kappa_{n-1}(\widetilde{c}q,\widetilde{a}q)},
\end{aligned}
\tag{17}
$$

[3]We have chosen $\widetilde{\rho}(z)$ in such a way that $\int_{s=\widetilde{c}q}^{\widetilde{a}q}\widetilde{\rho}(z)d_qz = 1$, i.e., to be a probability measure.

where

$$\mathrm{K}_n(z_1,z_2) := \sum_{k=0}^{n} \frac{P_n(s_1,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(s_2,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_k^2}, \tag{18}$$

are the corresponding kernels. Moreover,

$$(\widetilde{d}_n^{A,B})^2 = \widetilde{d}_n^2 + \frac{A(P_n(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q))^2\{1+B\mathrm{K}_{n-1}(\widetilde{a}q,\widetilde{a}q)\} + B(P_n(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q))^2\{1+A\mathrm{K}_{n-1}(\widetilde{c}q,\widetilde{c}q)\}}{\kappa_{n-1}(\widetilde{c}q,\widetilde{a}q)}$$
$$- \frac{2ABP_n(\widetilde{c}q;\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(\widetilde{a}q;\widetilde{a},\widetilde{b},\widetilde{c};q)\mathrm{K}_{n-1}(\widetilde{c}q,\widetilde{a}q)}{\kappa_{n-1}(\widetilde{c}q,\widetilde{a}q)}$$

where $\widetilde{d}_n^2$ denotes the squared norm of the n-th big q-Jacobi polynomials (see Table 1 in [4]) and

$$\begin{aligned}\kappa_m(s,t) &= 1 + A\mathrm{K}_m(s,s) + B\mathrm{K}_m(t,t)\\ &\quad + AB\left\{\mathrm{K}_m(s,s)\mathrm{K}_m(t,t) - (\mathrm{K}_m(s,t))^2\right\},\end{aligned} \tag{19}$$

being $\mathrm{K}_m(s,t)$ the kernels defined by (18). They can be written [5] as

$$\begin{aligned}P_n^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) &= P_n(z,\widetilde{a},\widetilde{b},\widetilde{c};q)_q - AP_n^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)\mathrm{K}_{n-1}(z,\widetilde{c}q)\\ &\quad - BP_n^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)\mathrm{K}_{n-1}(z,\widetilde{a}q).\end{aligned} \tag{20}$$

The big q-Jacobi-Krall-type polynomials satisfy the TTRR [4]

$$zP_n^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) = \widetilde{\alpha}_n^{A,B}P_{n+1}^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\beta}_n^{A,B}P_n^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\gamma}_n^{A,B}P_{n-1}^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q)$$

where

(21)

$$\begin{aligned}\widetilde{\alpha}_n^{A,B} &= 1,\\ \widetilde{\beta}_n^{A,B} &= \widetilde{\beta}_n - A\left(\frac{P_n^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_{n-1}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_{n-1}^2} - \frac{P_{n+1}^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(\widetilde{c}\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_n^2}\right)\\ &\quad - B\left(\frac{P_n^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_{n-1}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_{n-1}^2} - \frac{P_{n+1}^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_n^2}\right),\\ \widetilde{\gamma}_n^{A,B} &= \widetilde{\gamma}_n\frac{1+\widetilde{\Delta}_n^{A,B}}{1+\widetilde{\Delta}_{n-1}^{A,B}},\ \widetilde{\Delta}_n^{A,B} = \frac{AP_n^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_n^2} + \frac{BP_n^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_n^2}.\end{aligned}$$

The monic dual q-Hahn polynomials are defined by [8]

$$R_n(s)_q := R_n(x(s),\gamma,\delta,N;q) = (\gamma q, q^{-N};q)_n\, {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-s}, \gamma\delta q^{s+1} \\ \gamma q, q^{-N}\end{matrix}\,\middle|\, q\,,\, q\right). \tag{22}$$

They are orthogonal with respect to the positive weight function [8, (14.7.2)] supported on the points $x(s) = q^{-s} + \gamma\delta q^{s+1}$, $s = 0,1,...,N$, for $0 < \gamma q < 1$, $0 < \delta q < 1$ or for $\gamma > q^{-N}$, $\delta > q^{-N}$, i.e.,

$$\sum_{s=0}^{N} R_n(s)_q R_m(s)_q \widetilde{\rho}(s)\Delta x(s-\tfrac{1}{2}) = \widetilde{d}_n^2, \quad \Delta x(s-\tfrac{1}{2}) = (-\kappa_q)q^{-s}(1-\gamma\delta q^{2s+1}) \tag{23}$$

$$\begin{aligned}\widetilde{\rho}(s) &= \frac{(\gamma q)^N(\delta q;q)_N}{(\gamma\delta q^2;q)_N}\frac{q^{Ns-\binom{s}{2}}}{(-\kappa_q)(1-\gamma\delta q)(-\gamma)^s}\frac{(\gamma q,\gamma\delta q,q^{-N};q)_s}{(q,\gamma\delta q^{N+2},\delta q;q)_s},\\ \widetilde{d}_n^2 &= (\gamma\delta)^n(q,q^{-N},\gamma q,\delta^{-1}q^{-N};q)_n.\end{aligned}$$

Finally, we introduce the monic q-Hahn polynomials [8]

$$h_n(s)_q := h_n^{\widetilde{\alpha},\widetilde{\beta}}(x(s);N|q) = \frac{(q^{-N},\widetilde{\alpha}q;q)_n}{(\widetilde{\alpha}\widetilde{\beta}q^{n+1};q)_n}\,{}_3\varphi_2\left(\begin{array}{c} q^{-n},\widetilde{\alpha}\widetilde{\beta}q^{n+1},x(s),\\ q^{-N},\widetilde{\alpha}q \end{array}\middle|\, q\,,\,q\right), \tag{24}$$

which are orthogonal with respect to a positive weight function [8, (14.6.2)] supported on the points $x(s) = q^{-s}$, $s = 0, 1, ..., N$, for $0 < \alpha q < 1$, $0 < \beta q < 1$ or for $\alpha > q^{-N}$, $\beta > q^{-N}$, i.e.,

$$\sum_{s=0}^{N} h_n(s)_q h_m(s)_q \widetilde{\rho}(s)\Delta x(s-\tfrac{1}{2}) = \widetilde{d}_n^2, \tag{25}$$

where $\Delta x(s-\frac{1}{2}) = -\kappa_q q^{-s}$,

$$\widetilde{\rho}(s) = \frac{(\alpha\beta)^{-s}}{(-\kappa_q)}\frac{(\alpha q)^N(\beta q;q)_N}{(\alpha\beta q^2;q)_N}\frac{(\alpha q,q^{-N};q)_s}{(q,\beta^{-1}q^{-N};q)_s},$$
$$\widetilde{d}_n^2 = (-\alpha q)^n q^{\binom{n}{2}-Nn}\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}\frac{(q,\alpha q,\beta q,q^{-N},\alpha\beta q^{N+2};q)_n}{(\alpha\beta q,\alpha\beta q^{n+1},\alpha\beta q^{n+1};q)_n}.$$

3. Limit relation between non-standard q-Racah and big q-Jacobi polynomials

In this section we establish a limit formula from the non-standard q-Racah polynomials (1) to big q-Jacobi polynomials (12) that preserves the orthogonality relation as well as the TTRR.

Theorem 1. *Let*

$$\widetilde{\mu}(s) = q^{N+1}\widetilde{a}q^{a+1}\left[\frac{\mu(s+a)-c_3}{c_1}\right],\ q^{\alpha}=\widetilde{a},\ q^{\beta}=\widetilde{b},\ q^{a-b}=q^{-N-1},\ q^{a+b}=\frac{\widetilde{c}}{\widetilde{a}}. \tag{26}$$

Then, the following limit formula between the non-standard q-Racah and big q-Jacobi polynomials holds

$$\lim_{N\to\infty} C_n u_n^{\alpha,\beta}(\widetilde{\mu}(s),a,b)_q = P_n(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q), \tag{27}$$

where

$$C_n = (q^N)^{n/2}(q\widetilde{a})^n\left(\frac{\widetilde{c}}{\widetilde{a}}\right)^{\frac{n}{2}}\kappa_q^{2n}\ \ \textit{and}\ \ \{\widetilde{x}(s)\} = \{q^{s+1}\widetilde{c}\}_{s=0}^{\infty}\bigcup\{q^{s+1}\widetilde{a}\}_{s=0}^{\infty}. \tag{28}$$

Moreover, the orthogonality relation of the non-standard q-Racah polynomials (3) becomes into the one of big q-Jacobi polynomials (13).

Proof. From (2) it follows that for $s = 0, 1, ..., N$,

$$q^{N+1}\widetilde{a}q^{a+1}\left[\frac{\mu(s+a)-c_3}{c_1}\right] = q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\left[c_1(q^{s+a}+q^{-s-a-1})+c_3-c_3\right]$$
$$= q^{N+1}\widetilde{a}\left[q^{-s}+q^{2a}q^{s+1}\right] = q^{N+1}\widetilde{a}q^{-s}+\widetilde{c}q^{s+1} = \widetilde{\mu}(s).$$

Following [7] we remark that for certain M depending on N such that $M < N$, the set of points $\{\widetilde{\mu}(s)\}_{s=0}^{N}$ can be written as the union of the increasing sequence of non positive points

$$\left\{q\widetilde{c}+q^{N+1}\widetilde{a},q^2\widetilde{c}+q^N\widetilde{a},...,q^M\widetilde{c}+q^{N-M+2}\widetilde{a}\right\}$$

and the decreasing sequence of non negative points

$$\left\{q\widetilde{a}+q^{N+1}\widetilde{c},q^2\widetilde{a}+q^N\widetilde{c},...,q^{M+1}\widetilde{a}+q^{N-M+1}\widetilde{c}\right\},$$

which tend to the union of the sequence of negative points $\{q^{s+1}\widetilde{c}\}_{s=0}^{\infty}$ and the sequence of positive points $\{q^{s+1}\widetilde{a}\}_{s=0}^{\infty}$ as $N\to\infty$, i.e., to the set $\{\widetilde{x}(s)\}$. Notice that it is precisely the support of the orthogonality measure of the big q-Jacobi polynomials (see (13)).

Next, we rewrite (1) by using the identity (see e.g. [5])

$$(q^{s_1-s};q)_k(q^{s_1+s+\xi};q)_k=(-1)^k q^{k(s_1+\xi+\frac{k-1}{2})}\prod_{i=0}^{k-1}\Big[\frac{\mu(s)-c_3}{c_1}-q^{-\frac{\xi}{2}}\Big(q^{s_1+i+\frac{\xi}{2}}+q^{-s_1-i-\frac{\xi}{2}}\Big)\Big].$$

In fact, setting $s_1=a,\xi=1$ and making the transformation (26), it becomes

$$\begin{aligned}(q^{-s};q)_k(q^{s+2a+1};q)_k&=(-1)^k\frac{q^{k(a+1+\frac{k-1}{2})}}{q^{(N+1)k}\widetilde{a}^k q^{k(a+1)}}\\&\quad\times\prod_{i=0}^{k-1}\Big[q^{N+1}\widetilde{a}q^{a+1}\frac{\mu(s+a)-c_3}{c_1}-\widetilde{c}q^{i+1}-\widetilde{a}q^{N+1-i}\Big].\end{aligned}$$

Then,

$$\begin{aligned}u_n^{\alpha,\beta}(s+a,a,b)_q&=\Big(q^{-N}\frac{\widetilde{c}}{\widetilde{a}}\Big)^{-n/2}\frac{(q^{-N},\widetilde{b}q,\widetilde{c}q;q)_n}{\kappa_q^{2n}(\widetilde{a}\widetilde{b}q^{n+1};q)_n}\sum_{k=0}^{n}\frac{(q^{-n},\widetilde{a}\widetilde{b}q^{n+1};q)_k}{(\widetilde{b}q,\widetilde{c}q,q;q)_k}q^k\\&\quad\times\left\{\frac{1}{(q^{-N};q)_k}\frac{(-1)^k q^{k\frac{(k-1)}{2}}}{q^{(N+1)k}\widetilde{a}^k}\prod_{i=0}^{k-1}\Big[\widetilde{\mu}(s)-\widetilde{c}q^{i+1}-\widetilde{a}q^{N+1-i}\Big]\right\}.\end{aligned}$$

Next we need to take the limit $N\to\infty$. Notice that the set $\{\widetilde{\mu}(s)\}_{s=0}^{N}$ becomes into the set $\{\widetilde{x}(s)\}:=\{q^{s+1}\widetilde{c}\}_{s=0}^{\infty}\bigcup\{q^{s+1}\widetilde{a}\}_{s=0}^{\infty}$. Then,

$$\begin{aligned}\lim_{N\to\infty}C_n u_n^{\alpha,\beta}(\widetilde{\mu}(s),a,b)_q&=(-\widetilde{a})^n q^{\frac{n(n+1)}{2}}\frac{(\widetilde{b}q,\widetilde{c}q;q)_n}{(\widetilde{a}\widetilde{b}q^{n+1};q)_n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(q^{-n},\widetilde{a}\widetilde{b}q^{n+1};q)_k}{(\widetilde{b}q,\widetilde{c}q,q;q)_k}(\widetilde{a}^{-1}\widetilde{x}(s))^k\prod_{i=0}^{k-1}\Big[1-\frac{\widetilde{c}q^{i+1}}{\widetilde{x}(s)}\Big]\end{aligned}$$

where $\prod_{i=0}^{k-1}\Big[1-\widetilde{c}q^{i+1}/\widetilde{x}(s)\Big]=(q\widetilde{c}/\widetilde{x}(s);q)_k$, from where (27) follows. A similar analysis was done in [7] but starting from the standard q-Racah polynomials.

To show that the orthogonality relation of the non-standard q-Racah polynomials becomes into the one of the big q-Jacobi polynomials we rewrite (3) using the transformation (26). This yields

$$\begin{aligned}\sum_{s=0}^{N}f(q^s)\Delta\widetilde{\mu}(s-\tfrac{1}{2})&=\sum_{s=0}^{M-1}f(q^s)\Delta[q^{N+1}\widetilde{a}q^{s+\frac{1}{2}}+\widetilde{c}q^{s+\frac{1}{2}}]\\&\quad-\sum_{s=0}^{N-M}f(q^{N-s})\Delta[\widetilde{a}q^{s+\frac{1}{2}}+q^{N+1}\widetilde{c}q^{-s+\frac{1}{2}}]=d_n^2\delta_{mn},\end{aligned}\tag{29}$$

where M, as before, depends on N, $M<N$ and

$$f(q^s)=\frac{1}{C_nC_m}[C_nu_n^{\alpha,\beta}(s+a)_q][C_mu_m^{\alpha,\beta}(s+a)_q]\frac{c_1(-\kappa_q)}{q^{N+1}\widetilde{a}q^{a+1}}\rho(s+a).$$

If we take the limit $N \to \infty$ and use that

$$
\begin{aligned}
&\lim_{N\to\infty} C_n^2 d_n^2 = \widetilde{d}_n^2, \quad \lim_{N\to\infty} \frac{c_1(-\kappa_q)}{q^{N+1}\widetilde{a}q^{a+1}}\rho(s+a) = (1-q)\widetilde{\rho}(\widetilde{c}q^{s+1}),\\
&\lim_{N\to\infty} \frac{c_1(-\kappa_q)}{q^{N+1}\widetilde{a}q^{a+1}}\rho(N-s+a) = (1-q)\widetilde{\rho}(\widetilde{a}q^{s+1}),
\end{aligned} \tag{30}
$$

where $\widetilde{\rho}$ and $\widetilde{d}_n$ are the weight function and the norm of the big q-Jacobi polynomials, respectively, then (29) becomes into

$$
\begin{aligned}
&(1-q)(-\widetilde{c})\sum_{s=0}^{\infty} P_n(\widetilde{c}q^{s+1})P_m(\widetilde{c}q^{s+1})\widetilde{\rho}(\widetilde{c}q^{s+1})q^{s+1}\\
&+(1-q)\widetilde{a}\sum_{s=0}^{\infty} P_n(\widetilde{a}q^{s+1})P_m(\widetilde{a}q^{s+1})\widetilde{\rho}(\widetilde{a}q^{s+1})q^{s+1} = \widetilde{d}_n^2\delta_{mn},
\end{aligned}
$$

which is the orthogonality relation of the big q-Jacobi polynomials (13). □

To conclude this part let us show that the limit procedure stated in Theorem 1 transforms also the TTRR of the non-standard q-Racah polynomials (5) into the TTRR of the monic big q-Jacobi polynomials. Using the transformation (26) in the TTRR (5), we get

$$
\begin{aligned}
\widetilde{\mu}(s)u_n^{\alpha,\beta}(s+a)_q =& q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}u_{n+1}^{\alpha,\beta}(s+a)_q + q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n - c_3]u_n^{\alpha,\beta}(s+a)_q\\
&+ q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\gamma_n u_{n-1}^{\alpha,\beta}(s+a)_q.
\end{aligned}
$$

Multiplying the above equality by the normalization constant C_n (28), taking the limit $N \to \infty$ and using the relation (27), we get

$$
\widetilde{x}(s)P_n(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) = P_{n+1}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\beta}_n P_n(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\gamma}_n P_{n-1}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q),
$$

where

$$
\widetilde{\beta}_n = \lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n - c_3], \quad \widetilde{\gamma}_n = \lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n. \tag{31}
$$

Notice that

$$
\begin{aligned}
q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n - c_3] = q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\Bigg[&c_1(q^a + q^{-a-1}) - q^{-1/2}(1+q)\kappa_q^{-2}\\
&- \frac{q^{-a-\frac12}(1-q^{\alpha+\beta+n+1})(1-q^{a-b+n+1})(1-q^{\beta+n+1})(1-q^{a+b+\alpha+n+1})}{\kappa_q^2(1-q^{\alpha+\beta+2n+1})(1-q^{\alpha+\beta+2n+2})}\\
&- \frac{q^{a+\frac12}(1-q^{\alpha+n})(1-q^{b-a+\alpha+\beta+n})(1-q^{-a-b+\beta+n})(1-q^n)}{q^{-\frac12[\alpha+\beta+2n+\alpha+\beta+2n+1]}\kappa_q^2(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+2n+1})} + q^{-1/2}(1+q)\kappa_q^{-2}\Bigg],
\end{aligned}
$$

which becomes, by using the transformation (26), into

$$
\begin{aligned}
q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n - c_3] = q\widetilde{c} + q^{N+1}\widetilde{a} &- \frac{(1-\widetilde{a}\widetilde{b}q^{n+1})(1-q^{n-N})(1-\widetilde{b}q^{n+1})(1-\widetilde{c}q^{n+1})}{q^{-N-1}\widetilde{a}^{-1}(1-\widetilde{a}\widetilde{b}q^{2n+1})(1-\widetilde{a}\widetilde{b}q^{2n+2})}\\
&- \frac{q\widetilde{c}(1-\widetilde{a}q^n)(1-\widetilde{a}\widetilde{b}q^{n+N+1})(1-\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q^n)(1-q^n)}{(1-\widetilde{a}\widetilde{b}q^{2n})(1-\widetilde{a}\widetilde{b}q^{2n+1})}.
\end{aligned}
$$

Finally, by taking the limit as $N\to\infty$, we obtain

$$\widetilde{\beta}_n=\lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n-c_3]=q\widetilde{c}+\widetilde{a}q^{n+1}\frac{(1-\widetilde{a}\widetilde{b}q^{n+1})(1-\widetilde{b}q^{n+1})(1-\widetilde{c}q^{n+1})}{(1-\widetilde{a}\widetilde{b}q^{2n+1})(1-\widetilde{a}\widetilde{b}q^{2n+2})} - q\widetilde{c}\frac{(1-\widetilde{a}q^n)(1-\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q^n)(1-q^n)}{(1-\widetilde{a}\widetilde{b}q^{2n})(1-\widetilde{a}\widetilde{b}q^{2n+1})},$$

which is equivalent to $\widetilde{\beta}_n$ in (15). In a complete analogous way, one can obtain

$$\widetilde{\gamma}_n=\lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n=-\widetilde{a}\widetilde{c}q^{n+1}(1-\widetilde{a}q^n)(1-\widetilde{b}q^n)(1-\widetilde{c}q^n) \times\frac{(1-q^n)(1-\widetilde{a}\widetilde{b}q^n)(1-\widetilde{a}\widetilde{b}\widetilde{c}^{-1}q^n)}{(1-\widetilde{a}\widetilde{b}q^{2n-1})(1-\widetilde{a}\widetilde{b}q^{2n})^2(1-\widetilde{a}\widetilde{b}q^{2n+1})},$$

which is the coefficient $\widetilde{\gamma}_n$ of the TTRR of the big q-Jacobi polynomials (15).

A similar analysis can be done but starting with the standard q-Racah polynomials [8, page 422].

4. Limit relation between non-standard q-Racah and q-Hahn polynomials

In this section we consider two more limit cases: namely, the limits to q-Hahn (22) and dual q-Hahn (24) polynomials. In these two cases the situation is more simple since these two families are finite families (the orthogonality measure is supported on a finite set) as the non-standard q-Racah polynomials.

The limit formula between non-standard q-Racah and dual q-Hahn polynomials is stated in the following theorem.

Theorem 2. *Let*

$$\widetilde{\mu}(s)=q^{a+1}\frac{\mu(s+a)-c_3}{c_1},\quad q^{a-b}=q^{-N-1},\quad q^{\beta}=\gamma,\quad q^{2a}=\gamma\delta. \tag{32}$$

Then

$$\lim_{q^\alpha\to 0} C_n u_n^{\alpha,\beta}(\widetilde{\mu}(s),a,b)=R_n(x(s),\gamma,\delta,N;q),\quad C_n=(\gamma\delta q)^{n/2}\kappa_q^{2n},\quad x(s)=q^{-s}+\gamma\delta q^{s+1}.$$

Moreover, the orthogonality relation of the non-standard q-Racah polynomials (3) becomes into the one of dual q-Hahn polynomials (23).

Proof. We present here only the sketch of the proof. First of all, notice that by using the transformation (32) it follows from (2) that

$$\widetilde{\mu}(s)=q^{a+1}\frac{\mu(s+a)-c_3}{c_1}=q^{-s}+\gamma\delta q^{s+1}=x(s).$$

Taking into account that the weight function of the dual q-Hahn polynomials is also supported on a finite set of points, then it is straightforward to see that the orthogonality relation (23) can be derived from the one of the non-standard q-Racah polynomials (3) by taking the limit procedure defined in Theorem 2. □

Moreover, applying the same transformation to the TTRR for non-standard q-Racah polynomials (5) and taking the limit $q^\alpha\to 0$, we get

$$x(s)R_n(s)_q=R_{n+1}(s)_q+\widetilde{\beta}_nR_n(s)_q+\widetilde{\gamma}_nR_{n-1}(s)_q,$$

where

$$\widetilde{\beta}_n = \lim_{q^\alpha\to 0} \frac{q^{a+1}}{c_1}[\beta_n - c_3] = 1 + \gamma\delta q - (1-\gamma q^{n+1})(1-q^{n-N}) - \gamma q(\delta - q^{n-N-1})(1-q^n),$$

$$\widetilde{\gamma}_n = \lim_{q^\alpha\to 0} \frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n = \gamma q(1-q^n)(\delta - q^{n-N-1})(1-q^{n-N-1})(1-\gamma q^n),$$

which is the TTRR for the dual q-Hahn polynomials.

Theorem 3. *Let*

$$\widetilde{\mu}(s) = q^{a+1}\frac{\mu(s+a)}{c_1},\ q^\alpha = \widetilde{\beta},\ q^\beta = \widetilde{\alpha},\ q^{-b} = q^{-N-1-a}. \tag{33}$$

Then,

$$\lim_{q^a\to 0} C_n u_n^{\alpha,\beta}(\widetilde{\mu}(s),a,b) = h_n^{\widetilde{\alpha},\widetilde{\beta}}(x(s);N|q), \quad C_n = q^{\frac{n}{2}(2a+1)}\kappa_q^{2n}, \quad x(s) = q^{-s}.$$

Proof. We here only sketch the proof. Notice that the limit procedure stated in Theorem 3 yields $\lim_{q^a\to 0}\widetilde{\mu}(s) = q^{-s} = x(s)$. Moreover, the orthogonality relation of the q-Hahn polynomials (25) easily follows from the orthogonality relation of the non-standard q-Racah polynomials (3) by taking the limit defined in the Theorem 3. □

Finally, let us mention that by using the transformation stated in Theorem 3, the recurrence relation for the q-Racah polynomials (5) transforms into the TTRR for q-Hahn polynomials [8]:

$$x(s)h_n(s)_q = h_{n+1}(s)_q + \widetilde{\beta}_n h_n(s)_q + \widetilde{\gamma}_n h_{n-1}(s)_q,$$

$$\begin{aligned}\widetilde{\beta}_n = \lim_{q^a\to 0}\frac{q^{a+1}}{c_1}\beta_n &= 1 - \frac{(1-\widetilde{\alpha}\widetilde{\beta}q^{n+1})(1-\widetilde{\alpha}q^{n+1})(1-q^{n-N})}{(1-\widetilde{\alpha}\widetilde{\beta}q^{2n+1})(1-\widetilde{\alpha}\widetilde{\beta}q^{2n+2})} \\ &\quad + \frac{\widetilde{\alpha}q^{n-N}(1-q^n)(1-\widetilde{\beta}q^n)(1-\widetilde{\alpha}\widetilde{\beta}q^{n+N+1})}{(1-\widetilde{\alpha}\widetilde{\beta}q^{2n})(1-\widetilde{\alpha}\widetilde{\beta}q^{2n+1})}, \\ \widetilde{\gamma}_n = \lim_{q^a\to 0}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n &= -\widetilde{\alpha}q^{n-N}\frac{(1-\widetilde{\alpha}\widetilde{\beta}q^n)(1-\widetilde{\alpha}q^n)(1-\widetilde{\beta}q^n)(1-\widetilde{\alpha}\widetilde{\beta}q^{n+N+1})}{(1-\widetilde{\alpha}\widetilde{\beta}q^{2n-1})(1-\widetilde{\alpha}\widetilde{\beta}q^{2n})^2(1-\widetilde{\alpha}\widetilde{\beta}q^{2n+1})} \\ &\quad \times (1-q^n)(1-q^{n-N}).\end{aligned}$$

5. The limit relation for the Krall-type polynomials

As a consequence of the limit formula between non-standard q-Racah and big q-Jacobi polynomials (see Theorem 1) one can obtain the big q-Jacobi-Krall-type polynomials from the q-Racah-Krall-type polynomials.

In order to get this kind of limit formula for the Krall-type polynomials, we apply the transformation (26) to the formula (8). So, multiplying (8) by C_n^2 in (26) we have the expression

$$C_n u_n^{\alpha,\beta,A,B}(s)_q = C_n u_n^{\alpha,\beta}(s)_q - AC_n u_n^{\alpha,\beta,A,B}(a)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,a) - BC_n u_n^{\alpha,\beta,A,B}(b-1)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,b-1).$$

But $\lim_{N\to\infty} C_n u_n^{\alpha,\beta}(s)_q = P_n(z,\widetilde{a},\widetilde{b},\widetilde{c};q)$ (27) where C_n is defined in (28). The kernel (6) becomes

$$\mathrm{K}_n^{\alpha,\beta}(s_1,s_2) := \lim_{N\to\infty}\sum_{k=0}^{n}\frac{\left[C_k u_k^{\alpha,\beta}(s_1)_q\right]\left[C_k u_k^{\alpha,\beta}(s_2)_q\right]}{C_k^2 d_k^2}$$
$$= \sum_{k=0}^{n}\frac{P_n(z_1,\widetilde{a},\widetilde{b},\widetilde{c};q)P_n(z_2,\widetilde{a},\widetilde{b},\widetilde{c};q)}{\widetilde{d}_k^2} = \mathrm{K}_n(z_1,z_2)$$

whereas

$$\lim_{N\to\infty} C_n u_n^{\alpha,\beta}(a)_q = P_n(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q), \quad \lim_{N\to\infty} C_n u_n^{\alpha,\beta}(b-1)_q = P_n(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q).$$

Then, straightforward calculations lead to the following expressions for the values of the q-Racah-Krall-type polynomials at the points a and $b-1$

$$\lim_{N\to\infty} C_n u_n^{\alpha,\beta,A,B}(a)_q = P_n^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q), \quad \lim_{N\to\infty} C_n u_n^{\alpha,\beta}(b-1)_q = P_n^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q),$$

and the representation formula for the q-Racah-Krall-type polynomials (8) transforms into the representation formula for the big q-Jacobi polynomials defined in (20),

$$P_n^{A,B}(z,\widetilde{a},\widetilde{b},\widetilde{c};q) = P_n(z,\widetilde{a},\widetilde{b},\widetilde{c};q)_q - AP_n^{A,B}(\widetilde{c}q,\widetilde{a},\widetilde{b},\widetilde{c};q)\mathrm{K}_{n-1}(z,\widetilde{c}q) - BP_n^{A,B}(\widetilde{a}q,\widetilde{a},\widetilde{b},\widetilde{c};q)\mathrm{K}_{n-1}(z,\widetilde{a}q).$$

Notice also that using the transformation (26) into the orthogonality relation (7) yields to

$$\begin{aligned}
&\sum_{s=0}^{N}\left[C_n u_n^{\alpha,\beta,A,B}(s)_q\right]\left[C_m u_m^{\alpha,\beta,A,B}(s)_q\right]\rho(s)\Delta\widetilde{\mu}(s-\tfrac12) + AC_n u_n^{\alpha,\beta,A,B}(a)_q C_m u_m^{\alpha,\beta,A,B}(a)_q\\
&+ BC_n u_n^{\alpha,\beta,A,B}(b-1)_q C_m u_m^{\alpha,\beta,A,B}(b-1)_q = \sum_{s=0}^{M-1} f(q^s)\Delta[q^{N+1}\widetilde{a}q^{s+\frac12} + \widetilde{c}q^{s+\frac12}]\\
&- \sum_{s=0}^{N-M} f(q^{N-s})\Delta[\widetilde{a}q^{s+\frac12} + q^{N+1}\widetilde{c}q^{-s+\frac12}] + AC_n u_n^{\alpha,\beta,A,B}(a)_q C_m u_m^{\alpha,\beta,A,B}(a)_q\\
&+ BC_n u_n^{\alpha,\beta,A,B}(b-1)_q C_m u_m^{\alpha,\beta,A,B}(b-1)_q = C_n C_m \delta_{n,m}(d_n^{A,B})^2,
\end{aligned} \tag{34}$$

where M depends on N such that $M < N$ and

$$f(q^s) = [C_n u_n^{\alpha,\beta,A,B}(s+a)_q][C_m u_m^{\alpha,\beta,A,B}(s+a)_q]\frac{c_1(-\kappa_q)}{q^{N+1}\widetilde{a}q^{a+1}}\rho(s+a).$$

If we now take the limit $N\to\infty$ and use (28) and (30) we get

$$\lim_{N\to\infty}\left(C_n d_n^{A,B}\right)^2 = (\widetilde{d}_n^{A,B})^2.$$

Using now (2), expression (34) becomes into

$$\begin{aligned}
&(1-q)(-\widetilde{c})\sum_{s=0}^{\infty} P_n(\widetilde{c}q^{s+1})P_m(\widetilde{c}q^{s+1})\widetilde{\rho}(\widetilde{c}q^{s+1})q^{s+1}\\
&+ (1-q)\widetilde{a}\sum_{s=0}^{\infty} P_n(\widetilde{a}q^{s+1})P_m(\widetilde{a}q^{s+1})\widetilde{\rho}(\widetilde{a}q^{s+1})q^{s+1} + AP_n^{A,B}(\widetilde{c}q)_q P_m^{A,B}(\widetilde{c}q)_q\\
&+ BP_n^{A,B}(\widetilde{a}q)_q P_m^{A,B}(\widetilde{a}q)_q = (\widetilde{d}_n^{A,B})^2\delta_{mn},
\end{aligned}$$

which is the orthogonality relation of the big q-Jacobi-Krall-type polynomials (16).

To conclude this section we last deal with the TTRR of the q-Racah-Krall-type polynomials (11) considering the transformation defined in (26) which leads to

$$\begin{aligned}\widetilde{\mu}(s)C_n u_n^{\alpha,\beta,A,B}(s+a)_q &= q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\alpha_n^{A,B}C_n u_{n+1}^{\alpha,\beta,A,B}(s+a)_q\\ &+ q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n^{A,B}-c_3]C_n u_n^{\alpha,\beta,A,B}(s+a)_q\\ &+ q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\gamma_n^{A,B}C_n u_{n-1}^{\alpha,\beta,A,B}(s+a)_q.\end{aligned}$$

taking the limit $N\to\infty$ and using the relation (27), we get

$$\widetilde{x}(s)P_n^{A,B}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) = P_{n+1}^{A,B}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\beta}_n^{A,B}P_n^{A,B}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q) + \widetilde{\gamma}_n^{A,B}P_{n-1}^{A,B}(\widetilde{x}(s),\widetilde{a},\widetilde{b},\widetilde{c};q),$$

where

$$\begin{aligned}\widetilde{\beta}_n^{A,B} &= \lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n^{A,B}-c_3] = \lim_{N\to\infty}\Big\{q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}[\beta_n-c_3]\\ &- A\Big(q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_{n-1}}{C_n}\frac{C_n u_n^{\alpha,\beta,A,B}(a)_q C_{n-1}u_{n-1}^{\alpha,\beta}(a)_q}{C_{n-1}^2 d_{n-1}^2}\\ &- q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n+1}}\frac{C_{n+1}u_{n+1}^{\alpha,\beta,A,B}(a)_q C_n u_n^{\alpha,\beta}(a)_q}{C_n^2 d_n^2}\Big)\\ &- B\Big(q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_{n-1}}{C_n}\frac{C_n u_n^{\alpha,\beta,A,B}(b-1)_q C_{n-1}u_{n-1}^{\alpha,\beta}(b-1)_q}{C_{n-1}^2 d_{n-1}^2}\\ &- q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n+1}}\frac{C_{n+1}u_{n+1}^{\alpha,\beta,A,B}(b-1)_q C_n u_n^{\alpha,\beta}(b-1)_q}{C_n^2 d_n^2}\Big)\Big\},\\ \widetilde{\gamma}_n^{A,B} &= \lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n^{A,B} = \lim_{N\to\infty} q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_n}{C_{n-1}}\gamma_n\frac{1+\Delta_n^{A,B}}{1+\Delta_{n-1}^{A,B}},\\ \Delta_n^{A,B} &= \frac{AC_n u_n^{\alpha,\beta,A,B}(a)_q C_n u_n^{\alpha,\beta}(a)_q}{C_n^2 d_n^2} + \frac{BC_n u_n^{\alpha,\beta,A,B}(b-1)_q C_n u_n^{\alpha,\beta}(b-1)_q}{C_n^2 d_n^2}.\end{aligned}$$

Notice from (31) and $q^{N+1}\widetilde{a}\frac{q^{a+1}}{c_1}\frac{C_{n-1}}{C_n} = 1$ that

$$\begin{aligned}\widetilde{\beta}_n^{A,B} &= \widetilde{\beta}_n - A\Big(\frac{P_n^{A,B}(\widetilde{c}q)_q P_{n-1}(\widetilde{c}q)_q}{\widetilde{d}_{n-1}^2} - \frac{P_{n+1}^{A,B}(\widetilde{c}q)_q P_n(\widetilde{c}q)_q}{\widetilde{d}_n^2}\Big)\\ &- B\left(\frac{P_n^{A,B}(\widetilde{a}q)_q P_{n-1}(\widetilde{a}q)_q}{\widetilde{d}_{n-1}^2} - \frac{P_{n+1}^{A,B}(\widetilde{a}q)_q P_n(\widetilde{a}q)_q}{\widetilde{d}_n^2}\right),\\ \widetilde{\gamma}_n^{A,B} &= \widetilde{\gamma}_n\frac{1+\widetilde{\Delta}_n^{A,B}}{1+\widetilde{\Delta}_{n-1}^{A,B}},\ \widetilde{\Delta}_n^{A,B} = \frac{AP_n^{A,B}(\widetilde{c}q)_q P_n(\widetilde{c}q)_q}{\widetilde{d}_n^2} + \frac{BP_n^{A,B}(\widetilde{a}q)_q P_n(\widetilde{a}q)_q}{\widetilde{d}_n^2}\end{aligned}$$

which are the the coefficients of the TTRR for the big q-Jacobi-Krall-type polynomials (see (21)).

Since the limit relation from the non-standard q-Racah-Krall polynomials to q-Hahn polynomials is quite similar to this one, we will omit it here.

Concluding remarks

In the present work we have presented some limit formulas from the non-standard q-Racah polynomials to other families of q-polynomials of the q-Askey scheme [8] and from the q-Racah Krall-type polynomials to big q-Jacobi-Krall-type polynomials such that the orthogonality property remains present while the limit is approached. Also we show that under these limits the TTRR of the non-standard q-Racah polynomials becomes into the TTRR of the corresponding families of q-Hahn, dual q-Hahn and big q-Jacobi polynomials. Since the non-standard q-Racah polynomials $u_n^{\alpha,\beta}(s)_q$ are multiples of the standard q-Racah polynomials $R_n(x(s-a);q^\beta,q^\alpha,q^{a-b},q^{a+b}|q)$, $b-a=N$ [8, page 422] (see [3] for more details), then the orthogonality property of the standard q-Racah polynomials [8, Eq. (14.2.2) page 422] becomes into the one of the big q-Jacobi polynomials, as well as, the TTRR of the standard q-Racah polynomials [8, Eq. (14.2.3) page 423] becomes into the TTRR of the big q-Jacobi polynomials. In such a way we have completed the work by Koornwinder [7] and extend it to the non-standard q-Racah polynomials introduced in [11] as well as to the corresponding Krall-type polynomials obtained via the addition of two mass points to the weight function of this polynomial [4, 5].

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