On the q-polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$.

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Abstract

The main goal of the present paper is to continue the study of the q-polynomials on non-uniform lattices by using the approach introduced by Nikiforov and Uvarov in 1983. We consider the q-polynomials on the non-uniform exponential lattice $x(s) = c_1 q^s + c_3$ and study some of their properties (differentiation formulas, structure relations, representation in terms of hypergeometric and basic hypergeometric functions, etc). Special emphasis is given to a q-analogue of the Charlier orthogonal polynomials. For these polynomials (Charlier) we compute the main data, i.e., the coefficients of the three-term recurrence relation, structure relation, the square of the norm, etc, in the exponential lattices $x(s) = q^s$ and $x(s) = \frac{q^s - 1}{q - 1}$, respectively.

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1   Introduction.

In the last years the study of the discrete analogues of the classical special functions and, in particular, the orthogonal polynomials, has received an increasing interest (for a review see [20, 25, 26, 34]). Special emphasis was given to the q-analogues of the orthogonal polynomials or q-polynomials, which are closely related with different topics in other fields of actual science: Mathematics (e.g., continued fractions, eulerian series, theta functions, elliptic functions,...; see for instance [5, 24]) and Physics (e.g., q-Schrödinger equation and q-harmonic oscillators [8, 9, 10, 15, 16, 18, 30]). Moreover, the connection between the representation theory of quantum algebras (Clebsch-Gordan coefficients, 3j and 6j symbols) and the q-orthogonal polynomials is well known, (see [4, 27, 28, 31, 39, 41] among others.)

There exist several approaches to the study of these objects. The more standard one is based on the fact that these q-polynomials are special cases of the basic hypergeometric series [25] (see also [11, 26, 28]). Other approaches are the group-theoretical approach [23, 41], the algebraic approach [33], and the difference-equation in non-uniform lattices approach [12, 13, 14, 17, 22, 34, 35, 36, 37, 40]. The distribution of zeros of these polynomials has been considered in [2] and [21].

In most of the papers related to the last approach (Nikiforov et al.) the authors didn't study concrete families (up to the papers [19, 39]), although they are very useful for applications. In the present paper we will study some q-polynomials on the exponential lattice $x(s) = q^s$, connecting the results of different authors [19] (Meixner and Kravchuk), [39] (Hahn), and the general method described in [34, 37]. In particular, we will derive from the general formulas, obtained in [37], the hypergeometric representation of these polynomials which has not been considered in [19] and [39]. Particular emphasis will be given to a q-analogue of the Charlier polynomials, for which we will find all the characteristic formulas, including norm, three-term recurrence relation, structure relations, differentiation formulas, representation in terms of the basic hypergeometric series and so on.

It is important to remark that the above lattice $x(s) = q^s$ is a "bad" lattice in the sense that we can not recover the linear lattice $x(s) = s$ by taking limits in it. In fact, a more convenient lattice is the lattice $x(s) = \frac{q^s-1}{q-1}$. The election of such a lattice is determined by the fact that in the limit (when q tends to one) we recover the classical linear lattice, and then the new polynomials seem to be "good" analogues of the classical ones. In particular all the characteristics of the resulting polynomials will tend to the classical ones in the limit $q \to 1$. To show this we will study in details the $q-$analogues of the Charlier polynomials in both lattices $x(s) = q^s$ and $x(s) = \frac{q^s-1}{q-1}$.

The structure of the paper is as follows. In Section 2 we summarize some of the properties of the q-polynomials on the non-uniform lattices [34, 37] with special emphasis in the exponential lattice. In section 3 we derive some general relations for the q-polynomials on the exponential lattice. In particular, we will obtain the so-called differentiation formulas, the structure relations, and an expression involving the difference derivatives of the q-polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$ with the polynomial itself. All these relations are very useful for applications (see e.g. [3, 29]) and their knowledge allows to extend the results of [6, 38] for the orthogonal polynomials on the uniform lattice to the polynomials on the exponential one. In Section 4

we recover, from the general formulas by some limit processes, the q-polynomials considered in [19, 39] by using the approach by Nikiforov and Uvarov [34, 37]. Finally, in Section 5 we study in detail a q-analogue of the Charlier polynomials in the two aforesaid exponential lattices and find explicit expressions for some of their characteristics.

2 General properties of the q−polynomials on the exponential lattice.

2.1 The hypergeometric-type difference equation.

Let us start with the study of some general properties of orthogonal polynomials of a discrete variable on non-uniform lattices. Let

$$\tilde{\sigma}(x(s))\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)} + \frac{\tilde{\tau}(x(s))}{2}\left[\frac{\triangle y(s)}{\triangle x(s)} + \frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda y(s) = 0,$$

$$\nabla f(s) = f(s) - f(s-1), \triangle f(s) = f(s+1) - f(s),$$

(2.1)

be the *second order difference equation of hypergeometric type* for some lattice function $x(s)$, where $\nabla f(s)$ and $\triangle f(s)$, denote the backward and forward finite difference derivatives, respectively. Here $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. The equation (2.1) can be obtained from the classical differential *hypergeometric equation*

$$\tilde{\sigma}(x)y''(x) + \tilde{\tau}(x)y'(x) + \lambda y(s) = 0,$$

via the discretization of the first and second derivatives y' and y'' in an appropriate way [34, 36]. Following [34, 37] we will rewrite (2.1) in the equivalent form

$$\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)} + \tau(s)\frac{\triangle y(s)}{\triangle x(s)} + \lambda y(s) = 0,$$

$$\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2}\tilde{\tau}(x(s)) \triangle x(s-\frac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)).$$

(2.2)

It is known [12, 34, 37] that the above difference equations have polynomial solutions of hypergeometric type iff $x(s)$ is a function of the form

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q),$$

(2.3)

where c_1, c_2, c_3 and $q^\mu = \frac{c_1}{c_2}$ are constants which, in general, depend on q. A simple calculation shows that the exponential lattice belongs to this class. In fact we have

$$\lim_{c_2 \to 0} x(s) = \lim_{\mu \to \pm\infty} x(s) = c_1 q^{\pm s} + c_3,$$

(2.4)

where μ takes the appropriate sign according to $q > 1$ or $q < 1$.

The *k-order difference derivative* of a solution $y(s)$ of (2.1) , defined by

$$y_k(s)_q = \frac{\triangle}{\triangle x_{k-1}(s)}\frac{\triangle}{\triangle x_{k-2}(s)}\cdots\frac{\triangle}{\triangle x(s)}[y(s)] \equiv \triangle^{(k)}[y(s)],$$

where

$$x_m(s) = x(s + \tfrac{m}{2}),$$

(2.5)

3

also satisfy a difference equation of hypergeometric type

$$\sigma(s)\frac{\triangle}{\triangle x_k(s-\frac{1}{2})}\left[\frac{\nabla y_k(s)_q}{\nabla x_k(s)}\right] + \tau_k(s)\frac{\triangle y_k(s)_q}{\triangle x_k(s)} + \mu_k y_k(s)_q = 0, \qquad (2.6)$$

where [34, page 62, Eq. (3.1.29)]

$$\tau_k(s) = \frac{\sigma(s+k) - \sigma(s) + \tau(s+k)\,\triangle x(s+k-\frac{1}{2})}{\triangle x_{k-1}(s)}, \qquad (2.7)$$

and

$$\mu_k = \lambda_n + \sum_{m=0}^{k-1}\frac{\triangle \tau_m(s)}{\triangle x_m(s)}. \qquad (2.8)$$

Usually, the equations (2.2) and (2.6) are written in the compact form

$$\frac{\triangle}{\triangle x(s-\frac{1}{2})}\left[\sigma(s)\rho(s)\frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda\rho(s)y(s) = 0, \qquad (2.9)$$

and

$$\frac{\triangle}{\triangle x_k(s-\frac{1}{2})}\left[\sigma(s)\rho_k(s)\frac{\nabla y_k(s)}{\nabla x_k(s)}\right] + \mu_k \rho_k(s)y_k(s) = 0, \qquad (2.10)$$

where $\rho(s)$ and $\rho_k(s)$ are the solution of the Pearson-type difference equations [34]

$$\frac{\triangle}{\triangle x(s-\frac{1}{2})}[\sigma(s)\rho(s)] = \tau(s)\rho(s), \qquad \frac{\triangle}{\triangle x_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)] = \tau_k(s)\rho_k(s), \quad (2.11)$$

respectively, and $\rho_k(s)$ is given by

$$\rho_k(s) = \rho(s+k)\prod_{i=1}^{k}\sigma(s+i). \qquad (2.12)$$

2.2 The Rodrigues-type formula.

It is well known [34, 37], that the polynomial solutions of equation (2.2), denoted by $P_n(s)_q \equiv P_n(x(s))_q$, are determined, up to a normalizing factor B_n, by the difference analog of the Rodrigues formula [34, page 66, Eq. (3.2.19)]

$$P_n(s)_q = \frac{B_n}{\rho(s)}\nabla^{(n)}[\rho_n(s)], \qquad \nabla^{(n)} \equiv \frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}, \qquad (2.13)$$

where the function $\rho_n(s)$ is given in (2.12). These polynomial solutions correspond to some values of λ_n –the eigenvalues of equation (2.2)– and can be computed by substituting the polynomial $P_n(s)_q$ in (2.1) and equating the coefficients of the greatest power $x^n(s)$ (see [34, page 104] and [37]). Then

$$\lambda_n = -[n]_q\left\{\frac{1}{2}\left(q^{n-1} + q^{-n+1}\right)\tilde{\tau}' + [n-1]_q\frac{\tilde{\sigma}''}{2}\right\}, \qquad (2.14)$$

where (see Eq. (2.2))

$$\tilde{\sigma}(s) = \frac{\tilde{\sigma}''}{2}x(s)^2 + \tilde{\sigma}'(0)x(s) + \tilde{\sigma}(0), \quad \text{and} \quad \tilde{\tau}(s) \equiv \tau'x(s) + \tau(0).$$

Here and throughout the paper $[n]_q$ denotes the so called *q-numbers*

$$[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sinh(hn)}{\sinh(h)}, \quad q = e^{\frac{h}{2}}, \quad \text{and} \quad \kappa_q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}.$$

4

Also for the difference derivatives $y_{kn}(s)_q$ of the polynomial solution $P_n(s)_q$ a Rodrigues-type formula is valid

$$y_{kn}(s)_q = \triangle^{(k)} P_n(s)_q = \frac{A_{nk} B_n}{\rho_k(s)} \nabla_k^{(n)} [\rho_n(s)], \qquad (2.15)$$

where the operator $\nabla_k^{(n)}$ is defined by

$$\nabla_k^{(n)} f(s) = \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)} [f(s)],$$

$B_n = \frac{\triangle^{(n)} P_n}{A_{nn}}$, the eigenvalues λ_n are given by (2.14) and

$$A_{nk} = \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left\{ \left(\frac{q^{\frac{1}{2}(n+m-1)} + q^{-\frac{1}{2}(n+m-1)}}{2} \right) \tilde{\tau}' + [n+m-1]_q \frac{\tilde{\sigma}''}{2} \right\}. \qquad (2.16)$$

The Rodriges-type formula (2.13) can be written also in the form [13, 34]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \left[\frac{\delta}{\delta x(s)} \right]^n [\rho_n(s - \tfrac{n}{2})], \qquad \left[\frac{\delta}{\delta x(s)} \right]^n \equiv \overbrace{\frac{\delta}{\delta x(s)} \frac{\delta}{\delta x(s)} \cdots \frac{\delta}{\delta x(s)}}^{n \text{ times}}, \qquad (2.17)$$

where $\delta f(s) = \nabla f(s + \tfrac{1}{2}) \equiv f(s + \tfrac{1}{2}) - f(s - \tfrac{1}{2})$. Analogously,

$$\triangle^{(k)} P_n(s)_q = \frac{A_{nk} B_n}{\rho_k(s)} \left[\frac{\delta}{\delta x(s - \frac{k}{2})} \right]^{n-k} [\rho_n(s - \tfrac{n}{2} + \tfrac{k}{2})]. \qquad (2.18)$$

Sometimes, for the exponential lattice, it is better to rewrite the (2.13) in the equivalent form [3]

$$P_n(s)_q = \frac{q^{-\frac{n(n+1)}{4}} B_n}{\rho(s)} \left[\frac{\nabla}{\nabla x(s)} \right]^n [\rho_n(s)], \qquad \left[\frac{\nabla}{\nabla x(s)} \right]^n = \overbrace{\frac{\nabla}{\nabla x(s)} \cdots \frac{\nabla}{\nabla x(s)}}^{n\text{-times}}. \qquad (2.19)$$

To obtain the above formula the linearity of the operator $\nabla^{(n)}$ as well as the identity $\nabla x_k(s) = q^{\frac{k}{2}} \nabla x(s)$, have been used.

2.3 Integral representation and explicit formula.

For the polynomial solutions $P_n(s)_q$ of the difference equation (2.2) the following integral representation holds [12]

$$P_n(s)_q = \frac{[n]_q! B_n}{\rho(s)} \frac{1}{2\pi i} \int_C \frac{\rho_n(z) x_n'(z)}{[x_n(z) - x_n(s)]^{(n+1)}} dz, \qquad (2.20)$$

where

$$[x_k(z) - x_k(s)]^{(m)} = \prod_{j=0}^{m-1} [x_k(z) - x_k(s-j)], \quad m = 0, 1, 2..., \qquad (2.21)$$

are the so-called *generalized powers*. In this paper we suppose that the function $\rho_n(z)$ is analytic on and inside the closed contour C of the complex plane containing the points $z = s, s - 1, ..., s - n$. From the above expression we can obtain an explicit formula for the polynomials $P_n(s)_q$. In fact, if we integrate (2.20) by calculating the residues

(the only singularities inside the contour C are simple poles located at $z = s - l$, $l = 0, 1, \cdots, n$), we find the following expression for the polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$,

$$P_n(s)_q = \frac{B_n q^{-ns + \frac{n}{4}(n+1)}}{c_1^n (q-1)^n} \sum_{m=0}^{n} \frac{(-1)^{m+n}[n]_q! \, q^{-\frac{m}{2}(n-1)}}{[m]_q![n-m]_q!} \frac{\rho_n(s-n+m)}{\rho(s)}, \tag{2.22}$$

or, using the Pearson-type equation (2.11), rewritten in its equivalent form

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s) \triangle x(s - \frac{1}{2})}{\sigma(s+1)}, \tag{2.23}$$

$$P_n(s)_q = \frac{B_n q^{-ns + \frac{n}{4}(n+1)}}{c_1^n (q-1)^n} \sum_{m=0}^{n} \frac{[n]_q! \, q^{-\frac{m}{2}(n-1)}(-1)^{m+n}}{[m]_q![n-m]_q!} \times$$

$$\times \prod_{l=0}^{n-m-1} [\sigma(s-l)] \prod_{l=0}^{m-1} [\sigma(s+l) + \tau(s+l) \triangle x(s+l-\tfrac{1}{2})], \tag{2.24}$$

where it is assumed that $\prod_{l=0}^{-1} f(l) \equiv 1$. This is an explicit formula for the polynomials $P_n(s)_q$ in terms on the polynomials σ and τ of the difference equation (2.2).

2.4   Hypergeometric representation.

Let us consider the most general solution of (2.1) in the lattice $x(s) = c_1 q^s + c_3$. In [37] has been proved that the most general case in the exponential lattice is the one when

$$\sigma(s) = \bar{A}(q^{s-s_1} - 1)(q^{s-s_2} - 1),$$

$$\sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2}) = \bar{A}(q^{s-\bar{s}_1} - 1)(q^{s-\bar{s}_2} - 1). \tag{2.25}$$

In this case, the eigenvalues of the corresponding difference equation is given by

$$\lambda_n = -\frac{\bar{A}}{c_1^2} q^{-\frac{1}{2}(s_1+s_2+\bar{s}_1+\bar{s}_2)}[n]_q[s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1]_q, \tag{2.26}$$

and the polynomial solutions of (2.1) can be represented as q-hypergeometric functions [37]

$$P_n(s)_q = \left(\frac{\bar{A}\kappa_q}{c_1}\right)^n B_n q^{-\frac{n}{2}(\bar{s}_1+\bar{s}_2-\frac{n-1}{2})}(s_1 - \bar{s}_1|q)_n(s_1 - \bar{s}_2|q)_n \times$$

$$\times {}_3F_2\left(\begin{array}{c} -n, s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1, s_1 - s \\ s_1 - \bar{s}_1, s_1 - \bar{s}_2 \end{array} ; q, q^{\frac{1}{2}(s-s_2)} \right) = \tag{2.27}$$

$$= \left(\frac{-\bar{A}\kappa_q}{c_1}\right)^n B_n q^{-\frac{n}{2}(s_1+s_2+\frac{n-1}{2})}(s_1 - \bar{s}_1|q)_n(s_2 - \bar{s}_1|q)_n \times$$

$$\times {}_3F_2\left(\begin{array}{c} -n, s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1, s - \bar{s}_1 \\ s_1 - \bar{s}_1, s_2 - \bar{s}_1 \end{array} ; q, q^{\frac{1}{2}(s-\bar{s}_2)} \right), \tag{2.28}$$

or, in terms of the basic hypergeometric series [37]

$$
P_n(s)_q = B_n \left(\frac{\bar{A}}{c_1(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \right)^n q^{-\frac{n(n-1)}{4} - ns_1} (q^{s_1 - \bar{s}_1}; q)_n (q^{s_1 - \bar{s}_2}; q)_n \times
$$

$$
\times {}_3\varphi_2 \left(\begin{array}{c} q^{-n}, q^{s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1}, q^{s_1 - s} \\ q^{s_1 - \bar{s}_1}, q^{s_1 - \bar{s}_2} \end{array} ; q, q^{s - s_2 + 1} \right) =
$$

(2.29)

$$
= B_n \left(-\frac{\bar{A}}{c_1(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \right)^n q^{-\frac{3n(n-1)}{4} - n(s_1 + s_2 - \bar{s}_1)} (q^{s_1 - \bar{s}_1}; q)_n \times
$$

$$
\times (q^{s_2 - \bar{s}_1}; q)_n \, {}_3\varphi_2 \left(\begin{array}{c} q^{-n}, q^{s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1}, q^{s - \bar{s}_1} \\ q^{s_1 - \bar{s}_1}, q^{s_2 - \bar{s}_1} \end{array} ; q, q \right) .
$$

(2.30)

Here, the q-hypergeometric function is defined by

$$
{}_rF_p \left(\begin{array}{c} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_p \end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k (a_2|q)_k \cdots (a_r|q)_k}{(b_1|q)_k (b_2|q)_k \cdots (b_p|q)_k} \frac{z^k}{(1|q)_k} \left[\kappa_q^{-k} q^{\frac{1}{4}k(k-1)} \right]^{p-r+1} ,
$$

(2.31)

where $(a|q)_k$ is a q-analogue of the Pochhammer symbol

$$
(a|q)_k = \prod_{m=0}^{k-1} [a + m]_q = \frac{\tilde{\Gamma}_q(a+k)}{\tilde{\Gamma}_q(a)} ,
$$

(2.32)

and $\tilde{\Gamma}_q(x)$ is the q-analogue of the gamma function introduced in [34, 37]. Notice that the above definition for the q-hypergeometric function is different from the one introduced in [37] by a factor $\left(\kappa_q^{-k} q^{\frac{1}{4}k(k-1)} \right)^{p-r+1}$, which should be included in order to get "good" limit transitions. Notice also that it coincides with the Nikiforov-Uvarov one when $r = p + 1$.

The q-basic hypergeometric series is defined by [25]

$$
{}_r\varphi_p \left(\begin{array}{c} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_p \end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1} , \quad (2.33)
$$

where

$$
(a; q)_k = \prod_{m=0}^{k-1} (1 - aq^m) ,
$$

(2.34)

These two functions are related to each other by formula [37]

$$
{}_{p+1}\varphi_p \left(\begin{array}{c} q^{a_1}, q^{a_2}, ..., q^{a_{p+1}} \\ q^{b_1}, q^{b_2}, ..., q^{b_p} \end{array} ; q, z \right) = {}_{p+1}F_p \left(\begin{array}{c} a_1, a_2, ..., a_{p+1} \\ b_1, b_2, ..., b_p \end{array} ; q, t \right) \Bigg|_{t = t_0} ,
$$

where $t_0 = z \, q^{\frac{1}{2}\left(\sum_{i=1}^{p+1} a_i - \sum_{i=1}^{p} b_i - 1 \right)}$. The Gamma function $\tilde{\Gamma}_q(s)$, is closed related with the classical $q-$Gamma function $\Gamma_q(s)$ [25] defined by

$$
\Gamma_q(s) = \begin{cases} (1-q)^{1-s} \dfrac{\displaystyle\prod_{k=0}^{\infty} (1 - q^{k+1})}{\displaystyle\prod_{k=0}^{\infty} (1 - q^{s+k})}, & 0 < q < 1 \\[6mm] q^{\frac{(s-1)(s-2)}{2}} \Gamma_{q^{-1}}(s), & q > 1 \end{cases} ,
$$

(2.35)

by the expression [34]

$$\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s). \tag{2.36}$$

Notice that $\tilde{\Gamma}_q(s+1) = [s]_q\tilde{\Gamma}_q(s)$, whereas $\Gamma_q(s+1) = \dfrac{q^s-1}{q-1}\Gamma_q(s) = q^{\frac{s-1}{2}}[n]_q\Gamma_q(s)$.

The above representations (2.27)-(2.30) also follow from the Eq. (2.24) substituting (2.25) and doing some operations similar to those in [37].

2.5   The orthogonality property and three-term recurrence relation.

2.5.1   The orthogonality relation

In [22, 34, 37] has been shown that the polynomial solutions $P_n(x(s))_q \equiv P_n(s)_q$ of the difference equation (2.2) satisfy an orthogonality property of the form

$$\sum_{s_i=a}^{b-1} P_n(x(s_i))_q P_m(x(s_i))_q \rho(s_i) \triangle x(s_i - \tfrac{1}{2}) = \delta_{nm} d_n^2, \quad s_{i+1} = s_i + 1, \tag{2.37}$$

where $\rho(x)$ is a solution of the Pearson-type equation (2.11) or (2.23), and it is a non-negative function (weight function), i.e.,

$$\rho(s_i) \triangle x(s_i - \tfrac{1}{2}) > 0 \quad (a \le s_i \le b-1),$$

supported on a countable subset of the real line $[a, b]$ (a, b can be $\pm\infty$), providing that the condition

$$\sigma(s)\rho(s)x^k\left(s - \tfrac{1}{2}\right)\bigg|_{s=a,b} = 0, \quad k = 0, 1, 2, \dots, \tag{2.38}$$

holds [37]. In fact, substituting in (2.23) the expressions (2.25), we obtain for the weight function (formally) the expression

$$\rho(s) = \frac{\Gamma_q(s - \bar{s}_1)\Gamma_q(s - \bar{s}_2)}{\Gamma_q(s - s_1 + 1)\Gamma_q(s - s_2 + 1)},$$

i.e., $\rho(s)$ is a ratio of the Gamma functions $\tilde{\Gamma}_q(s - s_i + 1)$ and $\tilde{\Gamma}_q(s + \bar{s}_i + 1)$ ($i = 1, 2$), so it is a function which has only simple poles, and then we can always find a contour C in order to apply the Cauchy formula in (2.20).

In (2.37) d_n^2 denotes the square of the norm of the corresponding orthogonal polynomials (with discrete orthogonality relations). In order to calculate it we can use the expression deduced in [34, Chapter 3, Section **3.7.2**, pag. 104] (see also [1, Theorem 5.4])

$$d_n^2 = (-1)^n A_{nn} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \triangle x_n(s - \tfrac{1}{2}). \tag{2.39}$$

If there exists a contour Γ such that, instead of (2.38), the condition

$$\int_\Gamma \triangle[\rho(z)\sigma(z)x^k(z - \tfrac{1}{2})]\, dz = 0, \qquad \forall k = 0, 1, 2, \dots, . \tag{2.40}$$

holds, then, the polynomials satisfy a *continuous* orthogonality relation of the form [14, 17, 34, 37]

$$\int_\Gamma P_n(z)P_m(z)\rho(z) \triangle x(z - \tfrac{1}{2})\, dz = 0, \qquad n \ne m.$$

In some cases it is possible to choose the contour Γ in such a way that the above relation becomes an orthogonality relation on the real line. Using the above method, Atakishiyev and Suslov proved in a very easy way the orthogonality of the Askey-Wilson polynomials [17]. In this work we will not consider such a kind of orthogonality. For more details see [11, 13, 14, 17, 26, 37], among others.

2.5.2 The three-term recurrence relation.

A simple consequence of (2.37) is the fact that the q-orthogonal polynomials satisfy a three-term recurrence relations (TTRR) of the form

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \tag{2.41}$$

with the initial conditions $P_{-1}(s)_q = 0$ and $P_0(s)_q = 1$.

In order to calculate the coefficients α_n, β_n, and γ_n, we will substitute the polynomial $P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots$ in (2.41) (a_n is usually called the leading coefficient of the polynomial $P_n(s)_q$). Then, equating the coefficients for the powers $x^n(s)$ and $x^{n-1}(s)$, we find expressions for α_n and β_n, respectively. To obtain γ_n, we are constrained to use the orthogonality property of the polynomials $P_n(s)_q$,

$$\gamma_n = \frac{\displaystyle\sum_{s_k=a}^{b-1} x(s_k)P_n(s_k)P_{n-1}(s_k)\rho(s_k)\,\triangle\, x(s_k - \tfrac{1}{2})}{d_{n-1}^2}.$$

So we obtain the expressions

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2}. \tag{2.42}$$

Sometimes, the computation of b_n and then the β_n can be very cumbersome, so if we know α_n and γ_n and $P_n(a) \neq 0$ for all n, we can use (2.41) which gives us

$$\beta_n = \frac{x(a)P_n(a) - \alpha_n P_{n+1}(a) - \gamma_n P_{n-1}(a)}{P_n(a)}.$$

3 Some consequences of the Rodrigues-type formula.

3.1 The connection between the leading coefficient a_n and the Rodrigues coefficient B_n.

First of all, we will obtain an expression for the leading coefficient a_n of the polynomial $P_n(s)_q$. Notice that

$$\frac{\triangle x^n(s)}{\triangle x(s)} = [n]_q x^{n-1}(s + \tfrac{1}{2}) + \cdots, \quad \text{so} \quad \triangle^{(n)}[x^n(s)] = [n]_q!\,.$$

Then, we have in one hand, $\triangle^{(n)}P_n^{(n)}(s) = [n]_q!a_n$, and on the other, by using (2.15), $\triangle^{(n)}P_n(s)_q = B_n A_{nn}$, so taking into account expression (2.16) we finally obtain ($a_0 = B_0$)

$$a_n = B_n \prod_{k=0}^{n-1}\left\{\left(\frac{q^{\frac{1}{2}(n+k-1)} + q^{-\frac{1}{2}(n+k-1)}}{2}\right)\tilde{\tau}' + [n+k-1]_q\frac{\tilde{\sigma}''}{2}\right\}. \tag{3.1}$$

3.2 The differentiation formulas.

Let us now obtain the differentiation formulas for the q-polynomials. From formulas (2.11) and (2.12), we find

$$\frac{\nabla \rho_{n+1}(s)}{\nabla x_{n+1}(s)} = \frac{\nabla[\rho_n(s+1)\sigma(s+1)]}{\nabla x_n(s+\frac{1}{2})} = \frac{\triangle[\sigma(s)\rho_n(s)]}{\triangle x_n(s-\frac{1}{2})} = \tau_n(s)\rho_n(s).$$

Then by using the Rodrigues-type formula (2.17), we obtain

$$P_{n+1}(s)_q = \frac{B_{n+1}}{\rho(s)} \nabla^{(n+1)}[\rho_n(s)] = \frac{B_{n+1}}{\rho(s)} \nabla^{(n)} \frac{\nabla \rho_{n+1}(s)}{\nabla x_{n+1}(s)} =$$

$$= \frac{B_{n+1}}{\rho(s)} \nabla^{(n)}[\tau_n(s)\rho_n(s)] = \frac{B_{n+1}}{\rho(s)} \left[\frac{\delta}{\delta x(s)}\right]^n [\tau_n(s-\frac{n}{2})\rho_n(s-\frac{n}{2})].$$

$$(3.2)$$

In order to obtain an expression for $\left[\frac{\delta}{\delta x(s)}\right]^n [\tau_n(s)\rho_n(s)]$, we can use the q-analogue of the Leibnitz formula on the non-uniform lattices

$$\left[\frac{\delta}{\delta x(s)}\right]^n [f(s)g(s)] = \sum_{k=0}^{n} \frac{[n]_q!}{[k]_q![n-k]_q!} \times$$

$$\times \left\{ \left[\frac{\delta}{\delta x(s+\frac{n-k}{2})}\right]^k [f(s+\frac{n-k}{2})] \right\} \left\{ \left[\frac{\delta}{\delta x(s-\frac{k}{2})}\right]^{n-k} [g(s-\frac{k}{2})] \right\},$$

$$(3.3)$$

which can be proved by induction. Next, since

$$\frac{\delta \tau_n(s-\frac{n}{2}+\frac{n-1}{2})}{\delta x(s+\frac{n-1}{2})} = \tau_n', \quad \left[\frac{\delta}{\delta x(s+\frac{n-k}{2})}\right]^k [\tau_n(s-\frac{n}{2}+\frac{n-k}{2})] = 0, \quad \forall k \geq 2,$$

then, Eq. (3.2) transforms

$$P_{n+1}(s)_q = \frac{B_{n+1}}{\rho(s)} \times$$

$$\times \left(\tau_n(s) \left[\frac{\delta}{\delta x(s)}\right]^n [\rho_n(s-\frac{n}{2})] + [n]_q \tau_n' \left[\frac{\delta}{\delta x(s-\frac{1}{2})}\right]^{n-1} [\rho_n(s-\frac{n}{2}-\frac{1}{2})] \right).$$

$$(3.4)$$

Now we use the Rodrigues-type formula (2.18)

$$\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\triangle P_n(s-1)_q}{\triangle x(s-1)} = \frac{-\lambda_n B_n}{\rho_1(s-1)} \nabla_1^{(n)} \rho_n(s-1) =$$

$$= \frac{-\lambda_n B_n}{\sigma(s)\rho(s)} \left[\frac{\delta}{\delta x(s-\frac{1}{2})}\right]^{n-1} [\rho_n(s-\frac{n}{2}-\frac{1}{2})].$$

This leads to the expression,

$$P_{n+1}(s)_q = \frac{B_{n+1}\tau_n(s)}{B_n} P_n(s)_q - \frac{[n]_q B_{n+1}\tau_n'\sigma(s)}{\lambda_n B_n} \frac{\nabla P_n(s)_q}{\nabla x(s)},$$

and then, the following *differentiation formula* holds

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q\tau_n'} \left[\tau_n(s)P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \tag{3.5}$$

This proof generalizes the one presented in [3] to any lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. Notice that for proving (3.5) we have used only the Rodriges-type formula (2.13). Formula (3.5) has been also obtained in [13] by using the integral representation for the q-polynomials as well as some boundary conditions.

In order to obtain the second differentiation formula, we can make use of the identity

$$\triangle \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)} = \frac{\triangle P_n(s)_q}{\triangle x(s)} - \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)}. \tag{3.6}$$

Then, by using the difference equation (2.2), the equation (3.5) gives

$$[\sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2})] \frac{\triangle P_n(s)_q}{\triangle x(s)} =$$

$$= \frac{\lambda_n}{[n]_q \tau_n'} \left[(\tau_n(s) - \triangle x(s - \tfrac{1}{2})[n]_q \tau_n') P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \tag{3.7}$$

3.3 The structure relations in the exponential lattice.

Let us obtain the structure relations for the polynomial solutions of (2.2) in the lattice $x(s) = c_1 q^s + c_3$. In order to do this we substitute the power expansion of $\tau_n(s)$

$$\tau_n(s) = \tau_n' x_n(s) + \tau_n(0) = \tau_n' q^{\frac{n}{2}} x(s) + \tau_n(0) - \tau_n' c_3 (q^{\frac{n}{2}} - 1),$$

in (3.5). Then, using the TTRR (2.41) we obtain the *first structure relation*

$$\sigma(s) \frac{\bigtriangledown P_n(s)_q}{\bigtriangledown x(s)} = \tilde{S}_n P_{n+1}(s)_q + \tilde{T}_n P_n(s)_q + \tilde{R}_n P_{n-1}(s)_q, \tag{3.8}$$

where

$$\tilde{S}_n = \frac{\lambda_n}{[n]_q} \left[q^{\frac{n}{2}} \alpha_n - \frac{B_n}{\tau_n' B_{n+1}} \right], \quad \tilde{T}_n = \frac{\lambda_n}{[n]_q} \left[q^{\frac{n}{2}} \beta_n + \frac{\tau_n(0)}{\tau_n'} - c_3 (q^{\frac{n}{2}} - 1) \right],$$

$$\tilde{R}_n = \frac{\lambda_n q^{\frac{n}{2}} \gamma_n}{[n]_q}. \tag{3.9}$$

To obtain the *second structure relation* we transform (3.8) with the help of (3.6). Then, using the fact that $\triangle x(s - \tfrac{1}{2}) = \kappa_q x(s) - c_3 \kappa_q$, as well as the difference equation (2.2) and the TTRR (2.41) one gets

$$[\sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2})] \frac{\triangle P_n(s)_q}{\triangle x(s)} = S_n P_{n+1}(s)_q + T_n P_n(s)_q + R_n P_{n-1}(s)_q, \tag{3.10}$$

where

$$S_n = \tilde{S}_n - \alpha_n \lambda_n \kappa_q, \quad T_n = \tilde{T}_n - \beta_n \lambda_n \kappa_q + c_3 \lambda_n \kappa_q, \quad R_n = \tilde{R}_n - \gamma_n \lambda_n \kappa_q, \tag{3.11}$$

or, using (3.9),

$$S_n = \frac{\lambda_n}{[n]_q} \left[q^{-\frac{n}{2}} \alpha_n - \frac{B_n}{\tau_n' B_{n+1}} \right], \quad T_n = \frac{\lambda_n}{[n]_q} \left[q^{-\frac{n}{2}} \beta_n + \frac{\tau_n(0)}{\tau_n'} - c_3 (q^{-\frac{n}{2}} - 1) \right],$$

$$R_n = \frac{\lambda_n q^{-\frac{n}{2}} \gamma_n}{[n]_q}. \tag{3.12}$$

11

3.4   A difference-recurrence relation in the exponential lattice.

In the present section we will prove that the $q-$polynomials on the exponential lattice satisfy a relation of the form

$$P_n(s)_q = L_n \frac{\triangle P_{n+1}(s)_q}{\triangle x(s)} + M_n \frac{\triangle P_n(s)_q}{\triangle x(s)} + N_n \frac{\triangle P_{n-1}(s)_q}{\triangle x(s)} , \qquad (3.13)$$

where L_n, M_n and N_n are some constants.

To prove (3.13) we apply the operator $\frac{\triangle}{\triangle x(s)}$ on both sides of (3.8), and then use the second order difference equation (2.2) in which we will change the operator $\frac{\triangle}{\triangle x(s-\frac{1}{2})}$ by the equivalent one (only in the exponential lattice) $q^{\frac{1}{2}} \frac{\triangle}{\triangle x(s)}$. This is possible because for the exponential lattice the following identity holds $q^{-\frac{1}{2}} \triangle x(s) = \triangle x(s - \frac{1}{2})$. Thus,

$$\left[q^{\frac{1}{2}} \frac{\triangle \sigma(s)}{\triangle x(s)} - \tau(s) \right] \frac{\triangle P_n(s)_q}{\triangle x(s)} - \lambda_n P_n(s)_q =$$
$$= q^{\frac{1}{2}} \tilde{S}_n \frac{\triangle P_{n+1}(s)_q}{\triangle x(s)} + q^{\frac{1}{2}} \tilde{T}_n \frac{\triangle P_n(s)_q}{\triangle x(s)} + q^{\frac{1}{2}} \tilde{R}_n \frac{\triangle P_{n-1}(s)_q}{\triangle x(s)}. \qquad (3.14)$$

Next, we use the fact that $\sigma(s)$ and $\tau(s)$ are polynomials of second and first degree, respectively. Let

$$\sigma(s) = \frac{\sigma''}{2} x^2(s) + \sigma'(0)x(s) + \sigma(0), \quad \text{and} \quad \tau(s) = \tau' x(s) + \tau(0). \qquad (3.15)$$

Since, $x(s + 1) = q\, x(s) - c_3(q - 1)$, and using (3.15) we conclude that $\frac{\triangle \sigma(s)}{\triangle x(s)}$ is a polynomial of first degree in $x(s)$. Then

$$\left[q^{\frac{1}{2}} \frac{\triangle \sigma(s)}{\triangle x(s)} - \tau(s) \right] = Ax(s) + B,$$

where

$$A = \frac{\sigma''}{2}(1 + q)q^{\frac{1}{2}} - \tau', \quad \text{and} \quad B = q^{\frac{1}{2}} \sigma'(0) - \frac{\sigma''}{2} c_3 q^{\frac{1}{2}}(q - 1) - \tau(0), \qquad (3.16)$$

and (3.14) becomes

$$Ax(s) \frac{\triangle P_n(s)_q}{\triangle x(s)} - \lambda_n P_n(s)_q =$$
$$= q^{\frac{1}{2}} \tilde{S}_n \frac{\triangle P_{n+1}(s)_q}{\triangle x(s)} + q^{\frac{1}{2}} \tilde{T}_n \frac{\triangle P_n(s)_q}{\triangle x(s)} + q^{\frac{1}{2}} \tilde{R}_n \frac{\triangle P_{n-1}(s)_q}{\triangle x(s)} - B \frac{\triangle P_n(s)_q}{\triangle x(s)}. \qquad (3.17)$$

Now we use the TTRR (2.41) to eliminate the term $x(s)\frac{\triangle P_n(s)_q}{\triangle x(s)}$. In fact, if we apply the operator $\frac{\triangle}{\triangle x(s)}$ to both sides of (2.41) and use again the identity $x(s+1) = q\, x(s) - c_3(q - 1)$, we get

$$q x(s) \frac{\triangle P_n(s)_q}{\triangle x(s)} = \alpha_n \frac{\triangle P_{n+1}(s)_q}{\triangle x(s)} + [\beta_n + c_3(q - 1)] \frac{\triangle P_n(s)_q}{\triangle x(s)} + \gamma_n \frac{\triangle P_{n-1}(s)_q}{\triangle x(s)} - P_n(s)_q.$$

If we now multiply (3.17) by q and make use of the above equation, we finally obtain

$$[A + q\lambda_n] P_n(s)_q = \left[\alpha_n A - q^{\frac{3}{2}} \tilde{S}_n\right] \frac{\triangle P_{n+1}(s)_q}{\triangle x(s)} +$$

$$+ \left[\beta_n A + c_3 A(q-1) + qB - q^{\frac{3}{2}} \tilde{T}_n\right] \frac{\triangle P_n(s)_q}{\triangle x(s)} + \left[\gamma_n A - q^{\frac{3}{2}} \tilde{R}_n\right] \frac{\triangle P_{n-1}(s)_q}{\triangle x(s)}, \tag{3.18}$$

which is of the form (3.13) if $A + q\lambda_n \neq 0$.

To conclude this section notice that the expression (3.13) can also be obtained by using, with some modifications, the method given in [32].

4 The q-analogues of the classical discrete polynomials on the exponential lattice $x(s) = q^s$.

In this section we will consider some q-analogues of the Hahn, Meixner and Kravchuk polynomials in the exponential lattice $x(s) = q^s$. These polynomials were considered by other authors [19, 39] using the Nikiforov et al. approach. In this section we will introduce them in an unify way from Eqs. (2.27)-(2.30) and their limits (see below for more details), i.e., we will identify these q-Hahn, q-Meixner and q-Kravchuk with certain q-hypergeometric series which have not been done in [19, 39]. The case of a q-analogue of the Charlier polynomials will be considered with more details in the next section.

4.1 The q-Hahn polynomials $h_n^{\alpha,\beta}(s,N)_q$.

The q-Hahn polynomials, from the point of view of [34], have been studied in [39]. If we choose in (2.27) and (2.29) the parameters

$$x(s) = q^s \to c_1 = 1, \quad c_3 = 0, \qquad B_n = \frac{(-1)^n}{q^{\frac{n}{2}} \kappa_q^n [n]_q!},$$

$$s_1 = 0, \quad s_2 = N + \alpha, \quad \bar{s}_1 = -\beta - 1, \quad \bar{s}_2 = N - 1, \quad \bar{A} = -q^{N+\alpha},$$

we recover the q-Hahn polynomials [26, 39]

$$h_n^{\alpha,\beta}(s,N)_q = \frac{(\beta+1|q)_n(1-N|q)_n}{q^{-\frac{n}{2}(2\alpha+\beta+N+\frac{n+1}{2})}[n]_q!} {}_3F_2\left(\begin{array}{c} -n, -s, n+\alpha+\beta+1 \\ \beta+1, 1-N \end{array} ; q, q^{\frac{1}{2}(s-N-\alpha)}\right) =$$

$$= \frac{(\beta+1|q)_n(N+\alpha+\beta+1|q)_n}{q^{\frac{n}{2}(N+\alpha+\frac{n(n+1)}{2})}[n]_q!} {}_3F_2\left(\begin{array}{c} -n, s+\beta+1, n+\alpha+\beta+1 \\ \beta+1, N+\alpha+\beta+1 \end{array} ; q, q^{\frac{1}{2}(s+1-N)}\right),$$

or,

$$h_n^{\alpha,\beta}(s,N)_q = \frac{(-1)^n q^{n(\alpha+N)}(q^{\beta+1};q)_n(q^{1-N};q)_n}{\kappa_q^n(q;q)_n} {}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{-s}, q^{n+\alpha+\beta+1} \\ q^{\beta+1}, q^{1-N} \end{array} ; q, q^{s-N-\alpha+1}\right) =$$

$$= \frac{(q^{\beta+1};q)_n(q^{N+\alpha+\beta+1};q)_n}{q^{\frac{n}{2}(2\beta+n+1)}\kappa_q^n(q;q)_n} {}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{s+\beta+1}, q^{n+\alpha+\beta+1} \\ q^{\beta+1}, q^{N+\alpha+\beta+1} \end{array} ; q, q\right).$$

In this case, using (2.26), we find for λ_n the expression

$$\lambda_n = q^{\frac{1}{2}(\alpha+\beta+2)}[n]_q[n+\alpha+\beta+1]_q,$$

and (2.37) holds in the interval $[0, N-1]$ where

$$\rho(s) = q^{\frac{\alpha}{4}(\alpha+2N+2s-3)+\frac{\beta}{4}(\beta+2s-1)}\frac{\tilde{\Gamma}_q(\alpha+N-s)\tilde{\Gamma}_q(\beta+s+1)}{[N-s-1]_q![s]_q!}, \quad \alpha,\beta \geq -1\,,\ n \leq N-1\,,$$

and

$$d_n^2 = \frac{q^{\frac{1}{2}N(N-1)+\frac{1}{2}(N-1)(2\alpha+\beta+N)}}{q^{-\alpha-N-\frac{1}{4}\beta(\beta+1)-\frac{1}{2}n(\alpha+\beta-2)}}\frac{\kappa_q\tilde{\Gamma}_q(\alpha+n+1)\tilde{\Gamma}_q(\beta+n+1)\tilde{\Gamma}_q(\alpha+\beta+N+n+1)}{[n]_q![N-n-1]_q!\tilde{\Gamma}_q(\alpha+\beta+n+1)\tilde{\Gamma}_q(\alpha+\beta+2n+2)}.$$

4.2 The q-Meixner, q-Kravchuk polynomials.

From (2.27) or (2.29), and taking the limits $q^{\pm s_i} \to 0$ or $q^{\pm \bar{s}_i} \to 0$, $i = 1, 2$, it is possible to find different families of q-polynomials. In [37], 9 different possibilities have been considered. Here we will study only 2 of them, i.e., the cases corresponding to the q-Meixner, q-Kravchuk, respectively. The q-analogue of the Charlier polynomial will be introduced in the next section.

Taking the limit $q^{s_2} \to 0$, $q^{\bar{s}_2} \to 0$, and choosing $\bar{A} = A_1 q^{s_2}$, $s_2 - \bar{s}_2 = \delta + 1$, the expression (2.25) becomes

$$\sigma(s) = A_1 q^s(q^{s-s_1} - 1), \qquad \sigma(s) + \tau(s)\,\triangle\,x(s - \tfrac{1}{2}) = A_1 q^{\delta+1}q^s(q^{s-\bar{s}_1} - 1). \qquad (4.1)$$

Then, (2.27) and (2.29) lead us to

$$P_n(s)_q = \left(\frac{A_1}{c_1}\right)^n B_n q^{\frac{n(n+1)}{2}+n\delta}(s_1 - \bar{s}_1|q)_n \times$$
$$\times \ \ _3\mathrm{F}_1\left(\begin{array}{c} -n, s_1 - \bar{s}_1 + \delta + n, s_1 - s \\ s_1 - \bar{s}_1 \end{array} ; q, q^{\frac{1}{2}(s-s_1-\delta-1)}\right), \qquad (4.2)$$

and

$$P_n(s)_q = B_n\left(\frac{-A_1 q^{\frac{(n-1)}{4}+\delta+1}}{c_1(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}\right)^n (q^{s_1-\bar{s}_1}; q)_n \ _3\varphi_1\left(\begin{array}{c} q^{-n}, q^{s_1-\bar{s}_1+\delta+n}, q^{s_1-s} \\ q^{s_1-\bar{s}_1} \end{array} ; q, q^{s-s_1-\delta}\right),$$

respectively. In this case, taking limits on the expression (2.26) we find

$$\lambda_n = -\frac{A_1}{c_1^2}q^{-\frac{1}{2}(s_1+\bar{s}_1-\delta-1)}[n]_q[s_1 - \bar{s}_1 + \delta + n]_q. \qquad (4.3)$$

4.2.1 The q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$.

Let $A_1 = 1$, $B_n = \mu^{-n}$, $s_1 = 0$, $\bar{s}_1 = -\gamma$, $q^\delta = \mu$. Then Eqs.(4.2) and (4.2) define a q-analogue of the Meixner polynomials

$$m_n^{\gamma,\mu}(s,q) = q^{\frac{n(n+1)}{2}}(\gamma|q)_n \ _3\mathrm{F}_1\left(\begin{array}{c} -n, \gamma + \delta + n, -s \\ \gamma \end{array} ; q, q^{\frac{1}{2}(s-\delta-1)}\right),$$

$$m_n^{\gamma,\mu}(s,q) = (-1)^n \left(\frac{q^{\frac{(n+3)}{4}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \right)^n (q^\gamma;q)_n {}_3\varphi_1 \left(\begin{array}{c} q^{-n}, \mu q^{\gamma+n}, q^{-s} \\ q^\gamma \end{array} ; q, \mu^{-1}q^s \right).$$

In this case (4.1) and (4.3) become

$$\sigma(s) = q^s(q^s - 1), \qquad \sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2}) = \mu q^{s+1}(q^{s+\gamma} - 1), \qquad (4.4)$$

and

$$\lambda_n = [n]_q \frac{q^{-\frac{(n-1)}{2}} - \mu q^{\frac{(n+1)}{2}+\gamma}}{\kappa_q}, \qquad (4.5)$$

respectively. Finally, the Pearson-type equation leads us to the weight function

$$\rho(s) = \frac{\mu^s \Gamma_q(\gamma + s)}{\Gamma_q(\gamma)\Gamma_q(s + 1)}, \quad \gamma > 0, \ 0 < \mu < 1,$$

so that, (2.37) holds in the interval $[0, \infty)$ and

$$d_n^2 = q^{\frac{1}{2}n(6-n)} \frac{\kappa_q^{n+1}(\gamma|q)_n \Gamma_q(\gamma + \delta + 2n)\Gamma_q(n + 1)(-\mu q^{2n+\gamma+1};q)_\infty}{\mu^n \Gamma_q(\gamma + \delta + n)(-\mu q^{n+1};q)_\infty}.$$

These Meixner $q-$polynomials coincide with those studied in [19]. Notice also that these $q-$analogues of the Meixner polynomials $m_n^{\gamma,\mu}(s,q)$ are the q-little Jacobi $p_n(x; a, b|q)$ polynomials [25, 26], i.e., $m_n^{\gamma,\mu}(s, q) = p_n(q^s; \mu, q^{\gamma-1}|q)$.

4.2.2 The q-Kravchuk polynomials $k_n^{(p)}(s, q)$.

Analogously, from (4.2) and (4.2), but now choosing

$$A_1 = -1, \quad B_n = (-1)^n \frac{(1 - p)^n}{[n]_q!}, \quad s_1 = 0, \quad \bar{s}_1 = N, \quad q^\delta = -\frac{p}{1-p}q^N,$$

we find a q-analogue of the Kravchuk polynomials

$$k_n^{(p)}(s, q) = (-1)^n p^n\, q^{\frac{n(n+1)}{2}+Nn} \frac{(-N|q)_n}{[n]_q!} \times$$

$$\times {}_3F_1 \left(\begin{array}{c} -n, -N + \delta + n, -s \\ -N \end{array} ; q, q^{\frac{1}{2}(s-\delta-1)} \right),$$

$$k_n^{(p)}(s, q) = (-1)^n \left(p\, q^{\frac{(n+3)}{4}+N} \right)^n \frac{(q^{-N};q)_n}{(q;q)_n} {}_3\varphi_1 \left(\begin{array}{c} q^{-n}, \frac{p}{p-1}q^n, q^{-s} \\ q^{-N} \end{array} ; q, \frac{p-1}{p}q^{N+s} \right).$$

Notice that

$$(-1)^n \left(p\, q^{\frac{(n+3)}{4}+N} \right)^n \frac{(q^{-N};q)_n}{(q;q)_n} = \left(p\, q^{\frac{3(n+1)}{4}} \right)^n \frac{\Gamma_q(N + 1)}{[n]_q!\Gamma_q(N - n - 1)}.$$

In this case we have

$$\sigma(s) = q^s(1 - q^s), \qquad \sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2}) = \tfrac{p}{1-p}q^{s+1+N}(1 - q^{s-N}), \qquad (4.6)$$

$$\lambda_n = -[n]_q \frac{\frac{p}{1-p}q^{\frac{(n+1)}{2}} + q^{-\frac{(n-1)}{2}}}{\kappa_q}, \qquad (4.7)$$

15

and (2.37) holds in the interval $[0, N]$ where

$$\rho(s) = \left(\frac{p}{1-p}\right)^s \frac{q^{\frac{1}{2}s(s+1)}[N]_q!}{\Gamma_q(N+1-s)\Gamma_q(s+1)}, \quad 0 < p < 1,$$

and

$$d_n^2 = q^{\frac{1}{2}n(n+\theta+2)} \frac{\kappa_q p^n (1-p)^n [n+\theta-1]_q! [4n+2\theta-2]_q!! (\frac{p}{1-p}q^{2n+1};q)_{(N-n)}}{[n]_q! [2n+\theta-1]_q! [2n+2\theta-2]_q!! \Gamma_q(N-n+1)}, \quad q^\theta = \frac{p}{1-p}.$$

These polynomials where studied in detail in [19].

5 q-Charlier polynomials.

In the previous section we have obtained some $q-$analogues of the Hahn, Meixner and Kravchuk classical polynomials on the general exponential lattice with special emphasis in the lattice $x(s) = q^s$. Here we will consider the fourth family, i.e., a $q-$analogue of the Charlier polynomials.

Firstly, we take the limits $q^{s_2} \to 0$, $q^{\bar{s}_1} \to 0$, $q^{-\bar{s}_2} \to 0$, and choose the others parameters as $\bar{A} = A_1 q^{s_2}$, $s_2 - \bar{s}_1 = \delta + 1 + \frac{i\pi}{\log q}$. Then, (2.25) becomes

$$\sigma(s) = A_1 q^s (q^{s-s_1} - 1), \qquad \sigma(s) + \tau(s) \bigtriangleup x(s - \tfrac{1}{2}) = A_1 q^\delta q^{s+1}, \qquad (5.1)$$

and (2.27) and (2.29) transforms into

$$P_n(s)_q = \left(\frac{A_1}{\kappa_q c_1}\right)^n B_n q^{\frac{n(n+3)}{4}+n\delta} {}_2F_0 \left(\begin{array}{c} -n, s_1-s \\ - \end{array}; q, -q^{\frac{1}{2}(s-s_1-n-1)-\delta}\right), \qquad (5.2)$$

$$P_n(s)_q = \left(\frac{A_1}{\kappa_q c_1}\right)^n B_n q^{\frac{n(n+3)}{4}+n\delta} {}_2\varphi_0 \left(\begin{array}{c} q^{-n}, q^{s_1-s} \\ - \end{array}; q, -q^{s-s_1-\delta}\right), \qquad (5.3)$$

Finally, from (2.26) one gets

$$\lambda_n = \frac{A_1}{\kappa_q c_1^2} q^{-s_1-\frac{n-1}{2}} [n]_q. \qquad (5.4)$$

5.1   The q-Charlier polynomials in the lattice $x(s) = q^s$.

Let us now introduce the q-Charlier polynomials in the lattice $x(s) = q^s$. In order to do that we choose

$$x(s) = q^s, \quad A_1 = 1, \quad B_n = \mu^{-n}, \quad s_1 = 0, \quad q^\delta = (q-1)\mu.$$

Then, formulas (5.2) and (5.3) give us

$$c_n^{(\mu)}(s, q) = q^{\frac{n(n+5)}{4}} {}_2F_0 \left(\begin{array}{c} -n, -s \\ - \end{array}; q, -\frac{q^{\frac{1}{2}(s-n-1)}}{(q-1)\mu}\right),$$

$$c_n^{(\mu)}(s, q) = q^{\frac{n(n+5)}{4}} {}_2\varphi_0 \left(\begin{array}{c} q^{-n}, q^{-s} \\ - \end{array}; q, -\frac{q^s}{(q-1)\mu}\right),$$

respectively. Moreover, (4.1) and (4.3) lead us to the expressions

16

$$\sigma(s) = q^s(q^s - 1), \quad \sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2}) = \mu(q-1)q^{s+1}, \tag{5.5}$$

and

$$\lambda_n = \frac{q^{-\frac{n-1}{2}}}{\kappa_q}[n]_q. \tag{5.6}$$

Finally, for the weight function $\rho(s)$, the Pearson-type difference equation (2.23) gives

$$\rho(s) = \frac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad 0 < (1-q)\mu < 1, \tag{5.7}$$

where $e_q[z]$ denotes the q- exponential function [25] defined by

$$e_q[z] = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad \text{and} \quad (z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k). \tag{5.8}$$

These polynomials were partially studied in [1].

Main characteristics of the $q-$Charlier polynomials in the lattice $x(s) = q^s$.

Using the formulas obtained in the above Sections we can find the main properties of the q-Charlier polynomials. They are given in Table 1.

Table 1: Main data for the q-Charlier polynomials in the lattice $x(s) = q^s$.

$P_n(s)_q$	$c_n^{(\mu)}(s,q), \quad x(s) = q^s$
(a,b)	$(0,\infty)$
$\rho(s)$	$\dfrac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad \mu > 0, \quad 0 < (1-q)\mu < 1$
$\sigma(s)$	$q^s(q^s - 1)$
$\tau(s)$	$\kappa_q^{-1}\mu q(q-1) + 1 - \kappa_q^{-1}q^s$
$\tau_n(s)$	$-\dfrac{q^{-n}}{\kappa_q}x_n(s) + \dfrac{\mu q^{n+\frac{3}{2}}\kappa_q + 1}{q^{\frac{n}{2}}\kappa_q}$
λ_n	$[n]_q q^{-\frac{(n-1)}{2}}\kappa_q^{-1}$
B_n	μ^{-n}
d_n^2	$(\{1-q\}\mu; q)_{n+1}\dfrac{\kappa_q[n]_q!}{q^{\frac{n}{4}(n-9)}\mu^n} = \dfrac{e_q[(1-q)q^{n+1}\mu]}{e_q[(1-q)\mu]}\dfrac{\kappa_q[n]_q!}{q^{\frac{n}{4}(n-9)}\mu^n}$
$\rho_n(s)$	$\dfrac{\mu^{s+n}\kappa_q^n q^{\frac{n}{2}(n+2s+2)}}{e_q[(1-q)\mu]\Gamma_q(s+1)}$
a_n	$\dfrac{(-1)^n}{\kappa_q^n\mu^n}q^{-\frac{3n}{4}(n-1)}$

Explicit Formula.

$$c_n^{(\mu)}(s,q) = \frac{(-1)^n[n]_q!}{\mu^n}\sum_{m=0}^{n}\frac{(-1)^m q^{\frac{m}{2}(n+1)-\frac{n}{4}(n-3)}\mu^m\Gamma_q(s+1)}{[m]_q![n-m]_q!\Gamma_q(s+1-n+m)}.$$

17

Table 1: Main data for the q-Charlier polynomials in the lattice $x(s) = q^s$ (cont.)

$P_n(s)_q$	$c_n^{(\mu)}(s, q), \quad x(s) = q^s$
α_n	$-\mu \kappa_q q^{\frac{3}{2}n}$
β_n	$1 + \kappa_q q^{2n+\frac{3}{2}} \left\{ \mu + q^{-\frac{3(n+1)}{2}} [n]_q \left(1 - \mu(1-q)q^n\right) \right\}$
γ_n	$-q^{n+1} \kappa_q [n]_q \{1 - \mu(1-q)q^n\}$
$\tilde{S}_n$	$\mu q^{\frac{n+1}{2}} (1 - q^n)$
$\tilde{T}_n$	$[n]_q q^{\frac{n+1}{2}} \{1 - \mu q^n (1-q)\} - \mu q^{n+2} (1 - q^n)$
$\tilde{R}_n$	$-q^{n+\frac{3}{2}} [n]_q \left(1 - \mu(1-q)q^n\right)$
S_n	0
T_n	$-\mu(1-q)[n]_q q^{\frac{n+1}{2}}$
R_n	$-q^{\frac{3}{2}} [n]_q \left(1 - \mu(1-q)q^n\right)$
L_n	$-\dfrac{\mu q^n \kappa_q}{[n+1]_q}$
M_n	$\dfrac{\mu q^{\frac{n+1}{2}} (q^{n+1} - 1)\kappa_q}{[n+1]_q}$
N_n	0

Special Values

$$c_n^{(\mu)}(0, q) \quad = q^{\frac{n}{4}(n+5)}.$$

Differentiation formulas.

$$(q^s - 1) \bigtriangledown c_n^{(\mu)}(s, q) = \mu(q-1)q^{\frac{n-1}{2}} c_{n+1}^{(\mu)}(s, q) - \left[1 + \mu(q-1)q^{n+1} - q^s\right] c_n^{(\mu)}(s, q).$$

$$\mu(q - 1) \bigtriangleup c_n^{(\mu)}(s, q) = \mu q^{\frac{n-1}{2}} c_{n+1}^{(\mu)}(s, q) - \left[1 + \mu(q-1)q^{n+1} + q^{s-n}\right] c_n^{(\mu)}(s, q).$$

Notice that if we take the limit $q \to 1$, we will not recover the main data for the classical Charlier polynomials. For this reason we will introduce a different $q-$analogue of the Charlier polynomials, but in the lattice $x(s) = \frac{q^s - 1}{q - 1}$.

5.2 The $q-$Charlier polynomials in the lattice $x(s) = \frac{q^s - 1}{q - 1}$.

To avoid the problem commented below, we choose now

$$x(s) = \frac{q^s - 1}{q - 1}, \quad A_1 = \frac{1}{q - 1}, \quad c_1 = -c_3 = \frac{1}{q - 1} \quad B_n = \mu^{-n}, \quad s_1 = 0, \quad q^\delta = (q-1)\mu.$$

Then, formulas (5.2) and (5.3) give us

Table 2: Main data for the q-Charlier polynomials in the lattice $x(s) = \frac{q^s - 1}{q - 1}$.

$P_n(s)_q$	$c_n^{(\mu)}(s)_q$, $\quad x(s) = \frac{q^s - 1}{q - 1}$
(a, b)	$[0, \infty)$
$\rho(s)$	$\dfrac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad \mu > 0, \quad 0 < (1-q)\mu < 1$
$\sigma(s)$	$q^s x(s)$
$\tau(s)$	$\mu q^{\frac{3}{2}} - q^{\frac{1}{2}} x(s)$
$\tau_n(s)$	$-q^{-n+\frac{1}{2}} x_n(s) + \mu q^{\frac{n}{2}+\frac{3}{2}} + q^{-\frac{3}{4}n}[\frac{n}{2}]_q$
λ_n	$[n]_q q^{-\frac{(n-2)}{2}}$
B_n	μ^{-n}
d_n^2	$(\{1-q\}\mu; q)_{n+1} \dfrac{[n]_q!}{q^{\frac{n}{4}(n-9)+\frac{1}{2}}\mu^n} = \dfrac{e_q[(1-q)q^{n+1}\mu]}{e_q[(1-q)\mu]} \dfrac{[n]_q!}{q^{\frac{n}{4}(n-9)+\frac{1}{2}}\mu^n}$
$\rho_n(s)$	$\dfrac{\mu^{s+n} q^{\frac{n}{2}(n+2s+1)}}{e_q[(1-q)\mu]\Gamma_q(s+1)}$
a_n	$\dfrac{(-1)^n}{\mu^n} q^{-\frac{3n}{4}(n-1)+\frac{n}{2}}$

$$c_n^{(\mu)}(s)_q = q^{\frac{n(n+5)}{4}} \, _2F_0 \left(\begin{array}{c} -n, -s \\ - \end{array} ; q, -\frac{q^{\frac{1}{2}(s-n-1)}}{(q-1)\mu} \right),$$

and

$$c_n^{(\mu)}(s)_q = q^{\frac{n(n+5)}{4}} \, _2\varphi_0 \left(\begin{array}{c} q^{-n}, q^{-s} \\ - \end{array} ; q, -\frac{q^s}{(q-1)\mu} \right),$$

respectively. The functions $\sigma(s)$ and $\tau(s)$ and λ_n are defined now by relations

$$\sigma(s) = q^s \left(\frac{q^s - 1}{q - 1} \right), \quad \sigma(s) + \tau(s) \triangle x(s - \tfrac{1}{2}) = \mu q^{s+1}, \quad \lambda_n = q^{-\frac{n}{2}+1}[n]_q. \qquad (5.9)$$

Finally, for the weight function $\rho(s)$, the Pearson-type difference equation (2.11) gives

$$\rho(s) = \frac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad 0 < (1-q)\mu < 1. \qquad (5.10)$$

where $e_q[z]$ was given in (5.8).

Notice that with the present choice of parameters the hypergeometric representation remains the same but the coefficients of the second order difference equation $\sigma(s)$, $\tau(s)$ and λ_n tend, when $q \to 1$, to the corresponding coefficients of the classical Charlier polynomials, and the the lattice function $x(s) = \frac{q^s - 1}{q - 1}$ becomes the linear one.

Main characteristics of the q−Charlier polynomials in the lattice $x(s) = \frac{q^s - 1}{q - 1}$.

Again, using the formulas obtained in the above Sections we obtain the properties of the q-Charlier polynomials (see table 2).

Table 2: Main data for the q-Charlier polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$ (cont).

$P_n(s)_q$	$c_n^{(\mu)}(s)_q\,,\quad x(s) = \frac{q^s-1}{q-1}$
α_n	$-\mu q^{\frac{3}{2}n-\frac{1}{2}}$
β_n	$\mu q^{2n+1} + [n]_q\left\{1-\mu(1-q)q^n\right\} q^{\frac{n-1}{2}}$
γ_n	$-q^{n+\frac{1}{2}}[n]_q\{1-\mu(1-q)q^n\}$
$\tilde{S}_n$	$\mu q^{\frac{n+1}{2}}(1-q^n)$
$\tilde{T}_n$	$[n]_q q^{\frac{n+1}{2}}\{1-\mu(1-q)q^n\} - \mu q^{n+2}(1-q^n)$
$\tilde{R}_n$	$-q^{n+\frac{3}{2}}[n]_q\left\{1-\mu(1-q)q^n\right\}$
S_n	0
T_n	$-\mu(1-q)[n]_q q^{\frac{n+1}{2}}$
R_n	$-q^{\frac{3}{2}}[n]_q\left\{1-\mu(1-q)q^n\right\}$
L_n	$-\dfrac{\mu q^{n-\frac{1}{2}}}{[n+1]_q}$
M_n	$\dfrac{\mu q^{\frac{n}{2}}(q^{n+1}-1)}{[n+1]_q}$
N_n	0

Explicit Formula.

$$c_n^{(\mu)}(s,q) = \frac{(-1)^n[n]_q!}{\mu^n} \sum_{m=0}^{n} \frac{(-1)^m q^{\frac{m}{2}(n+1)-\frac{n}{4}(n-3)}\mu^m \Gamma_q(s+1)}{[m]_q![n-m]_q!\Gamma_q(s+1-n+m)}.$$

Special Values

$$c_n^{(\mu)}(0)_q \;= q^{\frac{n}{4}(n+5)}.$$

Differentiation formulas.

$$\frac{q^s-1}{q-1}\,\bigtriangledown c_n^{(\mu)}(s,q) = \mu q^{\frac{n-1}{2}} c_{n+1}^{(\mu)}(s,q) - \left[\mu q^{n+1} - \frac{q^s-1}{q-1}\right] c_n^{(\mu)}(s,q).$$

$$\mu \bigtriangleup c_n^{(\mu)}(s,q) = \mu q^{\frac{n-1}{2}} c_{n+1}^{(\mu)}(s,q) + \left[\frac{q^{s-n}-1}{q-1} - \mu q^{n+1}\right] c_n^{(\mu)}(s,q).$$

As we already pointed out, all characteristics of these $q-$Charlier polynomials tend to the corresponding characteristics of the classical ones when q tends to 1, which doesn't happens in the previous case (lattice $x(s) = q^s$, see Table 1). So, these polynomials are more natural $q-$analogues of the classical ones, and the exponential lattice $x(s) = \frac{q^s-1}{q-1}$ is a more natural lattice than the previous one $x(s) = q^s$. Moreover, the hypergeometric representation for the q-Charlier polynomias in both lattices is the

same, so, in this sense they are the same functions in s (obviously this happens if we use the approach based on the hypergeometric series [26]). But if we consider them as functions of $x(s)$ (i.e., as polynomials in $x(s)$), we obtain two complete different families and one of them (the one on $x(s) = \frac{q^s-1}{q-1}$) seem to be a more natural $q-$analogue than the other one. For this reason it is convenient to complete the study of the $q-$analogues of classical discrete polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$. This will be done in a forthcoming paper (see [7]).

Acknowledgements

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