On the Krall-type polynomials on q-quadratic lattices

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Abstract

In this paper, we study the Krall-type polynomials on non-uniform lattices. For these polynomials the second order linear difference equation, q-basic series representation and three-term recurrence relations are obtained. In particular, the q-Racah-Krall polynomials obtained via the addition of two mass points to the weight function of the non-standard q-Racah polynomials at the ends of the interval of orthogonality are considered in detail. Some important limit cases are also discussed.

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1. Introduction

The Krall-type polynomials are polynomials which are orthogonal with respect to a linear functional $\widetilde{\mathfrak{u}}$ obtained from a quasi-definite functional $\mathfrak{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures, i.e., $\widetilde{\mathfrak{u}}$ is the linear functional

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{k=1}^{N} A_k \delta_{x_k},$$

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where $A_k \in \mathbb{R}$, $x_1, \ldots, x_k \in \mathbb{R}$ and δ_a is the delta Dirac functional at the point a, i.e., $\langle \delta_a, p \rangle = p(a)$, where $p \in \mathbb{P}$.

These kind of polynomials appear as eigenfunctions of a fourth order linear differential operator with polynomial coefficients that do not depend on the degree of the polynomials. They were firstly considered by Krall in [27] (for a more recent reviews see [8] and [26, chapter XV]). In fact, H. L. Krall discovered that there are only three extra families of orthogonal polynomials apart from the classical polynomials of Hermite, Laguerre and Jacobi that satisfy such a fourth order differential equation which are orthogonal with respect to measures that are not absolutely continuous with respect to the Lebesgue measure. Namely, the Jacobi-type polynomials that are orthogonal with respect to the *weight* function $\rho(x) = (1-x)^\alpha + M\delta(x)$, $M > 0$, $\alpha > -1$ supported on $[0, 1]$, the Legendre-type polynomials orthogonal on $[-1, 1]$ with respect to $\rho(x) = \alpha/2 + \delta(x-1)/2 + \delta(x+1)/2$, $\alpha > 0$, and the Laguerre-type polynomials that are orthogonal with respect to $\rho(x)e^{-x} + M\delta(x)$, $M > 0$ on $[0, \infty)$. This result motivated the study of the polynomials orthogonal with respect to the more general weight functions [23, 25] that could contain more instances of orthogonal polynomials being eigenfunctions of higher-order differential equations (see also [26, chapters XVI, XVII]).

In the last years the study of such polynomials have attracted an increasing interest (see e.g. [4, 8, 19, 28] and the references therein) with a special emphasis on the case when the starting functional $\mathfrak{u}$ is a classical *continuous* linear functional (this case leads to the Jacobi-Krall, Laguerre-Krall, Hermite-Krall, and Bessel-Krall polynomials, see e.g. [7, 13, 14, 18, 23, 25]) or a classical *discrete* one (this leads to the Hahn-Krall, Meixner-Krall, Kravchuk-Krall, and Charlier-Krall polynomials, see e.g. [6, 7, 15]). Moreover, in [4] a general theory was developed for modifications of quasi-definite linear functionals that covers all the continuous cases mentioned above whereas in [9] the case when $\mathfrak{u}$ is a discrete semiclassical or q-semiclassical linear functional was considered in detail. But in [9] (see also [5]) only the linear type lattices (for a discussion on the linear type lattices see [3]) were considered. Here we go further and study the Krall-type polynomials obtained by adding delta Dirac functionals to the discrete functionals $\mathfrak{u}$ defined on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$.

Notice that since this lattice is not linear, the general results of [9] may not be applied in general, and therefore an appropriate method must be developed. In fact, the main aim of the present paper is to show that the method presented in [9] can be adapted for the more general lattice. For the sake of simplicity we focus on the case when the starting functional $\mathfrak{u}$ is a q-classical family on the q-quadratic lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$. In particular, we study the modifications of the non-standard q-Racah polynomials defined on the lattice $x(s) = [s]_q[s+1]_q$,

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad s \in \mathbb{C},$$

which were introduced in [29] and studied in detail in [2, 10]. Also, in section 4.2 we discuss some important limit cases of q-Racah, namely, the dual q-Hahn polynomials on the lattice $x(s) = [s]_q[s+1]_q$, the non-standard Racah polynomials on the lattice $x(s) = s(s+1)$ [29, page 108], and the q-Hahn polynomials on the lattice $x(s) = q^{-s}$ [24, page 445].

The structure of the paper is as follows. In Section 2, some preliminary results are presented as well as the representations for the kernels on the general lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$. In section 3, the general theory of the q-Krall polynomials on the general lattice is developed, and finally, in section 4, some concrete examples are considered.

2. Preliminary results

Here we include some results from the theory of classical polynomials on the general (q-quadratic) lattice (for further details and notations see e.g. [2, 29])

$$x(s) = c_1q^s + c_2q^{-s} + c_3 = c_1(q^s + q^{-s-\zeta}) + c_3. \tag{1}$$

The orthogonal polynomials on non-uniform lattices $P_n(s)_q := P_n(x(s))$ are the polynomial solutions of the second order linear difference equation (SODE) of hypergeometric type

$$\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)} + \tau(s)\frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0,$$
$$\Delta y(s) = y(s+1) - y(s), \quad \nabla y(s) = y(s) - y(s-1),$$

or, equivalently

$$\begin{aligned} &A_s y(s+1) + B_s y(s) + C_s y(s-1) + \lambda_n y(s) = 0, \\ &A_s = \frac{\sigma(s) + \tau(s)\Delta x(s-\frac{1}{2})}{\Delta x(s)\Delta x(s-\frac{1}{2})}, \;\; C_s = \frac{\sigma(s)}{\nabla x(s)\Delta x(s-\frac{1}{2})}, \;\; B_s = -A_s - B_s, \end{aligned} \tag{2}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of degree at most 2 and exactly 1, respectively, and λ_n is a constant, that are orthogonal with respect to the linear functional $\mathfrak{u}:\mathbb{P}_q \mapsto \mathbb{C}$, where $\mathbb{P}_q$ denotes the space of polynomials on the lattice (1) (compare with [29, Eq. (3.3.4) page 71])

$$\langle \mathfrak{u}, P_n P_m \rangle = \delta_{m\,n} d_n^2, \quad \langle \mathfrak{u}, P \rangle = \sum_{s=a}^{b-1} P(s)_q \rho(s) \Delta x(s-\tfrac{1}{2}). \tag{3}$$

In the above formula ρ is the *weight* function and $d_n^2 := \langle \mathfrak{u}, P_n^2 \rangle$.

Since the polynomials $P_n(s)_q$ are orthogonal with respect to a linear functional, they satisfy a three-term recurrence relation (TTRR) [2, 16]

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad n = 0, 1, 2, ..., \tag{4}$$

with the initial conditions $P_0(s)_q = 1, P_{-1}(s)_q = 0$, and also the differentiation formulas [2, Eqs. (5.65) and (5.67)] (or [11, Eqs. (24) and (25)]

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \overline{\alpha}_n P_{n+1}(s)_q + \overline{\beta}_n(s) P_n(s)_q, \tag{5}$$

$$\Phi(s)\frac{\Delta P_n(s)_q}{\Delta x(s)} = \widehat{\alpha}_n P_{n+1}(s)_q + \widehat{\beta}_n(s) P_n(s)_q, \tag{6}$$

where $\Phi(s) = \sigma(s) + \tau(s)\Delta x(s-\frac{1}{2})$, and

$$\overline{\alpha}_n = \widehat{\alpha}_n = -\frac{\alpha_n \lambda_{2n}}{[2n]_q}, \quad \overline{\beta}_n(s) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'}, \quad \widehat{\beta}_n(s) = \overline{\beta}_n(s) - \lambda_n \Delta x(s-\tfrac{1}{2}).$$

Notice that from (6) and the TTRR (4) it follows that

$$P_{n-1}(s)_q = \Theta(s,n) P_n(s)_q + \Xi(s,n) P_n(s+1)_q, \tag{7}$$

where

$$\Theta(s,n) = \frac{\alpha_n}{\widehat{\alpha}_n \gamma_n}\left[\frac{\Phi(s)}{\Delta x(s)} - \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n) + \widehat{\beta}_n(s)\right], \ \Xi(s,n) = -\frac{\alpha_n}{\widehat{\alpha}_n \gamma_n}\frac{\Phi(s)}{\Delta x(s)}.$$

From the TTRR (4) the Christoffel-Darboux formula for the n-th reproducing kernel follows (see [16, 29])

$$\mathrm{K}_n(s_1, s_2) := \sum_{k=0}^{n} \frac{P_k(s_1)_q P_k(s_2)_q}{d_k^2} = \frac{\alpha_n}{d_n^2}\frac{P_{n+1}(s_1)_q P_n(s_2)_q - P_{n+1}(s_2)_q P_n(s_1)_q}{x(s_1) - x(s_2)}. \tag{8}$$

Let us obtain an explicit representation of the kernels for the special values when $\sigma(s_0)=0$ and $\Phi(s_0)=0$, respectively.

1. The case $\sigma(s_0)=0$. Using (5) to eliminate P_{n+1} in (8) yields

$$\mathrm{K}_n(s,s_0)=\frac{\alpha_n P_n(s_0)_q}{\overline{\alpha}_n d_n^2}\left\{\frac{\overline{\beta}_n(s_0)-\overline{\beta}_n(s)}{x(s)-x(s_0)}P_n(s)_q+\frac{\sigma(s)}{x(s)-x(s_0)}\frac{\nabla P_n(s)_q}{\nabla x(s)}\right\}. \tag{9}$$

2. The case $\Phi(s_0)=0$. In an analogous way, but now using (6) we obtain

$$\mathrm{K}_n(s,s_0)=\frac{\alpha_n P_n(s_0)_q}{\widehat{\alpha}_n d_n^2}\left\{\frac{\widehat{\beta}_n(s_0)-\widehat{\beta}_n(s)}{x(s)-x(s_0)}P_n(s)_q+\frac{\Phi(s)}{x(s)-x(s_0)}\frac{\Delta P_n(s)_q}{\Delta x(s)}\right\}. \tag{10}$$

3. Representation formula and some of their consequences

To obtain the general representation formula for the polynomials orthogonal with respect to the perturbed functional $\widetilde{\mathfrak{u}}:\mathbb{P}_q\mapsto\mathbb{C}$, $\widetilde{\mathfrak{u}}=\mathfrak{u}+\sum_{k=1}^{M}A_k\delta_{x(a_k)}$, i.e.,

$$\langle\widetilde{\mathfrak{u}},P\rangle=\langle\mathfrak{u},P\rangle+\sum_{k=1}^{M}A_kP(a_k)_q, \tag{11}$$

we can use the ideas of [9, §2.1].

Let $\widetilde{P}_n(s)_q$ be the polynomials orthogonal with respect to $\widetilde{\mathfrak{u}}$ and $P_n(s)_q$ the polynomials orthogonal with respect to $\mathfrak{u}$. We assume that $\widetilde{\mathfrak{u}}$ is quasi-definite and therefore there exists a sequence of monic polynomials $(\widetilde{P}_n)_n$ orthogonal with respect to $\widetilde{\mathfrak{u}}$. Thus we can consider the Fourier expansion

$$\widetilde{P}_n(s)_q=P_n(s)_q+\sum_{k=0}^{n-1}\lambda_{n,k}P_k(s)_q,\quad n\in\{0\}\cup\mathbb{N}. \tag{12}$$

Then, for $0\le k\le n-1$,

$$\lambda_{n,k}=\frac{\langle\mathfrak{u},\widetilde{P}_n(s)_qP_k(s)_q\rangle}{\langle\mathfrak{u},P_k^2(s)_q\rangle}=-\sum_{i=1}^{M}A_i\widetilde{P}_n(a_i)\frac{P_k(a_i)_q}{\langle\mathfrak{u},P_k^2(s)_q\rangle},$$

and the following representation formulas hold [30] (see also [22, §2.9])

$$\widetilde{P}_n(s)_q = P_n(s)_q - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i)_q \mathrm{K}_{n-1}(s,a_i), \tag{13}$$

where $\mathrm{K}_n(x,y)$ is given by (8). In the following we denote by d_n^2 the quantity $d_n^2 = \langle \mathfrak{u}, P_n^2(s)_q \rangle$, i.e., the squared norm of the polynomials $P_n(s)_q$ whereas $\widetilde{d}_n^2$ denotes the value $\widetilde{d}_n^2 = \langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(s)_q \rangle$. Furthermore,

$$\widetilde{d}_n^2 := \langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(s)_q \rangle = d_n^2 + \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i)_q P_n(a_i)_q. \tag{14}$$

We assume that the leading coefficients of $\widetilde{P}_n(s)_q$ and $P_n(s)_q$ are the same, and for the sake of simplicity we consider monic polynomials, i.e., $P_n(s)_q = x^n(s) + b_n x^{n-1}(s) + \cdots$, and $\widetilde{P}_n(s)_q = x^n(s) + \widetilde{b}_n x^{n-1}(s) + \cdots$. Then, from (12) it follows that

$$\widetilde{b}_n = b_n + \lambda_{n,n-1} = b_n - \frac{1}{d_{n-1}^2} \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i)_q P_{n-1}(a_i)_q. \tag{15}$$

Evaluating (13) at the points a_k, $k = 1, 2, \ldots, M$, we obtain the following system of M linear equations in the M unknowns $(\widetilde{P}_n(a_k))_{k=1}^{M}$

$$\widetilde{P}_n(a_k)_q = P_n(a_k)_q - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i)_q \mathrm{K}_{n-1}(a_k,a_i), \quad k = 1, 2, \ldots, M. \tag{16}$$

Therefore, in order to assure the existence and uniqueness of the solution of the above linear system (16) (and then, the existence of the system of orthogonal polynomials $(\widetilde{P}_n)_n$) the matrix of the system (16) should be nonsingular, which is stated in the following proposition.

Proposition 1. *The linear functional $\widetilde{\mathfrak{u}}$ defined in (11) is a quasi-definite linear functional if and only if the matrix of the system (16) is not singular for every $n \in \mathbb{N}$, i.e., for $n = 1, 2, 3, \ldots$ $\det \mathcal{K}_n \neq 0$, where*

$$\mathcal{K}_n = \begin{vmatrix} 1 + A_1 K_{n-1}(a_1,a_1) & A_2 K_{n-1}(a_1,a_2) & \cdots & A_M K_{n-1}(a_1,a_M) \\ A_1 K_{n-1}(a_2,a_1) & 1 + A_2 K_{n-1}(a_2,a_2) & \cdots & A_M K_{n-1}(a_2,a_M) \\ \vdots & \vdots & \ddots & \vdots \\ A_1 K_{n-1}(a_M,a_1) & A_2 K_{n-1}(a_M,a_2) & \cdots & 1 + A_M K_{n-1}(a_M,a_M) \end{vmatrix}.$$

Remark 2. *Let us point out here that for finite sequences of orthogonal polynomials $(p_n)_{n=0}^N$, as the case of q-Racah polynomials, the conditions of the above proposition should be changed by* $\det \mathcal{K}_n \neq 0$, $n = 1, 2, \dots, N$.

If we multiply (13) by $\phi(s) = \prod_{i=1}^M (x(s) - x(a_i))$, and use the Christoffel-Darboux formula (8), then we obtain the following general representation formula

$$\phi(s)\widetilde{P}_n(s)_q = A(s;n)P_n(s)_q + B(s;n)P_{n-1}(s)_q, \tag{17}$$

where $A(s;n)$ and $B(s;n)$ are polynomials in $x(s)$ of degree bounded by a number independent of n and at most M and $M-1$, respectively, given by formulas

$$\begin{aligned} A(s;n) &= \phi(s) - \frac{\alpha_{n-1}}{d_{n-1}^2}\sum_{i=1}^M A_i \widetilde{P}_n(a_i)_q P_{n-1}(a_i)_q \phi_i(s), \\ B(s;n) &= \frac{\alpha_{n-1}}{d_{n-1}^2}\sum_{i=1}^M A_i \widetilde{P}_n(a_i)_q P_n(a_i)_q \phi_i(s), \end{aligned} \tag{18}$$

where $\phi_i(s) := \frac{\phi(s)}{x(s)-x(a_i)} = \prod_{k=1, k\neq i}^M (x(s) - x(a_k))$.

From the above formula (17) and the expression (7) the following useful representation follows

$$\phi(s)\widetilde{P}_n(s)_q = a(s;n)P_n(s)_q + b(s;n)P_n(s+1)_q, \tag{19}$$

where

$$a(s;n) = A(s;n) + B(s;n)\Theta(s;n), \qquad b(s;n) = B(s;n)\Xi(s;n).$$

Since the family $(\widetilde{P}_n)_n$ is orthogonal with respect to a linear functional, they satisfy a TTRR

$$x(s)\widetilde{P}_n(s)_q = \widetilde{\alpha}_n \widetilde{P}_{n+1}(s)_q + \widetilde{\beta}_n \widetilde{P}_n(s)_q + \widetilde{\gamma}_n \widetilde{P}_{n-1}(s)_q, \qquad n \in \mathbb{N}, \tag{20}$$

with the initial conditions $\widetilde{P}_{-1}(s)_q = 0$, $\widetilde{P}_0(s)_q = 1$. The values of the coefficients can be computed as usual (see e.g. [16, 29])

$$\begin{aligned} &\widetilde{\alpha}_n = \alpha_n = 1, \quad \widetilde{\gamma}_n = \alpha_{n-1}\frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x)\rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_{n-1}^2(x)\rangle} = \gamma_n \frac{1 + \Delta_n^{A_1, A_2, \dots, A_M}}{1 + \Delta_{n-1}^{A_1, A_2, \dots, A_M}}, \\ &\widetilde{\beta}_n = \widetilde{b}_n - \widetilde{b}_{n+1} = \beta_n - \sum_{i=1}^M \left(\frac{1}{d_{n-1}^2} A_i \widetilde{P}_n(a_i)_q P_{n-1}(a_i)_q - \frac{1}{d_n^2} A_i \widetilde{P}_{n+1}(a_i)_q P_n(a_i)_q \right), \end{aligned} \tag{21}$$

being $\widetilde{b}_n$ the coefficient given in (15), and

$$\Delta_n^{A_1,A_2,\ldots,A_M} = \sum_{i=1}^{M} A_i \frac{\widetilde{P}_n(a_i)_q P_n(a_i)_q}{d_n^2}.$$

To conclude this section let us prove the following proposition that is interesting by its own right and constitutes an extension of Theorem 2.1 in [9] to the polynomials on general non-uniform lattices.

Proposition 3. *Suppose that the polynomials* $(\widetilde{P}_n)_n$ *satisfy the relation*

$$\pi(s,n)\widetilde{P}_n(s)_q = a(s,n)P_n(s)_q + b(s,n)P_n(s+1)_q, \tag{22}$$

where the polynomial P_n *is a solution of a second order linear difference equation (SODE) of the form (2). Then, the family* $(\widetilde{P}_n)_n$ *satisfies a SODE of the form*

$$\widetilde{\sigma}(s,n)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla \widetilde{P}_n(s)_q}{\nabla x(s)} + \widetilde{\tau}(s,n)\frac{\Delta \widetilde{P}_n(s)_q}{\Delta x(s)} + \widetilde{\lambda}(s,n)\widetilde{P}_n(s)_q = 0, \tag{23}$$

where $\widetilde{\sigma}$, $\widetilde{\tau}$ *and* $\widetilde{\lambda}(s,n)$ *are defined in (28).*

Proof: The proof is similar to the proof of Theorem 2.1 in [1]. First we change s by $s+1$ in (22) and use (2) to eliminate $P_n(s+2)_q$. This yields

$$u(s,n)\widetilde{P}_n(s+1)_q = c(s,n)P_n(s)_q + d(s,n)P_n(s+1)_q, \tag{24}$$

where $u(s,n) = A_{s+1}\pi(s+1,n)$, $c(s,n) = -C_{s+1}b(s+1,n)$, and $d(s,n) = A_{s+1}a(s+1,n) - b(s+1,n)(\lambda_n + B_{s+1})$. Next we change s by $s-1$ in (22) and use (2) to eliminate $P_n(s-2)_q$, thus

$$v(s,n)\widetilde{P}_n(s-1)_q = e(s,n)P_n(s)_q + f(s,n)P_n(s+1)_q, \tag{25}$$

where $v(s,n) = C_s\pi(s-1,n)$, $e(s,n) = C_s b(s-1,n) - a(s-1,n)(\lambda_n + B_s)$, and $f(s,n) = -A_s a(s-1,n)$. Then (22), (24) and (25) yield to

$$\begin{vmatrix} \pi(s,n)\widetilde{P}_n(s)_q & a(s,n) & b(s,n) \\ u(s,n)\widetilde{P}_n(s+1)_q & c(s,n) & d(s,n) \\ v(s,n)\widetilde{P}_n(s-1)_q & e(s,n) & f(s,n) \end{vmatrix} = 0. \tag{26}$$

Expanding the determinant (26) by the first column, we get

$$\widetilde{\phi}(s,n)\widetilde{P}_n(s-1)_q + \widetilde{\varphi}(s,n)\widetilde{P}_n(s)_q + \widetilde{\xi}(s,n)\widetilde{P}_n(s+1)_q = 0,$$

$$\begin{aligned}
\widetilde{\phi}(s,n) &= v(s,n)\Big[a(s,n)d(s,n) - b(s,n)c(s,n)\Big],\\
\widetilde{\varphi}(s,n) &= \pi(s,n)\Big[c(s,n)f(s,n) - d(s,n)e(s,n)\Big],\\
\widetilde{\xi}(s,n) &= u(s,n)\Big[b(s,n)e(s,n) - a(s,n)f(s,n)\Big].
\end{aligned} \tag{27}$$

Notice that formula (27) can be rewritten in the form (23) with coefficients

$$\begin{aligned}
\widetilde{\sigma}(s,n) &= \nabla x(s)\Delta x(s-\tfrac{1}{2})\widetilde{\phi}(s,n),\\
\widetilde{\tau}(s,n) &= \Delta x(s)\widetilde{\xi}(s,n) - \nabla x(s)\widetilde{\phi}(s,n),\\
\widetilde{\lambda}(s,n) &= \widetilde{\xi}(s,n) + \widetilde{\phi}(s,n) + \widetilde{\varphi}(s,n).
\end{aligned} \tag{28}$$

□

Notice that from (19) and the above proposition it follows that the modified polynomials $\widetilde{P}_n(s)_q$ satisfies a second order difference equation of type (22) where $\pi(s,n) = \phi(s)$ is independent of n.

3.1. Two particular examples

Let consider now the cases when we add two mass points. Let $\widetilde{\mathfrak{u}}$ be given by $\widetilde{\mathfrak{u}} = u + A\delta(x(s)-x(a)) + B\delta(x(s)-x(b))$, $a \neq b$. Then, the representation formula (13) yields

$$\widetilde{P}_n^{A,B}(s)_q = P_n(s)_q - A\widetilde{P}_n^{A,B}(a)_q \mathrm{K}_{n-1}(s,a) - B\widetilde{P}_n^{A,B}(b)_q \mathrm{K}_{n-1}(s,b) \tag{29}$$

and the system (16) becomes

$$\begin{aligned}
\widetilde{P}_n^{A,B}(a)_q &= P_n(a)_q - A\widetilde{P}_n^{A,B}(a)_q \mathrm{K}_{n-1}(a,a) - B\widetilde{P}_n^{A,B}(b)_q \mathrm{K}_{n-1}(a,b),\\
\widetilde{P}_n^{A,B}(b)_q &= P_n(b)_q - A\widetilde{P}_n^{A,B}(a)_q \mathrm{K}_{n-1}(b,a) - B\widetilde{P}_n^{A,B}(b)_q \mathrm{K}_{n-1}(b,b),
\end{aligned}$$

whose solution is

$$\begin{pmatrix} \widetilde{P}_n^{A,B}(a)_q \\ \widetilde{P}_n^{A,B}(b)_q \end{pmatrix} = \begin{pmatrix} 1 + A\mathrm{K}_{n-1}(a,a) & B\mathrm{K}_{n-1}(a,b) \\ A\mathrm{K}_{n-1}(b,a) & 1 + B\mathrm{K}_{n-1}(b,b) \end{pmatrix}^{-1} \begin{pmatrix} P_n(a)_q \\ P_n(b)_q \end{pmatrix}.$$

Notice that $\forall A, B > 0$ and $a \neq b$,

$$\kappa_{n-1}(a,b) := \det \begin{vmatrix} 1 + A\mathrm{K}_{n-1}(a,a) & B\mathrm{K}_{n-1}(a,b) \\ A\mathrm{K}_{n-1}(b,a) & 1 + B\mathrm{K}_{n-1}(b,b) \end{vmatrix} > 0. \tag{30}$$

Thus, from Proposition 1 the polynomials $\widetilde{P}_n^{A,B}(s)_q$ are well defined for all values $A, B > 0$. Furthermore, for the values $\widetilde{P}_n^{A,B}(a)_q$ and $\widetilde{P}_n^{A,B}(b)_q$ we obtain the expressions

$$\begin{aligned} \widetilde{P}_n^{A,B}(a)_q &= \frac{(1 + B\mathrm{K}_{n-1}(b,b))P_n(a)_q - B\mathrm{K}_{n-1}(a,b)P_n(b)_q}{\kappa_{n-1}(a,b)}, \\ \widetilde{P}_n^{A,B}(b)_q &= \frac{(1 + A\mathrm{K}_{n-1}(a,a))P_n(b)_q - A\mathrm{K}_{n-1}(b,a)P_n(a)_q}{\kappa_{n-1}(a,b)}, \end{aligned} \tag{31}$$

where $\kappa_{n-1}(a,b)$ is given in (30). For this case formula (14) becomes

$$\widetilde{d}_n^2 = \langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(x) \rangle = d_n^2 + A\widetilde{P}_n^{A,B}(a)_q P_n(a)_q + B\widetilde{P}_n^{A,B}(b)_q P_n(b)_q. \tag{32}$$

Remark 4. *If A, B are in general complex numbers then, according to Proposition 1, in order that there exists a sequence of orthogonal polynomials $(\widetilde{P}_n^{A,B}(s)_q)_n$ the condition $\kappa_{n-1}(a,b) \neq 0$, where $\kappa_{n-1}(a,b)$ is defined in (30), should be hold for all $n \in \mathbb{N}$, $A, B \in \mathbb{C}$.*

From formula (29) the representation formulas (17) and (19) follow. Moreover, using the expression (21) we obtain the following expressions for the coefficients of the TTRR (20)

$$\begin{aligned} \widetilde{\alpha}_n &= 1, \\ \widetilde{\beta}_n &= \beta_n - A\left(\frac{\widetilde{P}_n^{A,B}(a)_q P_{n-1}(a)_q}{d_{n-1}^2} - \frac{\widetilde{P}_{n+1}^{A,B}(a)_q P_n(a)_q}{d_n^2} \right) \\ &\qquad - B\left(\frac{\widetilde{P}_n^{A,B}(b)_q P_{n-1}(b)_q}{d_{n-1}^2} - \frac{\widetilde{P}_{n+1}^{A,B}(b)_q P_n(b)_q}{d_n^2} \right), \\ \widetilde{\gamma}_n &= \gamma_n \frac{1 + \Delta_n^{A,B}}{1 + \Delta_{n-1}^{A,B}}, \quad \Delta_n^{A,B} = \frac{A\widetilde{P}_n^{A,B}(a)_q P_n(a)_q}{d_n^2} + \frac{B\widetilde{P}_n^{A,B}(b)_q P_n(b)_q}{d_n^2}. \end{aligned} \tag{33}$$

Putting $B = 0$ in all the above formulas we recover the case of one mass point, namely

$$\widetilde{P}_n^A(s)_q = P_n(s)_q - A\widetilde{P}_n^A(a)_q \mathrm{K}_{n-1}(s,a), \quad \widetilde{P}_n^A(a)_q = \frac{P_n(a)_q}{1 + A\mathrm{K}_{n-1}(a,a)},$$

$$
\begin{aligned}
\widetilde{d}_n^2 &= \langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x) \rangle = d_n^2 + A \widetilde{P}_n^A(a)_q P_n(a)_q, \\
\widetilde{\alpha}_n = 1, \quad \widetilde{\beta}_n &= \beta_n - A \left(\frac{\widetilde{P}_n^A(a)_q P_{n-1}(a)_q}{d_{n-1}^2} - \frac{\widetilde{P}_{n+1}^A(a)_q P_n(a)_q}{d_n^2} \right), \\
\widetilde{\gamma}_n = \gamma_n \frac{1 + \Delta_n^A}{1 + \Delta_{n-1}^A}, \quad \Delta_n^A &= \frac{A \widetilde{P}_n^A(a)_q P_n(a)_q}{d_n^2}.
\end{aligned}
\tag{34}
$$

Notice that since in both cases we have the representation formula (19), it follows from Proposition 3 that the Krall-type polynomials on the general non-uniform lattices $\widetilde{P}_n^{A,B}(s)_q$ and $\widetilde{P}_n^A(s)_q$ considered here satisfy a SODE (23) (or (27)) whose coefficients are given by (28).

4. Examples of Krall-type polynomials on the q-quadratic lattice

In this section we present some examples of families of the Krall-type polynomials on the lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. More exactly we start from the family of non-standard q-Racah defined on the q-quadratic lattice $x(s) = [s]_q [s+1]_q$ by the following basic series (for the definition and properties of basic series see [17]) where $b - a \in \mathbb{N}$

$$
\begin{aligned}
u_n^{\alpha,\beta}(s)_q := u_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{q^{-\frac{n}{2}(2a+1)} (q^{a-b+1};q)_n (q^{\beta+1};q)_n (q^{a+b+\alpha+1};q)_n}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2n} (q^{\alpha+\beta+n+1};q)_n} \\
&\times {}_4\varphi_3 \left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \,\middle|\, q \,,\, q \right)
\end{aligned}
\tag{35}
$$

and modified their corresponding linear functional by adding two delta Dirac functionals.

We use the above non-standard family, introduced in [29] and studied in detail in [2, 11], instead of the standard q-Racah polynomials introduced by Askey and Wilson [12], $R_n(x(s); \alpha, \beta, q^{-N-1}, \delta | q)$, since, contrary to the standard Racah polynomials, that are defined on the lattice $x(s) = q^{-s} + \delta q^{-N} q^s$ (see [12] or the more recent book [24, page 422]), they are polynomials on a lattice that *does not* depend on the parameters of the polynomials. Nevertheless, let also mention that from (35) it follows that the polynomials $u_n^{\alpha,\beta}(s)_q$ are multiples of the standard q-Racah polynomials $R_n(x(s-a); q^\beta, q^\alpha, q^{a-b}, q^{a+b}|q)$, $b - a = N$ (see [11] for more details).

Table 1: Main data of the monic non-standard q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ [11]

$P_n(s)$	$u_n^{\alpha,\beta}(x(s),a,b)_q\,,\quad x(s)=[s]_q[s+1]_q\,,\quad \Delta x(s)=[2s+2]_q$
(a,b)	$[a,b-1],\quad b-a\in\mathbb{N}$
$\rho(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(\alpha+\beta+2)\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(a+b-\beta)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(\alpha+1)\widetilde{\Gamma}_q(\beta+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+1)\widetilde{\Gamma}_q(a+b+\alpha+1)}$ $-\frac{1}{2}<a\leq b-1, \alpha>-1, -1<\beta<2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q$
$\Phi(s)$	$[s+a+1]_q[b-s-1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q$
$\tau(s)$	$[\alpha+1]_q[a]_q[a-\beta]_q+[\beta+1]_q[b]_q[b+\alpha]_q-[\alpha+1]_q[\beta+1]_q-[\alpha+\beta+2]_q x(s)$
$\tau_n(s)$	$-[\alpha+\beta+2n+2]_q x(s+\frac{n}{2})+[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[\beta+\frac{n}{2}+1-a]_q[b+\alpha+\frac{n}{2}+1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[\beta+\frac{n}{2}-a]_q[b+\alpha+\frac{n}{2}]_q$
λ_n	$[n]_q[\alpha+\beta+n+1]_q$
B_n	$\dfrac{(-1)^n\widetilde{\Gamma}_q(\alpha+\beta+n+1)}{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}$
d_n^2	$\dfrac{(q;q)_n(q^{\alpha+1};q)_n(q^{\beta+1};q)_n(q^{b-a+\alpha+\beta+1};q)_n(q^{a+b+\alpha+1};q)_n(q^{a-b+1};q)_n(q^{\beta-a-b+1};q)_n}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{4n}(q^{\alpha+\beta+2};q)_{2n}(q^{\alpha+\beta+n+1};q)_n}$
β_n	$[a]_q[a+1]_q-\dfrac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$ $+\dfrac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$
γ_n	$\dfrac{[n]_q[\alpha+\beta+n]_q[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n-1]_q([\alpha+\beta+2n]_q)^2[\alpha+\beta+2n+1]_q}$
$\overline{\alpha}_n$	$-[\alpha+\beta+2n+1]_q$
$\overline{\beta}_n(s)$	$\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q}\Big\{[\alpha+\beta+2n+2]_q x\left(s+\frac{n}{2}\right)-[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[\beta+\frac{n}{2}+1-a]_q[b+\alpha+\frac{n}{2}+1]_q$ $+[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[\beta+\frac{n}{2}-a]_q[b+\alpha+\frac{n}{2}]_q\Big\}$

We also consider here the non-standard dual q-Hahn polynomials defined by [10], $b-a\in\mathbb{N}$

$$w_n^c(s)_q := w_n^c(x(s),a,b)_q = \frac{(q^{a-b+1};q)_n(q^{a+c+1};q)_n}{q^{\frac{n}{2}(2a+1)}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n}}\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{a+c+1}\end{matrix}\,\middle|\, q,q\right). \tag{36}$$

Let us point out also that the above polynomials are different from the standard dual q-Hahn introduced in [21] since they are also defined on the lattice $x(s)=[s]_q[s+1]_q$ that does not depend on the parameters of the polynomials (see also [24, page 450]). Notice also that if we put $\beta = a + c$, and take the limit $q^{\alpha} \to 0$ the non-standard q-Racah polynomials (35) becomes into (36). Therefore, we concentrate on the modifications of the non-standard q-Racah polynomials and we will obtain the properties of the Krall-type dual q-Hahn polynomials by taking appropriate limits.

Table 2: Main data of the monic dual q-Hahn polynomials $w_n^c(x(s), a, b)_q$ [2]

$P_n(s)$	$w_n^c(x(s), a, b)_q\,, \quad x(s) = [s]_q[s+1]_q, \quad \Delta x(s) = [2s+2]_q$
(a, b)	$[a, b-1], \quad b - a \in \mathbb{N}$
$\rho(s)$	$\dfrac{q^{(ab+bc-ac-a-c+b-1-s(s+1))/2}\widetilde{\Gamma}_q(b-c)\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+c+1)}{\widetilde{\Gamma}_q(a+c+1)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s-c+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(b-s)}$ $-\frac{1}{2} \leq a \leq b-1, \lvert c \rvert < a+1$
$\sigma(s)$	$q^{(s+c+a-b+2)/2}[s-a]_q[s+b]_q[s-c]_q$
$\Phi(s)$	$-q^{(c+a-b+1-s)/2}[s+a+1]_q[s-b+1]_q[s+c+1]_q$
$\tau(s)$	$q^{(a-b+c+1)/2}[a+1]_q[b-c-1]_q + q^{(c-b+1)/2}[b]_q[c]_q - x(s)$
$\tau_n(s)$	$-q^{-n-1/2}x(s+\frac{n}{2}) + q^{(c+a-b+1-\frac{n}{2})/2}[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[c+\frac{n}{2}+1]_q$ $-q^{(c+a-b+2-\frac{n}{2})/2}[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[c+\frac{n}{2}]_q$
λ_n	$q^{-(n-1)/2}[n]_q$
B_n	$(-1)^n q^{\frac{3}{4}n(n-1)}$
d_n^2	$\dfrac{q^{n(a+c-b+n+1)}\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(a+c+n+1)\widetilde{\Gamma}_q(b-c)\widetilde{\Gamma}_q(b-a)}{\widetilde{\Gamma}_q(b-c-n)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(a+c+1)}$
β_n	$q^{n+(c-b+1)/2}[b-a-n+1]_q[a+c+n+1]_q + q^{n+a+(c-b+1)/2}[n]_q[b-c-n]_q + [a]_q[a+1]_q$
γ_n	$q^{2n+c+a-b}[n]_q[n+a+c]_q[b-a-n]_q[b-c-n]_q$
$\overline{\alpha}_n$	$-q^{-n+1/2}$
$\overline{\beta}_n(s)$	$-q^{n/2+1}\Big[-q^{-n-1(2}x(s+\frac{n}{2}) + q^{(c+a-b+1-\frac{n}{2})/2}[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[c+\frac{n}{2}+1]_q$ $-q^{(c+a-b+2-\frac{n}{2})/2}[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[c+\frac{n}{2}]_q\Big]$

In the following we use the q-analog of the Γ function, $\widetilde{\Gamma}_q(x)$, introduced in [29, Eq. (3.2.24)], and related to the classical q-Gamma function, Γ_q, (see [24]) by formula

$$\widetilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}}(1-q)^{1-s}\frac{(q;q)_\infty}{(q^s;q)_\infty}, \quad 0 < q < 1,$$

as well as the identities ($k \in \mathbb{N}$)

$$\frac{\widetilde{\Gamma}_q(a+k)}{\widetilde{\Gamma}_q(a)} = \prod_{m=0}^{k-1}[a+m]_q = (-1)^k (q^a;q)_k (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-k} q^{-\frac{k}{4}(k-1)-\frac{ka}{2}},$$

$$\frac{\widetilde{\Gamma}_q(a-k)}{\widetilde{\Gamma}_q(a)} = \prod_{m=0}^{k-1}\frac{-1}{[-a+1+m]_q} = \frac{1}{(q^{-a+1};q)_k (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-k} q^{-\frac{k}{4}(k-1)-\frac{k(-a+1)}{2}}}.$$

Notice that from (35) it follows

$$u_n^{\alpha,\beta}(a)_q = \frac{(q^{a-b+1};q)_n (q^{\beta+1};q)_n (q^{a+b+\alpha+1};q)_n}{q^{\frac{n}{2}(2a+1)} (q^{\alpha+\beta+n+1};q)_n (q^{1/2}-q^{-1/2})^{2n}},$$

$$u_n^{\alpha,\beta}(b-1)_q = \frac{(q^{a-b+1};q)_n (q^{\alpha+1};q)_n (q^{\beta-a-b+1};q)_n}{q^{\frac{n}{2}(-2b+1)} (q^{\alpha+\beta+n+1};q)_n (q^{1/2}-q^{-1/2})^{2n}}.$$

We also need the following identity for the non-standard q-Racah polynomials (see (7) from above)

$$u_{n-1}^{\alpha,\beta}(s)_q = \Theta(s,n) u_n^{\alpha,\beta}(s)_q + \Xi(s,n) u_n^{\alpha,\beta}(s+1)_q, \tag{37}$$

where

$$\Theta(s,n) = -\frac{[\alpha+\beta+2n-1]_q ([\alpha+\beta+2n]_q)^2 ([\alpha+n]_q[\beta+n]_q[\alpha+\beta+n]_q)^{-1}}{[n]_q[a+b+\alpha+n]_q[a+b-\beta-n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q} \times$$
$$\Bigg\{\frac{[s+a+1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q[b-s-1]_q}{[2s+2]_q} - [\alpha+\beta+2n+1]_q[s-a]_q[s+a+1]_q$$
$$-\frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+2]_q}$$
$$+\frac{[n]_q[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q}{[\alpha+\beta+2n]_q} - \frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q}$$
$$\times\Big[-[\alpha+\beta+2n+2]_q[s+\tfrac{n}{2}]_q[s+\tfrac{n}{2}+1]_q + [a+\tfrac{n}{2}+1]_q[b-\tfrac{n}{2}-1]_q[\beta+\tfrac{n}{2}+1-a]_q$$
$$\times[b+\alpha+\tfrac{n}{2}+1]_q - [a+\tfrac{n}{2}]_q[b-\tfrac{n}{2}]_q[\beta+\tfrac{n}{2}-a]_q[b+\alpha+\tfrac{n}{2}]_q\Big] - [n]_q[\alpha+\beta+n+1]_q[2s+1]_q\Bigg\},$$

$$\Xi(s,n) = \frac{[\alpha+\beta+2n-1]_q([\alpha+\beta+2n]_q)^2}{[n]_q[\alpha+\beta+n]_q[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q} \times$$
$$\frac{[s+a+1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q[b-s-1]_q}{[b-a-n]_q[2s+2]_q[b-a+\alpha+\beta+n]_q}.$$

For the dual q-Hahn polynomials (36), we have

$$w_n^c(a)_q = \frac{(q^{a-b+1};q)_n (q^{a+c+1};q)_n}{q^{\frac{n}{2}(2a+1)}(q^{1/2}-q^{-1/2})^{2n}}, \quad w_n^c(b-1)_q = \frac{(q^{a-b+1};q)_n (q^{c-b+1};q)_n}{q^{\frac{n}{2}(-2b+1)}(q^{1/2}-q^{-1/2})^{2n}},$$

as well as the relation (7)

$$w_{n-1}^c(s)_q = \Theta(s,n) w_n^c(s)_q + \Xi(s,n) w_n^c(s+1)_q,$$

where

$$\Theta(s,n) = -\frac{q^{b-a-c-n-\frac{1}{2}}}{[n]_q[b-c-n]_q[a+c+n]_q[b-a-n]_q} \times$$
$$\Bigg\{ -\frac{q^{(a+c-b+1-s)/2}[s+a+1]_q[s+c+1]_q[s-b+1]_q}{[2s+2]_q} - q^{-(2n-1)/2}[s-a]_q[s+a+1]_q$$
$$+ q^{(c-b+2)/2}[b-a-n+1]_q[a+c+n+1]_q + q^{a+(c-b+2)/2}[n]_q[b-c-n]_q$$
$$+ q^{\frac{n}{2}+1}\Big(q^{-n-\frac{1}{2}}[s+\tfrac{n}{2}]_q[s+\tfrac{n}{2}+1]_q - q^{\frac{n}{2}(c+a-b+1-\frac{n}{2})}[a+\tfrac{n}{2}+1]_q[b-\tfrac{n}{2}-1]_q[c+\tfrac{n}{2}+1]_q$$
$$+ q^{\frac{1}{2}(c+a-b+2-\frac{n}{2})}[a+\tfrac{n}{2}]_q[b-\tfrac{n}{2}]_q[c+\tfrac{n}{2}]_q\Big) - q^{-\frac{n-1}{2}}[n]_q[2s+1]_q \Bigg\},$$
$$\Xi(s,n) = -\frac{q^{(b-a-c-2n-s)/2}}{[n]_q[b-c-n]_q[a+c+n]_q[b-a-n]_q}\frac{[s+a+1]_q[s+c+1]_q[s-b+1]_q}{[2s+2]_q}.$$

4.1. Modification of non-standard q-Racah polynomials

In this section we consider the modification of the non-standard q-Racah polynomials defined in (35) by adding two mass points, i.e., the polynomials orthogonal with respect to the functional $\widetilde{\mathfrak{u}} = u + A\delta(x(s)-x(a)) + B\delta(x(s)-x(b-1))$, where $\mathfrak{u}$ is defined in (3). In other words, we study the polynomials $u_n^{\alpha,\beta,A,B}(s)_q := u_n^{\alpha,\beta,A,B}(x(s),a,b)_q$ that satisfy the following orthogonality relation

$$\sum_{s=a}^{b-1} u_n^{\alpha,\beta,A,B}(s)_q u_m^{\alpha,\beta,A,B}(s)_q \rho(s)[2s+1]_q + A u_n^{\alpha,\beta,A,B}(a)_q u_m^{\alpha,\beta,A,B}(a)_q$$
$$+ B u_n^{\alpha,\beta,A,B}(b-1)_q u_m^{\alpha,\beta,A,B}(b-1)_q = \delta_{n,m}\widetilde{d}_n^2, \tag{38}$$

where ρ is the non-standard q-Racah weight function (see table 1[1]).

From (31) and (32) we obtain the following expressions for the the values at the points $s=a$ and $s=b-1$ and the norm $\widetilde{d}_n^2$ of the modified polynomials

[1]We have chosen $\rho(s)$ in such a way that $\sum_{s=a}^{b-1}\rho(s)[2s+1]_q = 1$, i.e., to be a probability measure.

$u_n^{\alpha,\beta,A,B}(s)_q$, respectively

$$\begin{aligned} u_n^{\alpha,\beta,A,B}(a)_q &= \frac{(1+B\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,b-1))u_n^{\alpha,\beta}(a)_q - B\mathrm{K}_{n-1}^{\alpha,\beta}(a,b-1)u_n^{\alpha,\beta}(b-1)_q}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \\ u_n^{\alpha,\beta,A,B}(b-1)_q &= \frac{-A\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,a)u_n^{\alpha,\beta}(a)_q + (1+A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a))u_n^{\alpha,\beta}(b-1)_q}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \end{aligned} \tag{39}$$

$$\begin{aligned} \widetilde{d}_n^2 = d_n^2 + &\frac{A(u_n^{\alpha,\beta}(a)_q)^2\{1+B\mathrm{K}_{n-1}^{\alpha,\beta}(b-1,b-1)\} + B(u_n^{\alpha,\beta}(b-1)_q)^2\{1+A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a)\}}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)} \\ &- \frac{2ABu_n^{\alpha,\beta}(a)_q u_n^{\alpha,\beta}(b-1)_q \mathrm{K}_{n-1}^{\alpha,\beta}(a,b-1)}{\kappa_{n-1}^{\alpha,\beta}(a,b-1)}, \end{aligned}$$

where we use a notation similar to the one introduced in the previous section,

$$\begin{aligned} \kappa_m^{\alpha,\beta}(s,t) = 1 &+ A\mathrm{K}_m^{\alpha,\beta}(s,s) + B\mathrm{K}_m^{\alpha,\beta}(t,t) \\ &+ AB\left\{\mathrm{K}_m^{\alpha,\beta}(s,s)\mathrm{K}_m^{\alpha,\beta}(t,t) - (\mathrm{K}_m^{\alpha,\beta}(s,t))^2\right\}, \end{aligned} \tag{40}$$

where $\mathrm{K}_m^{\alpha,\beta}(s,t)$ are the kernels $\mathrm{K}_m^{\alpha,\beta}(s,t) = \sum_{k=0}^m u_k^{\alpha,\beta}(s)_q u_k^{\alpha,\beta}(t)_q / d_k^2$, and d_n^2 denotes the squared norm of the n-th non-standard q-Racah polynomials (see table 1).

Representation formulas for $u_n^{\alpha,\beta,A,B}(s)_q$

To obtain the representation formulas we use (29) that yields

$$\begin{aligned} u_n^{\alpha,\beta,A,B}(s)_q = u_n^{\alpha,\beta}(s)_q &- Au_n^{\alpha,\beta,A,B}(a)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,a) \\ &- Bu_n^{\alpha,\beta,A,B}(b-1)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,b-1). \end{aligned} \tag{41}$$

Next we use the expressions (9) and (10) for the kernels. In fact, using the main data of non-standard q-Racah polynomial [11] (see Table 1), we find

$$\mathrm{K}_{n-1}^{\alpha,\beta}(s,a) = \varkappa_a^{\alpha,\beta}(s,n)u_{n-1}^{\alpha,\beta}(s)_q + \overline{\varkappa}_a^{\alpha,\beta}(s,n)\frac{\nabla u_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)}, \tag{42}$$

where

$$\begin{aligned} \varkappa_a^{\alpha,\beta}(s,n) = &\frac{-q^{(\alpha+\beta-2an+n+2a)/2}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-1}(q^{\alpha+\beta+2};q)_{2n-3}}{(q;q)_{n-1}(q^{\alpha+1};q)_{n-1}(q^{b-a+\alpha+\beta+1};q)_{n-1}(q^{\beta-a-b+1};q)_{n-1}} \\ &\times \frac{[\alpha+\beta+n]_q[s+a+n]_q}{[s+a+1]_q}, \end{aligned} \tag{43}$$

$$\overline{\varkappa}_a^{\alpha,\beta}(s,n) = \frac{q^{(\alpha+\beta-2an+n+2a)/2}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-1}(q^{\alpha+\beta+2};q)_{2n-3}}{(q;q)_{n-1}(q^{\alpha+1};q)_{n-1}(q^{b-a+\alpha+\beta+1};q)_{n-1}(q^{\beta-a-b+1};q)_{n-1}} \times \frac{[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q}{[s+a+1]_q}, \tag{44}$$

$$\mathrm{K}_{n-1}^{\alpha,\beta}(s,b-1) = \varkappa_b^{\alpha,\beta}(s,n)u_{n-1}^{\alpha,\beta}(s)_q + \overline{\varkappa}_b^{\alpha,\beta}(s,n)\frac{\Delta u_{n-1}^{\alpha,\beta}(s)_q}{\Delta x(s)}, \tag{45}$$

where

$$\varkappa_b^{\alpha,\beta}(s,n) = -\frac{q^{(\alpha+\beta+2bn+n-2b)/2}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-1}(q^{\alpha+\beta+2};q)_{2n-3}}{(q;q)_{n-1}(q^{\beta+1};q)_{n-1}(q^{b-a+\alpha+\beta+1};q)_{n-1}(q^{a+b+\alpha+1};q)_{n-1}} \times \frac{[\alpha+\beta+n]_q}{[s+b]_q}\left\{[s+b+n-1]_q - [n-1]_q\left(q^{(s+b)/2}+q^{-(s+b)/2}\right)\right\},$$

$$\overline{\varkappa}_b^{\alpha,\beta}(s,n) = -\frac{q^{(\alpha+\beta+2bn+n-2b)/2}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-1}(q^{\alpha+\beta+2};q)_{2n-3}}{(q;q)_{n-1}(q^{\beta+1};q)_{n-1}(q^{b-a+\alpha+\beta+1};q)_{n-1}(q^{a+b+\alpha+1};q)_{n-1}} \times \frac{[s+a+1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q}{[s+b]_q}.$$

Substituting (42) and (45) in formula (41) one finds

$$u_n^{\alpha,\beta,A,B}(s)_q = u_n^{\alpha,\beta}(s)_q + \overline{A}(s,n)u_{n-1}^{\alpha,\beta}(s)_q + \overline{B}(s,n)\frac{\nabla u_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)} + \overline{C}(s,n)\frac{\Delta u_{n-1}^{\alpha,\beta}(s)_q}{\Delta x(s)}, \tag{46}$$

$$\begin{aligned} \overline{A}(s,n) &= -Au_n^{\alpha,\beta,A,B}(a)_q\varkappa_a^{\alpha,\beta}(s,n) - Bu_n^{\alpha,\beta,A,B}(b-1)_q\varkappa_b^{\alpha,\beta}(s,n), \\ \overline{B}(s,n) &= -Au_n^{\alpha,\beta,A,B}(a)_q\overline{\varkappa}_a^{\alpha,\beta}(s,n), \\ \overline{C}(s,n) &= -Bu_n^{\alpha,\beta,A,B}(b-1)_q\overline{\varkappa}_b^{\alpha,\beta}(s,n) \end{aligned} \tag{47}$$

where $u_n^{\alpha,\beta,A,B}(a)_q$ and $u_n^{\alpha,\beta,A,B}(b-1)_q$ are given in (39). Notice that from formula (46) it is not easy to see that $u_n^{\alpha,\beta,A,B}(s)_q$ is a polynomial of degree n in $x(s)$ (which is a simple consequence of (41)). This is because in (46) the involved functions $\overline{A}$, $\overline{B}$ and $\overline{C}$ as well as $\nabla u_{n-1}^{\alpha,\beta}(s)_q/\nabla x(s)$ and $\Delta u_{n-1}^{\alpha,\beta}(s)_q/\Delta x(s)$ are not, in general, polynomials in $x(s)$.

Another representation follows from (17)

$$\phi(s)u_n^{\alpha,\beta,A,B}(s)_q = A(s;n)u_n^{\alpha,\beta}(s)_q + B(s;n)u_{n-1}^{\alpha,\beta}(s)_q, \tag{48}$$

$$\phi(s) = [s-a]_q[s+a+1]_q[s-b+1]_q[s+b]_q,$$

$$\begin{aligned} A(s,n) &= \phi(s) - \frac{1}{d_{n-1}^2}\Big\{Au_n^{\alpha,\beta,A,B}(a)_q u_{n-1}^{\alpha,\beta}(a)_q[s-b+1]_q[s+b]_q \\ &\qquad + Bu_n^{\alpha,\beta,A,B}(b-1)_q u_{n-1}^{\alpha,\beta}(b-1)_q[s-a]_q[s+a+1]_q\Big\}, \\ B(s,n) &= \frac{1}{d_{n-1}^2}\Big\{Au_n^{\alpha,\beta,A,B}(a)_q u_n^{\alpha,\beta}(a)_q[s-b+1]_q[s+b]_q \\ &\qquad + Bu_n^{\alpha,\beta,A,B}(b-1)_q u_n^{\alpha,\beta}(b-1)_q[s-a]_q[s+a+1]_q\Big\}, \end{aligned} \tag{49}$$

where $u_n^{\alpha,\beta,A,B}(a)_q$ and $u_n^{\alpha,\beta,A,B}(b-1)_q$ are given in (39). Substituting the relation (37) in (48) we obtain the following representation formula

$$\phi(s)u_n^{\alpha,\beta,A,B}(s)_q = a(s;n)u_n^{\alpha,\beta}(s)_q + b(s;n)u_n^{\alpha,\beta}(s+1)_q,$$

where, as in (19), $a(s;n) = A(s;n) + B(s;n)\Theta(s;n)$, $b(s;n) = B(s;n)\Xi(s;n)$, and A, B and Θ, Ξ are given by (49) and (37), respectively.

Therefore, by Proposition 3 one obtains the second order linear difference equation for the $u_n^{\alpha,\beta,A,B}(s)_q$ polynomials where the coefficients are given in (28).

Finally, using formulas (33) we obtain

$$\begin{aligned} \widetilde{\alpha}_n &= 1, \\ \widetilde{\beta}_n &= \beta_n - A\left(\frac{u_n^{\alpha,\beta,A,B}(a)_q u_{n-1}^{\alpha,\beta}(a)_q}{d_{n-1}^2} - \frac{u_{n+1}^{\alpha,\beta,A,B}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2}\right) \\ &\quad - B\left(\frac{u_n^{\alpha,\beta,A,B}(b-1)_q u_{n-1}^{\alpha,\beta}(b-1)_q}{d_{n-1}^2} - \frac{u_{n+1}^{\alpha,\beta,A,B}(b-1)_q u_n^{\alpha,\beta}(b-1)_q}{d_n^2}\right), \\ \widetilde{\gamma}_n &= \gamma_n \frac{1+\Delta_n^{A,B}}{1+\Delta_{n-1}^{A,B}},\ \Delta_n^{A,B} = \frac{Au_n^{\alpha,\beta,A,B}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2} + \frac{Bu_n^{\alpha,\beta,A,B}(b-1)_q u_n^{\alpha,\beta}(b-1)_q}{d_n^2} \end{aligned}$$

where we use the notations defined in (39) and (40).

Representation of $u_n^{\alpha,\beta,A,B}(s)_q$ in terms of basic series

In this section, we obtain some explicit formulas for the $u_n^{\alpha,\beta,A,B}(s,a,b)_q$ polynomials in terms of basic hypergeometric series. To obtain the first representation formula in terms of basic series of $u_n^{\alpha,\beta,A,B}(s,a,b)_q$, we rewrite (46) by using the identity [11]

$$\frac{\Delta u_n^{\alpha,\beta}(s,a,b)_q}{\Delta x(s)} = [n]_q u_{n-1}^{\alpha+1,\beta+1}(x(s+\tfrac{1}{2}), a+\tfrac{1}{2}, b-\tfrac{1}{2})_q, \tag{50}$$

and substitute (35) into the resulting expression. This yields

$$\begin{aligned} u_n^{\alpha,\beta,A,B}(s,a,b)_q =& \frac{(q^{a-b+2};q)_{n-2}(q^{\beta+2};q)_{n-2}(q^{a+b+\alpha+2};q)_{n-2}}{q^{\frac{n}{2}(2a+1)}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-4}(q^{\alpha+\beta+n+1};q)_{n-2}} \times \\ & \sum_{k=0}^{\infty} \frac{(q^{-n}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1};q)_k}{(q^{a-b+2}, q^{\beta+2}, q^{a+b+\alpha+2}, q;q)_k} q^k \Pi_4(q^k), \end{aligned}$$

where

$$\begin{aligned} & \tfrac{(1-q^{a-b+n})(1-q^{a+b+\alpha+n})(1-q^{\alpha+\beta+n+k})(1-q^{a-b+k+1})(1-q^{\beta+k+1})(1-q^{a+b+\alpha+k+1})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^4(1-q^{\beta+n})^{-1}(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+n})} \\ & +\overline{A}(s,n)\tfrac{q^{(2a+1)/2}(1-q^{-n+k})(1-q^{a-b+k+1})(1-q^{\beta+k+1})(1-q^{a+b+\alpha+k+1})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(1-q^{\alpha+\beta+n})(1-q^{-n})} \\ & +\overline{B}(s,n)\tfrac{q^{(2a+2-\frac{n}{2})}[n-1]_q(1-q^{-n+k})(1-q^{-n+k+1})(1-q^{\alpha+\beta+n+k})(1-q^{a-s+k})}{(1-q^{\alpha+\beta+n})(1-q^{-n})(1-q^{-n+1})(1-q^{a-s})} \\ & +\overline{C}(s,n)\tfrac{q^{(2a+2-\frac{n}{2})}(1-q^{-n+k})(1-q^{-n+k+1})(1-q^{\alpha+\beta+n+k})(1-q^{a+s+k+1})}{[n-1]_q^{-1}(1-q^{\alpha+\beta+n})(1-q^{-n})(1-q^{-n+1})(1-q^{a+s+1})} := \Pi_4(q^k), \end{aligned}$$

is a fourth degree polynomial in q^k and $\overline{A}(s,n)$, $\overline{B}(s,n)$ and $\overline{C}(s,n)$ are defined in (47). After some straightforward calculations, we have

$$\Pi_4(q^k) = \Upsilon_n^{\alpha,\beta,A,B}(s)(q^k-q^{\alpha_1})(q^k-q^{\alpha_2})(q^k-q^{\alpha_3})(q^k-q^{\alpha_4}),$$

where

$$\begin{aligned} \Upsilon_n^{\alpha,\beta,A,B}(s) = & \tfrac{q^{2a+2\alpha+2\beta+n+3}(1-q^{a-b+n})(1-q^{\beta+n})(1-q^{a+b+\alpha+n})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^4(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+n})} \\ & +\overline{A}(s,n)\tfrac{q^{(3a+\alpha+\beta-n+\frac{7}{2})}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(1-q^{\alpha+\beta+n})(1-q^{-n})} \\ & +\overline{B}(s,n)\tfrac{q^{(3a+\alpha+\beta-s-n+\frac{5}{2})}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(1-q^{\alpha+\beta+n})(1-q^{-n})(1-q^{a-s})} \\ & +\overline{C}(s,n)\tfrac{q^{(3a+\alpha+\beta+s-n+\frac{7}{2})}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(1-q^{\alpha+\beta+n})(1-q^{-n})(1-q^{a+s+1})} \end{aligned}$$

and q^{α_1}, q^{α_2}, q^{α_3} and q^{α_4} ($\alpha_{1,2,3,4} := \alpha_{1,2,3,4}(n, s; a, b, A, B)$) are the zeros of Π_4, that depend, in general, on s and n. Then, using the identity $(q^k - q^z)(q^{-z}; q)_k = (1 - q^z)(q^{1-z}; q)_k$ we find the following expression

$$u_n^{\alpha,\beta,A,B}(s)_q = D_n^{\alpha,\beta,\alpha_{1,2,3,4}}(s)\times \\ {}_8\varphi_7\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1}, q^{1-\alpha_1}, q^{1-\alpha_2}, q^{1-\alpha_3}, q^{1-\alpha_4} \\ q^{a-b+2}, q^{\beta+2}, q^{a+b+\alpha+2}, q^{-\alpha_1}, q^{-\alpha_2}, q^{-\alpha_3}, q^{-\alpha_4} \end{matrix} \,\middle|\, q\,,\,q \right), \tag{51}$$

where

$$D_n^{\alpha,\beta,\alpha_{1,2,3,4}}(s) = \frac{(q^{a-b+2}; q)_{n-2}(q^{\beta+2}; q)_{n-2}(q^{a+b+\alpha+2}; q)_{n-2}}{q^{\frac{n}{2}(2a+1)}(q^{\frac12} - q^{-\frac12})^{2n-4}(q^{\alpha+\beta+n+1}; q)_{n-2}} \Upsilon_n^{\alpha,\beta,A,B}(s) \prod_{l=1}^{4}(1-q^{\alpha_l}).$$

Notice that from (46) and (50) we can write the polynomial $u_n^{\alpha,\beta,A,B}(s)_q$ as a linear combination of four basic series, namely

$$u_n^{\alpha,\beta,A,B}(s,a,b)_q = D_n^{\alpha,\beta,a,b}\,{}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \,\middle|\, q\,,\,q \right)$$
$$+ \overline{A}(s,n) D_{n-1}^{\alpha,\beta,a,b}\,{}_4\varphi_3\left(\begin{matrix} q^{-n+1}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \,\middle|\, q\,,\,q \right)$$
$$+ \overline{B}(s,n)[n-1]_q D_{n-2}^{\alpha+1,\beta+1,a+\frac12,b-\frac12}\,{}_4\varphi_3\left(\begin{matrix} q^{-n+2}, q^{\alpha+\beta+n+1}, q^{a-s+1}, q^{a+s+1} \\ q^{a-b+2}, q^{\beta+2}, q^{a+b+\alpha+2} \end{matrix} \,\middle|\, q\,,\,q \right)$$
$$+ \overline{C}(s,n)[n-1]_q D_{n-2}^{\alpha+1,\beta+1,a+\frac12,b-\frac12}\,{}_4\varphi_3\left(\begin{matrix} q^{-n+2}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+2} \\ q^{a-b+2}, q^{\beta+2}, q^{a+b+\alpha+2} \end{matrix} \,\middle|\, q\,,\,q \right),$$

where $\quad D_n^{\alpha,\beta,a,b} = \dfrac{q^{-\frac{n}{2}(2a+1)}(q^{a-b+1}; q)_n(q^{\beta+1}; q)_n(q^{a+b+\alpha+1}; q)_n}{(q^{\frac12} - q^{-\frac12})^{2n}(q^{\alpha+\beta+n+1}; q)_n}$.

Remark 5. *Notice that from (41) it is easy to see that $u_n^{\alpha,\beta,A,B}(s)_q$ is a polynomial of degree n in $x(s)$, whereas from (51) it is not. This is beacuse $D_n^{\alpha,\beta,\alpha_{1,2,3,4}}(s)$ and the parameters q^{α_1}, q^{α_2}, q^{α_3}, and q^{α_4}, that appear in the formula (51) depend, in general, on s. A similar situation happens with the representation as a sum of four basic series.*

Let us obtain a more convenient representation in terms of the basic series. For doing that we use (48). In fact, substituting (35) into (48) we obtain

$$\phi(s)u_n^{\alpha,\beta,A,B}(s)_q = \frac{(q^{a-b+1}; q)_{n-1}(q^{\beta+1}; q)_{n-1}(q^{a+b+\alpha+1}; q)_{n-1}}{q^{\frac{n}{2}(2a+1)}(q^{\frac12} - q^{-\frac12})^{2n-2}(q^{\alpha+\beta+n}; q)_{n-1}} \times \\ \sum_{k=0}^{\infty} \frac{(q^{-n}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1}; q)_k}{(q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1}, q; q)_k} q^k \Pi_1(q^k),$$

where $\phi(s)$, $A(s,n)$ and $B(s,n)$ are given in (49) and

$$\begin{aligned}\Pi_1(q^k) &= A(s,n)\frac{(1-q^{a-b+n})(1-q^{\beta+n})(1-q^{a+b+\alpha+n})(1-q^{\alpha+\beta+n+k})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n})}\\ &\quad + B(s,n)\frac{q^{(2a+1)/2}(1-q^{-n+k})}{(1-q^{-n})}\\ &= -\frac{q^{(2a+1)/2}}{1-q^{-n}}\Big\{A(s,n)q^{\alpha+\beta+n}\vartheta_n^{a,b,\alpha,\beta}+B(s,n)q^{-n}\Big\}(q^k-q^{\beta_1}),\end{aligned} \tag{52}$$

being

$$\begin{aligned}q^{\beta_1} &= \frac{A(s,n)\vartheta_n^{a,b,\alpha,\beta}+B(s,n)}{A(s,n)q^{\alpha+\beta+n}\vartheta_n^{a,b,\alpha,\beta}+B(s,n)q^{-n}},\\ \vartheta_n^{a,b,\alpha,\beta} &= \frac{q^{-(2a+1)/2}(1-q^{a-b+n})(1-q^{\beta+n})(1-q^{a+b+\alpha+n})(1-q^{-n})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n})}.\end{aligned}$$

If we now use the same identity as before $(q^k-q^z)(q^{-z};q)_k=(1-q^z)(q^{1-z};q)_k$ we obtain

$$\phi(s)u_n^{\alpha,\beta,A,B}(s)_q = D_n^{\alpha,\beta,\beta_1}(s)\,{}_5\varphi_4\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1}, q^{1-\beta_1}\\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1}, q^{-\beta_1}\end{array}\,\middle|\, q,q\right) \tag{53}$$

where

$$\begin{aligned}D_n^{\alpha,\beta,\beta_1}(s) &= -\frac{(q^{a-b+1};q)_{n-1}(q^{\beta+1};q)_{n-1}(q^{a+b+\alpha+1};q)_{n-1}}{q^{\frac{n}{2}(2a+1)}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2n-2}(q^{\alpha+\beta+n};q)_{n-1}}(1-q^{\beta_1})\\ &\quad\times\frac{q^{(2a+1)/2}}{1-q^{-n}}\Big\{A(s,n)q^{\alpha+\beta+n}\vartheta_n^{a,b,\alpha,\beta}+B(s,n)q^{-n}\Big\}.\end{aligned}$$

Remark 6. *Notice that the left hand side of (53) $\phi(s)u_n^{\alpha,\beta,A,B}(s)_q$ is a polynomial of degree $n+2$ in $x(s)$ (this follows from (48) and (18)). To see that formula (53) gives a polynomial of degree $n+2$ it is sufficient to notice that the function Π_1 defined in (52) is a polynomial in $x(s)$, which follows from that fact that $A(s,n)$ and $B(s,n)$ are polynomial of degree 2 and 1 in $x(s)$, respectively (see (18)).*

It follows from the above remark that, contrary to the formula (51), the representation (53) is a very convenient way of writing the polynomials

$u_n^{\alpha,\beta,A,B}(s)_q$. Notice also that the direct substitution of (35) into (48) leads to the following representation formula

$$\phi(s)u_n^{\alpha,\beta,A,B}(s,a,b)_q = A(s,n)\Lambda_n^{\alpha,\beta,a,b}{}_4\varphi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{array}\Bigg| \, q\,,\, q\right)$$
$$+ B(s,n)\Lambda_{n-1}^{\alpha,\beta,a,b}{}_4\varphi_3\left(\begin{array}{c} q^{-n+1}, q^{\alpha+\beta+n}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{array}\Bigg| \, q\,,\, q\right),$$

where

$$\Lambda_n^{\alpha,\beta,a,b} = \frac{q^{-\frac{n}{2}(2a+1)}(q^{a-b+1};q)_n(q^{\beta+1};q)_n(q^{a+b+\alpha+1};q)_n}{(q^{\frac12}-q^{-\frac12})^{2n}(q^{\alpha+\beta+n+1};q)_n}.$$

4.1.1. The case of one mass point

In this section we consider the case of non-standard q-Racah polynomials but with one mass point at the value $s=a$. All the formulas follow from the ones in section 4.1 just putting $B=0$, then we include only the final results.

First of all, we have the representation formula (46)

$$u_n^{\alpha,\beta,A}(s)_q \quad = u_n^{\alpha,\beta}(s)_q + \overline{A}(s,n)u_{n-1}^{\alpha,\beta}(s)_q + \overline{B}(s,n)\frac{\nabla u_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)},$$

where now

$$\overline{A}(s,n) = -Au_n^{\alpha,\beta,A}(a)_q\varkappa_a^{\alpha,\beta}(s,n), \quad \overline{B}(s,n) = -Au_n^{\alpha,\beta,A}(a)_q\overline{\varkappa}_a^{\alpha,\beta}(s,n),$$

and

$$u_n^{\alpha,\beta,A}(a)_q = \frac{u_n^{\alpha,\beta}(a)_q}{1+A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a)}. \tag{54}$$

Here the values $\varkappa_a^{\alpha,\beta}(s,n)$, $\overline{\varkappa}_a^{\alpha,\beta}(s,n)$ are as in (43) and (44), respectively. The representation formula (17) takes the form

$$\phi(s)u_n^{\alpha,\beta,A}(s)_q = A(s;n)u_n^{\alpha,\beta}(s)_q + B(s;n)u_{n-1}^{\alpha,\beta}(s)_q,$$

where $\phi(s)=[s-a]_q[s+a+1]_q$,

$$\begin{aligned} A(s,n) &= \phi(s) - \frac{A}{d_{n-1}^2}u_n^{\alpha,\beta,A}(a)_q u_{n-1}^{\alpha,\beta}(a)_q, \\ B(s,n) &= \frac{A}{d_{n-1}^2}u_n^{\alpha,\beta,A}(a)_q u_n^{\alpha,\beta}(a)_q, \end{aligned} \tag{55}$$

and $u_n^{\alpha,\beta,A}(a)_q$ is given in (54). Finally, as in the previous case, we obtain the third representation formula

$$\phi(s)u_n^{\alpha,\beta,A}(s)_q = a(s;n)u_n^{\alpha,\beta}(s)_q + b(s;n)u_n^{\alpha,\beta}(s+1)_q, \tag{56}$$

where

$$a(s;n) = A(s;n) + B(s;n)\Theta(s;n), \qquad b(s;n) = B(s;n)\Xi(s;n),$$

being A, B and Θ, Ξ given by (55) and (37), respectively. Notice that, as for the two mass point cases, from the above representation formula (56) the SODE (23) follows.

The coefficients of the TTRR in this case are given by (34)

$$\widetilde{\alpha}_n = 1, \quad \widetilde{\beta}_n = \beta_n - A\left(\frac{u_n^{\alpha,\beta,A}(a)_q u_{n-1}^{\alpha,\beta}(a)_q}{d_{n-1}^2} - \frac{u_{n+1}^{\alpha,\beta,A}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2}\right),$$
$$\widetilde{\gamma}_n = \gamma_n \frac{1+\Delta_n^A}{1+\Delta_{n-1}^A}, \quad \Delta_n^A = \frac{A u_n^{\alpha,\beta,A}(a)_q u_n^{\alpha,\beta}(a)_q}{d_n^2},$$

and the norm is given by

$$\widetilde{d}_n^2 = d_n^2 + \frac{A(u_n^{\alpha,\beta}(a)_q)^2}{1 + A\mathrm{K}_{n-1}^{\alpha,\beta}(a,a)},$$

where d_n is the norm of the non-standard q-Racah polynomials $u_n^{\alpha,\beta}(s)_q$.

Finally, let us mention that putting $B = 0$ in the basic series representations formulas (51) and (53) we obtain the corresponding basic series representations for the q-Racah-Krall polynomials $u_n^{\alpha,\beta,A}(s)_q$.

4.2. Some limit cases

We start with the modification of dual q-Hahn polynomials defined in (36) by adding two mass points at the end of the interval of orthogonality. I.e., the polynomials $w_n^{c,A,B}(s)_q := w_n^{c,A,B}(x(s),a,b)_q$ satisfying the orthogonality relation

$$\sum_{s=a}^{b-1} w_n^{c,A,B}(s)_q w_m^{c,A,B}(s)_q \rho(s)[2s+1]_q + A w_n^{c,A,B}(a)_q w_m^{c,A,B}(a)_q + B w_n^{c,A,B}(b-1)_q w_m^{c,A,B}(b-1)_q = \delta_{n,m}\widetilde{d}_n^2. \tag{57}$$

To obtain the values of $w_m^{c,A,B}(a)_q$, $w_m^{c,A,B}(b-1)_q$ and the norm $\widetilde{d}_n^2$ we can use the formulas (31) and (32), respectively, that yield

$$\begin{aligned} w_n^{c,A,B}(a)_q &= \frac{(1+B\mathrm{K}_{n-1}^c(b-1,b-1))w_n^c(a)_q - B\mathrm{K}_{n-1}^c(a,b-1)w_n^c(b-1)_q}{\kappa_{n-1}^c(a,b-1)}, \\ w_n^{c,A,B}(b-1)_q &= \frac{-A\mathrm{K}_{n-1}^c(b-1,a)w_n^c(a)_q + (1+A\mathrm{K}_{n-1}^c(a,a))w_n^c(b-1)_q}{\kappa_{n-1}^c(a,b-1)}, \end{aligned} \tag{58}$$

$$\begin{aligned} \widetilde{d}_n^2 = d_n^2 + &\frac{A(w_n^c(a)_q)^2\{1+B\mathrm{K}_{n-1}^c(b-1,b-1)\} + B(w_n^c(b-1)_q)^2\{1+A\mathrm{K}_{n-1}^c(a,a)\}}{\kappa_{n-1}^c(a,b-1)} \\ &- \frac{2ABw_n^c(a)_q w_n^c(b-1)_q \mathrm{K}_{n-1}^c(a,b-1)}{\kappa_{n-1}^c(a,b-1)}, \end{aligned}$$

where

$$\kappa_m^c(s,t) = 1 + A\mathrm{K}_m^c(s,s) + B\mathrm{K}_m^c(t,t) + AB\left\{\mathrm{K}_m^c(s,s)\mathrm{K}_m^c(t,t) - (\mathrm{K}_m^c(s,t))^2\right\}, \tag{59}$$

$\mathrm{K}_m^c(s,t) = \sum_{k=0}^m w_k^c(s)_q w_k^c(t)_q / d_k^2$, and d_n^2 denotes the norm of the dual q-Hahn polynomials.

Notice that if we make the change $\beta = a + c$ in the orthogonality relation for the non-standard q-Racah polynomials (38), take the limit $q^\alpha \to 0$, and use that $\lim_{q^\alpha \to 0} u_n^{\alpha,a+c}(s)_q = w_n^c(s)_q$, we obtain the orthogonality relation (57), and therefore, it is straightforward to see that

$$\lim_{q^\alpha \to 0} u_n^{\alpha,a+c,A,B}(s)_q = w_n^{c,A,B}(s)_q.$$

Thus, all properties of the modified dual q-Hahn polynomials $w_n^{c,A,B}(s)_q$ can be obtained from the corresponding properties of the modified q-Racah polynomials $u_n^{\alpha,\beta,A,B}(s)_q$ by taking the appropriate limit. For this reason we will only include here the TTRR for the $w_n^{c,A,B}(s)_q$

$$\begin{aligned} \widetilde{\alpha}_n &= 1, \\ \widetilde{\beta}_n &= \beta_n - A\left(\frac{w_n^{c,A,B}(a)_q w_{n-1}^c(a)_q}{d_{n-1}^2} - \frac{w_{n+1}^{c,A,B}(a)_q w_n^c(a)_q}{d_n^2}\right) \\ &\quad - B\left(\frac{w_n^{c,A,B}(b-1)_q w_{n-1}^c(b-1)_q}{d_{n-1}^2} - \frac{w_{n+1}^{c,A,B}(b-1)_q w_n^c(b-1)_q}{d_n^2}\right), \\ \widetilde{\gamma}_n &= \gamma_n \frac{1+\Delta_n^{A,B}}{1+\Delta_{n-1}^{A,B}},\ \Delta_n^{A,B} = \frac{Aw_n^{c,A,B}(a)_q w_n^c(a)_q}{d_n^2} + \frac{Bw_n^{c,A,B}(b-1)_q w_n^c(b-1)_q}{d_n^2}, \end{aligned}$$

where we use the notation defined in (58) and (59).

To conclude this paper we consider two important limit cases of the q-Racah-Krall polynomials $u_n^{\alpha,\beta,A,B}(s)_q$.

The first one is when we take the limit $q \to 1$. In fact, if we take the limit $q \to 1$ in (35) we recover the non-standard Racah polynomials in the quadratic lattice $x(s) = s(s+1)$ [11, 29] (notice that they are different from the standard Racah polynomials [24, page 190]), i.e.,

$$\lim_{q\to 1} u_n^{\alpha,\beta}([s]_q[s+1]_q, a, b)_q = u_n^{\alpha,\beta}(s(s+1), a, b),$$

where

$$u_n^{\alpha,\beta}(s) = \frac{(a-b+1)_n(\beta+1)_n(a+b+\alpha+1)_n}{(\alpha+\beta+n+1)_n} \times \\ {}_4\mathrm{F}_3\left(\begin{array}{c} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{array} \middle| 1 \right), \quad b-a \in \mathbb{N}.$$

Straightforward calculations show that all the properties of the non-standard q-Racah polynomials $u_n^{\alpha,\beta}(s)_q$ becomes into the properties of the non-standard Racah ones (see e.g. [10]). Thus, we have the following limit relation

$$\lim_{q\to 1} u_n^{\alpha,\beta,A,B}([s]_q[s+1]_q, a, b)_q = u_n^{\alpha,\beta,A,B}(s(s+1), a, b),$$

where $u_n^{\alpha,\beta,A,B}(s(s+1), a, b)$ denotes the modification of the non-standard Racah polynomials by adding two delta Dirac masses at the points a and $b-1$.

Moreover, from the corresponding formulas of the q-Racah-Krall polynomials $u_n^{\alpha,\beta,A,B}(s)_q$ we can obtain the main properties of the Racah-Krall (not q) polynomials $u_n^{\alpha,\beta,A,B}(s)$. Thus, taking appropriate limits one can construct the analogue of the Askey Tableau but for the Krall type polynomials (not q).

Another important family of Krall-type polynomials are the so called q-Hahn-Krall tableau of orthogonal polynomials considered in [5, 9]. Let us show how we can obtain it from our case.

First of all notice that the non-standard q-Racah polynomials are defined on the lattice $x(s) = [s]_q[s+1]_q$ which is of the form (1) with $c_1 = q^{1/2}(q^{1/2} - q^{-1/2})^{-2}$, $c_3 = -q^{1/2} + q^{-1/2}(q^{1/2} - q^{-1/2})^{-2}$, $\zeta = 1$. Then, making the transformation $x(s) \to q^{-a-1}c_1 x(s)$, $q^\alpha = \mu$, $q^\beta = \gamma$, $q^{-b} = q^{-N-1-a}$

in (35), taking the limit $q^a \to 0$, and using the identity [2]

$$(q^{s_1-s};q)_k(q^{s_1+s+\zeta};q)_k = (-1)^k q^{k(s_1+\zeta+\frac{k-1}{2})} \prod_{i=0}^{k-1} \Big[\frac{x(s)-c_3}{c_1} - q^{-\frac{\zeta}{2}}(q^{s_1+i+\frac{\zeta}{2}} + q^{-s_1-i-\frac{\zeta}{2}})\Big],$$

where $s_1 = a$, we obtain

$$C_n u_n^{\alpha,\beta}(x(s),a,b)_q \overset{q^a\to 0}{\longrightarrow} h_n^{\gamma,\mu}(x(s);N|q), \tag{60}$$

where $h_n^{\gamma,\mu}(x(s);N|q)$ are the q-Hahn polynomials on the lattice $x(s) = q^{-s}$

$$h_n^{\gamma,\mu}(x(s);N|q) := \frac{(\gamma q;q)_n(q^{-N};q)_n}{(\gamma\mu q^{n+1};q)_n} {}_3\varphi_2\left(\begin{array}{c} q^{-n}, \gamma\mu q^{n+1}, x(s) \\ \gamma q, q^{-N} \end{array} \middle| \, q\,,\, q\right),$$

and

$$C_n := (q^{1/2} - q^{-1/2})^{2n} q^{\frac{n}{2}(2a+1)}. \tag{61}$$

The above limit relation allows us to obtain the q-Hahn-Krall polynomials. For doing that we use

$$C_n u_n^{\alpha,\beta}(a) \overset{q^a\to 0}{\longrightarrow} h_n^{\gamma,\mu}(x(0);N|q) \quad C_n u_n^{\alpha,\beta}(b-1) \overset{q^a\to 0}{\longrightarrow} h_n^{\gamma,\mu}(x(N);N|q), \tag{62}$$

and

$$C_k d_k^2 \overset{q^a\to 0}{\longrightarrow} \overline{d}_k^2 = (-\gamma q)^n q^{\binom{n}{2}-Nn} \frac{(q,\mu q,\gamma q,q^{-N},\gamma\mu q^{N+2};q)_n}{(\gamma\mu q^2;q)_{2n}(\gamma\mu q^{n+1};q)_n},$$

where d_k and $\overline{d}_k^2$ denote the norms for the non-standard q-Racah and the q-Hahn polynomials, respectively. Then, applying the aforesaid transformation to (8) we obtain the following limit relation for the kernels of the non-standard q-Racah and q-Hahn polynomials

$$\begin{aligned} \mathrm{K}_n^{\alpha,\beta}(s_1,s_2) &:= \sum_{k=0}^{n} \frac{C_k u_k^{\alpha,\beta}(s_1)_q C_k u_k^{\alpha,\beta}(s_2)_q}{C_k^2 d_k^2} \overset{q^a\to 0}{\longrightarrow} \\ &\sum_{k=0}^{n} \frac{h_k^{\gamma,\mu}(x(\overline{s}_1);N|q) h_k^{\gamma,\mu}(x(\overline{s}_2);N|q)}{\overline{d}_k^2} := \mathrm{K}_n^{\gamma,\mu}(\overline{s}_1,\overline{s}_2). \end{aligned} \tag{63}$$

From the limit relations (60), (62), and (63) and using (41) we obtain that

$$\lim_{q^a\to 0} C_n u_n^{\alpha,\beta,A,B}(s)_q = h_n^{\gamma,\mu,A,B}(x(s);N|q) := h_n^{\gamma,\mu,A,B}(s)_q,$$

where C_n is given in (61). I.e., we obtain the q-Hahn-Krall polynomials on the lattice $x(s) = q^{-s}$ which satisfy the orthogonality relation

$$\sum_{s=0}^{N} h_n^{\gamma,\mu,A,B}(s)_q h_m^{\gamma,\mu,A,B}(s)_q \rho(s)\Delta x(s-\tfrac{1}{2}) + A h_n^{\gamma,\mu,A,B}(0)_q h_m^{\gamma,\mu,A,B}(0)_q$$
$$+ B h_n^{\gamma,\mu,A,B}(N)_q h_m^{\gamma,\mu,A,B}(N)_q = \delta_{n,m}\overline{d}_k^2, \quad x(s) = q^{-s},$$

where ρ is the weight function of the q-Hahn polynomials (see [24, page 445]).

5. Concluding remarks

In the present work we have developed a method for constructing the Krall-type polynomials on the q-quadratic non-uniform lattices, i.e., lattices of the form $x(s) = c_1 q^s + c_2 q^s + c_3$. As a representative example the modification of the non-standard q-Racah polynomials was considered in detail. This is an important example for two reasons: 1) it is the first family of the Krall-type polynomials on a non-linear type lattice that has been studied in detail and 2) almost all modifications (via the addition of delta Dirac masses) of the classical and q-classical polynomials can be obtained from them by taking appropriate limits (as it is shown for the dual q-Hahn, the Racah, and the q-Hahn polynomials in section 4.2). Let us also mention here that an instance of the Krall-type polynomials obtained from the Askey-Wilson polynomials (with a certain choice of parameters), by adding two mass points at the end of the orthogonality has been mentioned in [20, §6, page 330]. This Askey-Wilson-Krall-type polynomials solve the so-called bi-spectral problem associated with the Askey-Wilson operator. Then, it is an interesting open problem to study the general Krall-type Askey-Wilson polynomials and to obtain their main properties. This will be considered in a forthcoming paper.

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