ON THE MODIFICATIONS OF CLASSICAL ORTHOGONAL POLYNOMIALS: THE SYMMETRIC CASE. [1]

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July 4, 1997

Key words and phrases: Hermite polynomials, Gegenbauer polynomials, discrete measures, zeros, symmetric functionals.
AMS (MOS) subject classification: **33A65**

Abstract

We consider the modifications of the monic Hermite and Gegenbauer polynomials via the addition of one point mass at the origin. Some properties of the resulting polynomials are studied: three-term recurrence relation, differential equation, ratio asymptotics, hypergeometric representation as well as, for large n, the behaviour of their zeros.

1 Introduction.

In 1940, H. L. Krall [19] obtained three new classes of polynomials orthogonal with respect to measures which are not absolutely continuous with respect to the Lebesgue measure. In fact, his study is related to an extension of the very well known characterization of classical orthogonal polynomials by S. Bochner. This kind of measures was not considered in [28]. Moreover, in his paper H. L. Krall obtain that these three new families of orthogonal polynomials satisfy a fourth order differential equation. The corresponding measures are given in the following table.

$\{P_n(x)\}$	weight function $d\mu$	$supp(\mu)$
Laguerre-type	$e^{-x}\,dx + M\delta(x), \quad M>0$	$[0,\infty)$
Legendre-type	$\frac{\alpha}{2}\,dx + \frac{\delta(x-1)}{2} + \frac{\delta(x+1)}{2}, \quad \alpha>0$	$[-1,1]$
Jacobi-type	$(1-x)^{\alpha}\,dx + M\delta(x), \quad M>0, \alpha>-1$	$[0,1]$

A different approach to this subject was presented in [18].

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The analysis of properties of polynomials orthogonal with respect to a perturbation of a measure via the addition of mass points was introduced by P.Nevai [23]. There the asymptotic properties of the new polynomials have been considered. In particular, he proved the dependence of such properties in terms of the location of the mass points with respect to the support of the measure. Particular emphasis was given to measures supported in $[-1,1]$ and satisfying some extra conditions in terms of the parameters of the three-term recurrence relation that the corresponding sequence of orthogonal polynomials satisfy.

The analysis of algebraic properties for such polynomials attracted the interest of several researchers (see [7] for positive Borel measures and [21] for a more general situation). From the point of view of differential equations see [22].

When two mass points are considered, the difficulties increase as shows [10]. An interesting application for the addition of two mass points at ± 1 to the Jacobi weight function was analyzed in [17].

In this work we will study a generalization of the Hermite and Gegenbauer polynomials. In fact, we will study the polynomials orthogonal with respect to a modification of a symmetric weight function via the addition of one Delta measure at $x = 0$. It is easy to see that the resulting linear functional is symmetric.

In Section 2 we include all the properties of the Hermite and Gegenbauer polynomials which will need. In Section 3 we study the generalized Hermite polynomials and Section 4 is devoted to the Gegenbauer case. In particular, we obtain their expression in terms of the classical polynomials, the hypergeometric representations, the ratio asymptotics, the second order differential equation and the three-term recurrerence relation that such generalized polynomials satisfy.

Finally, using the techniques developed in [6] and [5],[31], we deduce in Section 5 some moments of the distribution of zeros, as well as his semiclassical or WKB density.

2 Some Preliminary Results.

In this section we will enclose the basic characteristics of the Hermite and Gegenbauer monic orthogonal polynomials. For more details see, for instance, [8], [11], [24], [28].

2.1 Classical Hermite Polynomials.

The Hermite polynomials $H_n(x)$ are the polynomial solutions of the second order differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0. \tag{1}$$

They satisfy an orthogonal relation of the form

$$\int_{-\infty}^{\infty} H_n(x)H_k(x)e^{-x^2}dx = \delta_{nk}\ 2^{-n}n!\sqrt{\pi}\,,$$

as well as a three-term recurrence relation (TTRR)

$$xH_n(x) = H_{n+1}(x) + \frac{n}{2}H_{n-1}(x), \tag{2}$$

and the differentiation formula

$$(H_n(x))^{(\nu)} = \frac{n!}{(n-\nu)!} H_{n-\nu}(x), \qquad \nu = 1, 2, 3, \ldots \,. \tag{3}$$

Since they are orthogonal with respect to a symmetric weight function, then

$$H_n(-x) = (-1)^n H_n(x).$$

They are connected with the classical Laguerre polynomials by relations (see [24] and [28])

$$H_{2m}(x) = L_m^{-\frac{1}{2}}(x^2), \qquad H_{2m+1}(x) = x L_m^{\frac{1}{2}}(x^2) \;,$$

and

$$H_{2m}(0) = L_m^{-\frac{1}{2}}(0) = \frac{(-1)^m (2m)!}{2^{2m} m!}, \qquad H_{2m+1}(0) = 0, \qquad m = 0, 1, 2, \ldots \,. \tag{4}$$

2.2 Classical Gegenbauer Polynomials.

The Gegenbauer polynomials $G_n^\lambda(x)$ are defined

$$G_n^\lambda(x) \equiv P_n^{\lambda-\frac{1}{2}, \lambda-\frac{1}{2}}(x), \tag{5}$$

where $P_n^{\alpha,\beta}(x)$ denotes the classical Jacobi polynomials [24], [28].

They satisfy the second order differential equation

$$(1-x^2)y''(x) - (2\lambda+1)xy'(x) + n(2\lambda+n)y(x) = 0, \tag{6}$$

as well as an orthogonal relation of the form ($\lambda > -\frac{1}{2}$)

$$\int_{-1}^{1} G_n^\lambda(x) G_k^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx = \delta_{nk} \, \frac{\sqrt{\pi} n! \Gamma(n+\lambda+\frac{1}{2}) \Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)\Gamma(2n+2\lambda)}.$$

They satisfy a three-term recurrence relation (TTRR)

$$x G_n^\lambda(x) = G_{n+1}^\lambda(x) + \frac{n(2\lambda+n-1)}{4(\lambda+n)(\lambda+n-1)} G_{n-1}^\lambda(x), \tag{7}$$

and the differentiation formula

$$(G_n^\lambda(x))^{(\nu)} = \frac{n!}{(n-\nu)!} G_{n-\nu}^{\lambda+\nu}(x), \qquad \nu = 1, 2, 3, \ldots, \tag{8}$$

which is a consequence of the differentiation formula for the Jacobi polynomials [24], [28]

$$(P_n^{\alpha,\beta}(x))^{(\nu)} = \frac{n!}{(n-\nu)!} P_{n-\nu}^{\alpha+\nu, \beta+\nu}(x), \qquad \nu = 1, 2, 3, \ldots \,. \tag{9}$$

Since they are orthogonal with respect to a symmetric weight function, then

$$G_n^\lambda(-x) = (-1)^n G_n^\lambda(x).$$

They are connected with the classical Jacobi polynomials by (see [28])

$$G_{2m}^\lambda(x) = \frac{1}{2^m} P_m^{(\lambda-\frac{1}{2}, -\frac{1}{2})}(2x^2-1), \qquad G_{2m+1}^\lambda(x) = \frac{1}{2^m} x \, P_m^{(\lambda-\frac{1}{2}, \frac{1}{2})}(2x^2-1), \tag{10}$$

and

$$G_{2m}^\lambda(0) = 2^{-m} P_m^{(\lambda-\frac{1}{2}, -\frac{1}{2})}(-1) = \frac{(-1)^m (\frac{1}{2})_m}{(m+\lambda)_m}, \qquad G_{2m+1}^\lambda(0) = 0, \qquad m = 0, 1, 2, \ldots \,. \tag{11}$$

3 Generalized Hermite Polynomials.

Definition 3.1 *The* generalized monic Hermite polynomials $H_n^A(x)$ *are the polynomials orthogonal with respect to the linear functional* $\mathcal{U}$

$$< \mathcal{U}, P > \; = < \mathcal{H}, P > + AP(0), \qquad A \geq 0, \tag{12}$$

defined on the set of polynomials $\mathbb{P}$ *with real coefficients supported on the real line, where* $\mathcal{H}$ *denotes the Hermite functional*

$$< \mathcal{H}, P > \; = \int_{-\infty}^{\infty} e^{-x^2} P(x) dx \, . \tag{13}$$

In order to obtain the polynomials $H_n^A(x)$ we consider their Fourier expansion in terms of the classical ones, i.e.,

$$H_n^A(x) = \sum_{k=0}^{n} a_{n,k} H_k(x),$$

and use the orthogonality property in the same sense that in [2]-[4], [21]. Nevertheless, we will obtain them by another way.

Let us write the symmetric functional $\mathcal{U}$ in the form

$$< \mathcal{U}, P > \; = \int_{-\infty}^{\infty} e^{-x^2} P(x) \, dx + AP(0), \qquad A \geq 0. \tag{14}$$

We will decompose the polynomial $H_k^A(x)$ in two polynomials

$$H_k^A(x) = p_n^A(x^2) + x q_m^A(x^2), \quad k = max\{2n, 2m+1\}, \tag{15}$$

and substitute it in (14). Some straightforward computation gives us

$$2\int_0^{\infty} p_n^A(x^2) p_k^A(x^2) e^{-x^2} dx + A p_n^A(0) p_k^A(0) + 2\int_0^{\infty} q_m^A(x^2) q_l^A(x^2) x^2 e^{-x^2} dx.$$

If we introduce in the last expression the change of variables $\xi = x^2$ we obtain

$$\underbrace{\int_0^{\infty} p_n^A(\xi) p_k^A(\xi) \xi^{-\frac{1}{2}} e^{-\xi} d\xi + A p_n^A(0) p_k^A(0)}_{p_m(\xi) = C_m L_m^{-\frac{1}{2},A}(x^2).} + \underbrace{\int_0^{\infty} q_m^A(\xi) q_l^A(\xi) \xi^{\frac{1}{2}} e^{-\xi} d\xi}_{q_m^A = c_m L_m^{\frac{1}{2}}(x)} \, .$$

Then,

$$H_{2m}^A(x) = L_m^{-\frac{1}{2},A}(x^2), \quad H_{2m+1}^A(x) = x L_m^{\frac{1}{2}}(x^2), \quad m = 0, 1, 2, \dots \; , \tag{16}$$

where $L_m^{\alpha,A}(x)$ denotes the generalized Laguerre-Koornwinder polynomials [4], [12]-[15], i.e., the polynomials orthogonal with respect to the modification of the weight function $\rho(x) = x^\alpha e^{-x}$ via the addition of one delta Dirac measure at $x = 0$. By using the representation formulas for the monic polynomials $L_m^{\alpha,A}(x)$ (see [4])

$$L_n^{\alpha,A}(x) = L_n^\alpha(x) + \Gamma_n \frac{d}{dx} L_n^\alpha(x) = (I + \Gamma_n \frac{d}{dx}) L_n^\alpha(x), \tag{17}$$

$$\Gamma_n = \frac{A(\alpha+1)_n}{n!\Gamma(\alpha+1)\left(1 + A\frac{(\alpha+1)_n}{(n-1)!\Gamma(\alpha+2)}\right)},$$

we obtain the following representation formula for these generalized Hermite polynomials

Proposition 3.1 *The generalized Hermite polynomials $H_n^A(x)$ admit the following representations in terms of the classical polynomials*

1. *If $n = 2m$, $m = 0, 1, 2, \ldots$, then*

$$\begin{aligned} H_{2m}^A(x) &= L_m^{-\frac{1}{2}}(x^2) + B_m \frac{d}{dx^2} L_m^{-\frac{1}{2}}(x^2), \\ 2xH_{2m}^A(x) &= 2xH_{2m}(x) + B_m \frac{d}{dx} H_{2m}(x), \\ B_m &= \frac{A}{\left(1 + A\frac{2\Gamma(m+\frac{1}{2})}{\pi\Gamma(m)}\right)} \frac{\Gamma(m+\frac{1}{2})}{\pi\, m!}. \end{aligned} \tag{18}$$

2. *If $n = 2m - 1$, $m = 1, 2, \ldots$, then*

$$H_{2m-1}^A(x) = xL_{m-1}^{\frac{1}{2}}(x^2) = H_{2m-1}(x). \tag{19}$$

As we can see from the above proposition, the polynomials of odd degree coincide with the classical ones; then we will only study the polynomials of even degree.

3.1 The hypergeometric representation.

Proposition 3.2 *The generalized Hermite polynomials $H_{2m}^A(x)$ are, up to a multiplicative factor, an hypergeometric function ${}_2\mathrm{F}_2$. More precisely,*

$$H_{2m}(x) = (-1)^m (1 - 2mB_m)(\tfrac{1}{2})_m \, {}_2\mathrm{F}_2\left(\begin{matrix} -m, \gamma_0 + 1 \\ \frac{3}{2}, \gamma_0 \end{matrix} ; x^2 \right), \tag{20}$$

where $\gamma_0 = \frac{1-2mB_m}{2(1+B_m)}$ is, in general, a real number. In the case when γ_0 is a nonpositive integer we will take the analytic continuation of the hypergeometric series.

Proof: From the hypergeometric representation of Laguerre polynomials [24], [28]

$$L_n^\alpha(x) \quad = \quad (-1)^n (\alpha+1)_n \quad {}_1\mathrm{F}_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} ; x \right), \tag{21}$$

where the hypergeometric function ${}_p\mathrm{F}_q$ is defined by

$${}_p\mathrm{F}_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

and $(a)_k$ denotes the Pochammer symbol

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1), \qquad k = 1, 2, 3, \ldots \, .$$

From formula (18) we deduce

$$H_{2m}^A(x) = (-1)^m (\tfrac{1}{2})_m \left[\sum_{k=0}^{\infty} \frac{(-m)_k}{(\frac{1}{2})_k} \frac{\xi^k}{k!} + B_m \frac{(-m)_{k+1}}{(\frac{1}{2})_{k+1}} \frac{\xi^k}{k!} \right], \qquad \xi = x^2.$$

Using $(a)_{k+1} = (a+k+1)(a)_k$ we find

$$H_{2m}^A(x) = (-1)^m (\tfrac{1}{2})_m (1+B_m) \sum_{k=0}^{\infty} \frac{(-m)_k}{(\frac{3}{2})_k} \frac{\xi^k}{k!} \left[k + \frac{1-2mB_m}{1+B_m} \right], \qquad \xi = x^2.$$

Notice that the expression inside the quadratic brackets is a polynomial in m of degree 1 of the form $[k+\gamma_0]$, where $\gamma_0 = \frac{1-2mB_m}{2(1+B_m)}$. Then, from the identities

$$(a+1)_k = \frac{a+k}{a}(a)_k \quad \text{ or } \quad (k+a) = a\frac{(a+1)_k}{(a)_k}, \tag{22}$$

the last expression yields (20). ■

3.2 Asymptotic of the polynomials $H_{2m}^A(x)$.

In order to obtain the asymptotic properties of the polynomials $H_{2m}^A(x)$ for large enough m, we rewrite (18) in the form

$$\frac{H_{2m}^A(x)}{H_{2m}(x)} = 1 + B_m \frac{(L_m^{-\frac{1}{2}}(x^2))'}{L_m^{-\frac{1}{2}}(x^2)}, \tag{23}$$

where $(L_n^{-\frac{1}{2}}(x^2))'$ denotes the derivative with respect to x^2. If we use the asymptotic formula for the gamma function [1]

$$\Gamma(ax+b) \sim \sqrt{2\pi} e^{-ax} (ax)^{ax+b-\frac{1}{2}}, \qquad x >> 1,$$

the following asymptotic expression for the constant B_m holds

$$B_m \sim \frac{1}{2m}.$$

To obtain the asymptotic formula for the ratio $\frac{H_{2m}^A(z)}{H_{2m}(z)}$ we can use the Perron formula (see [29], Eq. (4.2.6) page 133 and [28], Theorem 8.22.3) for the ratio $\dfrac{1}{\sqrt{n}} \dfrac{(L_n^\alpha)'(z)}{L_n^\alpha(z)}$ of the Laguerre polynomials ($z \in \mathbb{C} \setminus \{[0,\infty)\}$)

$$\frac{1}{\sqrt{n}} \frac{(L_n^\alpha)'(z)}{L_n^\alpha(z)} = \frac{-1}{\sqrt{z}} \left\{ 1 + \frac{1}{\sqrt{n}} [C_1(\alpha+1,z) - C_1(\alpha,z) - \sqrt{-z}] \right\} + o\left(\frac{1}{\sqrt{n}}\right),$$

where $C_1(\alpha,z) = \dfrac{1}{4\sqrt{-z}} \left(-3z + \dfrac{1}{3}z^2 + \dfrac{1}{4} - \alpha^2 \right)$. Taking into account that in (23) we have the ratio $\dfrac{(L_n^{-\frac{1}{2}}(x^2))'}{L_n^{-\frac{1}{2}}(x^2)}$, we need to substitute in the previous expression $z \longleftrightarrow z^2$. But

$$C_1(\tfrac{1}{2}, z^2) = C_1(-\tfrac{1}{2}, z^2),$$

and then, for m large enough

$$\frac{H_{2m}^A(z)}{H_{2m}(z)} = 1 - \frac{1}{2\sqrt{m}\, iz} \left\{ 1 - \frac{iz}{\sqrt{m}} \right\} + o\left(\frac{1}{m}\right), \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{24}$$

3.3 Second order differential equation.

Here we will obtain an algorithm that allows us to deduce the second order differential equation (SODE) that the generalized polynomials satisfy. First of all, notice that both classical polynomials under consideration (Hermite and Gegenbauer) satisfy a SODE

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0.$$

In order to obtain the second order differential equation (SODE) that the generalized Hermite polynomials satisfy we will rewrite formula (18) in a more convenient form (notice that for Hermite polynomials $\sigma(x) = 1$)

$$2x\tilde{P}_{2m}^A(x) = 2x\tilde{C}P_{2m}(x) + \sigma(x)\tilde{B}P_{2m}'(x), \tag{25}$$

where $\tilde{P}_{2m}^A(x)$ denotes the generalized polynomial and $P_{2m}(x)$ denotes the classical one. For the Hermite polynomials it is easy to check that $\tilde{C} = 1$ and $\tilde{B} = B_m$. We will show later (see formula (43) from below), that there exists for the generalized Gegenbauer polynomials a similar representation (25), but with $\sigma(x) = 1-x^2$, $\tilde{C} = 1+mW_m^A$ and $\tilde{B} = W_m^A$.

Next, we will deduce the SODE for these generalized polynomials. First of all, notice that the SODE which satisfy the classical polynomials can be rewritten in the form

$$\sigma(x)P_{2m}''(x) = -\tau(x)P_{2m}'(x) - \lambda_{2m}P_{2m}(x).$$

Taking derivatives in (25), multiplying by x and using the above SODE we obtain

$$\begin{aligned} &\sigma(x)\frac{d}{dx}\tilde{P}_{2m}^A(x) = c(x)P_{2m}(x) + d(x)\frac{d}{dx}P_{2m}(x), \\ &c(x) = -x\tilde{B}\lambda_n, \qquad d(x) = x[2x\tilde{C} + \sigma'(x)\tilde{B}] - [\sigma(x) + x\tau(x)]. \end{aligned} \tag{26}$$

Taking derivatives in (26), multiplying by $x\sigma(x)$ and using (26), as well as the SODE for the $P_{2m}(x)$ we get

$$\begin{aligned} &\sigma(x)^2\frac{d^2}{dx^2}\tilde{P}_{2m}^A(x) = e(x)P_{2m}(x) + f(x)\frac{d}{dx}P_{2m}(x), \\ &e(x) = \sigma(x)[xc'(x) - 2c(x)] - x\lambda_n d(x), \\ &f(x) = x\sigma(x)[c(x) + d'(x)] - d(x)(2\sigma(x) + x\tau(x)]. \end{aligned} \tag{27}$$

The expressions (25),(26) and (27) lead to the condition

$$\begin{vmatrix} 2x\tilde{P}_{2m}^A(x) & a(x) & b(x) \\ 2x^2\sigma(x)\frac{d}{dx}\tilde{P}_{2m}^A(x) & c(x) & d(x) \\ 2x^3\sigma(x)\frac{d^2}{dx^2}\tilde{P}_{2m}^A(x) & e(x) & f(x) \end{vmatrix} = 0, \tag{28}$$

where $a(x) = 2x\tilde{C}$ and $b(x) = \sigma(x)\tilde{B}$. Expanding the determinant in (28) by the first column

$$\tilde{\sigma}_m(x)\frac{d^2}{dx^2}\tilde{P}_{2m}^A(x) + \tilde{\tau}_m(x)\frac{d}{dx}\tilde{P}_{2m}^A(x) + \tilde{\lambda}_m(x)\tilde{P}_{2m}^A(x) = 0, \tag{29}$$

where

$$\tilde{\sigma}_m(x) = \sigma(x)x^2[a(x)d(x) - c(x)b(x)],$$

$$\tilde{\tau}_m(x) = x[e(x)b(x) - a(x)f(x)], \tag{30}$$

$$\tilde{\lambda}_m(x) = c(x)f(x) - e(x)d(x).$$

If we apply this algorithm for the generalized Hermite polynomials, for which (see Eq. (18))

$$\tilde{C} = 1, \quad \tilde{B} = B_m, \quad \sigma(x) = 1,$$

we obtain

Proposition 3.3 *The generalized Hermite polynomials of even degree satisfy a second order differential equation*

$$\tilde{\sigma}_m(x)\frac{d^2}{dx^2}H_{2m}^A(x) + \tilde{\tau}_m(x)\frac{d}{dx}H_{2m}^A(x) + \tilde{\lambda}_m(x)H_{2m}^A(x) = 0, \tag{31}$$

where

$$\tilde{\sigma}_m(x) = x\,\left(-B_m + 2\,B_m^2\,m + 2\,x^2 + 2\,B_m\,x^2\right),$$

$$\tilde{\tau}_m(x) = 2\left(-B_m + 2\,B_m^2\,m + B_m\,x^2 - 2\,B_m^2\,m\,x^2 - 2\,x^4 - 2\,B_m\,x^4\right), \tag{32}$$

$$\tilde{\lambda}_m(x) = 4\,m\,x\,\left(-3\,B_m - 2\,B_m^2 + 2\,B_m^2\,m + 2\,x^2 + 2\,B_m\,x^2\right).$$

3.4 The three-term recurrence relation.

Proposition 3.4 *The generalized Hermite polynomials satisfy a three-term recurrence relation (TTRR)*

$$\begin{gathered} xH_n^A(x) = H_{n+1}^A(x) + \beta_n^A H_n^A(x) + \gamma_n^A P_{n-1}^A(x), \quad n \geq 0 \\ H_{-1}^A(x) = 0 \qquad \textit{and} \qquad H_0^A(x) = 1. \end{gathered} \tag{33}$$

This is a consequence of the orthogonality property with respect to a positive definite functional (see [8] or [24]). To obtain the TTRR's coefficients notice that the functional is symmetric and then $< \mathcal{U}, xH_n^A(x)H_n^A(x) >= 0$, i.e., $\beta_n^A = 0$. To obtain the coefficient γ_n^A we can analyze the two cases $n = 2m$ and $n = 2m-1$, separately. For the coefficients γ_n^A, $n = 2m-1$, if we evaluate (33) in $x = 0$ ($H_{2m-2}^A(0) \neq 0$) we obtain

$$\gamma_{2m-1}^A = -\frac{H_{2m}^A(0)}{H_{2m-2}^A(0)} = \frac{(2m-1)}{2}\frac{1 + \frac{2A}{\pi}\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m-1)}}{1 + \frac{2A}{\pi}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}}. \tag{34}$$

For the coefficients γ_n^A, $n = 2m$, this procedure in not valid because $H_{2m-1}^A(0) = 0$. For this reason we need to calculate it directly from the definition

$$\gamma_{2m}^A = \frac{< \mathcal{U}, xH_{2m}^A(x)H_{2m-1}^A(x) >}{< \mathcal{U}, [H_{2m-1}^A(x)]^2 >}.$$

Since $H_{2m-1}^A(x) = H_{2m-1}(x)$, then the denominator is the square norm of the classical Hermite polynomials d_{2m-1}^2. Let us to calculate the numerator in the above expression.

In order to do this we will use the TTRR for the classical Hermite polynomials, the differentiation formula (3) as well as formula (18). Then,

$$< \mathcal{U}, xH_{2m}^A(x)H_{2m-1}^A(x) >= \int_{-\infty}^{\infty} e^{-x^2} H_{2m-1}(x) \left[xH_{2m}(x) + \frac{B_m}{2} H'_{2m}(x) \right] dx =$$

$$= \int_{-\infty}^{\infty} e^{-x^2} xH_{2m-1}(x)H_{2m}(x)dx + mB_m d_{2m-1}^2,$$

from which we obtain

$$\gamma_{2m}^A = \gamma_{2m} + mB_m = m(1+B_m). \tag{35}$$

Now, notice that

$$\gamma_{2m+1}^A = -\frac{H_{2m+2}^A(0)}{H_{2m}^A(0)} = \frac{< \mathcal{U}, xH_{2m+1}^A(x)H_{2m}^A(x) >}{(d_{2m}^A)^2}.$$

If we calculate the numerator of the above expression we find

$$< \mathcal{U}, xH_{2m+1}^A(x)H_{2m}^A(x) >= \int_{-\infty}^{\infty} e^{-x^2} H_{2m+1}(x) \left[xH_{2m}(x) + \frac{B_m}{2} H'_{2m}(x) \right] dx =$$

$$= \int_{-\infty}^{\infty} e^{-x^2} xH_{2m+1}(x)H_{2m}(x)dx = \tfrac{1}{2}(2m+1)d_{2m}^2 = (d_{2m}^A)^2 \gamma_{2m+1}^A.$$

The above formula allows us to calculate the square norm of the generalized Hermite polynomials. In fact, from the last expression and (34) we obtain

1. If $n = 2m$, $m = 0, 1, 2, ...,$ then

$$(d_{2m}^A)^2 = \frac{1 + \frac{2A}{\pi}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)}}{1 + \frac{2A}{\pi}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}} \frac{(2m)!\sqrt{\pi}}{2^{2m}}. \tag{36}$$

2. If $n = 2m+1$, $m = 0, 1, 2, ...,$ then

$$(d_{2m+1}^A)^2 = d_{2m+1}^2 = \frac{(2m+1)!\sqrt{\pi}}{2^{2m+1}}. \tag{37}$$

Notice that, when $m = 0$, $[d_0^A]^2 = \sqrt{\pi} + A$. This follows from (36) considering the limit when $m \to 0$ and using that $\lim_{x\to 0} \Gamma(x) = \infty$.

4 The generalized Gegenbauer polynomials.

Definition 4.1 *The* generalized monic Gegenbauer polynomials $G_n^{\lambda,A}(x)$ *are the polynomials orthogonal with respect to the linear functional* $\mathcal{U}$

$$< \mathcal{U}, P > \; =< \mathcal{C}_{\mathcal{G}}, P > + AP(0), \qquad A \geq 0, \tag{38}$$

defined on the set of polynomials $\mathbb{P}$ *with real coefficients, supported on* $[-1,1]$*, where* $\mathcal{C}_{\mathcal{G}}$ *denotes the Gegenbauer functional*

$$< \mathcal{C}_{\mathcal{G}}, P > \; = \int_{-1}^{1} (1-x^2)^{\lambda - \frac{1}{2}} P(x)dx, \qquad \lambda > -\tfrac{1}{2}. \tag{39}$$

To obtain the polynomials $G_n^{\lambda,A}(x)$ we will follow the same method as before. First of all, we will rewrite the functional $\mathcal{U}$ in the form

$$<\mathcal{U},P> \; = \int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}P(x)dx + AP(0), \qquad A \geq 0. \tag{40}$$

We will decompose the polynomial $G_k^{\lambda,A}(x)$ in two polynomials not necessarily monics $(G_k^{\lambda}(x) = P_k^{\lambda-\frac{1}{2},\lambda-\frac{1}{2}}(x))$

$$G_k^{\lambda,A}(x) = p_n^A(2x^2-1) + xq_m^A(2x^2-1), \qquad k = max\{2n, 2m+1\}, \tag{41}$$

and substitute it in (40). Some straightforward calculation gives us

$$2\int_0^1 p_n^A(2x^2-1)p_k^A(2x^2-1)(1-x^2)^{\lambda-\frac{1}{2}}dx + Ap_n^A(0)p_k^A(0)+$$

$$+2\int_0^1 q_m^A(2x^2-1)q_l^A(2x^2-1)x^2(1-x^2)^{\lambda-\frac{1}{2}}dx.$$

If we consider in the last expression the change of variables $\xi = 2x^2-1$, we find

$$\underbrace{\frac{1}{2^\lambda}\int_{-1}^{1} p_n^A(\xi)p_k^A(\xi)(1+\xi)^{-\frac{1}{2}}(1-\xi)^{\lambda-\frac{1}{2}}d\xi + Ap_n^A(-1)p_k^A(-1)}_{p_m(\xi) = C_m P_m^{\lambda-\frac{1}{2},-\frac{1}{2},2^\lambda A,0}(2x^2-1).} +$$

$$+\underbrace{\frac{1}{2^\lambda}\int_{-1}^{1} q_m^A(\xi)q_l^A(\xi)(1+\xi)^{\frac{1}{2}}(1-\xi)^{\lambda-\frac{1}{2}}d\xi}_{q_m^A = c_m P_m^{\lambda-\frac{1}{2},\frac{1}{2},}(2x^2-1)}.$$

Then,

$$\begin{aligned} G_{2m}^{\lambda,A}(x) &= 2^{-m}P_m^{\lambda-\frac{1}{2},-\frac{1}{2},2^\lambda A,0}(2x^2-1), \\ G_{2m+1}^{\lambda,A}(x) &= 2^{-m}xP_m^{\lambda-\frac{1}{2},\frac{1}{2},}(2x^2-1), \qquad m = 0,1,2,\ldots \; , \end{aligned} \tag{42}$$

where $P_m^{\alpha,\beta,A,0}(x)$ denotes the generalized Jacobi-Koorwinder polynomials [4], [17], i.e., the polynomials orthogonal with respect to the modification of the weight function $\rho(x) = (1-x)^\alpha(1+x)^\beta$ via the addition of one delta Dirac measure at $x=-1$. By using the representation formulas for these polynomials ([4], [17]) as well as (10), we obtain

Proposition 4.1 *The generalized Gegenbauer polynomials $G_n^{\lambda,A}(x)$ have the following representations in terms of the Jacobi or Gegenbauer polynomials*

1. *If $n = 2m$, $m = 0,1,2,...$, then*

$$\begin{aligned} 2^m G_{2m}^{\lambda,A}(x) = & \; (1+W_m^A)P_m^{\lambda-\frac{1}{2},-\frac{1}{2}}(2x^2-1)+ \\ & +2(1-x^2)W_m^A\frac{d}{d\xi}P_m^{\lambda-\frac{1}{2},-\frac{1}{2}}(\xi)\Big|_{\xi=2x^2-1}, \end{aligned} \tag{43}$$

$$2xG_{2m}^{\lambda,A}(x) = \; 2x(1+mW_m^A)G_{2m}^{\lambda}(x) + W_m^A(1-x^2)\tfrac{d}{dx}G_{2m}^{\lambda}(x),$$

where

$$W_m^A = J_{A,0}^{m,\lambda-\frac{1}{2},-\frac{1}{2}} = \frac{A}{\left(1 + A\frac{2\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi\,(m-1)!\Gamma(m+\lambda-\frac{1}{2})}\right)}\frac{\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi\,m!\Gamma(m+\lambda+\frac{1}{2})}.$$

2. If $n = 2m-1$, $m = 1, 2, ...,$ then

$$2^m G_{2m+1}^{\lambda,A}(x) = xP_m^{\lambda-\frac{1}{2},+\frac{1}{2}}(2x^2-1) = 2^m G_{2m+1}^{\lambda}(x). \tag{44}$$

As we can see from the above proposition, the polynomials of odd degree coincide with the classical ones; then we will only study the polynomials of even degree.

Proposition 4.2 *The generalized Gegenbauer polynomials $G_{2m}^{\lambda,A}(x)$ are, up to a multiplicative factor, an hypergeometric function ${}_4\mathrm{F}_3$. More precisely,*

$$\begin{aligned} G_{2m}^{\lambda,A}(x) &= [(\lambda+\tfrac{1}{2})(1-mW_m^A) + m(m+\lambda)W_m^A(1-x^2)]\times \\ &\times \frac{2^m\Gamma(m+\lambda-\frac{1}{2})}{\Gamma(\lambda+\frac{3}{2})(m+\lambda)_m}\,{}_4\mathrm{F}_3\left(\begin{array}{c} -m, m+\lambda, \gamma_0+1, \gamma_1+1 \\ \lambda+\frac{3}{2}, \gamma_0, \gamma_1 \end{array}; 1-x^2\right), \end{aligned} \tag{45}$$

where γ_0, γ_1 are the roots of the quadratic equation in k

$$[(2k+2\lambda+1)(1-mW_m^A) + 2(m-k)(k+m+\lambda)W_m^A(1-x^2)] = 0.$$

They are, in general, complex numbers. In the case when γ_0, γ_1 are nonpositive integers we need to take the analytic continuation of the hypergeometric series.

The proof is quite similar to the previous one (Hermite case). If $A = 0$ some straightforward calculations give $W_m^A = 0$, $\gamma_0 = \lambda + \frac{1}{2}$. Since $\gamma_0\gamma_1 = [2W_m^A(1-x^2)]^{-1} \to \infty$ when $A \to 0$, then $\gamma_1 \to \infty$ and therefore we recover the classical case, i.e.,

$$\begin{aligned} G_{2m}^{\lambda,0}(x) &= \lim_{A\to 0}\left[(\lambda+\tfrac{1}{2})(1-mW_m^A) + m(m+\lambda)W_m^A(1-x^2)\right]\times \\ &\times \frac{2^m\Gamma(m+\lambda-\frac{1}{2})\Gamma(m+\lambda)}{\Gamma(\lambda+\frac{3}{2})\Gamma(2m+\lambda)}\,{}_4\mathrm{F}_3\left(\begin{array}{c} -m, m+\lambda, \gamma_0+1, \gamma_1+1 \\ \lambda+\frac{3}{2}, \gamma_0, \gamma_1 \end{array}; 1-x^2\right) = \\ &= \frac{2^m\Gamma(m+\lambda-\frac{1}{2})\Gamma(m+\lambda)}{\Gamma(\lambda+\frac{1}{2})\Gamma(2m+\lambda)}\,{}_2\mathrm{F}_1\left(\begin{array}{c} -m, m+\lambda \\ \lambda+\frac{1}{2} \end{array}; 1-x^2\right) = G_{2m}^{\lambda}(x). \end{aligned}$$

4.1 Asymptotic of the polynomials $G_{2m}^{\lambda,A}(x)$.

In order to study the asymptotic properties of the polynomials $G_{2m}^{\lambda,A}(x)$ for m sufficiently large, we will rewrite (43) in the form

$$2x\left(G_{2m}^{\lambda,A}(x) - G_{2m}^{\lambda}(x)\right) = 2xmW_m^A P_{2m}^{\lambda-\frac{1}{2},\lambda-\frac{1}{2}}(x) + 2m(1-x^2)W_m^A P_{2m-1}^{\lambda+\frac{1}{2},\lambda+\frac{1}{2}}(x), \tag{46}$$

$$\frac{G_{2m}^{\lambda,A}(x)}{G_{2m}^{\lambda}(x)} = (1+mW_m^A) + \frac{2m}{x}(1-x^2)W_m^A\frac{P_{2m-1}^{\lambda+\frac{1}{2},\lambda+\frac{1}{2}}(x)}{P_{2m}^{\lambda-\frac{1}{2},\lambda-\frac{1}{2}}(x)}. \tag{47}$$

Again, using the asymptotic formula for the gamma function we obtain the following asymptotic expression for the constant W_m^A

$$W_m^A \sim \frac{1}{2m^2}.$$

The asymptotic formula for the difference $G_{2m}^{\lambda,A}(\cos\theta) - G_{2m}^{\lambda}(\cos\theta)$ follows from the Darboux formula in $\theta \in [\varepsilon, \pi-\varepsilon]$, $0 < \varepsilon << 1$ (see [28], Theorem 8.21.8, page 196). Taking into account the last expression we obtain for the generalized Gegenbauer polynomials the following asymptotic formula valid for $\theta \in [\varepsilon, \frac{\pi}{2}-\varepsilon] \bigcup [\frac{\pi}{2}+\varepsilon, \pi-\varepsilon]$ ($0 < \varepsilon << 1$)

$$\begin{aligned} 2x\left(G_{2m}^{\lambda,A}(\cos\theta) - G_{2m}^{\lambda}(\cos\theta)\right) &= \frac{1}{\sqrt{2\pi\, m^3}}\left(\frac{2}{\sin\theta}\right)^{\lambda} \times \\ &\times [\cos\theta\cos(2m\theta+\lambda\theta-\tfrac{1}{2}\lambda\pi) + 2\sin\theta\sin(2m\theta+\lambda\theta-\tfrac{1}{2}\lambda\pi)] + O\left(\frac{1}{m^{\frac{5}{2}}}\right). \end{aligned} \tag{48}$$

When $x = \cos\frac{\pi}{2} = 0$ we can use the expression [4]

$$G_{2m}^{\lambda,A}(0) = \frac{G_{2m}^{\lambda}(0)}{1 + A\displaystyle\sum_{k=0}^{2m-1}\left[\frac{G_k^{\lambda}(0)}{d_k^G}\right]^2},$$

where d_n^G is the norm of the Gegenbauer polynomials (see section **2.2**), which yields

$$\frac{G_{2m}^{\lambda,A}(0)}{G_{2m}^{\lambda}(0)} = \frac{\pi}{2Am} + O\left(\frac{1}{m^2}\right).$$

Now we can deduce the asymptotic formula for such generalized polynomials off the interval of orthogonality. In this case, we will use

$$\frac{1}{n}\frac{P_n'^{\alpha,\beta}(z)}{P_n^{\alpha,\beta}(z)} = \frac{2}{\sqrt{z^2-1}} + o(1).$$

which is a consequence of the Darboux formula in $\mathbb{R}\backslash[-1,1]$ (see [28], Theorem 8.21.7, page 196). The last formula holds uniformly in the exterior of an arbitrary closed curve which enclosed the segment $[-1,1]$. Notice that, if $z \in \mathbb{R}$, $z > 1$, the right side expression of the above formula is a real function of z. Then, for the generalized Gegenbauer polynomials we obtain the following asymptotic formula in $\mathbb{C}\backslash[-1,1]$

$$\frac{G_{2m}^{\lambda,A}(z)}{G_{2m}^{\lambda}(z)} = 1 + \frac{2}{m}\left(\frac{1}{4} - \sqrt{1-\frac{1}{z^2}}\right) + o\left(\frac{1}{m}\right). \tag{49}$$

As before, this formula holds uniformly in the exterior of an arbitrary closed curve which enclosed the segment $[-1,1]$.

4.2 Second order differential equation.

In the previous section we developed an algorithm which allows us to obtain the SODE for the Hermite and Gegenbauer polynomials. First of all, note that the generalized Gegenbauer polynomials can be represented by formula (25) (notice that for Gegenbauer polynomials $\sigma(x) = 1 - x^2$), but now

$$\tilde{C} = 1 + mW_m^A, \quad \tilde{B} = W_m^A, \quad \sigma(x) = 1 - x^2.$$

Then, from (29) and (30)

Proposition 4.3 *The generalized Gegenbauer polynomials of even degree $G_{2m}^{\lambda,A}(x)$, satisfy a second order differential equation*

$$\tilde{\sigma}_m(x)\frac{d^2}{dx^2}\tilde{G}_{2m}^{\lambda,A}(x) + \tilde{\tau}_m(x)\frac{d}{dx}\tilde{G}_{2m}^{\lambda,A}(x) + \tilde{\lambda}_m(x)\tilde{G}_{2m}^{\lambda,A}(x) = 0, \tag{50}$$

where

$$\begin{aligned}
\tilde{\sigma}_m(x) &= x\,(1-x^2)\,\Big(W_m^A + m\,{W_m^A}^2 - 2\,m^2\,{W_m^A}^2 - 2\,m\,\nu\,{W_m^A}^2 - \\
&\quad -2\,x^2 - 4\,m\,W_m^A\,x^2 - 2\,\nu\,W_m^A\,x^2\Big)\,, \\
\tilde{\tau}_m(x) &= -2\,W_m^A - 2\,m\,{W_m^A}^2 + 4\,m^2\,{W_m^A}^2 + 4\,m\,\nu\,{W_m^A}^2 + 3\,W_m^A\,x^2 + \\
&\quad +2\,\nu\,W_m^A\,x^2 + 3\,m\,{W_m^A}^2\,x^2 - 6\,m^2\,{W_m^A}^2\,x^2 - 4\,m\,\nu\,{W_m^A}^2\,x^2 - \\
&\quad -4\,m^2\,\nu\,{W_m^A}^2\,x^2 - 4\,m\,\nu^2\,{W_m^A}^2\,x^2 - 2\,x^4 - 4\,\nu\,x^4 - \\
&\quad -4\,m\,W_m^A\,x^4 - 2\,\nu\,W_m^A\,x^4 - 8\,m\,\nu\,W_m^A\,x^4 - 4\,\nu^2\,W_m^A\,x^4\,, \\
\tilde{\lambda}_m(x) &= 4\,m\,(m+\nu)\,x\,\Big(-3\,W_m^A + {W_m^A}^2 - 3\,m\,{W_m^A}^2 + 2\,m^2\,{W_m^A}^2 - \\
&\quad -2\,\nu\,{W_m^A}^2 + 2\,m\,\nu\,{W_m^A}^2 + 2\,x^2 + 4\,m\,W_m^A\,x^2 + 2\,\nu\,W_m^A\,x^2\Big)\,.
\end{aligned} \tag{51}$$

4.3 The three-term recurrence relation.

Proposition 4.4 *The generalized Gegenbauer polynomials satisfy a three-term recurrence relation (TTRR) ($n \geq 0$)*

$$\begin{gathered}
xG_n^{\lambda,A}(x) = G_{n+1}^{\lambda,A}(x) + \beta_n^A G_n^{\lambda,A}(x) + \gamma_n^A G_{n-1}^{\lambda,A}(x), \\
G_{-1}^{\lambda,A}(x) = 0 \qquad \text{and} \qquad G_0^{\lambda,A}(x) = 1.
\end{gathered} \tag{52}$$

This is a consequence of the orthogonality property with respect to a positive definite functional (see [8] or [24]). To obtain the TTRR's coefficients we can do the same as in the previous case. For this reason we only will provide here the results of the calculations.

- Since $G_n^{\lambda,A}(x)$ are orthogonal with respect to a symmetric functional $\beta_n^A = 0$.

- Coefficients γ_{2m-1}^A, $m = 1, 2, 3, \dots$,

$$\gamma_{2m-1}^A = \frac{(2m-1)(m+\lambda-1)}{2(2m+\lambda-1)} \frac{1 + A\frac{2\Gamma(m-\frac{1}{2})\Gamma(m+\lambda-1)}{\pi\,(m-2)!\Gamma(m+\lambda-\frac{3}{2})}}{1 + A\frac{2\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi\,(m-1)!\Gamma(m+\lambda-\frac{1}{2})}}. \tag{53}$$

- Coefficients γ_{2m}^A, $m = 0, 1, 2, \dots$,

$$\gamma_{2m}^A = \frac{m(2m+2\lambda-1)}{2(2m+\lambda)(2m+\lambda-1)}\left[1 + W_m^A(m+\lambda)\right]. \tag{54}$$

Finally, for the square norms we have the expressions
• If $n = 2m$, $m = 0, 1, 2, ...,$

$$(d_{2m}^A)^2 = \frac{1+A\frac{2\Gamma(m+\frac{3}{2})\Gamma(m+\lambda+1)}{\pi\, m!\Gamma(m+\lambda+\frac{1}{2})}}{1+A\frac{2\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi\,(m-1)!\Gamma(m+\lambda-\frac{1}{2})}} \frac{\sqrt{\pi}(2m)!\Gamma(2m+\lambda+\frac{1}{2})\Gamma(2m+2\lambda)}{\Gamma(2m+\lambda+1)\Gamma(4m+2\lambda)}. \tag{55}$$

• If $n = 2m+1$, $m = 0, 1, 2, ...$

$$(d_{2m+1}^A)^2 = d_{2m+1}^2 = \frac{\sqrt{\pi}(2m+1)!\Gamma(2m+\lambda+\frac{3}{2})\Gamma(2m+2\lambda+1)}{\Gamma(2m+\lambda+2)\Gamma(4m+2\lambda+2)}.$$

Notice that when $m = 0$, $[d_0^A]^2 = \frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)} + A$, which follows from (55) when we take the limit $m \to 0$ and use that $\lim_{x\to 0} \Gamma(x) = \infty$.

5 The Distribution of zeros: the moments μ_r and the WKB density.

In this section we will study the distribution of zeros of the generalized Hermite and Gegenbauer polynomials. We will use a general method presented in [6] for the moments of low order and the WKB approximation in order to obtain the density of the distribution of zeros. First of all we point out that, since our polynomials are orthogonal with respect to a positive definite functional all its zeros are real, simple and located in the interior of the interval of orthogonality. This a necessary condition in order to apply the next algorithms.

5.1 The moments of the distribution of zeros.

The method presented in [6] allows us to compute the moments μ_r of the distribution of zeros $\rho_n(x)$ around the origin, i.e.,

$$\mu_r = \frac{1}{n} y_r = \frac{1}{n}\sum_{i=1}^{n} x_{n,i}^r, \qquad \rho_n = \frac{1}{n}\sum_{i=1}^{n} \delta(x - x_{n,i}).$$

Buendía, Dehesa and Gálvez [6] have obtained a general formula to find these quantities (see [6], Section II, Eq.(11) and (13), page 226). We will apply these two formulas to obtain the general expression for the moments μ_1 and μ_2, but firstly, let us to introduce some notations.

We will rewrite the SODE that such polynomials satisfy

$$\tilde{\sigma}_m(x)\frac{d^2}{dx^2}\tilde{P}_{2m}^A(x) + \tilde{\tau}_m(x)\frac{d}{dx}\tilde{P}_{2m}^A(x) + \tilde{\lambda}_m(x)\tilde{P}_{2m}^A(x) = 0$$

where now

$$\tilde{\sigma}(x) = \sum_{k=0}^{c_2} a_k^{(2)} x^k, \qquad \tilde{\tau}(x) = \sum_{k=0}^{c_1} a_k^{(1)} x^k, \qquad \tilde{\lambda}_n(x) = \sum_{k=0}^{c_0} a_k^{(0)} x^k, \tag{56}$$

and c_2, c_1, c_0 are the degrees of the polynomials $\tilde{\sigma}(x)$, $\tilde{\tau}(x)$ and $\tilde{\lambda}_n(x)$, respectively. Here the values $a_j^{(i)}$ can be found from (30) in a straightforward way. Let $\xi_0 = 1$ and $q = max\{c_2 - 2, c_1 - 1, c_0\}$. Then from [6], (Section II, Eq.(11) and (13), page 226)

$$\xi_1 = y_1, \qquad \xi_2 = \frac{y_1^2 - y_2}{2}, \tag{57}$$

and

$$\xi_s = -\frac{\displaystyle\sum_{m=1}^{s} (-1)^m \xi_{s-m} \sum_{i=0}^{2} \frac{(n-s+m)!}{(n-s+m-i)!} a_{i+q-m}^{(i)}}{\displaystyle\sum_{i=0}^{2} \frac{(n-s)!}{(n-s-i)!} a_{i+q}^{(i)}}. \tag{58}$$

In general $\xi_k = \dfrac{(-1)^k}{k!} \mathcal{Y}_k(-y_1, -y_2, -2y_3, ..., -(k-1)!k_n)$ where $\mathcal{Y}_k$-symbols denote the well known Bell polynomials in the number theory [26].

Let us now to apply these general formulas to obtain the first two central moments μ_1 and μ_2 of our polynomials. Equation (58) give the following values.

5.1.1 Hermite polynomials $H_n^A(x)$.

- If $n = 2m$, $m = 0, 1, 2, ...$, then

$$\xi_1 = 0, \qquad \xi_2 = \frac{(1 + 2\,B_n - 2m)\; m}{2},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{(2m - 1 - 2\,B_m)}{2}.$$

- If $n = 2m - 1$, $m = 1, 2, ...$, then, $H_{2m-1}^A(x) \equiv H_{2m-1}(x)$

$$\xi_1 = 0, \qquad \xi_2 = (1 - m)\; m,$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = (m - 1).$$

The asymptotic behavior of these two moments in both cases is

$$\mu_1 = 0 \text{ y } \mu_2 \sim \frac{n}{2} + O(n).$$

5.1.2 Gegenbauer polynomials $G_n^{\lambda,A}(x)$.

- If $n = 2m$, $m = 0, 1, 2, ...$, then

$$\xi_1 = 0, \qquad \xi_2 = \frac{m\;(-1 + 2m + W_m - n^2\,W_m - 2\,\lambda\,W_m)}{2\;(-1 + 2m + \lambda)\;(-1 + 2m\,W_m)},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{1 - 2m - W_m + 4m^2\,W_m + 2\,\lambda\,W_m}{2\;(-1 + 2m + \lambda)\;(-1 + 2m\,W_m)}.$$

• If $n = 2m-1$, $m = 1, 2, ...$, then, $G_{2m-1}^{\lambda,A}(x) \equiv G_{2m-1}\lambda(x)$

$$\xi_1 = 0, \qquad \xi_2 = \frac{2m\,(2-2m)}{4\,(-2+2m+\lambda)},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{2m-1}{2\,(2m-2+\lambda)}.$$

The asymptotic behavior of these two moments in both cases is

$$\mu_1 = 0 \text{ y } \mu_2 \sim \frac{1}{2} + O(n^{-1}).$$

All odd moments vanish because our functionals are symmetric. Notice that equation (58) and relation $\xi_k = \dfrac{(-1)^k}{k!}\mathcal{Y}_k(-y_1, -y_2, -2y_3, ..., -(k-1)!y_k)$ provide us a general method to obtain all the moments $\mu_r = \frac{1}{n}y_r$, but it is highly non-linear and cumbersome. This is a reason why we use it only for the computation of the moments of low order. We want to remark here that the method described above allows a recurrent computation of the moments of any desired order and it can be implemented in any computer algebra system. See, for instance, [27], [33] where the corresponding symbolic programs were used to compute the moment of polynomial solutions of fourth-order differential equations.

5.2 The semiclassical density distribution of of zeros.

Next, we will analyze the so-called semiclassical or WKB approximation (see [5],[31] and references contained therein). Denoting the zeros of $\tilde{P}_n^A(x)$ by $\{x_{n,k}\}_{k=1}^n$ we can define its distribution function as

$$\rho_n(x) = \frac{1}{n}\sum_{k=1}^{n}\delta(x - x_{n,k}). \tag{59}$$

We will follow the method presented in [31] in order to obtain the WKB density of zeros, which is an approximate expression for the density of zeros of solutions of any second order linear differential equation with polynomial coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{60}$$

The main result is established in the following

Theorem 5.1 *Let $S(x)$ and $\epsilon(x)$ be the functions*

$$S(x) = \frac{1}{4a_2^2}\{2a_2(2a_0 - a_1') + a_1(2a_2' - a_1)\}, \tag{61}$$

$$\epsilon(x) = \frac{1}{4[S(x)]^2}\left\{\frac{5[S'(x)]^2}{4[S(x)]} - S''(x)\right\} = \frac{P(x,n)}{Q(x,n)}, \tag{62}$$

where $P(x,n)$ and $Q(x,n)$ are polynomials in x as well as in n. If the condition $\epsilon(x) << 1$ holds, then, the semiclassical or WKB density of zeros of the solutions of (60) is given by

$$\rho_{WKB}(x) = \frac{1}{\pi}\sqrt{S(x)}, \qquad x \in I \subseteq \mathbb{R}, \tag{63}$$

in every interval I where the function $S(x)$ is positive.

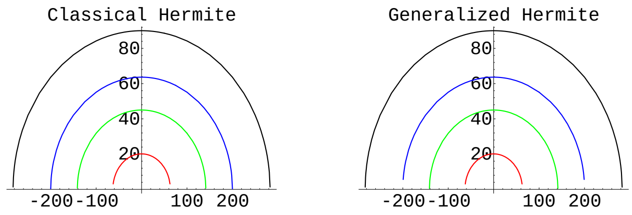

Figure 1: WKB density of zeros of the $H_n^A(x)$.

The proof of this Theorem can be found in [5], [31].

Now we can apply this result to our differential equation. Using the coefficients of the equation (56) we obtain that for sufficiently large n, $\epsilon(x) \sim n^{-1}$. From the above Theorem the corresponding WKB density of zeros of the polynomials $\tilde{P}_n^A(x)$ follows. The computations are very long and cumbersome. For this reason we provide a little program using *Mathematica* [30] and some graphics representation for the $\rho_{WKB}(x)$ function. We will analyze only the polynomials of even degree, i.e., $\tilde{P}_{2m}^A(x)$.

5.2.1 Hermite polynomials $H_{2m}^A(x)$.

In this case from (61) and (63)

$$\rho_{wkbclas}(x) = \frac{\sqrt{R(x)}}{(-B_m + 2\,B_m^2\,m + 2\,x^2 + 2\,B_m\,x^2)},$$

$$\begin{aligned} R(x) \;&= -6\,B_m - 3\,B_m^2 + 24\,B_m^2\,m + 8\,B_m^3\,m - 32\,B_m^3\,m^2 - 4\,B_m^4\,m^2 + 16\,B_m^4\,m^3 - \\ &-8\,B_m\,x^2 - 9\,B_m^2\,x^2 - 32\,B_m\,m\,x^2 - -32\,B_m^2\,m\,x^2 + 4\,B_m^3\,m\,x^2 + \\ &+32\,B_m^2\,m^2\,x^2 + 32\,B_m^3\,m^2\,x^2 - 4\,B_m^4\,m^2\,x^2 + 4\,x^4 + 12\,B_m\,x^4 - 8\,B_m^2\,x^4 + \\ &+16\,m\,x^4 + 32\,B_m\,m\,x^4 + 8\,B_m^2\,m\,x^4 - 8\,B_m^3\,m\,x^4 - 4\,x^6 - 8\,B_m\,x^6 - 4\,B_m^2\,x^6. \end{aligned}$$

If we take the limit $A \to 0$, we recover the classical expression [31], [32]

$$\rho_{wkb}^{\lambda}(x) = \frac{\sqrt{1 + 4\,m - x^2}}{\pi}.$$

Notice that, since $B_m \sim \frac{1}{2m}$, $\rho_{wkb}(x)$ has the asymptotic form

$$\rho_{wkb}^{asymp}(x) = \frac{\sqrt{-2 + x^2 + 4\,m\,x^2 - x^4}}{\pi\,x}.$$

In Figure 1 we represent the WKB density of zeros for our generalized Hermite polynomials. We have plotted the Density function for different values of n (from top to bottom) $n = 2 \times 10^4, 1.5 \times 10^4, 10^4, 10^3$. Notice that the value of the mass doesn't play a crucial role, since for $n >> 1$ $B_m \sim \frac{1}{2m}$, independently of A. It is important to take into account that our generalized polynomials have a lot of zeros near the origin. This follows from the fact that $\rho_{wkb}(x)$ have, asymptotically, a singular point at $x = 0$.

5.2.2 Gegenbauer polynomials $G_{2m}^{\lambda,A}(x)$.

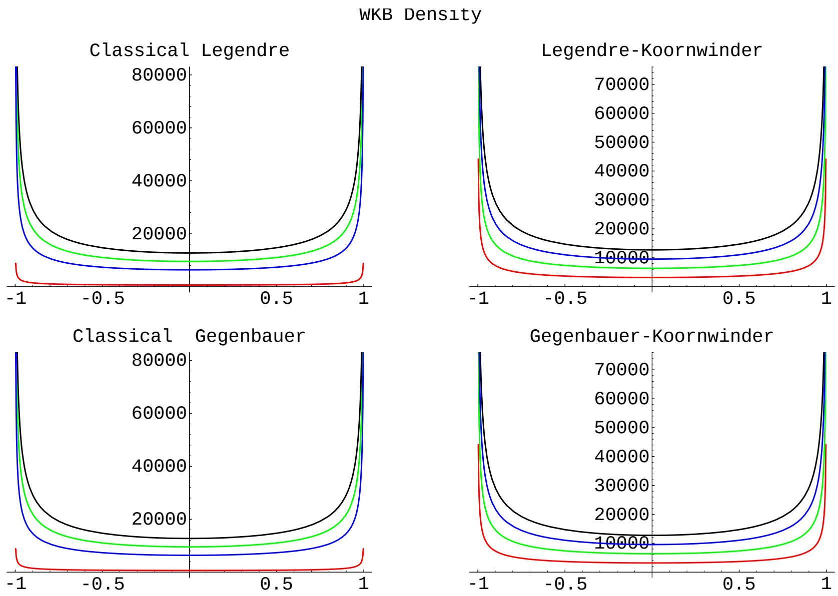

Figure 2: WKB density of zeros of the $G_n^{\lambda,A}(x)$.

In this case the expression is very large and we will provide only the limit case when $A \to 0$. In this case we recover the classical expression [31], [32]

$$\rho_{wkb}^{\lambda}(x) = \frac{\sqrt{2 + 16\,m^2 + 4\lambda + 16\,m\lambda + x^2 - 16\,m^2\,x^2 - 16\,m\lambda\,x^2 - 4\,\lambda^2\,x^2}}{2\pi(1-x^2)}.$$

For Legendre generalized polynomials we have, asymptotically,

$$\rho_{wkb}^{asymp}(x) = \frac{\sqrt{R^{asymp}(x)}}{\pi\ (1-x)\ (1+x)\ (1 - x^2 + m\,x^2 - 4\,m^2\,x^2 + 4\,m^3\,x^2)},$$

where

$$\begin{aligned} R^{asymp}(x) = \quad & 7 - 8\,m + 4\,m^2 - 12\,x^2 + 2\,m\,x^2 - 8\,m^2\,x^2 - 16\,m^3\,x^2 + 64\,m^4\,x^2 - 32\,m^5\,x^2 + \\ & + 7\,x^4 + 7\,m\,x^4 + 6\,m^2\,x^4 + 44\,m^3\,x^4 - 112\,m^4\,x^4 + 80\,m^5\,x^4 - 96\,m^6\,x^4 + 64\,m^7\,x^4 - \\ & - 2\,x^6 - 10\,m^2\,x^6 - 4\,m^3\,x^6 + 16\,m^4\,x^6 - 32\,m^5\,x^6 + 96\,m^6\,x^6 - 64\,m^7\,x^6. \end{aligned}$$

In Figure 2 we represent the WKB density of zeros for our generalized Gegenbauer polynomials. Notice that the value of the mass doesn't play a crucial role, since for $n >> 1$, $W_m \sim \frac{1}{2m^2}$, independently of A. We have plotted the Density function for different values of the degree of the polynomials (from top to bottom) $n = 2\times 10^4, 1.5\times 10^4, 10^4, 10^3$ for two different cases: the generalized Legendre polynomials ($\lambda = \frac{1}{2}$) and the generalized Gegenbauer with $\lambda = 5$.

5.2.3 Numerical Experiments.

As we can see in Figures 1 and 2, the zeros of the classical and generalized polynomials have the same behaviour. In order to convince ourself that really the influence of the masses is very small we compare the number N of zeros in a small interval, say $[-\frac{1}{10}, \frac{1}{10}]$, by computing the quantity

$$N = \int_{-\frac{1}{10}}^{\frac{1}{10}} \rho(x)\, dx \approx \int_{-\frac{1}{10}}^{\frac{1}{10}} \rho_{wkb}(x)\, dx,$$

which gives us (approximately) the number of zeros on such an interval. The result of such a calculation for the Legendre $L_{2m}^A(x)$ and Hermite $H_{2m}^A(x)$ families is given in table 1.

Table 1. The number of zeros in $[-\frac{1}{10}, \frac{1}{10}]$ of the $L_{2m}^A(x)$ and $H_{2m}^A(x)$ polynomials.

$2m$	$L_{2m}(x)$	$L_{2m}^A(x)$	$H_{2m}(x)$	$H_{2m}^A(x)$
100000	12753.7	12753.8	40.2634	38.8898
200000	25507.5	25507.5	56.941	55.4299
300000	38261.2	38261.2	69.7382	68.047
400000	51014.9	51014.9	80.5268	78.6623
500000	63768.6	63768.7	90.0317	88.0054
600000	76522.3	76522.4	98.6247	96.4471
700000	89276.	89276.1	106.527	104.207
800000	102030.	102030.	113.882	111.428
900000	114783.	114784.	120.79	118.208
1000000	127537.	127537	127.324	124.62

ACKNOWLEDGEMENTS

The research of the authors was supported by Dirección General de Investigación Científica y Técnica (DGICYT) of Spain under grant PB 93-0228-C02-01. The authors thank to the referee for its useful suggestions and remarks and also for pointing out references [27] and [33].

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