SPECIALIZATIONS OF MACMAHON SYMMETRIC FUNCTIONS AND THE POLYNOMIAL ALGEBRA.

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ABSTRACT. A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group. We use a combinatorial construction of the different bases of the vector space of MacMahon symmetric functions found by the author to obtain their image under the principal specialization: the powers, risings and falling factorials. Then, we compute the connection coefficients of the different polynomial bases in a combinatorial way.

1. INTRODUCTION

The aim of this paper is to compute the connection coefficients between the different polynomial bases by using specializations of MacMahon symmetric functions.

MacMahon symmetric functions were systematically studied by MacMahon [7], Vol. II, section XI, p. 281–332, who applied them to the problem of placing balls into boxes and to the theory of Latin squares. Later, they have been used by Gessel [5] and Haiman [6] in connection with enumerative combinatorics, by Gelfand and Dikii [4], and Olver and Shakiban [9] in connection with the theory of partial differential equations, by Rota and Stein [13] and Olver [8] in connection to classical invariant theory, and by Adem, Maginnis and Milgram [1] in the study of the cohomology of the symmetric group.

In this article we show how the combinatorial construction of the MacMahon symmetric functions obtained in [10] allows us to obtain their image under the principal specialization in a combinatorial way. Joni, Rota, and

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Sagan [3] studied the powers, rising, and falling factorial bases of the algebra of polynomials. They defined three different posets and then used the Möbius function to find the transition matrices between any two of them. They claimed that the only step in their approach that was not clear was how to find the right poset. We show that using specializations of MacMahon symmetric functions their choice of posets becomes transparent.

2. Combinatorial construction.

Let u be a vector in $\mathbf{N}^k$, where $\mathbf{N}$ is the set of nonnegative integers. A vector partition λ of u is an unordered sequence of vectors summing to u. The length of λ, written $l(\lambda)$, is the number of nonzero vectors (parts) of λ. We can write λ using block notation: $\lambda = \cdots (a_i, b_i, \ldots, c_i)^{m_i} \cdots$, with part $(a_i, b_i, , \cdots, c_i)$ appearing m_i times in λ and with i is running over all different parts of λ. In this article λ will always be a vector partition of $(a, b, \cdots, c)$, where $a + b + \cdots + c = n$.

Let $S_\infty = \cup_{i \geq 1} S_i$, where the symmetric group S_i acts on the first i letters. The symmetric group S_∞ acts diagonally on f in $\mathbf{Q}[[X, Y, \cdots, Z]]$ sending $w(f)$ to $f(x_{w(1)}, y_{w(1)}, \cdots, z_{w(1)}, x_{w(2)}, y_{w(2)}, \cdots, z_{w(2)}, \cdots)$. A formal power series f in $\mathbf{Q}[[X, Y, \cdots, Z]]$ is called a MacMahon symmetric function in k systems of indeterminates if the degree of f is bounded and if it is invariant under the diagonal action of S_∞. We say that f has homogeneous multidegree $u = (a, b, \cdots, c)$ if in each monomial term of f there are a letters in alphabet X, b letters in alphabet Y, and so on. We denote by $\mathfrak{M}_u$ the vector space of MacMahon symmetric functions of multihomogeneous degree u.

Let $\mathbf{x}^\lambda$ be $x_1^{a_1} y_1^{b_1} \cdots z_1^{c_1} x_2^{a_2} y_2^{b_2} \cdots z_2^{c_2} \cdots x_l^{a_l} y_l^{b_l} \cdots z_l^{c_l}$. Then, the monomial MacMahon symmetric function m_λ is the sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation w in S_∞ acting diagonally.

The elementary MacMahon symmetric function indexed by vector partition λ is $e_\lambda = e_{(a_1, b_1, \cdots, c_1)} e_{(a_2, b_2, \cdots, c_2)} \cdots$, where $e_{(a,b,\cdots,c)}$ is defined by the generating function:

$$\sum_{a,b,\cdots,c \geq 0} e_{(a,b,\cdots,c)} s^a t^b \cdots u^c = \prod_{i \geq 1} (1 + x_i s + y_i t + \cdots + z_i u).$$

The complete homogeneous MacMahon symmetric function indexed by vector partition λ is $h_\lambda = h_{(a_1,b_1,\cdots,c_1)}h_{(a_2,b_2,\cdots,c_2)}\cdots$, where $h_{(a,b,\cdots,c)}$ is defined by the generating function:

$$\sum_{a,b,\cdots,c\geq 0} h_{(a,b,\cdots,c)}s^a t^b \cdots u^c = \prod_{i\geq 1} \frac{1}{1 - x_i s - y_i t - \cdots - z_i u}.$$

The power sum MacMahon symmetric function indexed by vector partition λ is $p_\lambda = p_{(a_1,b_1,\cdots,c_1)}p_{(a_2,b_2,\cdots,c_2)}\cdots$, where $p_{(a,b,\cdots,c)}$ is $\sum_{i\geq 1} x_i^a y_i^b \cdots z_i^c$.

In the ring of MacMahon Symmetric Functions there is an involution defined by $\omega(e_\lambda) = h_\lambda$. The forgotten MacMahon symmetric functions are defined by $\omega(m_\lambda) = (\operatorname{sign}\lambda)\, f_\lambda$. To λ we associate the partition of the (number) weight of λ defined by $(a_1 + \cdots + c_1, a_2 + \cdots + c_2, \cdots) = (1^{n_1}2^{n_2}\cdots)$. Then, the sign of λ is defined as $(-1)^{n_2+2n_3+3n_4+\cdots}$.

MacMahon symmetric functions are the generating function for orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of the symmetric group [10, 12]. This construction allows us to describe the effect of the principal specialization in the different bases for the ring of polynomials in a combinatorial way, very much in the spirit of the work of Gian-Carlo Rota and his school.

A vector partition is unitary if it is a partition of $(1)^n = (1, 1, \cdots , 1)$. Similarly, a monomial (elementary, etc.) MacMahon symmetric function is unitary if it is indexed by a unitary vector partition. Unitary vector partitions can be identified with set partitions: To $\pi = \{B_1, B_2, \cdots , B_l\}$ we associate the unitary vector partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ where λ_j has its ith coordinate 1 if i is in B_j and 0 otherwise.

The Young subgroup S_u of S_n is defined by

$$S_u = S_{\{1,2,\cdots,a\}} \times S_{\{a+1,a+2,\cdots,a+b\}} \times \cdots \times S_{\{n-c+1,n-c+2,\cdots,n\}}.$$

There is a canonical action of S_u on $[n]$. It partitions $[n]$ into equivalence classes that we order using the smallest element in each of them.

The type of a set partition $\pi = B_1|B_2|\cdots|B_l$ under the action of S_u, denoted $\operatorname{type}_u(\pi)$, is the vector partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$, where λ_k is the vector whose i coordinate is the number of elements of B_k in the ith equivalence class. If u equals (n), we may omit the subindex (n).

Let F_n be the set of all functions from $[n]$ to $\mathbf{P}$, the set of positive integers.. Each f in F_n defines a set partition $\ker f$, where n_1 and n_2 are in the same block of $\ker f$ if and only if $f(n_1)$ equals $f(n_2)$. We read the expression $f(i) = j$ as saying that ball i has been placed on box j.

Let f be in F_n. Suppose that the Young subgroup S_u is acting on $[n]$. We weight f by $\gamma_u(f) = \prod_{d \in [n]} c(d)_{f(d)}$ where $c(d)$ denotes the equivalence class of d and we use variables $x, y, \cdots, z$ to denote the equivalence classes. In the particular case where $u = (1)^n$, we denote γ_u by γ. To a set of functions T we associate the generating function: $\gamma_u(T) = \sum_{f \in T} \gamma_u(f)$.

A disposition is an arrangement of the balls (that is, the elements of $[n]$) into the boxes (that is, the positive numbers, $\mathbf{P}$), where we may impose some condition on the way the balls are placed. In particular, a function is a disposition where there is no condition on the way the balls are placed. The underlying function of a disposition p is the function obtained from p if we forget about the extra data condition on the balls. The weight of a disposition is defined as the weight of its underlying function. The kernel of a disposition p, written as $\ker p$, is the kernel of its underlying function.

Definition 1 (The projection map)**.** *Let S_u be a Young subgroup of S_n acting on $[n]$. Given any function $f : [n] \to \mathbf{P}$. Let $\gamma_u(f)$ be defined as $\gamma_u(f_u(i, c(i)))$, where $f_u(i, c(i)) = f(i)$, and $c(i)$ is the equivalence class of i under S_u. Given a set of functions T going from $[n]$ to $\mathbf{P}$ we define $\gamma_u(T)$ as $\sum_{f \in T} \gamma_u(f)$.*

The map sending $\gamma(T)$ to $\gamma_u(T)$ is called the projection map and denoted ρ_u. Given any set of dispositions, the projection map is defined on their underlying functions.

Definition 2 (Doubilet)**.** *Let π be a set partition of $[n]$.*

(1) $\mathcal{M}_\pi = \{f : f \in F_n,\ \ker f = \pi\}$, *and let m_π be $\gamma(\mathcal{M}_\pi)$.*
(2) $\mathcal{P}_\pi = \{f : f \in F_n,\ \ker f \geq \pi\}$, *and let p_π be $\gamma(\mathcal{P}_\pi)$.*
(3) $\mathcal{E}_\pi = \{f : f \in F_n,\ \ker f \wedge \pi = \hat{0}\}$, *and let e_π be $\gamma(\mathcal{E}_\pi)$.*
(4) *Let $\mathcal{H}_\pi$ be the set of dispositions such that within each box the balls from the same block of π are linearly ordered, and let h_π be $\gamma(\mathcal{H}_\pi)$.*
(5) *Let $\mathcal{F}_\pi$ be the set of dispositions such that balls from the same block of π go into the same box, and within each box the blocks appearing are linearly ordered, and let f_π be $\gamma(\mathcal{F}_\pi)$.*

For any vector partition $\lambda = (a_1, b_1, \ldots, c_1) \cdots (a_l, b_l, \ldots, c_l)$, (or $\lambda = \cdots (a_i, b_i, \cdots c_i)^{m_i} \cdots$ when written in block notation), define $|\lambda| = \prod_i m_i!$, and $\lambda! = a_1!\, b_1! \cdots c_1!\, a_2!\, b_2! \cdots c_2! \cdots a_l!\, b_l! \cdots c_l!$.

Theorem 3. *Let S_u be a Young subgroup of S_n. Let π be a set partition of $[n]$ and let λ be the type π under S_u. Then, under the projection map ρ_u*

$$m_\pi \mapsto |\lambda|\, m_\lambda \qquad p_\pi \mapsto p_\lambda \qquad e_\pi \mapsto \lambda!\, e_\lambda$$
$$h_\pi \mapsto \lambda!\, h_\lambda \qquad f_\pi \mapsto |\lambda|\, f_\lambda.$$

In particular, $\rho_u : \mathfrak{M}_{(1)^n} \to \mathfrak{M}_u$.

Proof. As an illustration we show that $m_\pi \mapsto |\lambda|\, m_\lambda$.

If f is in $\mathcal{M}_\pi$, then, $\gamma_u(f) = x_{i_1}^{a_1} y_{i_1}^{b_1} \cdots z_{i_1}^{c_1}\, x_{i_2}^{a_2} \cdots z_{i_2}^{c_2} \cdots\, x_{i_l}^{a_l} y_{i_l}^{b_l} \cdots z_{i_l}^{c_l}$.

$$\rho_u(m_\pi) = \sum_{\substack{i_1, i_2, \cdots, i_l \geq 1 \\ \text{different}}} x_{i_1}^{a_1} y_{i_1}^{b_1} \ldots z_{i_1}^{c_1} \cdot x_{i_2}^{a_2} y_{i_2}^{b_2} \ldots z_{i_2}^{c_2} \cdots x_{i_l}^{a_l} y_{i_l}^{b_l} \ldots z_{i_l}^{c_l}.$$

Any monomial appears $m_1!\, m_2! \ldots m_l!$ times. Therefore, we have that $\rho_u(m_\pi) = |\lambda| m_\lambda$. The full proof appears in [10]. □

3. The principal specialization

We study the effect of the principal specialization ps_k^1 on the different bases of the algebra of symmetric functions.

Let f be a MacMahon symmetric function. We follow Stanley [14] and define the principal specialization ps_k^1 by setting $x_i = y_i = \cdots = z_i = 1$, if $i \leq k$ and $x_i = y_i = \cdots = z_i = 0$, otherwise. The principal specialization defines an algebra morphism from the MacMahon symmetric functions to the polynomials in the variable k.

Other generalizations of the principal specialization having several independent parameters to distinguish between the different alphabets do not seem to work as nicely.

Let λ be a partition of n of length l. We use the following notation

$$(k)_n = k(k-1) \cdots (k-n+1)$$
$$(k)^n = k(k+1) \cdots (k+n-1)$$
$$(k)_\lambda = (k)_{\lambda_1} (k)_{\lambda_2} \cdots (k)_{\lambda_l}$$
$$(k)^\lambda = (k)^{\lambda_1} (k)^{\lambda_2} \cdots (k)^{\lambda_l}$$

Theorem 4. *Let π be in the partition lattice Π_n, let λ be the partition defined by the size of its blocks of π, and let $l(\pi)$ be the number of blocks of π. Then*

$$\mathrm{ps}_k^1(m_\pi) = (k)_{l(\pi)} \qquad \mathrm{ps}_k^1(h_\pi) = (k)^\lambda$$
$$\mathrm{ps}_k^1(p_\pi) = k^{l(\pi)} \qquad \mathrm{ps}_k^1(e_\pi) = (k)_\lambda$$
$$\mathrm{ps}_k^1(f_\pi) = (k)^{l(\pi)}$$

Proof. Let T be any subset of F_n and let t be its generating function. Then $\mathrm{ps}_k^1(t)$ is equal to the number of functions in T such that their image is contained in $[k]$. Then, from Definition 2 we obtain the following results.

- There are $(k)_{l(\pi)}$ functions from $[n]$ to $[k]$ with kernel π.
- There are $(k)^n$ dispositions from $[n]$ to $[k]$.
- There are $k^{l(\pi)}$ functions from $[n]$ to $[k]$ that are constant on the blocks of π.
- There are $(k)_\lambda$ functions from $[n]$ to $[k]$ that are injective on the blocks of π.
- There are $(k)^\lambda$ dispositions from the set of blocks of $\ker \pi$ to k.

To count the number of disposition from $[n]$ to $[k]$ we proceed as follows. The image of 1 can be chosen in k different ways, the image of 2 can be chosen in $k+1$ ways. (There are k possible images for 2, but in the case where $f(1) = f(2)$ we must also choose the order of 1 and 2). Using induction, we see that the number of dispositions is $(k)^n = k(k+1)\cdots(k+n-1)$.

□

Applying the projection map $\rho_{(n)}$ to the equations obtained in Theorem 4, we obtain the classical result for the effect of the principal specialization on symmetric functions.

Corollary 5. *Let λ be a partition of number n. Then,*

$$\mathrm{ps}_k^1(m_\lambda) = \frac{(k)_{l(\lambda)}}{|\lambda|} \qquad \mathrm{ps}_k^1(h_\lambda) = \frac{(k)^\lambda}{\lambda!}$$
$$\mathrm{ps}_k^1(p_\lambda) = k^{l(\lambda)} \qquad \mathrm{ps}_k^1(e_\lambda) = \frac{(k)_\lambda}{\lambda!}$$
$$\mathrm{ps}_k^1(f_\lambda) = \frac{(k)^{l(\lambda)}}{|\lambda|}$$

Proof. It follows from Theorem 3 and Theorem 4. □

We use the results that we have obtained so far to compute the transition matrices between the different bases of the algebra of polynomials $\mathbf{C}[k]$, in a combinatorial way.

Theorem 6. *The connection constant formulae for the polynomial bases* $\{k^n\}$, $\{(k)_n\}$, *and* $\{(k)^n\}$ *are*

$$k^n = \sum_{i\geq 0} S(n,i)(k)_i \qquad (k)_n = \sum_{i\geq 0} s(n,i)k^i$$

$$(k)^n = \sum_{i\geq 0} \frac{n!}{i!}\binom{n-1}{i-1}(k)_i \qquad (k)_n = \sum_{i\geq 0}(-1)^{n-i}\frac{n!}{i!}\binom{n-1}{i-1}(k)^i$$

$$(k)^n = \sum_{i\geq 0} |s(n,i)|k^i \qquad k^n = \sum_{i\geq 0}(-1)^{n-i}S(n,i)(k)^i$$

Proof. From the definitions, it is immediate that F_n is equal to $\bigcup_{\pi\in\Pi_n} M_\pi$. Apply the homomorphism ps^1_k to their generating functions to obtain that

$$k^n = \sum_{\pi\in\Pi_n} (k)_{l(\pi)} = \sum_{i\geq 0} S(n,i)(k)_i, \tag{1}$$

because $S(n,i)$, the Stirling number of the second kind, counts the number of partitions of an n–set into k nonempty blocks.

It is well-known that $S(n,k)$ are the Whitney numbers of the second kind for the poset Π_n. Moreover, the Stirling numbers of the first kind, $s(n,k)$, are the corresponding Whitney numbers of the first kind. Therefore, if we apply Möbius inversion to (1) we obtain

$$(k)_n = \sum_{\pi\in\Pi_n} \mu(\pi)k^{l(\pi)} = \sum_{i} k^i \sum_{\substack{\pi\in\Pi_n\\ l(\pi)=i}} \mu(\pi) = \sum_{i\geq 0} s(n,i)k^i.$$

Define a linear partition to be a partition of n, together with a total order on each block. Let $\mathcal{L}_n$ be the poset of linear partitions of $[n]$.

To any disposition $p : [n] \to \mathbf{P}$ we associate a linear partition defined by $\ker p$ together with the ordering of the balls. Hence, we have the following equation:

$$h_{[n]} = \sum_{\sigma\in\mathcal{L}_n} m_\sigma, \tag{2}$$

Apply homomorphism ps_k^1 to both sides of equation (2) to obtain

$$(k)^n = \sum_{\sigma\in\mathcal{L}_n} (k)_{l(\sigma)} = \sum_{i\geq 0} \frac{n!}{i!}\binom{n-1}{i-1}(k)_i.$$

because the number of linear partitions of $[n]$ into i blocks is $\frac{n!}{i!}\binom{n-1}{i-1}$. These numbers are known as the Lah numbers [3]. Apply Möbius inversion to equation (2) to obtain $(k)_n = \sum_{\lambda\in\mathcal{L}_n}\mu(\lambda)(k)^{l(\lambda)}$.

For all λ in $\mathcal{L}_n$ we have that $\mu(\lambda) = (-1)^{n-l(\lambda)}$ because $[0,\lambda]$ is a Boolean lattice. Therefore,

$$(k)_n = \sum_{\sigma\in\mathcal{L}_n}(-1)^{n-l(\lambda)}(k)^{l(\lambda)} = \sum_{i\geq 0}(-1)^{n-i}\frac{n!}{i!}\binom{n-1}{i-1}(k)^i.$$

To each partition $\sigma = B_1|B_2|\cdots|B_l$, we can associate in a canonical way $\mu(\hat{0},\sigma) = (B_1-1)!(B_2-1)!\cdots(B_l-1)!$ sets of Lyndon words. Given one of such sets of Lyndon words, for each $\sigma' \geq \sigma$ we obtain one different linear partition. Hence,

$$h_{[n]} = \sum_{\sigma\in\Pi_n}|\mu(\hat{0},\sigma)|p_\sigma.$$

Apply the homomorphism ps_k^1 to both sides of the previous equation to obtain that $(k)^n = \sum_{\sigma\in\Pi_n}|\mu(\hat{0},\sigma)|k^{l(\sigma)}$. Then, since $|\mu(\hat{0},\sigma)|$ counts sets of Lyndon words, we obtain

$$(k)^n = \sum_{\sigma\in S_n} k^{\mathrm{cycles}(\sigma)}, \tag{3}$$

where cycles(σ) is the number of cycles of σ. Finally, we get

$$(k)^n = \sum_{i\geq 0}|s(n,i)|k^i, \tag{4}$$

because $|s(n,i)|$, the signless Stirling number of the second kind, counts the number of permutations in S_n with i cycles.

We give S_n a poset structure induced by the refinement order on partitions. We say that $\sigma \leq \tau$ if each cycle of σ, written with the smallest element first, is composed of a string of consecutive integers from the cycles of τ. For instance, (123)(4) is smaller that (1234), but (124)(3) is not.

If we apply Möbius inversion to the equation 3 we obtain that k^n equals

$$\sum_{\sigma\in S_n}\mu(\sigma)(k)^{\mathrm{cycles}(\sigma)} = \sum_i(-1)^{n-i}(k)^i \sum_{\substack{\sigma \text{ increasing} \\ \mathrm{cycles}(\sigma)=i}} 1 = \sum_{i\geq 0}(-1)^{n-i}S(n,i)(k)^i.$$

Because $\mu(\sigma)$ is zero unless each cycle of σ increases from left to right, in which case $\mu(\sigma) = (-1)^{n-\text{cycles}(\sigma)}$, and the number of increasing permutations with i cycles is $S(n,i)$, the number of partitions of $[n]$ with i blocks.

All three Möbius inversion arguments are fully explained in [3]. □

Finally, we use these results to describe the product of elements of the different polynomial bases. First, we introduce some notation.

Definition 7. *Let* $\text{disj}(\pi, i)$ *be the number of partitions* σ *of length* i *such that* $\sigma \wedge \pi = \hat{0}$.

Corollary 8. *Let* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ *be a partition of* n.

Let $\pi = 12\cdots\lambda_1|\cdots|n-\lambda_l+1\cdots n$. *Then*

$$(k)_{\lambda_1}(k)_{\lambda_2}\cdots(k)_{\lambda_l} = \sum_{i\geq 0}\text{disj}(\pi,i)(k)_i.$$

$$(k)^{\lambda_1}(k)^{\lambda_2}\cdots(k)^{\lambda_l} = \sum_{i\geq 0}(-1)^{n-i}\,\text{disj}(\pi,i)(k)^i.$$

Proof.

- Apply the principal specialization ps_k^1 to both sides of equation $e_\pi = \sum_{\sigma:\sigma\wedge\pi=\hat{0}} m_\sigma$.
- Apply the principal specialization ps_k^1 to both sides of equation $h_\pi = \sum_{\sigma:\sigma\wedge\pi=\hat{0}} \text{sign}(\sigma) f_\sigma$.

□

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