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# BOLLETTINO UNIONE MATEMATICA ITALIANA

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RAÚL M. FALCÓN, JUAN NÚÑEZ VALDÉS

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*Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. 8-B (2005),*  
n.2, p. 437–452.

Unione Matematica Italiana

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## Isorings and Related Isostructures.

RAÚL M. FALCÓN - JUAN NÚÑEZ VALDÉS

**Sunto.** – *Lo scopo principale di questo articolo è dare una fondazione matematica, seria e costante a delle parti dell'isoteoria di Santilli. Noi studiamo i sollevamenti isotopici di anelli, sottoanelli ed ideali. Usando il modello della isomoltiplicazione, condizioni necessarie che assicurano l'esistenza di tali isostrutture sono date. Tali condizioni sono basate sulle leggi interne che originano gli elementi associati di sollevamenti isotopici. Questi elementi permetteranno di estendere, da un punto di vista diverso lo studio di Santilli di teoria generalizzata e non-lineare. Molti esempi di queste isostrutture sono dati. Noi troviamo infine le differenze tra un isoanello quoziente ed un anello quoziente provenienti da un isoanello e da uno dei suoi isoideali.*

**Summary.** – *The main goal of this paper is to give a mathematical foundation, serious and consistent, to some parts of Santilli's isotheory. We study the isotopic liftings of rings, subrings and ideals, and we also introduce the concept of quotient isoring. By using the isoproduct model, necessary conditions assuring the existence of such isostructures are given. Such conditions are based on the inner laws which originate the associated elements of isotopy. These elements will allow to extend, from a different point of view, the Santilli's study of non-linear generalized theory. Several examples of these isostructures are also shown. We finally find the differences between a quotient isoring and a quotient ring coming from an isoring and one of its isoideals.*

### Introduction.

In 1978, the Italian-American theoretical physicist and mathematic Ruggero Maria Santilli proposes a generalization of conventional Lie's theory by using the concept of *isotopy* (in the Greek sense of being «axiom-preserving», also called *isotopic lifting*), which implies the origin of the actually known like *Santilli's isotheory* (see [4]). To do this, he extends the basic unit  $I = (+1, \text{diag}(+1, \dots, +1), \dots)$  of the initial structure to a generalized unit  $\hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, \dots, \mu, \tau, \dots)$ , called *isounit*, which depends on the coordinate  $x$  and its derivatives, on the density  $\mu$ , on the temperature  $\tau$  and, in general, on any magnitude of the physic environment of the system in which we are. By using it, Santilli does a step-by-step generalization of the more important mathematical structures, obtaining other new ones, characterized by the fact of having

the same properties as the initial ones, while the new units satisfy more general conditions than the verified by the initial ones. Santilli gives the name of *mathematical isostructures* to these new structures. In this way, he studies *isogroups*, *isorings*, *isofields*, *isovectorspaces* and *isoalgebras* (see [5], [6], [7] and [13], for instance). Isonumber theory has been also dealt by Jiang, in [2].

It allowed him to get in a fast way some development of physical applications, principally in Quantum Mechanics and Dynamical Problems of particles and antiparticles. Santilli's isotopies allow to map any given and fixed linear, local and canonical structure into its most general possible non-linear, non-local and non-canonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within the fixed inertial coordinates of the observer.

However, in the last years, Santilli has found some mathematical inconsistencies in his early formulation of the isothory. Due to it, Santilli and other mathematicians have studied the isotopic liftings of Functional Analysis and Differential Calculus (see [3] and [8]). It has allowed to get some important applications in Physics (see [9], [10], [11] and [12], for instance).

So, as Santilli's isothory needs even a consistent mathematical foundation, Santilli himself has proposed several subjects of research to the international mathematical-scientific community. One of them consists on proving the existence of isostructures corresponding to the lifting of structures already known, although they do not have a practical application in Physic. Santilli thinks that it would be good to convince the scientific about the relevance of his research, which would give bigger consistence and reliance to his isothory.

So, we are trying to partially response to Santilli's petition. Indeed, in a recent paper (see [1]) we have begun to settle the mathematical foundations of this theory by dealing with the simplest algebraic structure: the groups. Indeed, we studied in that paper the isotopic liftings of groups and subgroups.

In this paper we continue this research by considering the following structure in an increasing order of importance: the rings. We firstly study the isotopic lifting of a ring and we deal next with ideals and quotient rings.

To do this, we previously give in Section 1 some basic definitions related to isotopic liftings. Section 2 is devoted to the study of isoring. Section 3 deals with isoideals, giving some examples. Finally, we introduce the concept of quotient isoring and we distinguish between it and a quotient ring coming from an isoring and one of its isoideals.

## 1. – Preliminaries.

Remember that for a given and fixed mathematical structure, an *isotopy* or *isotopic lifting* is any lifting of it, which gives a new mathematical structure

verifying the same basic axioms (or properties) as the first. This new structure is called *isotopic structure* or *isostructure* (see [4]).

In 1978 (see [4]), Santilli proposes a possible model of isotopy, which he calls *Santilli's isotopy*, which allows to construct the named *mathematical isostructure*, based on an isounit  $I$ . This isounit can be obtained starting from the following definition:

Let  $E$  be any mathematical structure, defined on a set of elements  $C$ . Let  $V \supseteq C$  be a set with an inner law  $*$  and an unit element  $I$ . Such a set  $V$  is said to be the *general set of the isotopy*. Let  $\hat{I} \in V$  be such that its inverse  $T = \hat{I}^{-1}$ , with respect to the law  $*$ , exists.  $I$  will be called *isotopic unit* or *isounit* and it will be the basic unit in the lifting of the structure  $E$ .  $T$  will be the *isotopic element*. Finally,  $\hat{I}$  and  $*$  are the *elements of isotopy*.

Then, Santilli proposes to reach an isostructure  $\widehat{E}$  starting from the structure  $E$ , by considering the following construction levels:

a) **Conventional level:** (see [4]) It is the initial mathematical structure, formed by a set of elements and the laws defined among them. In this level appear the usual mathematical structures with respect to usual units:  $E = E(a, +, \times, \dots)$ .

b) **General level:** It is the general set  $V$ , in which are, particularly, the isotopy elements used in the isoproduct construction model, that is,  $V = V(a, *, \star, \dots)$ . It is important to note that  $E_* = E(a, *, \star, \dots)$  (the restriction of  $V$  to  $E$ ) must verify the same axioms as the initial structure  $E$ .

c) **Isotopic level:** (see [4]) It is the mathematical isostructure obtained when lifting, that is  $\widehat{E} = \widehat{E}(\widehat{a}, \widehat{+}, \widehat{\times}, \dots)$ .

It is formed by an isotopic set and the isolaws on it. Elements of such set, which are usually denoted by using a hat, are given with respect to the isounit of  $\widehat{E}$ . So, fixed and given the isostructure  $(\widehat{E}, \widehat{\times})$ , with isounit  $\hat{I}$ , where  $I$  is the unit of  $E$  with respect to  $*$ , these elements are  $\widehat{a} = \widehat{a} \widehat{\times} \hat{I}$ , where Santilli defines the law  $\widehat{\times}$  as  $\widehat{a} \widehat{\times} \widehat{b} = \widehat{a * b}$ . See then that,  $\widehat{a} \widehat{\times} \hat{I} = \widehat{a * I} = \widehat{a} = \hat{I} \widehat{\times} \widehat{a}$ , which implies that  $\hat{I}$  is the unit element of  $\widehat{E}$  with respect to  $\widehat{\times}$ .

It is immediate to check that the mapping  $I : E \rightarrow \widehat{E} : a \rightarrow \widehat{a}$  is a bijection, because it is onto by construction and it is also injective, due to  $\widehat{a} \neq \widehat{b}$ , for all  $a, b \in E$  such that  $a \neq b$ . Indeed, in  $\widehat{E}$ ,  $\widehat{a} = \widehat{a} \widehat{\times} \hat{I} \neq \widehat{b} \widehat{\times} \hat{I} = \widehat{b}$  with respect to the isounit  $\hat{I}$  of  $\widehat{E}$ ; in the same way as  $a = a \times e \neq b \times e = b$  in  $E$ , where  $e$  is the unit element of  $E$  with respect to  $\times$ .

d) **Projection level:** (see [8]) It appears when we consider the mathematical isostructure  $\widehat{E}$  referred to the isotopy elements used in its construction. Its elements are denoted by a line superposed to the hat  $\widehat{\cdot}$  of elements of  $\widehat{E}$ , that is,  $\overline{\widehat{\cdot}}$ .

In this way, if we use the isotopy elements  $*$  (with unit  $I$ ) and  $\hat{I}$  to con-

struct  $\widehat{E}$ , then we obtain a structure  $\overline{\widehat{E}}$  in the projection level, whose elements are referred to the unit  $I$ :  $\overline{\widehat{a}} = a * I = (a * I) * I$ .

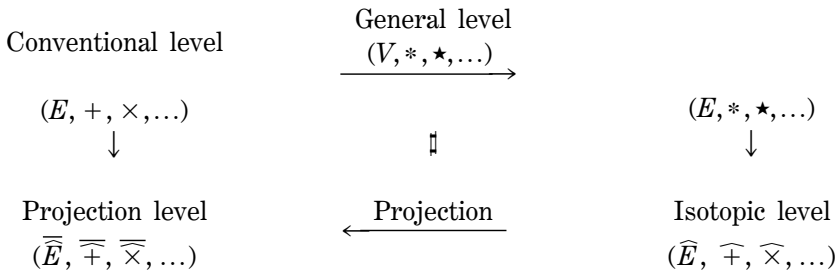
The mapping  $\pi : \widehat{E} \rightarrow \overline{\widehat{E}} : \widehat{a} \rightarrow \pi(\widehat{a}) = \overline{\widehat{a}}$  is named *projection*. In general, we say that an element of  $\widehat{E}$  is *projected* on its corresponding associated element belonging to  $\overline{\widehat{E}}$ . Note that, by construction, the mapping  $\pi$  is onto.

In a first stage,  $\overline{\widehat{E}}$  is only doted with laws when  $\pi$  is linear with respect to the isolaws associated with  $\widehat{E}$ . So, fixed an isostructure  $(\widehat{E}, \widehat{\times})$ , if  $\pi$  is linear with respect to  $\widehat{\times}$ , the law  $\overline{\widehat{\times}}$  is defined on  $\overline{\widehat{E}}$  by  $\overline{\widehat{a}} \overline{\widehat{\times}} \overline{\widehat{b}} = \overline{\widehat{a \times b}}$ . In such a case,  $\pi : (\widehat{E}, \widehat{\times}) \mapsto (\overline{\widehat{E}}, \overline{\widehat{\times}})$  is an onto morphism.

Therefore, this projection level is the most important in practice, because it allows to obtain some mathematical models which would be no possible under usual units.

There exists still another level which joins both conventional and isotopic levels. It is the *axiomatic level* ([4]), which identifies every mathematical structure verifying the same axioms.

So, in a schematic way, as the different construction levels appearing in an isotopic lifting, as the relations among them can be observed in the following diagram:



Finally, we will say that an isotopic lifting of the structure  $E$  is *injective* if  $X = Y$ , for all  $X, Y \in E$  such that  $\overline{\widehat{X}} = \overline{\widehat{Y}}$ . It is equivalent, by construction, to say that the projection  $\pi : \widehat{E} \rightarrow \overline{\widehat{E}} : \widehat{a} \rightarrow \pi(\widehat{a}) = \overline{\widehat{a}}$  is an injective mapping. Therefore, as a consequence, if the isotopic lifting of  $E$  is injective, then  $\pi : \widehat{E} \mapsto \overline{\widehat{E}}$  will be an isomorphism.

**2. – Isorings.**

In this Section we show the way in which rings can be isotopically lifted. We start introducing a more general definition than the given by Tsagas and Sourlas, in [13]:

DEFINITION 2.1. – Let  $(A, \circ, \bullet)$  be a ring with unit element  $e$ . An isoring  $\widehat{A}$  is an isotopy of  $A$ , with two new inner composition laws,  $\widehat{\circ}$  and  $\widehat{\bullet}$ , the sec-

ond of them with unit element  $\widehat{I} \in \widehat{A}$  (which is called isounit, not necessarily belonging to  $A$ ), verifying rings properties.

We already know that every isotopy is determined by an isounit and a law  $*$ . However, on this occasion, we dispose of two isounits,  $\widehat{S}$  and  $\widehat{I}$ , in the isostructure. So, we ask ourselves if two different isotopies are needed to construct an isoring.

To answer this question, we fixe, in the first place, an associative law  $*$  and an isounit (which we denote by  $\widehat{I}$ , because by construction we hope that it is the same as the isounit with respect to  $\widehat{\circ}$ ). We now construct in an explicit way the isotopic set  $\widehat{A}$  associated with  $A$ , where  $\widehat{A} = \{\widehat{a} = a * \widehat{I} \mid a \in A\}$ .

Next, we have to construct the lifting of the corresponding laws  $\circ$  and  $\bullet$ . To do this, we will begin with the isotopic lifting of the second law,  $\bullet$ , to use the isoproduct construction model, as follows. If  $I$  is the unit of  $*$  and  $T = \widehat{I}^{-1}$ , then the isoproduct will be the law  $\widehat{\bullet}$ , defined by

$$\widehat{a} \widehat{\bullet} \widehat{b} = \widehat{a * b} = I(a * b), \quad \forall \widehat{a}, \widehat{b} \in \widehat{A}.$$

So, if the isotopy is injective, we define in the projection level

$$\widehat{a} \bullet \widehat{b} = \widehat{\widehat{a} \widehat{\bullet} \widehat{b}} = (a * b) * \widehat{I} = \widehat{a} * T * \widehat{b}, \quad \forall a, b \in A.$$

Besides, we get in this way that  $\widehat{\bullet}$  is associative, due to  $*$  is associative by hypothesis. Moreover, if we demand that  $I \in A$ , then we have  $\widehat{I} \in \widehat{A}$ , and  $\widehat{I}$  will be the isounit with respect to  $\widehat{\bullet}$ , pointed out in the previous definition.

We have still to lift the first law  $\circ$ . If we proceed in a similar way as before, we would need an isounit  $\widehat{S}$  and a law  $\star$ , similar to  $\widehat{I}$  and  $*$ , respectively. However, by taking into consideration that the isotopic set  $\widehat{A}$  has been already constructed, the associated with  $\star$  should coincide with the first one. That is,  $\{a * \widehat{I} : a \in A\} = \{\alpha \star \widehat{S} : \alpha \in A\}$ . So, if  $\widehat{a} \in \widehat{A}$ , then one can find two elements  $a$  and  $\alpha$  in  $A$ , with  $a * \widehat{I} = \widehat{a}$  and  $\alpha \star \widehat{S} = \widehat{a}$ .

In this way, to simplify, you can consider the general set associated with  $*$  and  $\star$  as the same. So, the elements of this set are related with these two laws.

On the other way, as  $(\widehat{A}, \widehat{\circ})$  should be an isogroup, then we need that  $\widehat{S} \in \widehat{A}$ , and so, it should be  $\widehat{S} = S * \widehat{I} = \sigma \star \widehat{S}$ , with  $S, \sigma \in A$ . Apart from that, if the law  $\star$  has an unit  $s$ , we already know that to obtain an isogroup we must demand that  $(A, \star)$  is a group with  $s \in A$ . Moreover, it will be  $s \star \widehat{S} = \widehat{S} = \sigma \star \widehat{S}$  and so,  $\sigma = s$ .

We will also impose that  $\star$  is associative and that the distributive property on  $\widehat{A}$  (related to  $\star$  and  $*$ ) is satisfied.

Now, to have an idea to construct the isosum  $\widehat{\circ}$  on  $\widehat{A}$ , we can firstly construct  $\widehat{\circ}$  on  $\widehat{A}$ , in a similar way as we constructed  $\widehat{\bullet}$ . So, we need the element of

isotopy  $\widehat{S}^{-s}$ . In general, it is not in  $\widehat{A}$ . However, we will see later that it is sufficient to suppose it, because all isotopies of rings can be studied under such model. So, if  $\widehat{S}^{-s} = \widehat{R} \in \widehat{A}$  (where  $\widehat{R} = R * \widehat{I}$ , with  $R \in A$ ) is supposed, we can construct the isosum  $\widehat{\circ}$  as follows:

$$\begin{aligned} \overline{\widehat{a}} \widehat{\circ} \overline{\widehat{b}} &= \overline{\widehat{a}} \star \overline{\widehat{R}} \star \overline{\widehat{b}} = ((a * \widehat{I}) \star (R * \widehat{I})) \star \overline{\widehat{b}} = \\ &= ((a \star R) * \widehat{I}) \star (b * \widehat{I}) = (a \star R \star b) * \widehat{I} \text{ for all } \overline{\widehat{a}}, \overline{\widehat{b}} \in \overline{\widehat{A}}. \end{aligned}$$

Finally, we can define the isosum  $\widehat{\circ}$  on  $\widehat{A}$  as:

$$\widehat{a} \widehat{\circ} \widehat{b} = a \star \widehat{R} \star b = I(a \star R \star b)$$

for all  $\widehat{a}, \widehat{b} \in \widehat{A}$ . Note that this isosum so defined is an inner law, due to  $a \star R \star b \in A$ , since  $(A, \star)$  is a group.

So,  $(\widehat{A}, \widehat{\circ})$  is an isogroup and thus, to finish the construction, we would only need that  $(\widehat{A}, \widehat{\circ}, \bullet)$  had the distributive property (in both senses). However, this property is not going to be satisfied, in general. To see it, let consider  $\widehat{a}, \widehat{b}, \widehat{c} \in \widehat{A}$ . Then,

$$\begin{aligned} \widehat{a} \bullet (\widehat{b} \widehat{\circ} \widehat{c}) &= \widehat{a} \bullet I(b \star R \star c) = I(a * (b \star R \star c)) = \\ &= I((a * b) \star (a * R) \star (a * c)) \neq I((a * b) \star R \star (a * c)) = (\widehat{a} \bullet \widehat{b}) \widehat{\circ} (\widehat{a} \bullet \widehat{c}) \end{aligned}$$

and, similarly, the left-distributivity is not satisfied either.

In the general case, as the distributive property is not satisfied (whereas the rest of them are), the lifting obtained in this case is named *pseudoisotopy*. In this way, a new kind of structure is obtained. It is named *pseudoisostucture*. So, in this case, we would have obtained a *pseudoisoring* (see [7]).

However, left and right distributivity can be satisfied. It is possible if and only if  $a * R = R = R * a, \forall a \in A$ . Thus, we would have already constructed the isoring, coming from the isotopy of main elements  $\widehat{I}$  and  $*$ , and secondary elements  $\widehat{S}$  and  $\star$ .

So, the following result is proved:

PROPOSITION 2.1. – *Let  $(A, \circ, \bullet)$  be a ring and  $\widehat{I}, \widehat{S}, *$  and  $\star$  be isotopic elements as in the Definition given in Preliminaries, where  $I$  and  $s$  are the respective units of  $*$  and  $\star$ , being  $S \in A$ . Under these conditions, if  $(A, \star, *)$  has a ring structure with respective units  $s, I \in A$ , then the isotopic lifting  $(\widehat{A}, \widehat{\circ}, \bullet)$  constructed by the isoproduct model, corresponding to the isotopy of main elements  $\widehat{I}$  and  $*$ , and secondary elements  $\widehat{S}$  and  $\star$ , has a structure of isoring with respect to the multiplication if and only if  $a * R = R = R * a, \forall a \in A$ , where  $R \in A$  is such that  $\widehat{R} = \widehat{S}^{-s} = R * \widehat{I}$ . ■*

This proposition can be improved with the following lemmas:



LEMMA 2.1. – *Under the conditions of Proposition 2.1,  $s * a = s = a * s$ , for all  $a \in A$ .*

PROOF. – Fixed  $a \in A$ , by using that  $(A, \star, *)$  must be a ring, we have that  $s * a = (s \star s) * a = (s * a) \star (s * a)$ . So,  $s = (s * a)^{-s} \star (s * a) \star (s * a) = s * a$ .

Similarly,  $a * s = s$ , which completes the proof. ■

LEMMA 2.2. – *Under the conditions of Proposition 2.1,  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  is an isoring with respect to the multiplication, if and only if  $\widehat{R} = \widehat{S}^{-s} = \widehat{s}$ .*

PROOF. – a) By Proposition 2.1,  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  is an isoring with respect to the multiplication if and only if  $a * R = R = R * a$ , for all  $a \in A$ . In this way, if  $a = s$ , Lemma 2.1 implies that  $s = s * R = R = R * s$ . So,  $R = s$  and thus,  $\widehat{R} = \widehat{s}$ .

b) We must prove  $a * R = R = R * a$ , for all  $a \in A$ . But, in our case,  $R = s$ . So, Lemma 2.1 implies the thesis. ●

Particularly, Lemma 2.2 involves on  $\widehat{A}$  the law  $\widehat{a} \widehat{\circ} \widehat{b} = \widehat{a \star R \star b} = \widehat{a \star s \star b} = \widehat{a \star b}$ , for all  $a, b \in A$ . Moreover,  $\widehat{S} = \widehat{S \star s} = \widehat{S} \widehat{\circ} \widehat{s} = \widehat{s}$ . So,  $S = s$ .

So, we have proved the following:

THEOREM 2.1. – *Let  $(A, \circ, \bullet)$  be a ring and  $\widehat{I}, \widehat{S}, *$  and  $\star$  be isotopic elements as in the Definition given in Preliminaries, where  $S \in A$  and  $I$  and  $s$  are the respective units of  $*$  and  $\star$ . Under these conditions, if  $(A, \star, *)$  has a ring structure with respective units  $s, I \in A$ , then the isotopic lifting  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  constructed by the isoproduct model, corresponding to the isotopy of main elements  $\widehat{I}$  and  $*$ , and secondary elements  $\widehat{S}$  and  $\star$ , has a structure of isoring with respect to the multiplication if and only if  $\widehat{S}^{-s} = \widehat{S} = \widehat{s}$ .*

Particularly, we define  $\widehat{a} \widehat{\circ} \widehat{b} = a \star b = I(a \star b)$  for all  $a, b \in A$ . ■

As a consequence, It is easy to prove the following:

COROLLARY 2.1. – *Under the conditions of Theorem 2.1 the following asserts are verified:*

a) *The map  $I_* : (A, \star, *) \rightarrow (\widehat{A}, \widehat{\circ}, \widehat{\bullet}) : a \rightarrow I_*(a) = \widehat{a}$  is an isomorphism of rings.*

b) *If the isotopy is injective, the followings maps are isomorphisms of rings:*

b.1)  $\pi : (\widehat{A}, \widehat{\circ}, \widehat{\bullet}) \rightarrow (\overline{\overline{A}}, \overline{\overline{\circ}}, \overline{\overline{\bullet}})$ .

b.2)  $\pi \circ I_* : (A, \star, *) \rightarrow (\overline{\overline{A}}, \overline{\overline{\circ}}, \overline{\overline{\bullet}})$ . ■

We will see next three examples of isorings:

EXAMPLE 2.1. – Let  $(\mathbf{Z}, +, \times)$  be the ring of integers, with the usual sum and product. We consider now the elements of isotopy  $\widehat{I} = 1$ ,  $* \equiv \times$ ,  $\widehat{S} = 0$  and  $\star \equiv +$ .

In this way,  $I = 1$  and  $s = 0$  are the units of  $*$  and  $\star$ , respectively. We can see that these two units belong to  $\mathbf{Z}$ . Besides,  $(\mathbf{Z}, \star, *) \equiv (\mathbf{Z}, +, \times)$  and thus,  $(\mathbf{Z}, \star, *)$  is a ring. Finally,  $\widehat{S}^{-s} = 0^{-0} = 0 = \widehat{S}$ . So, we can use Theorem 2.1 to say that the isotopic lifting of  $(\mathbf{Z}, +, \times)$ , obtained by the isoproduct model, with main elements  $\widehat{I}$  and  $*$  and secondary elements  $\widehat{S}$  and  $\star$  is an isoring.

If we want to obtain this isoring, we must firstly construct the isotopic set  $\widehat{\mathbf{Z}}$ . To do it, we study the projection level and so we get  $\widehat{\mathbf{Z}} = \{a * \widehat{I} = a \times 1 = a : a \in \mathbf{Z}\} = \mathbf{Z} = \{\alpha \star \widehat{S} = \alpha + 0 = \alpha : \alpha \in \mathbf{Z}\}$ .

Now, it is sufficient to define the isoproduct generated by  $*$  and  $\star$  on the elements of  $\widehat{\mathbf{Z}} = \mathbf{Z}$ . But, by fixing  $a, b \in \mathbf{Z}$ , we already know that these laws are defined by

$$\widehat{a} \widehat{+} \widehat{b} = \widehat{a \star b} = \widehat{a + b}$$

$$\widehat{a} \widehat{\times} \widehat{b} = \widehat{a * b} = \widehat{a \times b}.$$

Moreover, in the projection level, we have:

$$\overline{\widehat{a} \widehat{+} \widehat{b}} = \overline{\widehat{a}} + \overline{\widehat{b}}$$

$$\overline{\widehat{a} \widehat{\times} \widehat{b}} = \overline{\widehat{a}} \times \overline{\widehat{b}}.$$

Note that we obtain in this way a trivial isotopy of the starting rings, which was predictable because if the initial laws and unit do not change in an isotopic lifting, then the initial structure does not change either. ■

EXAMPLE 2.2. – We continue considering the ring  $(\mathbf{Z}, +, \times)$ . Take the law  $* \equiv \times$  and the isounit  $\widehat{I} = -1$ , where  $T = \widehat{I}^{-1} = (-1)^{-1} = -1 = \widehat{I}$ , since the unit with respect to  $*$  is  $I = 1$ . We try to construct an isoring with respect to the product. We could have  $\widehat{\mathbf{Z}}_{-1}$  as the isotopic set, being in the projection level  $\overline{\widehat{\mathbf{Z}}_{-1}} = \{\overline{\widehat{a}} = a \times (-1) = -a : a \in \mathbf{Z}\} = \mathbf{Z}$ .

Then, the isoproduct is defined by  $\widehat{a} \widehat{\times} \widehat{b} = \widehat{a * b} = \widehat{a \times b}$  for all  $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Z}}_{-1}$ . Moreover, in the projection level,  $\overline{\widehat{a} \widehat{\times} \widehat{b}} = \overline{\widehat{a}} * T \overline{\widehat{b}} = -(\overline{\widehat{a}} \times \overline{\widehat{b}}) = -((-a) \times (-b)) = -(a \times b) = a \times b$  for all  $a, b \in \mathbf{Z}$ .

We try now to lift the law  $+$  in such a way that it remains invariant. To do this, it would be sufficient to use  $\widehat{S} = 0$  as the secondary isounit and  $+$  itself as the law  $\star$ , as we made in the previous example. Then, as the unit of  $\star$  would be  $s = 0$ , we would have that  $\widehat{S}^{-s} = 0^{-0} = 0$ . So,  $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$  is a ring verifying hypotheses of Theorem 2.1 and thus, the isotopic lifting  $(\widehat{\mathbf{Z}}_{-1}, \widehat{+}, \widehat{\times})$  results to be an isoring with respect to the product. ■

EXAMPLE 2.3. – *By using the same ring  $(\mathbf{Z}, +, \times)$ , we are going to deal with a new case, not previously considered until now: when the isounit  $\widehat{I}$  belongs to the obtained isotopic set but the isotopic element does not. To do this, it is sufficient to consider  $* \equiv \times$  and  $\widehat{I} = 2$ , for instance. So, we would have the isotopic set  $\widehat{\mathbf{Z}}_2$ , where  $\widehat{\mathbf{Z}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Z}\} = \mathbf{P} = \mathbf{Z}_2$ . On the other hand,  $T = \widehat{I}^{-1} = 2^{-1} = \frac{1}{2} \notin \mathbf{P}$ , since the unit with respect to  $*$  is  $I = 1$ . Then, the isoproduct would be defined in the isotopic level by  $\widehat{a} \widehat{\times} \widehat{b} = \widehat{a \times b}$  and in the projection level by  $\overline{\widehat{a} \widehat{\times} \widehat{b}} = \overline{\widehat{a}} * \frac{1}{2} * \overline{\widehat{b}} = (a \times 2) \times \frac{1}{2} \times (b \times 2) = (a \times b) \times 2 = a \times \overline{\widehat{b}}$ , for all  $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Z}}_2$ .*

*In any case, if we want the law  $+$  to remain invariant, we could only take  $\widehat{S} = 0$  as the secondary unit, because it is the unique element such that  $\widehat{S}^{-s} = \widehat{S}^{-0} = \widehat{S}$ , which is a necessary condition for constructing the isoring, according to Theorem 2.1. So, by completing the isotopy with the secondary elements  $\widehat{S} = 0$  and  $\star \equiv +$ , we would finally deduce that  $(\widehat{\mathbf{Z}}, \widehat{+}, \widehat{\times})$  is an isoring with respect to the product. ■*

Finally, we must prove that the model used to construct an isoring can be used for all cases.

PROPOSITION 2.2. – *Any isotopy  $I : (A, \circ, \bullet) \rightarrow (\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  can be studied as following the isoproduct model that we have seen. That is, any isoring is it respect to the isomultiplication.*

PROOF. – It is sufficient to consider the set  $(A, \star, *)$  in the general level, where any law  $\sharp$  is associated with the appropriate  $\widehat{\sharp}$  on  $\widehat{A}$ , by defining  $a \sharp b = I^{-1}(\widehat{a \widehat{\sharp} b})$ . It has sense because the map  $I : A \rightarrow \widehat{A} : a \rightarrow \widehat{a}$  is bijective by construction. So, we get  $\widehat{\sharp}$  defined as  $\widehat{a \widehat{\sharp} b} = I(a \sharp b) = \widehat{a \sharp b}$ . That is, we get the same result as when the isoproduct model is used.

Besides, by linearity we get that  $(A, \star, *)$  is a ring. Moreover, the unit with respect to  $\sharp$  will be by construction the element in  $A$  such that the isounit of  $\widehat{\sharp}$  is lifted of its.

So, we have the conditions for isoproduct model can be applied, obtaining by construction the same mathematical structure which we firstly had. ■

Note that this proposition has very importance in the study of isotopies because it proves that we can always base our study of isorings on the isoproduct model.

Next, we are going to study isotopies of the substructures related to rings: the subrings.

### 3. – Isosubrings.

In the first place, we introduce the definition of *isosubring* and secondly, we study if it is possible to apply such a definition to the construction model of isotopies which we are considering.

**DEFINITION 3.1.** – *Let  $(A, \circ, \bullet)$  be a ring and let  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be an associated isoring with unit  $\widehat{I}$  with respect to  $\widehat{\bullet}$ . Let  $B$  be a subring of  $A$ . We say that  $\widehat{B}$  is an isosubring of  $\widehat{A}$  if, being an isotopy of  $B$ ,  $(\widehat{B}, \widehat{\circ}, \widehat{\bullet})$  is a subring of  $\widehat{A}$ , that is, if the following conditions are verified:*

1.  $\widehat{\circ}$  and  $\widehat{\bullet}$  are inner on  $\widehat{B}$ , and verify associativity and distributivity.
2.  $(\widehat{B}, \widehat{\circ})$  is an isosubgroup of  $(\widehat{A}, \widehat{\circ})$ .
3.  $\widehat{I} \in \widehat{B}$ .

To see if the construction model of isotopies is compatible with this definition, we consider the ring  $(A, \circ, \bullet)$  and the isoring  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ , constructed starting from that ring by the isotopy of main elements  $\widehat{I}$  and  $*$  (with unit element  $I$ ), and secondary elements  $\widehat{S}$  and  $\star$  (with unit element  $s$ ), verifying hypotheses of Theorem 2.1.

According to this model, the laws on the isosubring  $\widehat{B}$  have to be the same as those on the ring  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ . So, the main and secondary elements must be also the same as the used to lift  $A$ . In this way, a necessary condition for  $\widehat{B}$  to be subring, which is  $\widehat{B} \subseteq \widehat{A}$ , is obtained. Moreover, by demanding that  $(B, \star, *)$  has an structure of ring, we will obtain condition (1) of Definition 3.1, by construction. Note that distributivity is also satisfied, because we are under hypotheses of Theorem 2.1 and thus  $(B, \star, *)$  inherits this property from  $(A, \star, *)$ .

On the other hand, condition (3) is obtained by imposing  $I \in B$ , from which we deduce  $\widehat{I} \in \widehat{B}$ . Finally, as  $(B, \star)$  has a structure of group, due to  $(B, \star, *)$  is a ring, if we also impose that  $S \in B$ , we will obtain that  $\widehat{S} \in \widehat{B}$ , which is sufficient for condition (2) to be satisfied.

From these reasons, it is deduced the following:

**THEOREM 3.1.** – *Let  $(A, \circ, \bullet)$  be a ring and  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be the associated isoring corresponding to the isotopy of main elements  $\widehat{I}$  and  $*$  (with unit  $I$ ), and secondary elements  $\widehat{S}$  and  $\star$  (with unit  $S$ ), under hypotheses of Theorem 2.1. Let  $B$  be a subring of  $A$ . If  $(B, \star, *)$  is a subring of  $(A, \star, *)$  with  $I \in B$  and  $S \in B$ , then the isotopic lifting  $(\widehat{B}, \widehat{\circ}, \widehat{\bullet})$ , corresponding to the isotopy on the same elements, is an isosubring of  $\widehat{A}$ . ■*

We now give an example of isosubring:

EXAMPLE 3.1. – Let consider the ring  $(\mathbf{Q}, +, \times)$  of rational numbers with the usual sum and product. Take the isounit  $\widehat{I} = 2$  and the law  $* \equiv \times$ . The isotopic set will be  $\widehat{\mathbf{Q}}_2$ , where  $\widehat{\mathbf{Q}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Q}\} = \mathbf{Q}$ . As  $* \equiv \times$ , we have that the unit with respect to  $*$  is  $I = 1 \in \mathbf{Q}$  and thus,  $T = \widehat{I}^{-1} = 2^{-1} = \frac{1}{2}$ . The isoproduct so obtained is  $\widehat{a} \widehat{\times} \widehat{b} = \widehat{a \times b}$  in the isotopic level and  $\widehat{a} \widehat{\times} \widehat{b} = \widehat{a} * \frac{1}{2} * \widehat{b} = \widehat{a * b} = \widehat{a * b} * 2 = (a * b) \times 2 = (a \times b) \times 2$  in the projection level, for all  $a, b \in \mathbf{Q}$ .

If we now consider the secondary elements  $\widehat{S} = 0$  and  $\star \equiv +$  (then  $s = S = 0$ ), we will obtain, in a similar way as in the Example 2.3, that  $(\widehat{\mathbf{Q}}_2, \widehat{+}, \widehat{\times}) = (\mathbf{Q}, +, \widehat{\times})$  is an isoring.

We consider now the subring of integers  $(\mathbf{Z}, +, \times)$  of  $(\mathbf{Q}, +, \times)$ , and we try to construct the isotopy of this subring, with the same elements as the used in the construction of the isoring  $(\mathbf{Q}, +, \widehat{\times})$ .

Then, we would have that the isotopic set is  $\widehat{\mathbf{Z}}_2$ , where  $\widehat{\mathbf{Z}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Z}\} = \mathbf{P}$ . Therefore, since  $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$  has a structure of subring of  $(\mathbf{Q}, \star, *) = (\mathbf{Q}, +, \times)$ , with unit  $I = 1 \in \mathbf{Z}$  with respect to  $*$  (the same as for  $(\mathbf{Q}, \star, *)$ ) and  $s = S = 0 \in \mathbf{Z}$ , we deduce from Theorem 3.1 that  $(\widehat{\mathbf{Z}}_2, \widehat{+}, \widehat{\times})$  is an isosubring of  $\widehat{\mathbf{Q}}_2$ , with  $(\widehat{\mathbf{Z}}_2, \widehat{+}, \widehat{\times}) = (\mathbf{P}, +, \widehat{\times})$  in the projection level.

So, by considering Example 2.3, we observe that  $(\mathbf{P}, +, \widehat{\times})$  can be equipped with an structure as of isoring as of isosubring, with respect to the same isotopy mentioned. ■

Now, we can ask ourselves if every subring gives rise to an isosubring under a determined isotopy or if every subring of a given isoring has a structure of isosubring. Observe that a possible counterexample could be found in those cases in which the conditions required by Theorem 3.1 were not satisfied. For instance, it would be sufficient for it to have a subring  $B$  of  $A$  such that either  $S \notin A$  or  $I \notin B$ .

Similarly, if we had a subring  $\widehat{B}$  of the isoring  $\widehat{A}$ , with  $\widehat{S} \in \widehat{B}$ , such that  $S \notin C$ , for all  $C$ , subring of  $A$ , we could not give a structure of isosubring to  $\widehat{B}$ , because we could not find any subring in  $A$  which gave  $\widehat{B}$  after making the corresponding isotopic lifting. So, we can conjecture that, in general, a subring could not be isotopically lifted to an isosubring by a fixed isotopy and moreover, that a subring of an isoring could not have a structure of isosubring by the isotopy corresponding with such an isoring.

#### 4. – Isoideals of an isoring.

We introduce in this section the definition and some properties of the *isoideals* of an isoring.

DEFINITION 4.1. – Let  $(A, \circ, \bullet)$  be a ring and  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be an associated isoring. Let  $\mathfrak{S} \subseteq A$  be an ideal of  $A$ . It is said that  $\widehat{\mathfrak{S}}$  is an isoideal of  $\widehat{A}$  if, being an isotopy of  $\mathfrak{S}$ ,  $\widehat{\mathfrak{S}}$  has a structure of ideal with respect to  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ .

We continue with the isotopy model which we are using. Suppose that we have a ring  $(A, \circ, \bullet)$  and the corresponding isoring  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  obtained by the isotopy of main elements  $\widehat{I}$  and  $*$ , and secondary elements  $\widehat{S}$  and  $\star$ . As we already made, we only deal with isorings with respect to the product.

We continue demanding that given an ideal  $\mathfrak{S}$  of  $A$ , the lifted isoideal has the same associated laws as in  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ . Then, if we use the isoproduct construction model, we will have the same main and secondary elements, respectively, as the used to construct the isoring  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ . In this way, we will have that  $\widehat{\mathfrak{S}} \subseteq \widehat{A}$ , due to  $\mathfrak{S} \subseteq A$ .

Under these suppositions, it will be sufficient to impose  $\widehat{\mathfrak{S}}$  to be an ideal of the ring  $(A, \star, *)$ , because in this way we will have that  $\widehat{\mathfrak{S}} \bullet \widehat{a} = x * a \in \widehat{\mathfrak{S}}$ , for all  $\widehat{x} \in \widehat{\mathfrak{S}}$  and  $\widehat{a} \in \widehat{A}$ , since  $x * a \in \mathfrak{S}$  due to  $\mathfrak{S}$  is an ideal of  $(A, \star, *)$ .

So, we finally get that  $\widehat{\mathfrak{S}}$  is an isoideal of  $\widehat{A}$ , and thus, the following result has been proved:

THEOREM 4.1. – Let  $(A, \circ, \bullet)$  be a ring and  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be the associated isoring corresponding to the isotopy of main elements  $\widehat{I}$  and  $*$ , and secondary elements  $\widehat{S}$  and  $\star$ , under hypotheses of Theorem 2.1. Let  $\mathfrak{S}$  be an ideal of  $(A, \circ, \bullet)$ . If  $\mathfrak{S}$  is an ideal of  $(A, \star, *)$ , being  $(\mathfrak{S}, \star)$  a subgroup of  $(A, \star)$ , then the isotopic lifting  $(\widehat{\mathfrak{S}}, \widehat{\circ}, \widehat{\bullet})$  corresponding to the isotopy of elements the above mentioned is an isoideal of  $\widehat{A}$ . ■

We give now some examples of isoideals:

EXAMPLE 4.1. – Let consider the ring  $(\mathbf{Z}, +, \times)$  and the associated isoring  $(\widehat{\mathbf{Z}}, \widehat{+}, \widehat{\times})$  from Example 2.2. Take  $\mathbf{P} = \mathbf{Z}_2$  as an ideal of  $(\mathbf{Z}, +, \times)$ . So, according to the notations of that example,  $(\mathbf{P}, +)$  is a subgroup of  $(\mathbf{Z}, +)$ , with unit  $0 \in \mathbf{P}$ , being  $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$  an ideal of  $(\mathbf{Z}, +, \times)$ . Theorem 4.1 involves that  $(\widehat{\mathbf{P}}_{-1}, \widehat{+}, \widehat{\times})$  is an isoideal of  $(\widehat{\mathbf{Z}}, \widehat{+}, \widehat{\times})$ . Then, as  $\widehat{\mathbf{P}}_{-1} = \{\widehat{a} = a * (-1) = a \times (-1) = -a : a \in \mathbf{P}\} = \mathbf{P}$ , we have that in the projection level,  $(\overline{\mathbf{P}}_{-1}, +, \overline{\times}) = (\mathbf{P}, +, \overline{\times})$  is the required isoideal. ■

EXAMPLE 4.2. – Let consider the ring  $(\mathbf{Z}, +, \times)$  and the associated isoring  $(\widehat{\mathbf{Z}}, \widehat{+}, \widehat{\times})$ , as in Example 2.3. Taking again the ideal  $(\mathbf{P}, +, \times)$  of  $(\mathbf{Z}, +, \times)$ , we would have that  $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$ , which is also an ideal of  $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$ , being  $(\mathbf{P}, \star)$  a subgroup of  $(\mathbf{Z}, \star)$  with unit  $I = 0 \in \mathbf{P} \cap \mathbf{Z}$ . Therefore, Theorem 4.1 implies that  $(\widehat{\mathbf{P}}_2, \widehat{+}, \widehat{\times})$  is an isoideal of  $\widehat{\mathbf{Z}}$ . In other way,  $\overline{\mathbf{P}}_2 = \{\widehat{a} = a * 2 = a \times 2 \mid a \in \mathbf{P}\} = \mathbf{Z}_4$  and thus,

in the projection level,  $(\widehat{\mathbf{P}}_2, +, \widehat{\times}) = (\mathbf{Z}_4, +, \widehat{\times})$  is the required isoideal. ■

We will now introduce the concept of *isosubideal*:

DEFINITION 4.2. – Let  $(A, \circ, \bullet)$  be a ring,  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be an associated isoring and  $\mathfrak{S}$  an ideal of  $A$ , such that the corresponding isotopic lifting  $\widehat{\mathfrak{S}}$  is an isoideal. Let  $J$  be a subideal of  $\mathfrak{S}$ . We say that  $\widehat{J}$  is an isosubideal of  $\widehat{\mathfrak{S}}$  if, being an isotopy of  $J$ ,  $(\widehat{J}, \widehat{\circ}, \widehat{\bullet})$  is a subideal of  $\widehat{\mathfrak{S}}$  with respect to  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ .

By using the same isotopic construction model, it is easy to prove the following:

THEOREM 4.2. – Let  $(A, \circ, \bullet)$  be a ring and  $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$  be the associated isoring, corresponding to the isotopy of main elements  $\widehat{I}$  and  $*$  and secondary elements  $\widehat{S}$  and  $\star$ , under hypotheses of Theorem 2.1. Let  $\mathfrak{S}$  be an ideal of  $A$  such that the corresponding isotopic lifting,  $(\widehat{\mathfrak{S}}, \widehat{\circ}, \widehat{\bullet})$ , is an isoideal of  $\widehat{A}$ . Let  $J$  be a subideal of  $\mathfrak{S}$ . If  $(J, \star, *)$  is a subideal of  $(\mathfrak{S}, \star, *)$ , then the corresponding isotopic lifting  $(\widehat{J}, \widehat{\circ}, \widehat{\bullet})$  is an isosubideal of  $\widehat{\mathfrak{S}}$ . ■

Observe that this construction model allows that  $\widehat{J} \subseteq \widehat{\mathfrak{S}}$ , since  $J \subseteq \mathfrak{S}$  and the used elements of isotopy are the same as the ones used to construct  $\widehat{\mathfrak{S}}$ .

We show the following example of a subideal:

EXAMPLE 4.3. – In the Example 4.1, let consider the subideal  $(\mathbf{Z}_6, +, \times)$  of  $(\mathbf{P}, +, \times)$ . We have that  $(\mathbf{Z}_6, \star, *) = (\mathbf{Z}_6, +, \times)$  is a subideal of  $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$ . Then, from Theorem 4.2  $(\widehat{\mathbf{Z}}_6, \widehat{+}, \widehat{\times})$  is an isosubideal of  $(\widehat{\mathbf{P}}, \widehat{+}, \widehat{\times})$ , being  $\widehat{\mathbf{Z}}_{6-1} = \{\widehat{a} = a * (-1) = a \times (-1) = -a \mid a \in \mathbf{Z}_6\} = \mathbf{Z}_6$ . So, in the projection level,  $(\mathbf{Z}_6, +, \widehat{\times})$  is the required isosubideal. ■

In the following section we will complete the study of the isotopic liftings of the rings by introducing the concept of *quotient isoring*.

### 5. – Quotient isorings.

DEFINITION 5.1. – Let  $(A, \circ, \bullet)$  be a ring,  $\mathfrak{S}$  be an ideal of  $A$  and  $A/\mathfrak{S}$  be the quotient ring, with the usual structure  $(A/\mathfrak{S}, +, \times)$ . We say that  $\widehat{A/\mathfrak{S}}$  is a quotient isoring if, being an isotopy of  $A/\mathfrak{S}$ ,  $(\widehat{A/\mathfrak{S}}, \widehat{+}, \widehat{\times})$  has a structure of quotient ring, that is, if there exists a ring  $(B, \square, \diamond)$  and an ideal  $J$  of  $B$ , such that  $\widehat{A/\mathfrak{S}} = B/J$ , being  $\widehat{+}$  and  $\widehat{\times}$  the usual laws of quotient rings, coming from  $\square$  and  $\diamond$ , respectively.

Note that this definition allows that the ring  $B$  and its ideal  $J$  do not have

to be, in general, the isotopic liftings of the ring  $A$  and of its ideal  $\mathfrak{S}$ , which at once allows to set a difference between the concepts of quotient isoring and quotient ring constructed starting from an isoring and one of its isoideals. In fact, it is possible from a theoretician point of view that either  $B$  or  $J$  only is an isotopic lifting of  $A$  or  $\mathfrak{S}$ , respectively. In this way, although all of them had a structure of quotient ring, there would have to distinguish among the possible sets  $B/J$ ,  $\widehat{A}/J$ ,  $B/\widehat{\mathfrak{S}}$  and  $\widehat{A}/\widehat{\mathfrak{S}}$ .

Apart from that, although the quotient ring  $\widehat{A}/\widehat{\mathfrak{S}}$  could be constructed, we know that the isotopies used to obtain  $\widehat{A}$  and  $\widehat{\mathfrak{S}}$  must have the same main and secondary elements, respectively. Then, as we use the isoring construction model to construct  $\widehat{A}/\widehat{\mathfrak{S}}$  starting from the quotient ring  $A/\mathfrak{S}$ , our isotopy should have two main elements and two secondary elements too. Naturally, as the rings  $A$  and  $A/\mathfrak{S}$  have different characteristics, the elements of isotopy will not be the same, in general, due to, particularly, the laws would be defined on different sets. However, we are now going to see in the following example that the equality  $\widehat{A}/\widehat{\mathfrak{S}} = \widehat{A}/\widehat{\mathfrak{S}}$  is possible in some case, although it is not verified in general.

EXAMPLE 5.1. – *Let consider the ring  $(\mathbf{Z}, +, \times)$  and its ideal  $(\mathbf{Z}_3, +, \times)$ , with the usual sum and product. By constructing the isotopy of main elements  $\widehat{I} = 2$  and  $* \equiv \times$  and secondary elements  $\widehat{S} = 0$  and  $\star \equiv +$ , we obtain the isoring  $(\mathbf{P}, +, \overline{\times})$  and its isoideal  $(\mathbf{Z}_6, +, \overline{\times})$  (we have to take into consideration that  $\overline{\mathbf{Z}}_{32} = \{\overline{a} = a \times 2 \mid a \in \mathbf{Z}_3\} = \mathbf{Z}_6$ ). In this way, we would construct the quotient ring  $\overline{\mathbf{Z}}_2/\overline{\mathbf{Z}}_{32} = \mathbf{P}/\mathbf{Z}_6 = \{0 + \mathbf{Z}_6, 2 + \mathbf{Z}_6, 4 + \mathbf{Z}_6\}$ , with the usual sum and product, coming from  $+$  and  $\overline{\times}$  (to simplify the notations, from now on, both laws will be denoted by the same symbols as before).*

*We will give in an explicit way the second of them, seen in the projection level:*

1.  $(0 + \mathbf{Z}_6) \overline{\times} (a + \mathbf{Z}_6) = (0 \overline{\times} a) + \mathbf{Z}_6 = \left(0 \times \frac{1}{2} \times a\right) + \mathbf{Z}_6 = 0 + \mathbf{Z}_6 = (a + \mathbf{Z}_6) \overline{\times} (0 + \mathbf{Z}_6)$ , for all  $a \in \{0, 2, 4\}$ .
2.  $(2 + \mathbf{Z}_6) \overline{\times} (2 + \mathbf{Z}_6) = (2 \overline{\times} 2) + \mathbf{Z}_6 = \left(2 \times \frac{1}{2} \times 2\right) + \mathbf{Z}_6 = 2 + \mathbf{Z}_6$
3.  $(2 + \mathbf{Z}_6) \overline{\times} (4 + \mathbf{Z}_6) = \left(2 \times \frac{1}{2} \times 4\right) + \mathbf{Z}_6 = 4 + \mathbf{Z}_6 = (4 + \mathbf{Z}_6) \overline{\times} (2 + \mathbf{Z}_6)$ .
4.  $(4 + \mathbf{Z}_6) \overline{\times} (4 + \mathbf{Z}_6) = \left(4 \times \frac{1}{2} \times 4\right) + \mathbf{Z}_6 = 8 + \mathbf{Z}_6 = 2 + \mathbf{Z}_6$ .

*In this way we deduce that the unit element of  $\mathbf{P}/\mathbf{Z}_6 = \mathbf{Z}_2/\mathbf{Z}_6 = \overline{\mathbf{Z}}_2/\overline{\mathbf{Z}}_{32}$  is  $2 + \mathbf{Z}_6$ .*

*On the other hand, let consider now the quotient ring  $\mathbf{Z}/\mathbf{Z}_3$ , with usual laws coming from the ring  $(\mathbf{Z}, +, \times)$ . We denote both laws by  $\circ$  and  $\bullet$ , respectively. We now make a similar lifting as before, with main elements  $\widehat{I} = 2 + \mathbf{Z}_3$  and  $* \equiv \bullet$ , and secondary elements  $\widehat{S} = 0 + \mathbf{Z}_3$  and  $\star \equiv \circ$ . Note that in*



this case,  $T = \widehat{I}^{-1} = \widehat{I}$ . So, in a similar way as previous examples, the isotopy so constructed is in the projection level the isoring  $(\mathbf{Z}/\mathbf{Z}_3, \circ, \overline{\bullet})$ , where the isoproduct  $\overline{\bullet}$  is defined by:

1.  $(0 + \mathbf{Z}_3) \overline{\bullet} (a + \mathbf{Z}_3) = ((0 + \mathbf{Z}_3) \times (2 + \mathbf{Z}_3) \times (a + \mathbf{Z}_3)) = 0 + \mathbf{Z}_3 = (a + \mathbf{Z}_3) \overline{\bullet} (0 + \mathbf{Z}_3)$ , for all  $a \in \{0, 1, 2\}$ .
2.  $(1 + \mathbf{Z}_3) \overline{\bullet} (1 + \mathbf{Z}_3) = ((1 + \mathbf{Z}_3) \times (2 + \mathbf{Z}_3) \times (1 + \mathbf{Z}_3)) = 2 + \mathbf{Z}_3$ .
3.  $(1 + \mathbf{Z}_3) \overline{\bullet} (2 + \mathbf{Z}_3) = ((1 + \mathbf{Z}_3) \times (2 + \mathbf{Z}_3)) \times (2 + \mathbf{Z}_3) = 1 + \mathbf{Z}_3 = (2 + \mathbf{Z}_3) \overline{\bullet} (1 + \mathbf{Z}_3)$ .
4.  $(2 + \mathbf{Z}_3) \overline{\bullet} (2 + \mathbf{Z}_3) = ((2 + \mathbf{Z}_3) \times (2 + \mathbf{Z}_3)) \times (2 + \mathbf{Z}_3) = 2 + \mathbf{Z}_3$ .

In this way, the isoring so obtained is a quotient isoring because it has been constructed starting from the ring  $(\mathbf{Z}, +, \times)$  and its ideal  $(\mathbf{Z}_3, +, \times)$ .

Finally, observe that although used isotopies have been very similar, we obtain that  $\overline{\mathbf{Z}/\mathbf{Z}_3} \neq \widehat{\mathbf{Z}/\mathbf{Z}_3}$  (we do not intentionally point out the isounits to which both sets are referred, to emphasize the difference between them). It proves that it is important to distinguish between a quotient isoring and a quotient ring coming from an isoring and one of its isoideals. ■

*Acknowledgement.* Authors thank to Professor R. M. Santilli for the useful help given.

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Dpto. de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla  
Aptdo 1160. 41080-Sevilla (España)  
E-mails: rafalga@us.es; jnvaldes@us.es