

On the controllability of the heat equation with nonlinear boundary Fourier conditions

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Abstract

In this paper we analyze the approximate and null controllability of the classical heat equation with nonlinear boundary conditions of the form $\frac{\partial y}{\partial n} + f(y) = 0$ and distributed controls, with support in a small set. We show that, when the function f is globally Lipschitz-continuous, the system is approximately controllable. We also show that the system is locally null controllable and null controllable for large time when f is regular enough and $f(0) = 0$. For the proofs of these assertions, we use controllability results for similar linear problems and appropriate fixed point arguments. In the case of the local and large time null controllability results, the arguments are rather technical, since they need (among other things) Hölder estimates for the control and the state.

Key words: Controllability, heat equation, nonlinear boundary conditions

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$. In the sequel, γ_0 will stand for the usual trace operator $\gamma_0 : H^1(\Omega) \mapsto H^{1/2}(\partial\Omega)$. On the other hand, we will denote by C, C_1, C_2, \dots generic positive constants (usually depending on Ω, \mathcal{O}, T and possibly other data).

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We will consider the heat equation with nonlinear Fourier (or Robin) conditions

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + f(y) = 0 & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (1)$$

Here, we assume that $v \in L^2(\mathcal{O} \times (0, T))$ (at least), $1_{\mathcal{O}}$ is the characteristic function of \mathcal{O} , $y^0 \in L^2(\Omega)$ and $f : \mathbb{R} \mapsto \mathbb{R}$ is a given function. In (1), $y = y(x, t)$ is the state and $v = v(x, t)$ is the control; it is assumed that we can act on the system only through $\mathcal{O} \times (0, T)$.

For the existence, uniqueness, regularity and general properties of the solutions to problems like (1), see for instance [1], [2] and [7]. An illustrative interpretation of the data and variables in (1) is the following. The function $y = y(x, t)$ can be viewed as the relative temperature of a body (with respect to the exterior surrounding air). The parabolic equation in (1) means that a heat source $v1_{\mathcal{O}}$ acts on a part of the body. On the boundary, $-\frac{\partial y}{\partial n}$ can be viewed as the *normal heat flux*, inwards directed, up to a positive coefficient. Thus, the equality

$$-\frac{\partial y}{\partial n} = f(y)$$

means that this flux is a (nonlinear) function of the temperature. Accordingly, it is reasonable to assume that f is nondecreasing and $f(0) = 0$.

Of course, the simplified linear model corresponds to the case

$$-\frac{\partial y}{\partial n} = ay,$$

where a is a constant. For the reasons above, it is natural to assume that $a > 0$.

The main goal of this paper is to analyze the controllability properties of (1).

System (1) is said to be *approximately controllable* in $L^2(\Omega)$ at time T if, for any $y^0, y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exist a control $v \in L^2(\mathcal{O} \times (0, T))$ and an associated solution $y \in C^0([0, T]; L^2(\Omega))$ satisfying

$$\|y(\cdot, T) - y^1\|_{L^2} \leq \varepsilon. \quad (2)$$

On the other hand, it will be said that system (1) is *null controllable* at time T if, for each $y^0 \in L^2(\Omega)$, there exist $v \in L^2(\mathcal{O} \times (0, T))$ and an associated

solution $y \in C^0([0, T]; L^2(\Omega))$ such that

$$y(x, T) = 0 \quad \text{in } \Omega. \quad (3)$$

The controllability properties of linear and semilinear time dependent systems have been studied intensively these last years, see for instance [8], [10], [15], [17], [22] and [24]. In this paper, we will be concerned with (1), where the nonlinearity is in the boundary condition. This is more difficult to analyze than the cases considered in [5], [8] and [10], where the boundary condition is linear and the equations are of the form

$$\frac{\partial y}{\partial t} - \Delta y + F(y) = v1_{\mathcal{O}}$$

or

$$\frac{\partial y}{\partial t} - \Delta y + F(y, \nabla y) = v1_{\mathcal{O}}.$$

In order to justify this assertion, let us consider the following relatively simple system, one-dimensional in space:

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = v1_{(\alpha, \beta)} \quad \text{in } (0, 1) \times (0, T), \\ \left(-\frac{\partial y}{\partial x} + a_0 y \right) (0, t) = \left(\frac{\partial y}{\partial x} + a_1 y \right) (1, t) = 0 \quad \text{for } t \in (0, T), \\ y(x, 0) = y^0(x) \quad \text{in } (0, 1). \end{array} \right. \quad (4)$$

Here, we assume that $0 < \alpha < \beta < 1$ and a_0 and a_1 are given in $C^0([0, T])$ (for instance). Let us introduce the function \tilde{a} , with

$$\tilde{a}(x, t) = -a_0(t)x + (a_0(t) + a_1(t))\frac{x^2}{2}$$

and the new variable z , with

$$z = e^{\tilde{a}(x, t)} y.$$

Then y solves (4) for some $v \in L^2((\alpha, \beta) \times (0, T))$ and $y^0 \in L^2(0, 1)$ if and only if z satisfies

$$\left\{ \begin{array}{l} \frac{\partial z}{\partial t} - Lz - \frac{\partial \tilde{a}}{\partial t} z = e^{\tilde{a}(x, t)} v1_{(\alpha, \beta)} \quad \text{in } (0, 1) \times (0, T), \\ \frac{\partial z}{\partial x}(0, t) = \frac{\partial z}{\partial x}(1, t) = 0 \quad \text{for } t \in (0, T), \\ z(x, 0) = e^{\tilde{a}(x, 0)} y^0(x) \quad \text{in } (0, 1), \end{array} \right. \quad (5)$$

where we have set

$$Lz = \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial \tilde{a}}{\partial x} \frac{\partial z}{\partial x} - \frac{\partial^2 \tilde{a}}{\partial x^2} z + \left(\frac{\partial \tilde{a}}{\partial x} \right)^2 z.$$

Therefore, the approximate (resp. null) controllability of (4) is equivalent to the approximate (resp. null) controllability of a linear heat equation with a possibly singular coefficient $\frac{\partial \tilde{a}}{\partial x}$ in the zero order term, completed with homogeneous Neumann conditions. This indicates that the case under study in this paper is indeed more intricate.

Remark 1 Recall that the linear heat equation completed with terms of the form $B \cdot \nabla y$ and Dirichlet boundary conditions has been considered in [13]. There, null controllability is established under the assumption $B \in L^\infty(Q)^N$. The proof relies on an appropriate Carleman estimate for the solutions of the adjoint equation

$$-\frac{\partial \varphi}{\partial t} - \Delta \varphi - \nabla \cdot (\varphi B) = 0.$$

Trying to apply the same techniques to (5), we readily see that what is needed is a Carleman estimate for the solutions to the equation

$$-\frac{\partial}{\partial t} ((1 + \tilde{a})\varphi) - L^* \varphi = 0 \quad \text{in } (0, 1) \times (0, T),$$

where L^* is the adjoint of L . But this seems much more complicate.

The first main result in this paper concerns the approximate controllability of (1). It is the following:

Theorem 2 *Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz-continuous and $T > 0$. Then (1) is approximately controllable in $L^2(\Omega)$ at time T .*

Notice that, under these assumptions, using standard arguments, it can be shown that for each $y^0 \in L^2(\Omega)$ and each $v \in L^2(\mathcal{O} \times (0, T))$ the nonlinear system (1) possesses exactly one solution y that satisfies:

$$y \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)), \quad \frac{\partial y}{\partial t} \in L^2(0, T; H^{-1}(\Omega)). \quad (6)$$

Remark 3 The global null controllability of (1) for a globally Lipschitz-continuous function f without any assumption on the size and regularity of y^0 is an open problem. In fact, at present, this is an unsolved question even for similar linear systems, when the nonlinear boundary Fourier condition in (1) is replaced by

$$\frac{\partial y}{\partial n} + a(x, t)y = 0 \quad \text{on } \Sigma. \quad (7)$$

Indeed, if the coefficient a is only assumed to be in $L^\infty(\Sigma)$ (and this seems to be the natural assumption), the null controllability of the system is unknown (see [11] and remark 15 in Section 3).

In order to state our second main result, it will be convenient to introduce some notation. For $\alpha, \beta \in [0, 1)$, $C^{\alpha, \beta}(\overline{Q})$ will stand for the space formed by all functions $u \in C^0(\overline{Q})$ such that

$$[u]_{\alpha, \beta} = \sup_{\overline{Q}} \frac{|u(x, t) - u(x', t)|}{|x - x'|^\alpha} + \sup_{\overline{Q}} \frac{|u(x, t) - u(x, t')|}{|t - t'|^\beta} < +\infty.$$

The *natural* norm in $C^{\alpha, \beta}(\overline{Q})$ is

$$\|u\|_{\alpha, \beta} = \|u\|_{L^\infty(Q)} + [u]_{\alpha, \beta}.$$

With this norm, $C^{\alpha, \beta}(\overline{Q})$ is a Banach space.

The second main result in this paper concerns the *local null controllability* of (1). It is the following:

Theorem 4 *Assume that $f \in C^3(\mathbb{R})$ and $f(0) = 0$. Then we can find a positive $\eta = \eta(\Omega, \mathcal{O}, \alpha, T)$ with the following property: If we have $y^0 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$, the compatibility condition*

$$\frac{\partial y^0}{\partial n} + f(y^0) = 0 \quad \text{on} \quad \partial\Omega \tag{8}$$

is fulfilled and $\|y^0\|_{C^{2+\alpha}(\overline{\Omega})} \leq \eta$, there exists a control $v \in C^{\alpha, \alpha/2}(\overline{Q})$ such that the associated solution y of (1) satisfies (3).

This theorem indicates that the nonlinear system (1) is *locally null controllable* when f is regular enough and vanishes at 0. It will be clear from the proof that the same local property holds when f is C^3 just in a neighbourhood of 0.

Our third main result deals with the case in which f is nondecreasing. It is a consequence of theorem 4 and reads as follows:

Theorem 5 *Assume that $f \in C^4(\mathbb{R})$, $f(0) = 0$ and $f'(s) \geq 0$ for all $s \in \mathbb{R}$. Then (1) is null controllable in large time intervals. In other words, for every $y^0 \in L^2(\Omega)$ there exist $T = T(y^0)$ and $v \in L^2(\mathcal{O} \times (0, T(y^0)))$ such the associated solution to (1) satisfies (3).*

Again, it will be noticed in the proof of this result that f has only to be C^4 in a neighborhood of 0.

The rest of this paper is organized as follows. In Section 2, we prove theorem 2. It will be seen that the proof relies on an approximate controllability result

for a linear system similar to (1) where the boundary condition is again of the kind (7) and an appropriate fixed point argument. In Section 3, we give the proof of theorem 4. In this case, we have to introduce and estimate controls in a much more regular space (in fact, this is the reason the argument works only when y^0 is sufficiently close to zero). Section 4 deals with the proof of theorem 5. This is achieved in several steps: we start from y^0 at $t = 0$ and we first choose a control such that the associated state becomes small in the $C^{2+\alpha}$ -norm at $t = T^*$ for T^* large enough; then we apply theorem 4 and we find a control that leads the state to zero at a time $T(y^0) > T^*$. Finally, in Section 5 we make some comments.

2 Proof of the approximate controllability result

This Section is devoted to prove theorem 2. As usual, the proof relies on an approximate controllability result for similar linear problems and a fixed point argument. This strategy was introduced in [22], in the framework of the controllability of the semilinear wave equation. See also [8] and [10] for similar results concerning the semilinear heat equation with Dirichlet boundary conditions.

2.1 The approximate controllability of similar linear problems

We consider the following linear system:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + a(x, t)y = 0 & \text{on } \Sigma, \\ y(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (9)$$

where the coefficient $a \in L^\infty(\Sigma)$. For each $v \in L^2(\mathcal{O} \times (0, T))$, (9) possesses exactly one solution y satisfying (6).

We have the following result:

Lemma 6 *Assume that $T > 0$ and $a \in L^\infty(\Sigma)$. Then (9) is approximately controllable in $L^2(\Omega)$ at time T . In other words, for each $z^1 \in L^2(\Omega)$ and each $\varepsilon > 0$, there exists a control $v \in L^2(\mathcal{O} \times (0, T))$ such that the corresponding*

solution of (9) satisfies

$$\|y(\cdot, T) - z^1\|_{L^2(\Omega)} \leq \varepsilon. \quad (10)$$

Furthermore, the control v can be found such that

$$\|v\|_{L^2(\mathcal{O} \times (0, T))} \leq C_1(\Omega, \mathcal{O}, T, \varepsilon, \|a\|_{L^\infty(\Sigma)}, \|z^1\|_{L^2}), \quad (11)$$

where $C_1(\Omega, \mathcal{O}, T, R, \|z^1\|_{L^2})$ is nondecreasing in R .

SKETCH OF THE PROOF: For the proof, we will adapt the arguments in [8] (more details are given in [4]).

Let $T > 0$ and $a \in L^\infty(\Sigma)$ be given. We will use the well known fact that the approximate controllability of the linear problem (9) is equivalent to the *unique continuation property* for the solutions to the following adjoint system (where $\varphi^0 \in L^2(\Omega)$):

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } Q, \\ \frac{\partial \varphi}{\partial n} + a(x, t)\varphi = 0 & \text{on } \Sigma, \\ \varphi(x, T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \quad (12)$$

That is to say, (9) is approximately controllable in $L^2(\Omega)$ at time T if and only if the following holds:

If $\varphi^0 \in L^2(\Omega)$, φ is the associated solution to (12) and we have $\varphi = 0$ in $\mathcal{O} \times (0, T)$, then $\varphi \equiv 0$.

It is clear that this property holds. Actually, we have a much stronger result in which the boundary conditions play no role:

If $\varphi \in L^2_{\text{loc}}(Q)$ (for instance), $\frac{\partial \varphi}{\partial t} - \Delta \varphi = 0$ in Q and we have $\varphi = 0$ in $\mathcal{O} \times (0, T)$, then $\varphi \equiv 0$.

In fact, this is also true for much more general parabolic equations, see for instance [21]. Thus, if z^1 is given in $L^2(\Omega)$ and $\varepsilon > 0$ is fixed, there exist controls $v \in L^2(\mathcal{O} \times (0, T))$ such that the corresponding solution of (9) verifies (10).

It is also clear that v can be chosen of minimal L^2 -norm. Let us introduce the functional $J_\varepsilon(\cdot; a, z^1)$, with

$$J_\varepsilon(\varphi^0; a, z^1) = \frac{1}{2} \iint_{\mathcal{O} \times (0, T)} |\varphi|^2 dx dt + \varepsilon \|\varphi^0\|_{L^2} - (z^1, \varphi^0)_{L^2} \quad (13)$$

for all $\varphi^0 \in L^2(\Omega)$, where φ is the associated solution of (12). This is a continuous and strictly convex functional on $L^2(\Omega)$. Furthermore, using the previous unique continuation property, it can be proved that $J_\varepsilon(\cdot; a, z^1)$ is coercive on $L^2(\Omega)$. Assume the minimum is attained at $\hat{\varphi}^0$. We can then take

$$\hat{v} = \hat{\varphi}|_{\mathcal{O} \times (0, T)},$$

where $\hat{\varphi}$ is the solution to (12) for $\varphi^0 = \hat{\varphi}^0$. This control \hat{v} is such that (10) holds. Moreover, \hat{v} is the unique control with the following property: If v is another control such that the solution of (9) verifies (10), then

$$\|\hat{v}\|_{L^2(\mathcal{O} \times (0, T))} \leq \|v\|_{L^2(\mathcal{O} \times (0, T))}.$$

We can now argue as in [8] to deduce that \hat{v} satisfies (11) for some $C_1 = C_1(\Omega, \mathcal{O}, T, R, \|z^1\|_{L^2})$ that is nondecreasing in R . In fact, we have the following stronger result, whose proof is essentially based on the arguments of [8]:

Lemma 7 *Let $\Phi : L^\infty(\Sigma) \times L^2(\Omega) \mapsto L^2(\Omega)$ be given by $\Phi(a, y^1) = \hat{\varphi}^0$, where $\hat{\varphi}^0$ is the unique minimizer of $J_\varepsilon(\cdot; a, y^1)$ in $L^2(\Omega)$. If B is a bounded subset of $L^\infty(\Sigma)$ and K is a compact subset of $L^2(\Omega)$, then $\Phi(B \times K)$ is a bounded subset of $L^2(\Omega)$. Moreover, if $a_\mu \rightarrow a$ weakly-* in $L^\infty(\Sigma)$ and $y_\mu^1 \rightarrow y^1$ strongly in $L^2(\Omega)$, then $\hat{\varphi}_\mu^0 \rightarrow \hat{\varphi}^0$ weakly in $L^2(\Omega)$.*

This ends the proof of lemma 6.

2.2 Proof of theorem 2. The fixed point argument

We will first consider the case in which f is C^1 in $(-1, 1)$. Let us take $y^0, y^1 \in L^2(\Omega)$ and $\varepsilon > 0$. We denote by g the following function:

$$g(s) = \begin{cases} \frac{f(s) - f(0)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \quad (14)$$

Then g is continuous and uniformly bounded (because f is globally Lipschitz-continuous) and we have

$$|g(s)| \leq L \quad \forall s \in \mathbb{R}. \quad (15)$$

Let us introduce the mapping $\Gamma : L^2(\Sigma) \mapsto L^2(\Sigma)$ as follows: For each $z \in L^2(\Sigma)$, we put $\Gamma(y_z) = \gamma_0 y_z$, where $y_z = u_z + w_z$, u_z is the solution of

$$\begin{cases} \frac{\partial u_z}{\partial t} - \Delta u_z = 0 & \text{in } Q, \\ \frac{\partial u_z}{\partial n} + g(z)u_z = -f(0) & \text{on } \Sigma, \\ u_z(x, 0) = y^0(x) & \text{in } \Omega \end{cases} \quad (16)$$

and w_z is (together with v_z) the solution to the approximate controllability problem

$$\begin{cases} \frac{\partial w_z}{\partial t} - \Delta w_z = v_z 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial w_z}{\partial n} + g(z)w_z = 0 & \text{on } \Sigma, \\ w_z(x, 0) = 0 & \text{in } \Omega, \\ \|w_z(\cdot, T) - (y^1 - u_z(\cdot, T))\|_{L^2} \leq \varepsilon \end{cases} \quad (17)$$

furnished by lemma 6 (thus, v_z is the unique minimal L^2 -norm control for which the inequality $\|w_z(\cdot, T) - (y^1 - u_z(\cdot, T))\|_{L^2} \leq \varepsilon$ is satisfied). We then have

$$\begin{cases} \frac{\partial y_z}{\partial t} - \Delta y_z = v_z 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y_z}{\partial n} + g(z)y_z = -f(0) & \text{on } \Sigma, \\ y_z(x, 0) = y^0(x) & \text{in } \Omega, \\ \|y_z(\cdot, T) - y^1\|_{L^2} \leq \varepsilon \end{cases}$$

and

$$\|v_z\|_{L^2(\mathcal{O} \times (0, T))} \leq C_1(\Omega, \mathcal{O}, T, \varepsilon, L, \|y^1 - u_z(\cdot, T)\|_{L^2}).$$

We will see that Schauder's theorem can be applied to Γ . This will serve to deduce that Γ possesses a fixed point and will suffice to prove theorem 2 in this case.

Let us first check that Γ is a compact mapping. The systems in (16) and (17) are linear. In view of (15), $g(z)$ is uniformly bounded in $L^\infty(\Sigma)$. Thanks to the regularizing effect of the heat equation, we can affirm that u_z belongs to a fixed compact set of $L^2(Q)$ and $u_z(\cdot, T)$ belongs to a fixed compact set of $L^2(\Omega)$ as z runs over $L^2(\Sigma)$.

Let us put $z^1 = y^1 - u_z(\cdot, T)$ and consider the functional $J_\varepsilon(\cdot; g(z), z^1)$ (given by (13) with $a = g(z)$). We have

$$v_z = \hat{\varphi}|_{\mathcal{O} \times (0, T)},$$

where $\hat{\varphi}$ is the solution of (12) associated to the final data $\hat{\varphi}^0$, the unique minimizer in $L^2(\Omega)$ of $J_\varepsilon(\cdot; g(z), z^1)$. In view of lemma 7, $\hat{\varphi}^0$ is uniformly bounded in $L^2(\Omega)$ (independently of z). Accordingly, the associated solution $\hat{\varphi}$ belongs to a compact set in $L^2(Q)$ and, in particular, v_z belongs to a compact set of $L^2(\mathcal{O} \times (0, T))$.

Since the right hand side of (17) is $v_z 1_{\mathcal{O}}$, we can affirm that the corresponding solution w_z belongs to a bounded set of $L^2(0, T; H^1(\Omega))$, with the time derivative $\partial w_z / \partial t$ in a bounded set of $L^2(0, T; H^{-1}(\Omega))$ (among other things). Thus, w_z belongs to a compact set of $L^2(Q)$.

For simplicity of notation, let us put

$$Y = \{ y \in L^2(0, T; H^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2(0, T; H^{-1}(\Omega)) \}.$$

Notice that Y is a Hilbert space for the *natural* norm

$$\|y\|_Y = \left(\|y\|_{L^2(H^1)}^2 + \|\frac{\partial y}{\partial t}\|_{L^2(H^{-1})}^2 \right)^{1/2}.$$

Taking into account that $y_z = u_z + w_z$, we deduce that y_z lies in a bounded set of Y . Since $H^1(\Omega)$ is compactly embedded in $H^s(\Omega)$ for all $s < 1$, the embedding $Y \hookrightarrow L^2(0, T; H^s(\Omega))$ is compact for all $s < 1$. Consequently,

$$y_z \text{ belongs to a compact set of } L^2(0, T; H^s(\Omega)) \text{ for all } s < 1.$$

We will now use the following results:

- If $w \in H^s(\Omega)$ with $s > 1/2$, we can define the trace $\gamma_0 w = w|_{\partial\Omega}$ as an element of $H^{s-1/2}(\partial\Omega)$ and we have that $w \mapsto \gamma_0 w$ is a linear continuous mapping from $H^s(\Omega)$ into $H^{s-1/2}(\partial\Omega)$, cf. [18].
- For each $s > 1/2$, the embedding $L^2(0, T; H^{s-1/2}(\partial\Omega)) \hookrightarrow L^2(\Sigma)$ is continuous.
- In particular, we deduce that $\gamma_0 y_z$ belongs to a compact set of the space $L^2(0, T; H^{s-1/2}(\partial\Omega))$ for each $s \in (1/2, 1)$.

This proves that Γ is a (compact) mapping that maps the whole space $L^2(\Sigma)$ into a compact set of $L^2(\Sigma)$.

Now, let us see that Γ is also continuous. Let $\{z_k\}$ be a sequence in $L^2(\Sigma)$ such that

$$z_k \rightarrow z \quad \text{in } L^2(\Sigma).$$

Our aim is to prove that

$$\Gamma(z_k) \rightarrow \Gamma(z) \quad \text{in } L^2(\Sigma).$$

Let us set $\Gamma(z_k) = \gamma_0 y_k$ for all k . Recall that $y_k = u_k + w_k$ is, together with some v_k , a solution to the controllability problem

$$\begin{cases} \frac{\partial y_k}{\partial t} - \Delta y_k = v_k 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y_k}{\partial n} + g(z_k) y_k = -f(0) & \text{on } \Sigma, \\ y_k(x, 0) = y^0(x) & \text{in } \Omega, \\ \|y_k(\cdot, T) - y^1\|_{L^2} \leq \varepsilon, \end{cases}$$

constructed as above. We are going to prove that $\gamma_0 y_k$ converges strongly in $L^2(\Sigma)$ to $\Gamma(z)$. Obviously, it will sufficient to check this for a subsequence.

Since z_k converges to z in $L^2(\Sigma)$ and the function g is continuous, we deduce that there exists a subsequence z_μ such that

$$\begin{aligned} z_\mu &\rightarrow z \quad \text{a.e. in } \Sigma, \\ g(z_\mu) &\rightarrow g(z) \quad \text{weakly-* in } L^\infty(\Sigma) \text{ and a.e.} \end{aligned} \tag{18}$$

On the other hand, at least for a subsequence $\{v_\mu\}$, we must also have

$$v_\mu \rightarrow v_z \quad \text{strongly in } L^2(\mathcal{O} \times (0, T)). \tag{19}$$

To prove this, it suffices to argue in a similar way as we did when the compactness of Γ was shown. More precisely, let us recall that $y_\mu = u_\mu + w_\mu$ and let us observe that, at least for a new subsequence, we have

$$\begin{aligned} u_\mu &\rightarrow u_z \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \frac{\partial u_\mu}{\partial t} &\rightarrow \frac{\partial u_z}{\partial t} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)) \end{aligned}$$

and

$$u_\mu(\cdot, T) \rightarrow u_z(\cdot, T) \quad \text{strongly in } L^2(\Omega). \quad (20)$$

Taking into account (18), (20) and lemma 7, we deduce at once that the corresponding $\hat{\varphi}_\mu^0$ satisfy

$$\hat{\varphi}_\mu^0 \rightarrow \hat{\varphi}_z^0 \quad \text{strongly in } L^2(\Omega).$$

Accordingly, the associated solutions of (12) satisfy

$$\hat{\varphi}_\mu \rightarrow \hat{\varphi}_z \quad \text{strongly in } L^2(Q),$$

which implies (19).

It is now clear that the functions w_μ satisfy

$$\begin{aligned} \overline{w_\mu} &\rightarrow \overline{w_z} \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \frac{\partial w_\mu}{\partial t} &\rightarrow \frac{\partial w_z}{\partial t} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned}$$

Thus, we have

$$\begin{aligned} y_\mu &\rightarrow y_z \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \frac{\partial y_\mu}{\partial t} &\rightarrow \frac{\partial y_z}{\partial t} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)) \end{aligned}$$

and

$$\gamma_0 y_\mu \rightarrow \gamma_0 y_z \quad \text{strongly in } L^2(\Sigma),$$

i.e. $\Gamma(z_\mu) \rightarrow \Gamma(z)$ strongly in $L^2(\Sigma)$. This proves that Γ is continuous.

In view of Schauder's theorem, the mapping Γ possesses at least one fixed point y satisfying

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + g(y)y = -f(0) & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & \text{in } \Omega, \quad \|y(\cdot, T) - y^1\|_{L^2} \leq \varepsilon \end{cases}$$

for some $v \in L^2(\mathcal{O} \times (0, T))$. Recall that

$$\|v\|_{L^2(\mathcal{O} \times (0, T))} \leq C_2$$

where C_2 only depends on Ω , \mathcal{O} , T , ε , $\|g\|_{L^\infty(\mathbb{R})}$, $|f(0)|$, $\|y^0\|_{L^2}$ and $\|y^1\|_{L^2}$. This proves the desired result when f is C^1 in $(-1, 1)$.

Let us now assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is (only) a globally Lipschitz-continuous function. Using the convolution product, we can easily construct a sequence of functions f_m which are C^1 in $(-1, 1)$, uniformly globally Lipschitz-continuous and satisfy

$$f_m \rightarrow f \quad \text{uniformly on the compact sets of } \mathbb{R}.$$

For each $m \geq 1$, we can argue as before. This provides controls $v_m \in L^2(\mathcal{O} \times (0, T))$ and states y_m satisfying

$$\begin{cases} \frac{\partial y_m}{\partial t} - \Delta y_m = v_m 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y_m}{\partial n} + f_m(y_m) = -f(0) & \text{on } \Sigma, \\ y_m(x, 0) = y^0(x) & \text{in } \Omega \end{cases} \quad (21)$$

and

$$\|y_m(\cdot, T) - y^1\|_{L^2} \leq \varepsilon.$$

Since the functions f_m are uniformly globally Lipschitz-continuous, it can be assumed that the controls v_m are uniformly bounded in $L^2(\mathcal{O} \times (0, T))$. Arguing as in the case of regular data, we deduce (eventually after extracting a subsequence) that

$$\begin{aligned} v_m &\rightarrow v && \text{weakly in } L^2(\mathcal{O} \times (0, T)), \\ y_m &\rightarrow y && \text{weakly in } L^2(0, T; H^1(\Omega)), \text{ weakly-* in } L^\infty(0, T; L^2(\Omega)), \\ \frac{\partial y_m}{\partial t} &\rightarrow \frac{\partial y}{\partial t} && \text{weakly in } L^2(0, T; H^{-1}(\Omega)), \\ \gamma_0 y_m &\rightarrow \gamma_0 y && \text{strongly in } L^2(\Sigma). \end{aligned}$$

Hence, passing to the limit in (21) as $m \rightarrow +\infty$, we find a control $v \in L^2(\mathcal{O} \times (0, T))$ such that (1) possesses a solution y satisfying (2). This ends the proof of theorem 2.

Remark 8 Many variants and generalizations of theorem 2 can be proved in a similar way:

- Thus, following the ideas in [8], we can construct *quasi bang-bang* controls that lead the solution to (1) from y^0 to a state as close as we want to y^1 .

- We can also consider systems of the form

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + F(y) = v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + f(y) = 0 & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & \text{in } \Omega, \end{cases}$$

where f and F are globally Lipschitz-continuous functions. With arguments similar to those above, it can be proved that this system is again approximately controllable in $L^2(\Omega)$ at any time $T > 0$.

- We can even permit in the previous equation nonlinear terms of the form $F(y, \nabla y)$.
- Another interesting generalization of theorem 2 concerns simultaneous finite dimensional and approximate controllability. More precisely, under the assumptions of theorem 2, the following holds: Let $E \subset L^2(\Omega)$ be a finite dimensional subspace and let us denote by Π the corresponding orthogonal projector; then, for any $y^0, y^1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exist a control $v \in L^2(\mathcal{O} \times (0, T))$ and an associated solution $y \in C^0([0, T]; L^2(\Omega))$ satisfying

$$\|y(\cdot, T) - y^1\|_{L^2} \leq \varepsilon \quad \text{and} \quad \Pi(y(\cdot, T)) = \Pi(y^1). \quad (22)$$

This controllability property was introduced and analyzed in [23] for semi-linear heat equations with Dirichlet boundary conditions. For the proof of the previous assertion, it suffices to adapt the arguments in that reference.

3 Proof of the local null controllability result

The main goal of this Section is to prove theorem 4. As in the previous Section, we will begin by analyzing the situation for similar linear problems.

3.1 Some previous results for a linear problem

We will consider here the linear system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + a(x, t)y = 0 & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & \text{in } \Omega, \end{cases} \quad (23)$$

where (at least) $a \in L^\infty(\Sigma)$ and $y^0 \in L^2(\Omega)$.

In the sequel, we will denote by a_t the time derivative of a . The null controllability of (23) is ensured by the following result:

Theorem 9 *Assume that $a \in L^\infty(\Sigma)$, $a_t \in L^\infty(\Sigma)$ and $y^0 \in L^2(\Omega)$. Then (23) is null controllable with controls $v \in C^\infty(\overline{Q})$ furthermore satisfying*

$$\|v\|_{C^\ell(\overline{Q})} \leq C_3(\Omega, \mathcal{O}, T, \ell, \|a\|_{L^\infty(\Sigma)}, \|a_t\|_{L^\infty(\Sigma)}) \|y^0\|_{L^2} \quad (24)$$

for all integer $\ell \geq 0$.

PROOF: The null controllability of (23) with controls in $L^2(\mathcal{O} \times (0, T))$ is essentially proved in [11]. In this reference, the authors assume in fact that $a \in C^1(\overline{\Sigma})$, but the argument works as well under the assumptions we have made above. We will provide here a different proof which leads to an improvement of the regularity of the control.

Our goal is to prove that, under the previous assumptions for a , (23) is null controllable with regular controls. For convenience, we will first perform a change of variable. Thus, let $\theta \in C^\infty([0, T])$ be such that

$$0 \leq \theta \leq 1, \theta = 1 \text{ near } t = 0 \text{ and } \theta = 0 \text{ near } t = T.$$

Let us put $y = \theta(t)q + w$, where q is the solution of

$$\begin{cases} \frac{\partial q}{\partial t} - \Delta q = 0 & \text{in } Q, \\ \frac{\partial q}{\partial n} + a(x, t)q = 0 & \text{on } \Sigma, \\ q(x, 0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (25)$$

Then we have

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = -\theta'(t)q(x, t) + v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial w}{\partial n} + a(x, t)w = 0 & \text{on } \Sigma, \\ w(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (26)$$

The control v which gives the null controllability of (26) also provides the null controllability of (23) (and viceversa). So, we want to find $v = v(x, t)$ with

support in $\mathcal{O} \times [0, T]$ such that

$$w(x, T) = 0 \quad \text{in } \Omega. \quad (27)$$

In a first step, we will construct a control \tilde{v} in $L^2(\mathcal{O} \times (0, T))$ with this property. Then, using the regularizing property of the heat equation, we will be able to find a more regular control v such that (27) also holds.

First of all, let us recall from [11] a global Carleman inequality for the adjoint system (12). To this end, let us introduce a nonempty open set \mathcal{O}_0 satisfying $\mathcal{O}_0 \subset\subset \mathcal{O}$ and a function $\alpha_0 = \alpha_0(x)$ satisfying $\alpha_0 \in C^4(\overline{\Omega})$ and

$$\alpha_0 > 0 \text{ in } \Omega, \quad \alpha_0 = 0 \text{ on } \partial\Omega \text{ and } \nabla\alpha_0 \neq 0 \text{ in } \overline{\Omega} \setminus \overline{\mathcal{O}_0}.$$

The existence of such a function α_0 is justified in [11]. One has the following:

Lemma 10 *Assume that $a \in L^\infty(\Sigma)$ and $a_t \in L^\infty(\Sigma)$. There exists a positive number λ_1 depending on $\Omega, \mathcal{O}, T, \|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$, with the following property: For each $\lambda \geq \lambda_1$, there exist positive constants C and s_1 , again depending on $\Omega, \mathcal{O}, T, \|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$, such that*

$$\begin{aligned} \iint_Q (e^{-2s\alpha} + e^{-2s\tilde{\alpha}}) t^{-3} (T-t)^{-3} |\varphi|^2 dx dt \\ \leq C \iint_{\mathcal{O}_0 \times (0, T)} (e^{-2s\alpha} + e^{-2s\tilde{\alpha}}) t^{-3} (T-t)^{-3} |\varphi|^2 dx dt \end{aligned}$$

for all $s \geq s_1$. Here, φ is the solution of (12) associated to $\varphi^0 \in L^2(\Omega)$ and the functions $\alpha = \alpha(x, t)$ and $\tilde{\alpha} = \tilde{\alpha}(x, t)$ are given by

$$\alpha(x, t) = \frac{e^{2\lambda\|\alpha_0\|_\infty} - e^{\lambda\alpha_0}}{t(T-t)}, \quad \tilde{\alpha}(x, t) = \frac{e^{2\lambda\|\alpha_0\|_\infty} - e^{-\lambda\alpha_0}}{t(T-t)}.$$

For the proof of this result, see [11]. We can now deduce an observability estimate for the solutions to (12) whose proof is postponed to the end of this paragraph:

Lemma 11 *There exist positive constants C_4 and M depending on $\Omega, \mathcal{O}, T, \|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$ such that*

$$\iint_Q e^{-\frac{M}{T-t}} |\varphi|^2 dx dt \leq C_4 \iint_{\mathcal{O}_0 \times (0, T)} |\varphi|^2 dx dt \quad (28)$$

for any $\varphi^0 \in L^2(\Omega)$.

Arguing as in [9], we can deduce from (28) that (26) is null controllable with L^2 -controls supported in $\overline{\mathcal{O}_0} \times [0, T]$. More precisely, let $y^0 \in L^2(\Omega)$ be given

and let us introduce the functional $K_\varepsilon(\cdot; a)$, with

$$\begin{cases} K_\varepsilon(\varphi^0; a) = \frac{1}{2} \iint_{\mathcal{O}_0 \times (0, T)} |\varphi|^2 dx dt + \varepsilon \|\varphi^0\|_{L^2} - \iint_Q \theta'(t) q \varphi dx dt \\ \forall \varphi^0 \in L^2(\Omega) \end{cases}$$

(recall that q is the solution to (25)). Then $K_\varepsilon(\cdot; a)$ is continuous, strictly convex and coercive in $L^2(\Omega)$. This is due to the unique continuation property of the solutions to the adjoint system (12).

Let φ_ε^0 be the unique minimizer of $K_\varepsilon(\cdot; a)$ and let φ_ε be the associated solution to (12). Then the control $v_\varepsilon = \varphi_\varepsilon|_{\mathcal{O}_0 \times (0, T)}$ is such that the corresponding solution w_ε to (26) (with \mathcal{O} replaced by \mathcal{O}_0) satisfies

$$\|w_\varepsilon(\cdot, T)\|_{L^2} \leq \varepsilon.$$

On the other hand, thanks to the fact that $\theta' = 0$ near $t = T$, we have

$$\left(\iint_Q e^{\frac{M}{T-t}} |\theta'(t) q|^2 dx dt \right)^{1/2} \leq C \|y^0\|_{L^2} \quad (29)$$

for some C depending only on Ω , \mathcal{O} , T , $\|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$. Then the optimality conditions satisfied by φ_ε^0 give

$$\begin{aligned} \iint_{\mathcal{O}_0 \times (0, T)} |\varphi_\varepsilon|^2 dx dt + \varepsilon \|\varphi_\varepsilon^0\|_{L^2} &= \iint_Q \theta'(t) q \varphi_\varepsilon dx dt \\ &\leq \left(\iint_Q e^{\frac{M}{T-t}} |\theta'(t) q|^2 dx dt \right)^{1/2} \left(\iint_Q e^{-\frac{M}{T-t}} |\varphi_\varepsilon|^2 dx dt \right)^{1/2}. \end{aligned}$$

Therefore, from the estimates (28) and (29), we easily find that

$$\|v_\varepsilon\|_{L^2(\mathcal{O}_0 \times (0, T))} = \left(\iint_{\mathcal{O}_0 \times (0, T)} |\varphi_\varepsilon|^2 dx dt \right)^{1/2} \leq C \|y^0\|_{L^2},$$

for a new constant C only depending on Ω , \mathcal{O} , T , $\|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$.

Thus, at least for a subsequence, we have $v_\varepsilon \rightarrow \tilde{v}$ weakly in $L^2(\mathcal{O}_0 \times (0, T))$. In this way, we have found a control \tilde{v} that vanishes outside $\mathcal{O}_0 \times (0, T)$, satisfies

$$\|\tilde{v}\|_{L^2(\mathcal{O}_0 \times (0, T))} \leq C(\Omega, \mathcal{O}, T, \|a\|_{L^\infty(\Sigma)}, \|a_t\|_{L^\infty(\Sigma)}) \|y^0\|_{L^2} \quad (30)$$

and is such that the solution to (26) associated to \tilde{v} satisfies (27). Obviously, this proves that (23) is null controllable with controls in $L^2(\mathcal{O}_0 \times (0, T))$.

Let us finally indicate the way we can obtain from \tilde{v} a second (regular) control v with similar properties.

Let us introduce a C^∞ function $\xi = \xi(x)$ such that

$$\xi = 1 \text{ in a neighborhood of } \mathcal{O}_0 \text{ and } \xi \in \mathcal{D}(\mathcal{O}).$$

Let us set $w = (1 - \xi)\tilde{w}$, where \tilde{w} is the solution to (26) associated to \tilde{v} . Then w is the solution of

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = -\theta'(t)q(x, t) + v1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial w}{\partial n} + a(x, t)w = 0 & \text{on } \Sigma, \\ w(x, 0) = 0, \quad w(x, T) = 0 & \text{in } \Omega, \end{cases}$$

where

$$v = \xi(x)\theta'(t)q + 2\nabla\xi \cdot \nabla\tilde{w} + (\Delta\xi)\tilde{w}.$$

We have therefore built a new control v which provides the null controllability of (23).

In view of the interior regularity properties for the solution of (25), we have

$$q \in C^\infty(\overline{\Omega' \times (\varepsilon, T)})$$

and

$$\|q\|_{C^\ell(\overline{\Omega' \times (\varepsilon, T)})} \leq C(\Omega, \Omega', \varepsilon, T, \ell, \|a\|_{L^\infty(\Sigma)}) \|y^0\|_{L^2} \quad (31)$$

for any integer $\ell \geq 0$, any $\varepsilon > 0$ and any open set $\Omega' \subset\subset \Omega$. Using this fact, the interior regularity properties satisfied by \tilde{w} (the solution to (26) for $v = \tilde{v}$) and the fact that ξ is constant in a neighborhood of \mathcal{O}_0 and outside \mathcal{O} , we have that $v \in C^\infty(\overline{Q})$, the estimates (24) hold and, obviously, the associated solution to (26) satisfies (27). This ends the proof of theorem 9.

PROOF OF LEMMA 11: Let us first apply lemma 10 in the time interval $[T/4, T]$ for fixed and sufficiently large λ and s . We obtain:

$$\begin{aligned} & \iint_{\Omega \times (T/4, T)} (e^{-2s\alpha} + e^{-2s\bar{\alpha}})t^{-3}(T-t)^{-3}|\varphi|^2 dx dt \\ & \leq C \iint_{\mathcal{O}_0 \times (0, T)} (e^{-2s\alpha} + e^{-2s\bar{\alpha}})t^{-3}(T-t)^{-3}|\varphi|^2 dx dt. \end{aligned} \quad (32)$$

In view of the form of the weight functions in (32), we can easily deduce that there exist positive constants K_1 and M depending only on Ω , \mathcal{O} , T , $\|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$ such that

$$\iint_{\Omega \times (T/4, T)} e^{-\frac{M}{T-t}}|\varphi|^2 dx dt \leq K_1 \iint_{\mathcal{O}_0 \times (0, T)} |\varphi|^2 dx dt. \quad (33)$$

On the other hand, multiplying (12) by φ and integrating in Ω , we get

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\varphi|^2 dx + \int_{\Omega} |\nabla \varphi|^2 dx &\leq \|a\|_{L^\infty(\Sigma)} \int_{\partial\Omega} |\varphi|^2 d\sigma \\ &\leq \int_{\Omega} |\nabla \varphi|^2 dx + C(\|a\|_{L^\infty(\Sigma)}) \int_{\Omega} |\varphi|^2 dx \end{aligned}$$

for every $t > 0$. From these inequalities, it is immediate that

$$\iint_{\Omega \times (0, T/4)} |\varphi|^2 dx dt \leq e^{C(T, \|a\|_{L^\infty(\Sigma)})} \iint_{\Omega \times (T/4, T/2)} |\varphi|^2 dx dt$$

and we also find that

$$\iint_{\Omega \times (0, T/4)} |\varphi|^2 dx dt \leq e^{C(T, \|a\|_{L^\infty(\Sigma)}) + 2M/T} \iint_{\Omega \times (T/4, T/2)} e^{-\frac{M}{T-t}} |\varphi|^2 dx dt.$$

Using (33), we see that

$$\iint_{\Omega \times (0, T/4)} |\varphi|^2 dx dt \leq K_2 \iint_{\mathcal{O}_0 \times (0, T)} |\varphi|^2 dx dt, \quad (34)$$

where $K_2 = K_1 \exp\left(C(T, \|a\|_{L^\infty(\Sigma)}) + 2M/T\right)$. Now, from (33) and (34), the desired observability estimate (28) follows with $C_4 = K_1 + K_2$. This ends the proof.

Remark 12 It is possible to find an estimate of the constant in (30) that is explicit in $\|a\|_{L^\infty(\Sigma)}$ and $\|a_t\|_{L^\infty(\Sigma)}$. This can be made arguing as in [9], using sharp estimates of the constants λ_1 and s_1 in the Carleman inequality in lemma 10. All this yields the following estimate of the cost $\mathcal{C}(y^0)$ of the null controllability of (23) with controls in $L^2(\mathcal{O} \times (0, T))$:

$$\mathcal{C}(y^0) \leq e^{C(\Omega, \mathcal{O}) \left(1 + T + \frac{1}{T} + \|a\|_{L^\infty(\Sigma)}^2 + \|a_t\|_{L^\infty(\Sigma)} + T \|a\|_{L^\infty(\Sigma)}^2\right)} \|y^0\|_{L^2}.$$

In this estimate, we find $\|a\|_{L^\infty(\Sigma)}$ and, unfortunately, also $\|a_t\|_{L^\infty(\Sigma)}$. This is the main reason we cannot give a positive answer to the global null controllability problem for (1) when f is Lipschitz-continuous (see remark 15 below for additional details). In fact, an estimate of the cost for problem (23) of the form

$$\mathcal{C}(y^0) \leq e^{C(\Omega, \mathcal{O}) \left(1 + T + \frac{1}{T} + \gamma(\|a\|_{L^\infty(\Sigma)}) + T \|a\|_{L^\infty(\Sigma)}^2\right)} \|y^0\|_{L^2},$$

where γ is a positive increasing function, would lead to the null controllability of (1) even when f is locally Lipschitz-continuous and slightly superlinear at infinity. Results of this kind were deduced in [10] when the nonlinearity is in the partial differential equation and we impose homogeneous Dirichlet conditions.

3.2 The local null controllability of the nonlinear problem

We will need the (Banach) spaces

$$\tilde{C}^{1+\alpha,1}(\overline{Q}) = \{ u \in C^1(\overline{Q}) : D_x^1 u \in C^{\alpha,\alpha/2}(\overline{Q}) \},$$

$$\tilde{C}^{1+\alpha,1/2+\alpha/2}(\overline{Q}) = \{ u \in C^{\alpha,1/2+\alpha/2}(\overline{Q}) : D_x^1 u \in C^{\alpha,\alpha/2}(\overline{Q}) \}$$

and

$$\tilde{C}^{2+\alpha,1+\alpha/2}(\overline{Q}) = \{ u \in C^0(\overline{Q}) : D_x^1 u \in C^{1+\alpha,1/2+\alpha/2}(\overline{Q}), \frac{\partial u}{\partial t} \in C^{\alpha,\alpha/2}(\overline{Q}) \}.$$

Here, we have used $D_x^m u$ to denote all space derivatives of u of order m put together. We will denote by $\tilde{C}^{n+\alpha,r+\beta}(\overline{\Sigma})$ the Banach space formed by the restrictions to $\overline{\Sigma}$ of the functions in $\tilde{C}^{n+\alpha,r+\beta}(\overline{Q})$.

For linear systems of the form

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z = k(x, t) & \text{in } Q, \\ \frac{\partial z}{\partial n} + a(x, t)z = 0 & \text{on } \Sigma, \\ z(x, 0) = z^0(x) & \text{in } \Omega, \end{cases} \quad (35)$$

one has the following result, whose proof is given in [14], p. 320:

Lemma 13 *Assume that $k \in C^{\alpha,\alpha/2}(\overline{Q})$, $a \in \tilde{C}^{1+\alpha,1/2+\alpha/2}(\overline{\Sigma})$, $z^0 \in C^{2+\alpha}(\overline{\Omega})$ and the following compatibility condition is satisfied:*

$$\frac{\partial z^0}{\partial n} + a(x, 0)z^0 = 0 \quad \text{on } \partial\Omega.$$

Then (35) possesses exactly one solution z , with $z \in \tilde{C}^{2+\alpha,1+\alpha/2}(\overline{Q})$ and

$$\begin{cases} \|z\|_{\tilde{C}^{2+\alpha,1+\alpha/2}(\overline{Q})} \\ \leq C(\Omega, T, \|a\|_{\tilde{C}^{1+\alpha,1/2+\alpha/2}(\overline{\Sigma})}) \left(\|k\|_{C^{\alpha,\alpha/2}(\overline{Q})} + \|z^0\|_{C^{2+\alpha}(\overline{\Omega})} \right). \end{cases} \quad (36)$$

Assume that f is of class C^3 , $f(0) = 0$ and $y^0 \in C^{2+\alpha}(\overline{\Omega})$ satisfies the compatibility condition (8). Let us introduce the function g , given by (14). Then g is a C^2 function and

$$g(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases}$$

Let us introduce the Banach space

$$Z = \tilde{C}^{1+\alpha,1}(\bar{\Sigma})$$

and the closed linear manifold

$$Z^0 = \{ z \in Z : z(x, 0) = y^0(x) \text{ on } \partial\Omega \}.$$

For each $z \in Z^0$, we will consider the null controllability problem for the linear system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y}{\partial n} + g(z(x, t))y = 0 & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (37)$$

This can be solved arguing as in the previous paragraph. Indeed, in view of theorem 9, there exist controls $v_z \in C^\infty(\bar{Q})$ satisfying

$$\|v_z\|_{C^{\alpha,\alpha/2}(\bar{Q})} \leq C_5(\Omega, \mathcal{O}, T, \|g(z)\|_Z) \|y^0\|_{L^2}, \quad (38)$$

such that the solution y_z to (37) with $v = v_z$ satisfies

$$y_z(x, T) = 0 \quad \text{in } \Omega. \quad (39)$$

Furthermore, the constant C_5 in (38) can be chosen nondecreasing with respect to the last argument $\|g(z)\|_Z$. From the compatibility condition (8), the fact that $z \in Z^0$ and lemma 5, we deduce that $y_z \in \tilde{C}^{2+\alpha,1+\alpha/2}(\bar{Q})$ and an estimate like (36) holds. Notice that, here, we are using the fact that g is twice continuously differentiable, which gives $g(z) \in \tilde{C}^{1+\alpha,1/2+\alpha/2}(\bar{\Sigma})$. This is why we need f of class C^3 .

Let $A(z)$ be the family formed by all the controls in $C^{\alpha,\alpha/2}(\bar{Q})$ such that (38) and (39) hold and let us set

$$\Lambda(z) = \{ \gamma_0 y_z : y_z \text{ is the solution of (37) associated to } v \in A(z) \}.$$

Notice that $\Lambda(z) \subset Z^0$ for all $z \in Z^0$. Then, for all $q \in \Lambda(z)$, we have

$$\|q\|_Z \leq C_6(\Omega, \mathcal{O}, \alpha, T, \|g(z)\|_Z) \|y^0\|_{C^{2+\alpha}(\bar{\Omega})} \quad (40)$$

and

$$\|q\|_{\tilde{C}^{2+\alpha, 1+\alpha/2}(\bar{\Sigma})} \leq C_7(\Omega, \mathcal{O}, \alpha, T, \|g(z)\|_Z) \|y^0\|_{C^{2+\alpha}(\bar{\Omega})} \quad (41)$$

for some constants C_6 and C_7 again nondecreasing in $\|g(z)\|_Z$.

We will consider the set-valued mapping $z \mapsto \Lambda(z)$. We will check that, for some $\eta(\Omega, \mathcal{O}, \alpha, T) > 0$, the inequality $\|y^0\|_{C^{2+\alpha}(\bar{\Omega})} \leq \eta$ is sufficient to ensure that Λ possesses at least one fixed point in Z . To this end, we will check that, under these conditions, Kakutani's fixed point theorem can be applied to Λ (for the statement and proof of this result, see for instance [3]).

Of course, this will imply the existence of a control $v \in C^{\alpha, \alpha/2}(\bar{Q})$ such that the corresponding solution to (1) satisfies (3).

Indeed, it is not difficult to see that $\Lambda(z)$ is, for each $z \in Z^0$, a nonempty closed convex set in Z^0 . Furthermore, from (41) and the compactness of the embedding $\tilde{C}^{2+\alpha, 1+\alpha/2}(\bar{\Sigma}) \hookrightarrow Z$, we deduce that for each $z \in Z^0$ there exists a compact set $K_z \subset Z^0$ such that

$$\Lambda(z) \subset K_z.$$

We also have the following result, whose proof is given below:

Lemma 14 *Under the assumptions of theorem 4 and with the previous notation, the set-valued mapping $z \rightarrow \Lambda(z)$ is upper hemicontinuous. In other words, for each bounded linear form $\xi \in Z'$, the real-valued function*

$$z \mapsto \sup_{q \in \Lambda(z)} \langle \xi, q \rangle$$

is upper semicontinuous.

Now, let $R > 0$ be given, let us assume that $z \in Z^0$ satisfies

$$\|z\|_Z \leq R$$

and let us denote by $M(R)$ the following quantity:

$$M(R) = \sup_{\|z\|_Z \leq R} C_6(\Omega, \mathcal{O}, \alpha, T, \|g(z)\|_Z)$$

Let us set $\eta = R/M(R)$ and let us assume that the initial state y^0 satisfies $\|y^0\|_{C^{2+\alpha}(\bar{\Omega})} \leq \eta$ (besides (8)). Let us put

$$K(y^0) = \{z \in Z^0 : \|z\|_Z \leq R\}.$$

Then $K(y^0)$ is a nonempty closed convex set in Z . In view of (40) and (41), Λ maps $K(y^0)$ into a fixed compact set $K \subset K(y^0)$. Consequently, all hypotheses of Kakutani's theorem are certainly satisfied and the existence of a fixed point of Λ in $K(y^0)$ is ensured.

This ends the proof of theorem 4.

PROOF OF LEMMA 14: Let us see that the set

$$B(\kappa, \xi) = \{ z \in Z^0 : \sup_{q \in \Lambda(z)} \langle \xi, q \rangle \geq \kappa \}$$

is closed for every $\kappa \in \mathbb{R}$ and every $\xi \in Z'$. Thus, assume that $z_m \in B(\kappa, \xi)$ for all m and

$$z_m \rightarrow z \quad \text{in } Z.$$

Our aim is to prove that $z \in B(\kappa, \xi)$. In view of the regularity of g , we have

$$g(z_m) \rightarrow g(z) \quad \text{in } Z.$$

Since all sets $\Lambda(z_m)$ are compact, for each m we must have

$$\kappa \leq \sup_{q \in \Lambda(z_m)} \langle \xi, q \rangle = \langle \xi, q_m \rangle \tag{42}$$

for some $q_m \in \Lambda(z_m) \subset K$. From the definitions of $\Lambda(z_m)$ and $A(z_m)$, there must exist controls $v_m \in C^{\alpha, \alpha/2}(\overline{Q})$ and associated states y_m satisfying

$$\begin{cases} \frac{\partial y_m}{\partial t} - \Delta y_m = v_m 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial y_m}{\partial n} + g(z_m(x, t))y_m = 0 & \text{on } \Sigma, \\ y_m(x, 0) = y^0(x), \quad y_m(x, T) = 0 & \text{in } \Omega \end{cases}$$

and $q_m = \gamma_0 y_m$. We also have

$$\|v_m\|_{C^{\alpha, \alpha/2}(\overline{Q})} \leq C_5(\Omega, \mathcal{O}, T, \|g(z_m)\|_Z) \|y_0\|_{L^2}$$

and

$$\|q_m\|_{\tilde{C}^{2+\alpha, 1+\alpha/2}(\overline{\Sigma})} \leq C_7(\Omega, \mathcal{O}, \alpha, T, \|g(z_m)\|_Z) \|y^0\|_{C^{2+\alpha}(\overline{\Omega})}.$$

Hence, q_m (resp. v_m) is uniformly bounded in $\tilde{C}^{2+\alpha, 1+\alpha/2}(\overline{\Sigma})$ (resp. $C^{\alpha, \alpha/2}(\overline{Q})$). Therefore, we can write the following at least for a subsequence:

$$\begin{aligned} q_m &\rightarrow \hat{q} \quad \text{strongly in } Z, \\ v_m &\rightarrow \hat{v} \quad \text{strongly in } C^0(\overline{Q}) \end{aligned}$$

and $\hat{v} \in C^{\alpha, \alpha/2}(\bar{Q})$.

Now, it is easy to deduce that $\hat{v} \in A(z)$ and $\hat{q} = \gamma_0 \hat{y}$, with

$$\begin{cases} \frac{\partial \hat{y}}{\partial t} - \Delta \hat{y} = \hat{v} 1_{\mathcal{O}} & \text{in } Q, \\ \frac{\partial \hat{y}}{\partial n} + g(z(x, t)) \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(x, 0) = y^0(x), \quad \hat{y}(x, T) = 0 & \text{in } \Omega. \end{cases}$$

In particular, we have $\hat{q} \in \Lambda(z)$. Now, we can take limits in (42) and this gives

$$\kappa \leq \langle \xi, \hat{q} \rangle \leq \sup_{q \in \Lambda(z)} \langle \xi, q \rangle,$$

that is to say, $z \in B(\kappa, \xi)$. This proves that $z \mapsto \Lambda(z)$ is upper hemicontinuous.

Remark 15 To prove a (global) null controllability result for (1), a natural strategy is a fixed point approach similar to the argument we have used in Section 2. But the requirement $a_t \in L^\infty(\Sigma)$, which seems to be necessary in the proofs of lemma 10 and theorem 9, is apparently too strong. Indeed, we would need in practice functions z such that the trace of the time derivative of $g(z)$ belongs to $L^\infty(\Sigma)$. Thus, we are not too far from

$$\frac{\partial z}{\partial t} \in L^\infty(0, T; W^{1, N+\kappa}(\Omega)),$$

with $\kappa > 0$. But the spaces of this kind seem to be too small to permit compactness and good estimates for the fixed point mapping. Hence, as we already mentioned at the end of Section 1, the global null controllability of (1) is an open question.

4 Proof of the large time null controllability result

This Section is devoted to prove theorem 5. To this end, we will argue as follows:

- Starting from an arbitrary large $y^0 \in L^2(\Omega)$, we first use the local feedback law $v = -y 1_{\mathcal{O}}$. This provides a first control v^1 for $t \in [0, T_1]$ which leads the system to a state $y^1 = y(\cdot, T_1)$ which is small in the H^1 -norm.
- Then, we simply take $v^2 = 0$ for $t \in [T_1, T_2]$. This leads to a second intermediate state $y^2 = y(\cdot, T_2)$ which is small in the H^2 -norm.

- Starting from y^2 at time $t = T_2$ and setting again $v^3 = 0$ for $t \in [T_2, T^*]$, we arrive now at a state $y^* = y(\cdot, T^*)$ such that

$$\|y^*\|_{C^{2+\alpha}} \leq \eta(\Omega, \mathcal{O}, \alpha, \varepsilon), \quad (43)$$

where η is the constant arising in theorem 4 and ε is arbitrarily small.

- Let us introduce $T = T^* + \varepsilon$. In view of (43) and theorem 4, we can find a control v^* defined for $t \in [T^*, T]$ such that the associated state y^* satisfies

$$y^*(x, T) = 0 \quad \text{in } \Omega. \quad (44)$$

Obviously, this ends the proof.

Let us now give more details. For simplicity, we will assume that $N \leq 4$. This assumption is not strictly necessary but will make the argument easier and will clarify the presentation we can give. We will use well known regularity results for linear and semilinear parabolic systems, see for instance [14] and [18].

Thus, let $y^0 \in L^2(\Omega)$ be given and let us choose $\alpha \in (0, 1)$ and $\varepsilon > 0$.

FIRST STEP: Consider the *closed-loop* controlled system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = -y1_{\mathcal{O}} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial y}{\partial n} + f(y) = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ y(x, 0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (45)$$

This semilinear system possesses exactly one solution \hat{y} , with

$$\hat{y} \in L^2(0, +\infty; H^1(\Omega)) \cap C^0([0, +\infty); L^2(\Omega)).$$

Furthermore, using standard techniques, we see at once that

$$\begin{cases} \frac{1}{2} \|\hat{y}(\cdot, t)\|_{L^2}^2 + \int_{\tau}^t \|\nabla \hat{y}(\cdot, s)\|_{L^2}^2 ds \\ \quad + \int_{\tau}^t \int_{\mathcal{O}} |\hat{y}(x, s)|^2 dx ds + \int_{\tau}^t \int_{\partial\Omega} f(\hat{y}) \hat{y} d\Gamma ds \\ \quad = \frac{1}{2} \|\hat{y}(\cdot, \tau)\|_{L^2}^2 \end{cases} \quad (46)$$

for all $t, \tau \in [0, +\infty)$ with $\tau < t$. Since $f(s)s \geq 0$ for all s , we deduce that

$$\|\hat{y}(\cdot, t)\|_{L^2}^2 + C \int_{\tau}^t \|\hat{y}(\cdot, s)\|_{H^1}^2 ds \leq \|\hat{y}(\cdot, \tau)\|_{L^2}^2 \quad (47)$$

for $0 \leq \tau < t < +\infty$ and also

$$\|\hat{y}(\cdot, t)\|_{L^2}^2 \leq e^{-Ct} \|y^0\|_{L^2}^2 \quad (48)$$

and

$$\int_t^{t+1} \|\hat{y}(\cdot, s)\|_{H^1}^2 ds \leq C e^{-Cs} \|y^0\|_{L^2}^2 \quad (49)$$

for all $t \geq 0$.

For each $\delta > 0$, we also have

$$\|\hat{y}(\cdot, t)\|_{L^\infty} \leq \|\hat{y}(\cdot, \delta)\|_{L^\infty} \leq C_\delta \|y^0\|_{L^2} \quad \forall t \geq \delta. \quad (50)$$

This last estimate can be easily deduced, for instance, by comparing in $\Omega \times (\delta, +\infty)$ the functions \hat{y} and $-\hat{y}$ with the solution w to the linear problem

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = -w1_{\mathcal{O}} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ w(x, 0) = |y^0(x)| & \text{in } \Omega. \end{cases}$$

We will choose $T_1 > 0$ large enough (to be precised below) and such that

$$\|\hat{y}(\cdot, T_1)\|_{H^1}^2 \leq e^{-CT_1} \|y^0\|_{L^2}^2. \quad (51)$$

In view of (49), many such times T_1 exist.

SECOND STEP: Let us set $y^1 = \hat{y}(\cdot, T_1)$ and let us consider the uncontrolled system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = 0 & \text{in } \Omega \times (T_1, +\infty), \\ \frac{\partial y}{\partial n} + f(y) = 0 & \text{on } \partial\Omega \times (T_1, +\infty), \\ y(x, T_1) = y^1(x) & \text{in } \Omega. \end{cases} \quad (52)$$

In view of the assumptions we have made on f , there exists a unique solution \tilde{y} to (52), with

$$\tilde{y} \in L^2(T_1, +\infty; H^2(\Omega)) \cap C^0([T_1, +\infty); H^1(\Omega)) \cap L^\infty(\Omega \times (T_1, +\infty)). \quad (53)$$

Indeed, if we multiply the equation in (52) (written for \tilde{y}) by the time derivative of \tilde{y} and we integrate with respect to x and t , we easily find that

$$\left\{ \begin{array}{l} \int_{\tau}^t \|\frac{\partial \tilde{y}}{\partial t}(\cdot, s)\|_{L^2}^2 ds + \frac{1}{2} \|\nabla \tilde{y}(\cdot, t)\|_{L^2}^2 + \int_{\partial\Omega} F(\tilde{y}(x, t)) d\Gamma \\ = \frac{1}{2} \|\nabla \tilde{y}(\cdot, \tau)\|_{L^2}^2 + \int_{\partial\Omega} F(\tilde{y}(x, \tau)) d\Gamma \end{array} \right. \quad (54)$$

for all $t, \tau \geq T_1$ with $\tau < t$. On the other hand, if we multiply the same equation by $-\Delta \tilde{y}$ and we integrate again with respect to x and t , we see that

$$\left\{ \begin{array}{l} \frac{1}{2} \|\nabla \tilde{y}(\cdot, t)\|_{L^2}^2 + \int_{\partial\Omega} F(\tilde{y}(x, t)) d\Gamma + \int_{\tau}^t \|\Delta \tilde{y}(\cdot, s)\|_{L^2}^2 ds \\ = \frac{1}{2} \|\nabla \tilde{y}(\cdot, \tau)\|_{L^2}^2 + \int_{\partial\Omega} F(\tilde{y}(x, \tau)) d\Gamma \end{array} \right. \quad (55)$$

for all these t and τ . In (54) and (55), F stands for the following function:

$$F(s) = \int_0^s f(\sigma) d\sigma \quad \forall s \in \mathbb{R}. \quad (56)$$

Since $F(s) \geq 0$ for all s , we easily deduce from (54), (55) and the estimates in the first step that

$$\|\tilde{y}(\cdot, t)\|_{H^1}^2 + \int_{T_1}^t \|\hat{y}(\cdot, s)\|_{H^2}^2 ds \leq C e^{-CT_1} \|y^0\|_{L^2}^2 \quad (57)$$

and

$$\int_t^{t+1} \|\tilde{y}(\cdot, s)\|_{H^2}^2 ds \leq C e^{-CT_1} \|y^0\|_{L^2}^2 \quad (58)$$

for all $t \geq T_1$. Consequently, we can choose $T_2 > T_1$ such that

$$\|\tilde{y}(\cdot, T_2)\|_{H^2}^2 \leq C e^{-CT_1} \|y^0\|_{L^2}^2. \quad (59)$$

In fact, (58) indicates that there are “many” T_2 with this property. Also, notice that T_2 can be chosen arbitrarily close to T_1 .

THIRD STEP: Let us set $y^2 = \tilde{y}(\cdot, T_2)$ and let us look at the restriction of \tilde{y} to the time interval $[T_2, +\infty)$. We have

$$\begin{cases} \tilde{y} \in L^2(T_2, +\infty; H^2(\Omega)) \cap C^0([T_2, +\infty); H^1(\Omega)), \\ \frac{\partial \tilde{y}}{\partial t} \in L^2(T_2, +\infty; H^1(\Omega)) \cap L^\infty(T_2, +\infty; L^2(\Omega)). \end{cases} \quad (60)$$

Indeed, if we compute the time derivative of the equation satisfied by \tilde{y} , we multiply by $\frac{\partial \tilde{y}}{\partial t}$ and we integrate in space and time, the following is found:

$$\begin{cases} \frac{1}{2} \left\| \frac{\partial \tilde{y}}{\partial t}(\cdot, t) \right\|_{L^2}^2 + \int_{\tau}^t \left\| \nabla \frac{\partial \tilde{y}}{\partial t}(\cdot, s) \right\|_{L^2}^2 ds \\ \quad + \int_{\tau}^t \left(\int_{\partial \Omega} f'(\tilde{y}) \left| \frac{\partial \tilde{y}}{\partial t} \right|^2 d\Gamma \right) ds \\ \quad = \frac{1}{2} \left\| \frac{\partial \tilde{y}}{\partial t}(\cdot, \tau) \right\|_{L^2}^2 \end{cases} \quad (61)$$

for all $t, \tau \geq T_2$ with $\tau < t$. Since

$$\frac{\partial \tilde{y}}{\partial t}(x, T_2) \equiv \Delta \tilde{y}(x, T_2)$$

and $f'(s) \geq 0$ for all s , we deduce from (59) and (61) that

$$\left\| \frac{\partial \tilde{y}}{\partial t}(\cdot, t) \right\|_{L^2}^2 + \int_{T_2}^t \left\| \nabla \frac{\partial \tilde{y}}{\partial t}(\cdot, s) \right\|_{L^2}^2 ds \leq C e^{-CT_1} \|y^0\|_{L^2}^2 \quad (62)$$

for all $t \geq T_2$.

We are now going to perform a classical *bootstrap* argument, using the fact that

$$\frac{\partial \tilde{y}}{\partial n} = -f(\tilde{y}) \quad \text{on} \quad \partial \Omega \times (T_2, +\infty). \quad (63)$$

Thus, let us set $\tilde{F} = f(\tilde{y})$ (a function defined in the whole cylinder $\Omega \times (T_2, +\infty)$) and let \tilde{f} be the ‘‘lateral’’ trace of \tilde{F} on Σ . Since $f \in C^4(\mathbb{R})$, we have:

$$\begin{cases} \tilde{F} \in L^2(T_2, +\infty; H^2(\Omega)) \cap C^0([T_2, +\infty); H^1(\Omega)) \\ \frac{\partial \tilde{F}}{\partial t} \in L^2(T_2, +\infty; H^1(\Omega)) \end{cases} \quad (64)$$

and

$$\begin{cases} \tilde{f} \in L^2(T_2, +\infty; H^{3/2}(\partial\Omega)) \cap C^0([T_2, +\infty); H^{1/2}(\partial\Omega)) \\ \frac{\partial \tilde{f}}{\partial t} \in L^2(T_2, +\infty; H^{1/2}(\partial\Omega)) \end{cases} \quad (65)$$

(here, we have used that $N \leq 4$).

Reading the boundary condition in (52) in the form (63), we deduce from (65) that

$$\begin{cases} \tilde{y} \in L^2(T_2, +\infty; H^3(\Omega)) \cap C^0([T_2, +\infty); H^2(\Omega)), \\ \frac{\partial \tilde{y}}{\partial t} \in L^2(T_2, +\infty; H^1(\Omega)) \cap L^\infty(T_2, +\infty; L^2(\Omega)), \end{cases}$$

with estimates of \tilde{y} and $\frac{\partial \tilde{y}}{\partial t}$ in these spaces bounded by $Ce^{-CT_1} \|y^0\|_{L^2}$.

Now, let us choose $T_{21} > T_2$ such that

$$\|\tilde{y}(\cdot, T_{21})\|_{H^3}^2 \leq Ce^{-CT_1} \|y^0\|_{L^2}^2.$$

Once more, it is clear that many such T_{21} exist. Again, taking into account that f is of class C^4 , we see that \tilde{F} is as regular as \tilde{y} for $t \in [T_{21}, +\infty)$ and

$$\begin{cases} \tilde{f} \in L^2(T_2, +\infty; H^{5/2}(\partial\Omega)) \cap C^0([T_2, +\infty); H^{3/2}(\partial\Omega)), \\ \frac{\partial \tilde{f}}{\partial t} \in L^2(T_2, +\infty; H^{1/2}(\partial\Omega)). \end{cases} \quad (66)$$

Consequently,

$$\begin{cases} \tilde{y} \in L^2(T_{21}, +\infty; H^4(\Omega)) \cap C^0([T_{21}, +\infty); H^3(\Omega)), \\ \frac{\partial \tilde{y}}{\partial t} \in L^2(T_{21}, +\infty; H^2(\Omega)), \end{cases} \quad (67)$$

with the norms bounded by $Ce^{-CT_1} \|y^0\|_{L^2}^2$.

At this moment, let us introduce T_{22} , with $T_{22} > T_{21}$ and such that

$$\|\tilde{y}(\cdot, T_{22})\|_{H^4}^2 \leq Ce^{-CT_1} \|y^0\|_{L^2}^2.$$

Again, it is clear that many such T_{22} exist. We have now

$$\begin{cases} \tilde{F} \in L^2(T_{22}, +\infty; W^{3,p_1}(\Omega)) \cap C^0([T_{22}, +\infty); W^{2,p_1}(\Omega)), \\ \frac{\partial \tilde{F}}{\partial t} \in L^2(T_{22}, +\infty; W^{1,p_1}(\Omega)), \end{cases} \quad (68)$$

where p_1 is the Sobolev embedding exponent for $H^1(\Omega)$, i.e.

$$p_1 = \begin{cases} \frac{2N}{N-2} & \text{if } N = 3 \text{ or } N = 4, \\ \text{arbitrary but finite} & \text{if } N = 2. \end{cases}$$

Hence, arguing as above we find that

$$\begin{cases} \tilde{y} \in L^2(T_{22}, +\infty; W^{4,p_1}(\Omega)) \cap C^0([T_{22}, +\infty); W^{3,p_1}(\Omega)), \\ \frac{\partial \tilde{y}}{\partial t} \in L^2(T_{22}, +\infty; W^{2,p_1}(\Omega)). \end{cases} \quad (69)$$

In this way, we can repeat the argument and find subsequent times T_{23}, T_{24}, \dots with $T_{22} < T_{23} < T_{24} \dots$ such that

$$\|\tilde{y}(\cdot, T_{2i})\|_{W^{4,p_{i-2}}}^2 \leq C e^{-CT_1} \|y^0\|_{L^2}^2$$

and \tilde{y} is as in (69) with T_{22} and p_1 respectively replaced by T_{2i} and p_{i-1} for $i = 3, 4, \dots$. Here, for each i , p_i is the Sobolev embedding exponent of $W^{1,p_{i-1}}(\Omega)$. Of course, we also have the norms of \tilde{y} and $\frac{\partial \tilde{y}}{\partial t}$ in the corresponding spaces bounded by $C e^{-CT_1} \|y^0\|_{L^2}$.

Obviously, for i large enough (only depending on N), we have

$$W^{3,p_{i-1}}(\Omega) \hookrightarrow C^{2+\alpha}(\overline{\Omega}),$$

whence $\tilde{y} \in C^0([T_{2i}, +\infty); C^{2+\alpha}(\overline{\Omega}))$ and

$$\|\tilde{y}(\cdot, t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_8 e^{-C_9 T_1} \|y^0\|_{L^2} \quad \forall t \geq T_{2i} \quad (70)$$

for some constants C_8 and C_9 . We will set $T^* = T_{2i}$ for this i . We will also set $y^* = \tilde{y}(\cdot, T^*)$.

FOURTH STEP: Let us assume that T_1 has been chosen in the first step such that

$$C_8 e^{-C_9 T_1} \|y^0\|_{L^2} \leq \eta(\Omega, \mathcal{O}, \alpha, \varepsilon), \quad (71)$$

where η is the constant furnished by theorem 4 and let us set $T = T^* + \varepsilon$. Then, in view of (70), we deduce that, for some $v^* = v^*(x, t)$, the solution y^* to the system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = v^* 1_{\mathcal{O}} & \text{in } \Omega \times (T^*, T), \\ \frac{\partial y}{\partial n} + f(y) = 0 & \text{on } \partial\Omega \times (T^*, T), \\ y(x, T^*) = y^3(x) & \text{in } \Omega. \end{cases}$$

satisfies (44). As explained above, the proof of theorem 5 is now achieved.

Remark 16 It is clear that, in the previous proof, the times T_2 , T^* and T can be chosen arbitrarily close to T_1 . It is also clear that T_1 can be chosen of the form

$$T_1 = C_{10} \log \|y^0\|_{L^2} + C_{11},$$

where C_{10} and C_{11} only depend on Ω , \mathcal{O} and f .

5 Some final comments and open questions

5.1 Null controllability

Assume that the function f in (1) is Lipschitz-continuous. In this case, we do not know at present whether or not (1) is null controllable in an arbitrarily small time interval.

What we would need to give a positive answer to this question is, essentially, a Carleman estimate like the one in lemma 10 valid for all $a \in L^\infty(\Sigma)$ (with constants λ_1 , s_1 and C only depending on Ω , \mathcal{O} , T and $\|a\|_{L^\infty(\Sigma)}$). As we have explained above, “good” estimates of λ_1 , s_1 and C would even lead to the null controllability of some slightly superlinear systems. But, unfortunately, this is unknown.

5.2 The role of blow-up

On the other hand, if the nonlinearity is too strong, it is expected that the system blows up in such a way that null controllability is impossible (unless the control acts in the whole domain). This was shown in [10] for semilinear parabolic equations completed with Dirichlet boundary conditions.

In order to clarify this point, let us consider the relatively simple case of a radial solution of the system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = 0 & \text{in } B_R \times (0, T), \\ \frac{\partial y}{\partial n} - h(y) = 0 & \text{on } \partial B_R \times (0, T), \\ y(x, 0) = y^0(x) & \text{in } B_R. \end{cases} \quad (72)$$

where B_R is the open ball in \mathbb{R}^N of radius R . We will assume here that $h \in C^1(\mathbb{R})$ is nondecreasing, $h(s) > 0$ for all $s > 0$ and

$$\int_0^{+\infty} \frac{1}{h(s)} ds < +\infty.$$

Assume that y^0 is a regular radial function such that $y_r^0(x) = \nabla y^0(x) \cdot x/|x| \geq 0$ and

$$m(0) = \int_{B_R} y^0(x) dx > 0.$$

Let us denote by y the associated solution to (72) and let us set

$$m(t) = \int_{B_R} y(x, t) dx$$

for all t . Then, following for instance [19], it is not difficult to prove that

$$\begin{aligned} m'(t) &= \int_{B_R} y_t(x, t) dx = \int_{B_R} \Delta y(x, t) dx \\ &= \int_{\partial B_R} \frac{\partial y}{\partial n}(x, t) d\Gamma = \int_{\partial B_R} h(y(x, t)) d\Gamma \\ &\geq Ah(B) \int_{B_R} y(x, t) dx \end{aligned}$$

and thus

$$m'(t) \geq Ah(Bm(t)) \quad (73)$$

for some positive constants A and B . Notice that we have used here the fact that $y_r(x, t) = \nabla y(x, t) \cdot x/|x| \geq 0$ for all t .

In particular, we deduce from (73) that $m(t) > 0$ for all positive t .

Let us introduce the functions H and L , with

$$H(s) = \int_s^{+\infty} \frac{1}{h(\sigma)} d\sigma \quad \text{and} \quad L = H^{-1}.$$

Then, it can be easily deduced from (73) that, for some $C > 0$, one has

$$H(Bm(0)) - H(Bm(t)) \geq Ct$$

and

$$m(t) \geq \frac{1}{B}L(H(Bm(0)) - Ct)$$

for all t . Since $L(H(Bm(0)) - Ct) \rightarrow +\infty$ as $t \rightarrow \frac{1}{C}H(Bm(0))$, we have blow-up before $t = T^* = \frac{1}{C}H(Bm(0))$.

Unfortunately, the arguments in [10] cannot be applied to a system of the kind (1), since they rely strongly on the fact that, there, the nonlinear term interacts with the elliptic operator $-\Delta$ in $\Omega \setminus \omega$.

Indeed, to apply the techniques in [10] in the context of (72), we have to introduce a cut-off function $\rho = \rho(x)$ with support in $\overline{B_R} \setminus \overline{\omega}$ and we have to analyze the evolution of

$$\tilde{m}(t) = \int_{B_R} \rho(x)y(x, t) dx.$$

It would be satisfactory to have for \tilde{m} a differential inequality of the kind (73). But this time we have

$$\begin{aligned} \tilde{m}'(t) &= \int_{B_R} \rho(x)y_t(x, t) dx = \int_{B_R} \rho(x)\Delta y(x, t) dx \\ &= \int_{B_R} \Delta \rho(x)y(x, t) dx + \int_{\partial B_R} \left(\rho(x)\frac{\partial y}{\partial n}(x, t) - \frac{\partial \rho}{\partial n}(x)y(x, t) \right) d\Gamma. \end{aligned}$$

By choosing ρ such that $\frac{\partial \rho}{\partial n}(x) = 0$ on ∂B_R , we find that

$$\tilde{m}'(t) = \int_{B_R} \Delta \rho(x)y(x, t) dx + \int_{\partial B_R} \rho(x)h(y(x, t)) d\Gamma,$$

but it seems complicate to bound from below the sum of these integrals by an expression of the form $Ah(B\tilde{m}(t)) - C$ (notice that we do not have now $y_r \geq 0$).

Thus, for systems like (1) a new argument is required and, for the moment, the question is open.

For other basic facts on the blow-up due to the presence of nonlinear boundary conditions, see for instance [6], [19], [20] and [16].

5.3 A variant for systems of the Stokes kind

Let us now consider the Stokes system with nonlinear slip boundary conditions

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + \nabla \pi = v 1_{\mathcal{O}}, & \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, & (\sigma(y, \pi) \cdot n)_{\text{tg}} + f(y)_{\text{tg}} = 0 & \text{on } \Sigma, \\ y(x, 0) = y^0(x) & & \text{in } \Omega, \end{cases} \quad (74)$$

where $v \in L^2(\mathcal{O} \times (0, T))^N$, $y^0 \in H$ and $f : \mathbb{R}^N \mapsto \mathbb{R}^N$ is globally Lipschitz-continuous. Here, we have used the following notation:

$a_{\text{tg}} = a - (a \cdot n)n$ is the tangential component of a ,

$\sigma(y, \pi) = -\pi \mathbf{Id} + (\nabla y + {}^t \nabla y)$ is the usual stress tensor,

$H = \{ v \in L^2(\Omega) : \nabla \cdot v = 0 \text{ in } \Omega, v \cdot n = 0 \text{ on } \partial\Omega \}$.

Arguing as in Section 2, it can be proved that (74) is approximately controllable in H for all $T > 0$. Furthermore, the control v can be chosen of the form $v = (v_1, v_2, 0)$, with $v_i \in L^2(\mathcal{O} \times (0, T))$ (see [4]).

However, the null controllability of (74) is an open problem. It seems reasonable to expect results similar to theorems 4, 5 and 9. But, again, this is unknown at present.

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