Insensitizing controls for a heat equation with a nonlinear term involving the state and the gradient ⋆

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Abstract

In this paper we present two results on the existence of insensitizing controls for a heat equation in a bounded domain of $\mathbb{R}^N$. We first consider a semilinear heat equation involving gradient terms with homogeneous Dirichlet boundary conditions. Then a heat equation with a nonlinear term $F(y)$ and linear boundary conditions of Fourier type is considered. The nonlinearities are assumed to be globally Lipschitz-continuous. In both cases, we prove the existence of controls insensitizing the L^2-norm of the observation of the solution in an open subset $\mathcal{O}$ of the domain, under suitable assumptions on the data. Each problem boils down to a special type of null controllability problem. General observability inequalities are proved for linear systems similar to the linearized problem. The proofs of the main results in this paper involve such inequalities and rely on the study of these linear problems and appropriate fixed point arguments.

Key words: controllability, nonlinear PDE of parabolic type, nonlinear gradient terms
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1 Setting the problems and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded connected open set with boundary $\partial\Omega \in C^2$. For $T > 0$, we denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Let ω and $\mathcal{O}$ be nonempty open subsets of Ω. We first consider the nonlinear heat equation:

$$\begin{cases} \partial_t y - \Delta y + f(y, \nabla y) = \xi + v\mathbf{1}_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) + \tau\, \hat{y}_0(x) \ \text{ in } \Omega, \end{cases} \tag{1}$$

where f is a C^1 globally Lipschitz-continuous function defined on $\mathbb{R} \times \mathbb{R}^N$, $\xi \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ are given, $\hat{y}_0 \in L^2(\Omega)$ is unknown with $|\hat{y}_0|_{L^2(\Omega)} = 1$, τ is a small unknown real number, and $v \in L^2(Q)$ is a control function to be determined. Here, ∂_t denotes the time derivative and $\mathbf{1}_\omega$ is the characteristic function of the set ω.

Let us define

$$\Phi(y(\cdot,\cdot;\tau,v)) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |y(x,t;\tau,v)|^2 \, dx\, dt, \tag{2}$$

where $y(\cdot,\cdot;\tau,v)$ is the solution of (1) associated to τ and v. A control function v is said to insensitize the functional Φ if

$$\left.\frac{\partial \Phi(y(\cdot,\cdot;\tau,v))}{\partial \tau}\right|_{\tau=0} = 0, \quad \forall \hat{y}_0 \in L^2(\Omega) \text{ with } |\hat{y}_0|_{L^2(\Omega)} = 1. \tag{3}$$

This problem, originally addressed by J.-L. Lions in [1], has been studied in the semilinear case for globally Lipschitz-continuous nonlinearities $f = f(y)$. In [2], the authors weakened the underlying problem, defining approximately insensitizing controls. They proved the existence of such controls for unknown data in both the initial and boundary conditions. In [3] two mains results are given. On one hand, the author proves that one cannot expect the existence of insensitizing controls for every $y_0 \in L^2(\Omega)$ when $\Omega \setminus \overline{\omega} \neq \emptyset$, even if $f \equiv 0$. On the other hand, for $y_0 = 0$ and suitable assumptions on ξ, L. de Teresa proves the existence of controls such that (3) holds (see Theorem 1 in [3]). This result is generalized in [4] and [5] to nonlinearities with certain superlinear growth at infinity. One of the purposes of this paper is to extend Theorem 1 in [3] to the case of a semilinear heat equation where the nonlinearity is allowed to depend on both the state y and its gradient. Then, an insensitivity result for a semilinear heat equation with a nonlinear term $F(y)$ and linear boundary conditions of Fourier type is given.

The first insensitivity result we present in this paper is the following one:

Theorem 1.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a C^1 globally Lipschitz–continuous function such that $f(0,0) = 0$. Then, there* *exists a positive constant $\mathcal{M}$ depending on Ω, ω, $\mathcal{O}$, T, and f such that for any $\xi \in L^2(Q)$ verifying*

$$\iint_Q \exp\left(\frac{\mathcal{M}}{t}\right)|\xi|^2\,dx\,dt < \infty, \tag{4}$$

one can find a control function $v \in L^2(Q)$ insensitizing the functional Φ given by (2)*.*

Adapting the computations in [1] and [2] to the present case, one gets that the existence of a control v such that (3) holds is equivalent to the existence of a control v such that the solution (y, q) of

$$\begin{cases} \partial_t y - \Delta y + f(y, \nabla y) = \xi + v\mathbf{1}_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) \ \text{ in } \Omega, \end{cases} \tag{5}$$

$$\begin{cases} -\partial_t q - \Delta q + \partial_s f(y, \nabla y) q - \nabla \cdot (\partial_p f(y, \nabla y) q) = y\mathbf{1}_{\mathcal{O}} \ \text{ in } Q, \\ q = 0 \ \text{ on } \Sigma, \quad q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{6}$$

verifies

$$q(x,0) = 0 \ \text{ in } \Omega. \tag{7}$$

Here we noted $(s,p) \mapsto f(s,p)$, $s \in \mathbb{R}$, $p \in \mathbb{R}^N$, $\partial_s f$ the derivative of f with respect to s, and $\partial_p f$ the gradient of f with respect to p. Thus, so as to prove Theorem 1.1, we will restrict our attention to solve the nonstandard null controllability problem (5)–(7) for $y_0 = 0$.

Let us now consider a semilinear heat equation with linear boundary conditions of Fourier type and partially known initial data:

$$\begin{cases} \partial_t y - \Delta y + F(y) = \xi + v\mathbf{1}_\omega \ \text{ in } Q, \\ \partial_n y + hy = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) + \tau\,\hat{y}_0(x) \ \text{ in } \Omega, \end{cases} \tag{8}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a C^1 globally Lipschitz-continuous function, $h \in L^\infty(\Sigma)$ (at least), ξ, y_0, τ, and $\hat{y}_0$ are as in (1), and $v \in L^2(Q)$ is again a control function to be determined. Here, ∂_n denotes the derivation with respect to the unit outward normal to $\partial\Omega$ and the norm in $L^\infty(\Sigma)$ will be denoted by $\|\cdot\|_{\infty;\Sigma}$. The next aim in this paper is to prove the existence of controls insensitizing the L^2–norm of the observation of the solution of (8) in the open set $\mathcal{O}$.

Theorem 1.2 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Let $F \in C^1(\mathbb{R})$ be a globally Lipschitz–continuous function (with Lipschitz constant $\mathcal{L} > 0$) satisfying $F(0) = 0$ and let $h \in L^\infty(\Sigma)$ be such that $\partial_t h \in L^\infty(\Sigma)$. Then, there exists a positive constant $\mathcal{N}$ (depending on Ω, ω, $\mathcal{O}$, T, $\mathcal{L}$, $\|h\|_{\infty;\Sigma}$, and $\|\partial_t h\|_{\infty;\Sigma}$)*

such that, for any $\xi \in L^2(Q)$ verifying

$$\iint_Q \exp\left(\frac{\mathcal{N}}{t}\right)|\xi|^2\,dx\,dt < \infty, \tag{9}$$

one can find a control function $v \in L^2(Q)$ insensitizing the functional defined in (2), $y(\cdot,\cdot;\tau,v)$ being the solution of (8) associated to τ and v.

In this case, there exists a control function v such that (3) holds if and only if there exists a control v such that the solution (y,q) of

$$\begin{cases} \partial_t y - \Delta y + F(y) = \xi + v\mathbf{1}_\omega \ \text{ in } Q, \\ \partial_n y + hy = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) \ \text{ in } \Omega, \end{cases} \tag{10}$$

$$\begin{cases} -\partial_t q - \Delta q + F'(y)q = y\mathbf{1}_\mathcal{O} \ \text{ in } Q, \\ \partial_n q + hq = 0 \ \text{ on } \Sigma, \quad q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{11}$$

verifies (7). To prove Theorem 1.2, it will then suffice to find an L^2–control solving this new null controllability problem for $y_0 = 0$.

As in [2] and [5], one can expect to choose a control function v such that the associated solution (y,q) of (5), (6) (with $y_0 = 0$), in addition to insensitize the functional Φ, it also verifies $y(x,T) = 0$ in Ω. This can be done with an extra assumption on ξ:

Theorem 1.3 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Let f be as in Theorem 1.1. Then, there exists $\mathcal{M} > 0$ (depending on Ω, ω, $\mathcal{O}$, T, and f) such that for any $\xi \in L^2(Q)$ verifying*

$$\iint_Q \exp\left(\frac{\mathcal{M}}{t(T-t)}\right)|\xi|^2\,dx\,dt < \infty,$$

one can find a control function $v \in L^2(Q)$ insensitizing the functional Φ given by (2) and such that the solution $y(\cdot,\cdot;\tau,v)|_{\tau=0}$ of (1) (with $y_0 = 0$) satisfies

$$y(x,T;\tau,v)|_{\tau=0} = 0 \quad \text{in } \ \Omega.$$

We will not give the proof of this result, since it is similar to the one of Theorem 1.1.

The rest of this paper is organized as follows: in section 2, we first prove an observability inequality that generalizes the one in [3]. This result is indeed one of the main results in this work and we will use it in other forthcoming papers (cf. [4], [5]). We also give an observability inequality for the case of linear Fourier boundary conditions, which will also be used in [7]. In section 3, we prove Theorems 1.1 and 1.2. We end with comments and conclusions.

2 The Observability Inequalities

In this section we first prove an observability inequality that is a generalization of the one given in [3] to the case of linear systems with first order terms. This inequality will be the main tool in the proof of Theorem 1.1. We also give an observability inequality for linear systems with linear boundary conditions of Fourier type, which will be essential to prove Theorem 1.2.

Let us consider φ and ψ solving the following systems:

$$\begin{cases} \partial_t \varphi - \Delta \varphi + c\varphi + D \cdot \nabla \varphi = 0 \ \text{ in } Q, \\ \varphi = 0 \ \text{ on } \Sigma, \quad \varphi(x,0) = \varphi^0(x) \ \text{ in } \Omega, \end{cases} \tag{12}$$

$$\begin{cases} -\partial_t \psi - \Delta \psi + a\psi - \nabla \cdot (B\psi) = \varphi \mathbf{1}_{\mathcal{O}} \ \text{ in } Q, \\ \psi = 0 \ \text{ on } \Sigma, \quad \psi(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{13}$$

with $a, c \in L^\infty(Q)$, $B, D \in L^\infty(Q)^N$, and $\varphi^0 \in L^2(\Omega)$. In the sequel, $\| \cdot \|_\infty$ will denote the norm in both $L^\infty(Q)$ and $L^\infty(Q)^N$. It is known (cf. [8], p. 356) that

$$\varphi, \psi \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \quad \partial_t \varphi, \partial_t \psi \in L^2(0,T; H^{-1}(\Omega)).$$

The main result in this section is the following one:

Theorem 2.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exist positive constants M and H such that, for every $\varphi^0 \in L^2(\Omega)$, the corresponding solution (φ, ψ) of* (12) *and* (13) *satisfies*

$$\iint_Q \exp\left(-\frac{M}{t}\right) |\psi|^2 \, dx \, dt \le H \iint_{\omega \times (0,T)} |\psi|^2 \, dx \, dt.$$

More precisely, $M = C\Big(1 + TM'\Big)$ and

$$H = \exp\left[C\left(M' + \frac{1}{T} + T \left(1 + \|a\|_\infty + \|c\|_\infty + \|B\|_\infty^2 + \|D\|_\infty^2 \right)\right)\right],$$

where $C = C(\Omega, \omega, \mathcal{O})$ and M' is given by

$$M' = 1 + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|a - c\|_\infty^{1/2} + \|B\|_\infty + \|B - D\|_\infty + \|B\|_\infty^2 + \|D\|_\infty^2.$$

The basic tool to prove this theorem is a global Carleman inequality for linear systems of the form

$$
\begin{cases} \partial_t z - \Delta z = F \ \text{ in } Q, \\ z = 0 \ \text{ on } \Sigma, \quad z(x,0) = z^0(x) \ \text{ in } \Omega, \end{cases} \tag{14}
$$

with $z^0 \in L^2(\Omega)$ and F in $L^2(Q)$ or in $L^2(0,T;H^{-1}(\Omega))$. For this we need to introduce an auxiliary function whose existence is guaranteed by the following result (see Lemma 1.1. in [9]):

Lemma 2.2 *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open subset. Then there exists a function $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$ and $|\nabla \eta^0| > 0$ in $\overline{\Omega \setminus \mathcal{B}}$.* □

For a fixed nonempty open subset $\mathcal{B} \subset\subset \Omega$, let us set

$$
\alpha_0(x) = e^{2C^*\|\eta^0\|_\infty} - e^{C^*\eta^0(x)}, \quad x \in \overline{\Omega} \tag{15}
$$

and

$$
\widetilde{\alpha}_0(x) = e^{2C^*\|\eta^0\|_\infty} - e^{-C^*\eta^0(x)}, \quad x \in \overline{\Omega}, \tag{16}
$$

C^* being an appropriate positive constant depending on Ω and $\mathcal{B}$. Using results in [9] and [10], one can prove the following

Lemma 2.3 *Let z be the solution of* (14) *associated to $z^0 \in L^2(\Omega)$. Let $\mathcal{B}$ be an open subset of Ω. There exist positive constants C_0, σ_0, and $\overline{\sigma}_0$ (depending only on Ω and $\mathcal{B}$) such that:*

(1) If $F \in L^2(Q)$, for every $s \geq s_0 = \sigma_0(\Omega, \mathcal{B})\,(T + T^2)$ one has

$$
\begin{aligned}
&\frac{1}{s} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t z|^2 + |\Delta z|^2 \right) + s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla z|^2 \\
&+ s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|z|^2 \leq C_0 \Bigg(s^3 \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|z|^2 \\
&\qquad + \iint_Q e^{-2s\alpha}|F|^2 \Bigg),
\end{aligned}
$$

with α defined by $\alpha(x,t) = \dfrac{\alpha_0(x)}{t(T-t)}$, $x \in \Omega$, $t \in (0,T)$, and α_0 given by (15).

(2) If $F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$*, with* $f_i \in L^2(Q)$*,* $i = 0, 1, \ldots, N$*, then*

$$\begin{aligned} &s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla z|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|z|^2 \\ &\leq C_0 \Bigg(s^3 \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|z|^2 + \iint_Q e^{-2s\alpha}|f_0|^2 \\ &\qquad + s^2 \sum_{i=1}^{N} \iint_Q e^{-2s\alpha} t^{-2}(T-t)^{-2}|f_i|^2 \Bigg), \end{aligned}$$

for $s \geq \overline{s}_0 = \overline{\sigma}_0(\Omega, \mathcal{B})\,(T + T^2)$*,* α *being as above.*

The explicit dependence of s_0 on T has been analyzed in [6]. Arguing in a similar way, we can obtain the precise way $\overline{s}_0$ depends on T (also see [11]). We will also need the following technical lemma, which proof will be given further for the sake of clarity.

Lemma 2.4 *Let* α_0 *and* α *be given as in Lemma 2.3,* $m_0 = \min_{\overline{\Omega}} \alpha_0$*, and* $M_0 = \max_{\overline{\Omega}} \alpha_0$.

(1) One has $s^4 e^{-2s\alpha} t^{-7}(T-t)^{-7} \leq 2^2 e^{-7} \left(\frac{7}{m_0}\right)^4 T^{-6}$*, for every* $s \geq \frac{7T^2}{2^3 m_0}$ *and* $(x,t) \in Q$.

(2) For $s \geq \frac{3T^2}{2M_0}$*, one has* $e^{-2s\alpha} t^{-3}(T-t)^{-3} \geq A_s \exp\left(-M_s/t\right)$*, for* $(x,t) \in \Omega \times (0, T/2)$*, with*

$$A_s = 2^6 T^{-6} \exp\left(-4M_0 s/T^2\right), \quad M_s = 2M_0 s/T. \tag{17}$$

(3) For every $s \geq 0$*, one has*

$$e^{2s\alpha} t^3 (T-t)^3 \leq 2^{-6} T^6 \exp\left(\frac{2^5 M_0 s}{3T^2}\right), \quad (x,t) \in \Omega \times (T/4, 3T/4)\,.$$

Proof of Theorem 2.1: The structure of the proof is similar to that of Proposition 2 in [3]. In the first place, using appropriate Carleman inequalities, we prove an inequality involving the functions φ and ψ which solve (12) and (13). This inequality allows us to bound the function φ in terms of ψ (see (32)). Combining it with energy estimates yields the result. Here we adapt the method exhibited in [3] to the lack of regularity in the term $\nabla\cdot(B\psi)$ in equation (13). Moreover, the constants in the inequalities are explicit.

Let us consider two open sets B_1 and B_2 such that $B_1 \subset\subset B_2 \subset \omega \cap \mathcal{O}$. Applying Lemma 2.3 to the solution φ of (12) with $F = -c\varphi - D\cdot\nabla\varphi$ and $\mathcal{B} = B_1$, there exist positive constants $C_1 = C_1(\Omega, B_1)$ and $\sigma_1 = \sigma_1(\Omega, B_1)$

such that

$$
\begin{aligned}
s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla\varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\
\leq C_1\, s^3 \iint_{B_1\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2,
\end{aligned}
\tag{18}
$$

for every $s \geq s_1$, with

$$
s_1 = \sigma_1(\Omega, B_1)\left(T + T^2 + T^2\|c\|_\infty^{2/3} + T^2\|D\|_\infty^2\right). \tag{19}
$$

Then applying Lemma 2.3 to the solution ψ of (13) with $\mathcal{B} = B_1 \subset B_2$ and $F = -a\psi + \nabla\cdot(B\psi) + \varphi\mathbf{1}_{\mathcal{O}}$, there exist positive constants $C_2 = C_2(\Omega, B_1)$ and $s_2 = \sigma_2(\Omega, B_1)\,(T + T^2 + T^2\|a\|_\infty^{2/3} + T^2\|B\|_\infty^2)$ such that, for $s \geq s_2$, one has

$$
\begin{aligned}
s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla\psi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\psi|^2 \\
\leq C_2\left(s^3 \iint_{B_2\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3} |\psi|^2 + \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}|\varphi|^2\right).
\end{aligned}
\tag{20}
$$

In a first step, we prove an inequality which bounds φ with respect to ψ. Consider a function $\xi_1 \in C_0^\infty(\Omega)$ such that

$$
0 \leq \xi_1 \leq 1 \text{ in } \Omega, \quad \xi_1 = 1 \text{ in } B_1, \quad \operatorname{supp}\xi_1 \subset B_2 \subset \omega\cap\mathcal{O}, \tag{21}
$$

$$
\Delta\xi_1/\xi_1^{1/2} \in L^\infty(\Omega), \quad \text{and} \quad \nabla\xi_1/\xi_1^{1/2} \in L^\infty(\Omega)^N. \tag{22}
$$

This is achieved by setting $\xi_1 = \zeta^4$, with $\zeta \in C_0^\infty(\Omega)$ verifying (21). To simplify notations, we set

$$
u = e^{-2s\alpha} s^3 t^{-3}(T-t)^{-3}. \tag{23}
$$

Let $s \geq s_1$, s_1 given by (19). Multiplying (13) by $\varphi\xi_1 u$, integrating over Q, and taking into account that $u(0)$ vanishes in Ω, we have

$$
\begin{aligned}
\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} s^3 t^{-3}(T-t)^{-3} |\varphi|^2 \xi_1 = \iint_Q (a-c)\varphi\psi\xi_1 u \\
+ \iint_Q (B-D)\cdot\nabla\varphi\,\psi\xi_1 u + \varphi\psi\Big(\xi_1\partial_t u - \Delta(\xi_1 u) + B\cdot\nabla(\xi_1 u)\Big) \\
-2\iint_Q \nabla(\xi_1 u)\cdot\nabla\varphi\,\psi := I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
\tag{24}
$$

Let us estimate each I_i, $1 \leq i \leq 6$. In the sequel, C will denote a positive constant depending only on Ω and B_1 (thus on B_2) which may change from one line to another. In the first place, using Hölder and Young inequalities, we have

$$
I_1 = \iint_Q (a-c)\varphi\psi\xi_1 u \leq \delta_1 \iint_Q \xi_1 u|\varphi|^2 + \frac{1}{4\delta_1}\|a-c\|_\infty^2 \iint_Q \xi_1 u|\psi|^2, \tag{25}
$$

for any $\delta_1 > 0$. Then,

$$I_2 = \iint_Q (B-D)\cdot\nabla\varphi\,\psi\xi_1 u \le \gamma_1 \iint_Q e^{-2s\alpha} s t^{-1}(T-t)^{-1}|\nabla\varphi|^2\xi_1 \\ + \frac{1}{4\gamma_1}\|B-D\|_\infty^2 \iint_Q e^{-2s\alpha} s^5 t^{-5}(T-t)^{-5}|\psi|^2\xi_1, \tag{26}$$

for any $\gamma_1 > 0$. Let us now observe that

$$|\partial_t u| \le T s^3 e^{-2s\alpha} t^{-5}(T-t)^{-5}\left(Cs + 3T^2/4\right) \le CTs^4 e^{-2s\alpha} t^{-5}(T-t)^{-5},$$

since $s \ge \sigma_1(\Omega, B_1)T^2$. Thus, we can estimate

$$I_3 \le \iint_Q |\varphi||\psi|\xi_1|\partial_t u| \le \iint_Q CTe^{-2s\alpha}s^4t^{-5}(T-t)^{-5}|\varphi||\psi|\xi_1 \\ \le \delta_2 \iint_Q e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 + \frac{CT^2}{\delta_2}\iint_Q e^{-2s\alpha}s^5t^{-7}(T-t)^{-7}|\psi|^2\xi_1 \\ \le \delta_2 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_2}\iint_Q e^{-2s\alpha}s^7t^{-7}(T-t)^{-7}|\psi|^2\xi_1\,, \tag{27}$$

for $\delta_2 > 0$, since $s \ge \sigma_1(\Omega, B_1)T$.

In order to estimate $I_4 = -\iint_Q \varphi\psi\Delta(\xi_1 u)$, let us observe that

$$\Delta(\xi_1 u) = s^3t^{-3}(T-t)^{-3}\left((\Delta\xi_1)e^{-2s\alpha} + 2\nabla\xi_1\cdot\nabla(e^{-2s\alpha}) + \xi_1\Delta(e^{-2s\alpha})\right),$$

with

$$|\nabla(e^{-2s\alpha})| = 2se^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\alpha_0| \le Cse^{-2s\alpha}t^{-1}(T-t)^{-1},$$

$$|\Delta(e^{-2s\alpha})| \le 2se^{-2s\alpha}t^{-2}(T-t)^{-2}\left(2s|\nabla\alpha_0|^2 + t(T-t)|\Delta\alpha_0|\right) \\ \le Cse^{-2s\alpha}t^{-2}(T-t)^{-2}(s+T^2) \le Cs^2e^{-2s\alpha}t^{-2}(T-t)^{-2}.$$

Taking these considerations and (22) into account, we have

$$I_4 \le C\Bigg(\iint_Q e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}|\varphi||\psi|\xi_1^{1/2} \\ + \iint_Q e^{-2s\alpha}s^4t^{-4}(T-t)^{-4}|\varphi||\psi|\xi_1^{1/2} + \iint_Q e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}|\varphi||\psi|\xi_1\Bigg).$$

We now use Hölder and Young inequalities and (21) to get

$$I_4 \le \delta_3 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_3}\iint_Q e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}|\psi|^2\mathbf{1}_{B_2} \\ + \frac{C}{\delta_3}\iint_Q e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{B_2} + \frac{C}{\delta_3}\iint_Q e^{-2s\alpha}s^7t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{B_2},$$

with $\delta_3 > 0$ to be fixed later. Notice that for any $n, m \in I\!N$ with $n \geq m$, we have

$$\begin{aligned} s^m t^{-m}(T-t)^{-m} &= s^m t^{-n}(T-t)^{-n}\left(t(T-t)\right)^{n-m} \\ &\leq s^m t^{-n}(T-t)^{-n}\left(T^2/4\right)^{n-m} \leq C s^n t^{-n}(T-t)^{-n}, \end{aligned} \tag{28}$$

since $s \geq \sigma_1(\Omega, B_1)T^2$. Then,

$$I_4 \leq \delta_3 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_3}\iint_Q e^{-2s\alpha}s^7t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{B_2}. \tag{29}$$

We estimate $I_5 = \iint_Q \varphi\psi B\cdot\nabla(\xi_1 u)$ in a similar way. Remarking that

$$\nabla(\xi_1 u) = (\nabla\xi_1)u - 2\xi_1 e^{-2s\alpha}s^4t^{-4}(T-t)^{-4}\nabla\alpha_0,$$

and proceeding as above, we get

$$I_5 \leq \delta_4 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_4}\|B\|_\infty^2 \iint_Q e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{B_2}, \tag{30}$$

for any $\delta_4 > 0$. We finally estimate $I_6 = -2\iint_Q \nabla(\xi_1 u)\cdot\nabla\varphi\,\psi$. We have

$$\begin{aligned} I_6 &\leq C\iint_Q e^{-2s\alpha}\left(s^3t^{-3}(T-t)^{-3}|\nabla\varphi||\psi|\xi_1^{1/2} + s^4t^{-4}(T-t)^{-4}|\nabla\varphi||\psi|\xi_1\right) \\ &\leq \gamma_2\iint_Q e^{-2s\alpha}st^{-1}(T-t)^{-1}|\nabla\varphi|^2\xi_1 \;+\; \frac{C}{\gamma_2}\iint_Q e^{-2s\alpha}s^7t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{B_2}, \end{aligned} \tag{31}$$

for $\gamma_2 > 0$, using (22), (23), and (28).

Let us now set $\delta_i = 1/8$, $1 \leq i \leq 4$, and $\gamma_1 = \gamma_2 = 1/4C_1$, with $C_1 = C_1(\Omega, B_1) > 0$ as in (18). Taking estimates (25)–(27), (29)–(31) to (24) and using (21), we obtain

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 &\leq \frac{1}{C_1}\iint_{B_2\times(0,T)} e^{-2s\alpha}st^{-1}(T-t)^{-1}|\nabla\varphi|^2 \\ &+C\iint_{B_2\times(0,T)} e^{-2s\alpha}\left[\|a-c\|_\infty^2 s^3t^{-3}(T-t)^{-3}|\psi|^2 + s^7t^{-7}(T-t)^{-7}|\psi|^2\right] \\ &+C\left(\|B\|_\infty^2 + \|B-D\|_\infty^2\right)\iint_{B_2\times(0,T)} e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}|\psi|^2. \end{aligned}$$

Thus, in view of (18) and (21), for $s \geq s_1$ we get

$$\begin{aligned} s^3\iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2 &\leq C\,\|a-c\|_\infty^2\iint_{B_2\times(0,T)} e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}|\psi|^2 \\ &+C\left(\|B\|_\infty^2 + \|B-D\|_\infty^2\right)\iint_{B_2\times(0,T)} e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}|\psi|^2 \\ &+C\iint_{B_2\times(0,T)} e^{-2s\alpha}s^7t^{-7}(T-t)^{-7}|\psi|^2. \end{aligned}$$

On the other hand, taking $s \geq CT^2 \left(\|a-c\|_\infty^{1/2} + \|B\|_\infty + \|B-D\|_\infty \right)$, this inequality rewrites into

$$\iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \;\leq\; C \iint_{B_2\times(0,T)} e^{-2s\alpha} s^4 t^{-7}(T-t)^{-7}|\psi|^2, \qquad (32)$$

for every $s \geq s_3$, with

$$s_3 = \sigma_3 \left(\left(T + T^2 \left(1 + \|c\|_\infty^{2/3} + \|a-c\|_\infty^{1/2} + \|B\|_\infty + \|B-D\|_\infty + \|D\|_\infty^2 \right) \right) \right) \qquad (33)$$

and $\sigma_3 = \sigma_3(\Omega, B_1)$.

Next we will prove a new Carleman type inequality for the function ψ which does not involve φ. For $s \geq CT^2$, we have

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}|\varphi|^2 &= \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} t^3(T-t)^3 t^{-3}(T-t)^{-3}|\varphi|^2 \\ &\leq C \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} s^3 t^{-3}(T-t)^{-3}|\varphi|^2. \end{aligned}$$

Thus, in view of (32), applying (20) yields

$$\begin{aligned} \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\psi|^2 \leq\; & C \left(\iint_{B_2\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\psi|^2 \right. \\ & \left. + \iint_{B_2\times(0,T)} e^{-2s\alpha} s^4 t^{-7}(T-t)^{-7}|\psi|^2 \right), \end{aligned}$$

hence

$$\iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\psi|^2 \;\leq\; C \iint_{B_2\times(0,T)} e^{-2s\alpha} s^4 t^{-7}(T-t)^{-7}|\psi|^2, \qquad (34)$$

for any $s \geq s_4$, with

$$s_4 = \sigma_4(\Omega, B_1) \left(T + T^2 M' \right), \qquad (35)$$

M' being as in the statement.

The final step of the proof will consist in combining energy estimates with inequalities (32) and (34). Applying classical estimates for the heat equation to systems (12) and (13), for $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $t \in [0,T]$, we have

$$\begin{aligned} |\varphi(t_2)|_{L^2(\Omega)}^2 &\leq \exp\Big((2\|c\|_\infty + \|D\|_\infty^2)\,(t_2 - t_1) \Big) |\varphi(t_1)|_{L^2(\Omega)}^2, \\ |\psi(t)|_{L^2(\Omega)}^2 &\leq \int_t^T \exp\left(\left(1 + 2\|a\|_\infty + \|B\|_\infty^2 \right)(s-t) \right) |\varphi(s)|_{L^2(\mathcal{O})}^2 \, ds. \end{aligned} \qquad (36)$$

In particular, we have

$$\left| \varphi\left(t + \frac{T}{4} \right) \right|_{L^2(\Omega)}^2 \leq \exp\left(\left(2\|c\|_\infty + \|D\|_\infty^2 \right) \frac{T}{4} \right) |\varphi(t)|_{L^2(\Omega)}^2, \quad \forall t \in [T/4, 3T/4],$$

and hence,

$$\iint_{\Omega\times(T/2,T)} |\varphi|^2 \leq \exp\left(\left(2\|c\|_\infty + \|D\|_\infty^2\right)\frac{T}{4}\right)\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2. \tag{37}$$

On the other hand, from (36) we easily deduce

$$\int_t^T |\psi(s)|_{L^2(\Omega)}^2\, ds \leq \ (T-t)\exp\left(\left(1+2\|a\|_\infty + \|B\|_\infty^2\right)\frac{T}{2}\right)\int_t^T |\varphi(s)|_{L^2(\mathcal{O})}^2\, ds,$$

for all $t\in[T/2,T]$, thus also

$$\iint_{\Omega\times(T/2,T)} |\psi|^2 \leq \exp\left(\left(2+2\|a\|_\infty + \|B\|_\infty^2\right)\frac{T}{2}\right)\iint_{\mathcal{O}\times(T/2,T)} |\varphi|^2. \tag{38}$$

Let $m_0 = \min_{\overline{\Omega}}\alpha_0$ and $M_0 = \max_{\overline{\Omega}}\alpha_0$ be, with α_0 as in Lemma 2.3 in page 6. Applying Lemma 2.4, for any $s \geq 3T^2/2M_0$ we get

$$\iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2 \geq A_s\iint_{\Omega\times(0,T/2)} \exp\left(-\frac{M_s}{t}\right)|\psi|^2,$$

with A_s and M_s given by (17). Bounding the right hand side of (34) using Lemma 2.4 again, we get

$$\iint_{\Omega\times(0,T/2)} \exp\left(-\frac{M_s}{t}\right)|\psi|^2 \leq C\exp\left(\frac{4M_0 s}{T^2}\right)\iint_{B_2\times(0,T)} |\psi|^2,$$

for $s\geq s_5$, s_5 given by

$$s_5 = \sigma_5(\Omega,B_1)\left(T+T^2M'\right), \tag{39}$$

where $\sigma_5(\Omega,B_1) = \max\left\{\sigma_4(\Omega,B_1), 3/(2M_0), 7/(2^3m_0)\right\}$, σ_4 being as in (35), and M' as in the statement of the Theorem. Then using (38), for $s\geq s_5$ we have

$$\begin{aligned}\iint_Q \exp\left(-\frac{M_s}{t}\right)|\psi|^2 &\leq \iint_{\Omega\times(0,T/2)} \exp\left(-\frac{M_s}{t}\right)|\psi|^2 + \iint_{\Omega\times(T/2,T)} |\psi|^2\\ &\leq C\exp\left(\frac{4M_0 s}{T^2}\right)\iint_{B_2\times(0,T)} |\psi|^2\\ &\quad+\exp\left((2+2\|a\|_\infty+\|B\|_\infty^2)\frac{T}{2}\right)\iint_{\Omega\times(T/2,T)} |\varphi|^2,\end{aligned} \tag{40}$$

with M_0 as above. In view of (32), (37), and Lemma 2.4, we can bound the last integral in (40) as follows:

$$\iint_{\Omega\times(T/2,T)} |\varphi|^2 \leq \exp\left(\left(2\|c\|_\infty + \|D\|_\infty^2\right)\frac{T}{4}\right)\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2$$

$$\leq \exp\left((2\|c\|_\infty + \|D\|_\infty^2)\frac{T}{4} + \frac{2^5 M_0 s}{3T^2}\right)\frac{T^6}{2^6}\iint_{\Omega\times(T/4,3T/4)} e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2$$

$$\leq C\exp\left((2\|c\|_\infty + \|D\|_\infty^2)\frac{T}{4} + \frac{2^5 M_0 s}{3T^2}\right)T^6\iint_{B_2\times(0,T)} e^{-2s\alpha}s^4 t^{-7}(T-t)^{-7}|\psi|^2$$

$$\leq C\exp\left((2\|c\|_\infty + \|D\|_\infty^2)\frac{T}{4} + \frac{2^5 M_0 s}{3T^2}\right)\iint_{B_2\times(0,T)} |\psi|^2,$$

for $s \geq \max\{s_3, 7T^2/(2^3 m_0)\}$, s_3 given by (33). Thus, combining this last estimate with (40), we obtain

$$\iint_Q \exp\left(-\frac{M_s}{t}\right)|\psi|^2 \leq C\exp\left(\frac{4M_0 s}{T^2}\right)\iint_{B_2\times(0,T)} |\psi|^2$$

$$+C\exp\left((1+\|a\|_\infty + \|c\|_\infty + \|B\|_\infty^2 + \|D\|_\infty^2)\,T + \frac{2^5 M_0 s}{3T^2}\right)\iint_{B_2\times(0,T)} |\psi|^2,$$

for any $s \geq s_5$, with s_5 defined by (39). Finally, setting $s = s_5$ in the previous inequality and recalling the definition of M_s in (17) and the fact that $B_2 \subset \omega$, we obtain the result. $\square$

For the sake of completeness, we now prove Lemma 2.4.

Proof of Lemma 2.4: Let α_0, α, m_0 and M_0 be as in the statement. Then

$$s^4 e^{-2s\alpha}t^{-7}(T-t)^{-7} \leq s^4\exp\left(-\frac{2m_0 s}{t(T-t)}\right)t^{-7}(T-t)^{-7} := f_s(t) = \frac{1}{g_s(t)},$$

for $s > 0$, $t \in (0,T)$. We will determine a lower bound of $g_s(t)$, for t in $(0,T)$. The computation of $g_s'(t)$ shows that, for $s \geq \dfrac{7T^2}{2^3 m_0}$, the function g_s is strictly decreasing in $(0,T/2)$ and strictly increasing in $(T/2,T)$. Thus, in this case, for $t \in (0,T)$ we have $f_s(t) \leq f_s(T/2) = 2^{14}T^{-14}G(s)$, with $G(s) = s^4\exp\left(-8m_0 s/T^2\right)$. The function $G(s)$ strictly decreases in $(T^2/2m_0, +\infty)$, which proves the first part of the lemma.

Now, writing $\dfrac{1}{t(T-t)} = \dfrac{1}{Tt} + \dfrac{1}{T(T-t)}, \forall t \in (0,T)$, we have

$$e^{-2s\alpha}t^{-3}(T-t)^{-3} \geq \exp\left(-\frac{2M_0 s}{Tt}\right)t^{-3}H_s(t),$$

and $H_s(t) = (T-t)^{-3}\exp\left(-\dfrac{2M_0 s}{T(T-t)}\right)$. It is easy to see that, for every

$s \geq 3T^2/2M_0$, H_s is decreasing in $(0,T/2)$. Therefore a simple computation gives the second part of the lemma.

Finally, observing that $\dfrac{1}{t(T-t)} \leq \dfrac{2^4}{3T^2}$ for all $t \in (T/4, 3T/4)$, we get

$$e^{2s\alpha} t^3 (T-t)^3 \leq \exp\left(\frac{2M_0 s}{t(T-t)}\right) 2^{-6} T^6 \leq 2^{-6} T^6 \exp\left(\frac{2^5 M_0 s}{3T^2}\right),$$

for $x \in \Omega$, $t \in (T/4, 3T/4)$, which ends the proof. □

We will devote the rest of this section to the case of Fourier boundary conditions. Let φ and ψ be the solutions of

$$\begin{cases} \partial_t \varphi - \Delta\varphi + c\varphi = 0 \ \text{ in } Q, \\ \partial_n \varphi + k\varphi = 0 \ \text{ on } \Sigma, \quad \varphi(x,0) = \varphi^0(x) \ \text{ in } \Omega, \end{cases} \tag{41}$$

$$\begin{cases} -\partial_t \psi - \Delta\psi + a\psi = \varphi \mathbf{1}_{\mathcal{O}} \ \text{ in } Q, \\ \partial_n \psi + h\psi = 0 \ \text{ on } \Sigma, \quad \psi(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{42}$$

with $a, c \in L^\infty(Q)$, $h, k \in L^\infty(\Sigma)$, and $\varphi^0 \in L^2(\Omega)$. One has

$$\varphi, \psi \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \quad \partial_t\varphi, \partial_t\psi \in L^2(0,T; H^{-1}(\Omega)).$$

In order to obtain the observability inequality required in this case, we will recall another global Carleman inequality, which can be found in [9]. Let us consider linear systems of the form

$$\begin{cases} \partial_t z - \Delta z = F \ \text{ in } Q, \\ \partial_n z + bz = 0 \ \text{ on } \Sigma, \quad z(x,0) = z^0(x) \ \text{ in } \Omega, \end{cases} \tag{43}$$

with $z^0 \in L^2(\Omega)$ and $F \in L^2(Q)$. The following result holds:

Lemma 2.5 *Assume that $b, \partial_t b \in L^\infty(\Sigma)$. Let z be the solution of (43) associated to $z^0 \in L^2(\Omega)$ and $F \in L^2(Q)$. Let $\mathcal{B}$ be a nonempty open subset of Ω. Then there exist positive constants C_0 and s_0 depending on Ω, $\mathcal{B}$, T, $\|b\|_{\infty;\Sigma}$ and $\|\partial_t b\|_{\infty;\Sigma}$ such that*

$$\frac{1}{s}\iint_Q \rho t(T-t)\left(|\partial_t z|^2 + |\Delta z|^2\right) + s\iint_Q \rho t^{-1}(T-t)^{-1}|\nabla z|^2$$
$$+ s^3 \iint_Q \rho t^{-3}(T-t)^{-3}|z|^2 \leq C_0\left(\iint_Q \rho|F|^2 + s^3\iint_{\mathcal{B}\times(0,T)} \rho t^{-3}(T-t)^{-3}|z|^2\right),$$

for $s \geq s_0$, with $\rho(x,t) = e^{-2s\alpha(x,t)} + e^{-2s\widetilde{\alpha}(x,t)}$, α as in Lemma 2.3, $\widetilde{\alpha}$ given by $\widetilde{\alpha}(x,t) = \dfrac{\widetilde{\alpha}_0(x)}{t(T-t)}$, $\quad x \in \Omega,\ t \in (0,T)$ and $\widetilde{\alpha}_0$ defined in (16).

Applying this Lemma and following the proof of Theorem 2.1, we can prove its equivalent for systems (41), (42):

Theorem 2.6 *Assume that* $\omega \cap \mathcal{O} \neq \emptyset$*,* $a, c \in L^\infty(Q)$*, and* $h, k, \partial_t h, \partial_t k \in L^\infty(\Sigma)$*. Then, there exist positive constants* N *and* K *(depending on* Ω*,* ω*,* $\mathcal{O}$*,* T*,* $\|a\|_\infty$*,* $\|c\|_\infty$*,* $\|h\|_{\infty;\Sigma}$*,* $\|k\|_{\infty;\Sigma}$*,* $\|\partial_t h\|_{\infty;\Sigma}$ *and* $\|\partial_t k\|_{\infty;\Sigma}$*) such that*

$$\iint_Q \exp\Big(-\frac{N}{t}\Big)|\psi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} |\psi|^2\,dx\,dt,$$

for every $\varphi^0 \in L^2(\Omega)$*, where* ψ *solves* (42)*,* φ *being the solution of* (41) *associated to* φ^0*.*

Proof: We consider two open sets B_1 and B_2 such that $B_1 \subset\subset B_2 \subset \omega \cap \mathcal{O}$. Applying Lemma 2.5 to the solution φ of (41) with $F = -c\varphi$ and $\mathcal{B} = B_1$, there exist two positive constants $C_1 = C_1(\Omega, B_1, T, \|k\|_{\infty;\Sigma}, \|\partial_t k\|_{\infty;\Sigma})$ and $s_1 = s_1(\Omega, B_1, T, \|c\|_\infty, \|k\|_{\infty;\Sigma}, \|\partial_t k\|_{\infty;\Sigma})$ such that

$$\begin{aligned} &s \iint_Q \rho t^{-1}(T-t)^{-1}|\nabla\varphi|^2 + s^3 \iint_Q \rho t^{-3}(T-t)^{-3}|\varphi|^2 \\ &\qquad \le C_1\, s^3 \iint_{B_1\times(0,T)} \rho t^{-3}(T-t)^{-3}|\varphi|^2, \end{aligned} \tag{44}$$

for every $s \ge s_1$. Secondly, we apply Lemma 2.5 to the solution ψ of (42) (with $F = -a\psi + \varphi \mathbf{1}_{\mathcal{O}}$ and $\mathcal{B} = B_1$). Then, there exist positive constants $C_2 = C_2(\Omega, B_1, T, \|h\|_{\infty;\Sigma}, \|\partial_t h\|_{\infty;\Sigma})$ and $s_2 = s_2(\Omega, B_1, T, \|a\|_\infty, \|h\|_{\infty;\Sigma}, \|\partial_t h\|_{\infty;\Sigma})$ such that, for any $s \ge s_2$, one has

$$\begin{aligned} &s \iint_Q \rho t^{-1}(T-t)^{-1}|\nabla\psi|^2 + s^3 \iint_Q \rho t^{-3}(T-t)^{-3}|\psi|^2 \\ &\le C_2 \left(s^3 \iint_{B_2\times(0,T)} \rho t^{-3}(T-t)^{-3}|\psi|^2 + \iint_{\mathcal{O}\times(0,T)} \rho|\varphi|^2 \right). \end{aligned} \tag{45}$$

Let $s \ge s_1$ and let $\xi_1 \in C_0^\infty(\Omega)$ be a function satisfying (21) and (22). Multiply the equation in (42) by $\varphi\xi_1(u + \widetilde{u})$, with u given by (23) and $\widetilde{u}$ defined by $\widetilde{u} = e^{-2s\widetilde{\alpha}} s^3 t^{-3}(T-t)^{-3}$ ($\widetilde{\alpha}$ as in Lemma 2.5). Integrating over Q, one gets

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} \rho s^3 t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 &= \iint_Q \big(-\partial_t\psi - \Delta\psi + a\psi\big)\varphi\xi_1(u+\widetilde{u}) \\ &:= I_1 + \widetilde{I}_1 + I_3 + \widetilde{I}_3 + I_4 + \widetilde{I}_4 + I_6 + \widetilde{I}_6, \end{aligned} \tag{46}$$

with I_i as in (24) and $\widetilde{I}_i$ defined as I_i, with u replaced by $\widetilde{u}$. Each $\widetilde{I}_i$ can be bounded exactly as the corresponding I_i was in the proof of Theorem 2.1. First, we have

$$I_1 + \widetilde{I}_1 = \iint_Q (a-c)\varphi\psi\xi_1(u+\widetilde{u}) \le \delta \iint_Q \xi_1(u+\widetilde{u})|\varphi|^2 + \frac{\|a-c\|_\infty^2}{4\delta} \iint_Q \xi_1(u+\widetilde{u})|\psi|^2,$$

for $\delta > 0$. We can also obtain the corresponding estimates for $I_i + \widetilde{I}_i$, $i = 3, 4, 6$, similar to (27), (29) and (31), respectively, and valid for any $s \geq \max\{s_1, C(T+T^2)\}$, with $C > 0$ depending only on Ω and B_1. Taking such estimates to (46) and using (44) (and (21)), we can estimate

$$s^3 \iint_Q \rho t^{-3}(T-t)^{-3}|\varphi|^2 \leq C\,\|a-c\|_\infty^2 \iint_{B_2\times(0,T)} \rho s^3 t^{-3}(T-t)^{-3}|\psi|^2$$
$$+C \iint_{B_2\times(0,T)} \rho s^7 t^{-7}(T-t)^{-7}|\psi|^2,$$

for $s \geq \max\{s_1, C(T+T^2))\}$. Then, if

$$s \geq s_3 = \max\left\{s_1, C\left(T+T^2+T^2\|a-c\|_\infty^{1/2}\right)\right\},$$

the following estimate for φ holds

$$\iint_Q \rho t^{-3}(T-t)^{-3}|\varphi|^2 \leq C_3 \iint_{B_2\times(0,T)} \rho s^4 t^{-7}(T-t)^{-7}|\psi|^2, \tag{47}$$

with $C_3 > 0$ depending on Ω, B_1, T, $\|k\|_{\infty;\Sigma}$, and $\|\partial_t k\|_{\infty;\Sigma}$. Now, using (45) and (47), a new estimate for ψ analogous to (34) is obtained. More precisely, there exists $C_4 = C_4\left(\Omega, B_1, T, \|h\|_{\infty;\Sigma}, \|\partial_t h\|_{\infty;\Sigma}, \|k\|_{\infty;\Sigma}, \|\partial_t k\|_{\infty;\Sigma}\right) > 0$ such that

$$\iint_Q \rho t^{-3}(T-t)^{-3}|\psi|^2 \leq\ C_4 \iint_{B_2\times(0,T)} \rho s^4 t^{-7}(T-t)^{-7}|\psi|^2, \tag{48}$$

for any $s \geq s_4$, with

$$s_4 = \max\left\{s_1, s_2, C\left(T+T^2+T^2\|a-c\|_\infty^{1/2}\right)\right\}. \tag{49}$$

On the other hand, multiplying the equation in (41) by φ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi(t)|^2 + \int_\Omega |\nabla\varphi(t)|^2 \leq \|k\|_{\infty;\Sigma}\int_{\partial\Omega} |\varphi(t)|^2\,d\sigma + \|c\|_\infty \int_\Omega |\varphi(t)|^2, \tag{50}$$

for t a.e. in $(0,T)$. We claim that

$$\frac{d}{dt}|\varphi(t)|^2_{L^2(\Omega)} \leq K_1|\varphi(t)|^2_{L^2(\Omega)}, \qquad \text{a.e. in } (0,T), \tag{51}$$

K_1 being a positive constant depending on $\|c\|_\infty$ and $\|k\|_{\infty;\Sigma}$. Indeed, in view of the chain of embeddings

$$H^1(\Omega) \rightrightarrows H^\gamma(\Omega) \hookrightarrow L^2(\Omega), \quad \gamma < 1,$$

the first one being compact, for any $\varepsilon > 0$ there exists $\mathcal{C}(\varepsilon) > 0$ such that

$$\|u\|^2_{H^\gamma(\Omega)} \leq \varepsilon \int_\Omega |\nabla u|^2\,dx + \mathcal{C}(\varepsilon)|u|^2_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Taking also into account the continuous embedding of $H^\gamma(\Omega)$ into $L^2(\partial\Omega)$, for $\gamma > 1/2$, there exists $\mathcal{C}(\|k\|_{\infty;\Sigma}) > 0$ such that

$$\|k\|_{\infty;\Sigma} \int_{\partial\Omega} |\varphi(t)|^2 \, d\sigma \leq \frac{1}{2} \int_\Omega |\nabla\varphi(t)|^2 + \mathcal{C}(\|k\|_{\infty;\Sigma}) |\varphi(t)|^2_{L^2(\Omega)},$$

for $1/2 < \gamma < 1$. Combining this estimate with (50), yields (51), with K_1 given by $K_1 = 2(\mathcal{C}(\|k\|_{\infty;\Sigma}) + \|c\|_\infty)$. Then

$$|\varphi\,(t + T/4)|^2_{L^2(\Omega)} \leq \exp\left(K_1 T/4\right) |\varphi(t)|^2_{L^2(\Omega)}, \quad \forall t \in (T/4, 3T/4)\,,$$

and hence

$$\iint_{\Omega\times(T/2,T)} |\varphi|^2 \leq \exp\left(K_1 T/4\right) \iint_{\Omega\times(T/4,3T/4)} |\varphi|^2. \tag{52}$$

Now, multiply the equation in (42) by ψ and integrate over Ω. Using again a compactness–uniqueness argument, we obtain

$$-\frac{d}{dt}|\psi(t)|^2_{L^2(\Omega)} \leq K_2 |\psi(t)|^2_{L^2(\Omega)} + |\varphi(t)|^2_{L^2(\mathcal{O})}, \qquad \text{a.e. in } (0,T),$$

with $K_2 = K_2(\|a\|_\infty, \|h\|_{\infty;\Sigma}) > 0$. Then

$$|\psi(t)|^2_{L^2(\Omega)} \leq \int_t^T \exp\left(K_2(s-t)\right) |\varphi(s)|^2_{L^2(\mathcal{O})} \, ds, \quad \forall t \in (0,T),$$

whence

$$\iint_{\Omega\times(T/2,T)} |\psi|^2 \leq \exp\left(K_2 T\right) \iint_{\mathcal{O}\times(T/2,T)} |\varphi|^2. \tag{53}$$

The form of the weight function ρ defined in Lemma 2.5 allows one to prove estimates similar to those in Lemma 2.4, with $e^{-2s\alpha}$ replaced by ρ, valid for $s \geq CT^2$. Let us fix $s = \max\{s_4, CT^2\}$, with s_4 given by (49). We can thus bound both sides of (48) and deduce

$$\iint_{\Omega\times(0,T/2)} \exp\left(-\frac{N}{t}\right) |\psi|^2 \leq C_5 \iint_{B_2\times(0,T)} |\psi|^2, \tag{54}$$

with $N > 0$ and $C_5 > 0$ depending on Ω, B_1, T, $\|a\|_\infty$, $\|c\|_\infty$, $\|h\|_{\infty;\Sigma}$, $\|k\|_{\infty;\Sigma}$, $\|\partial_t h\|_{\infty;\Sigma}$, and $\|\partial_t k\|_{\infty;\Sigma}$. In addition, due to (53), (52), and (47), we can also estimate (see a similar proof in page 12)

$$\iint_{\Omega\times(T/2,T)} \exp\left(-\frac{N}{t}\right) |\psi|^2 \leq \exp\left(K_1\frac{T}{4} + K_2 T + C_6\right) \iint_{B_2\times(0,T)} |\psi|^2, \tag{55}$$

with $C_6 > 0$ depending on Ω, T, $\|a\|_\infty$, $\|c\|_\infty$, $\|h\|_{\infty;\Sigma}$, $\|k\|_{\infty;\Sigma}$, $\|\partial_t h\|_{\infty;\Sigma}$, $\|\partial_t k\|_{\infty;\Sigma}$, and B_1, thus on ω and $\mathcal{O}$. Finally, gathering (54) and (55) yields the desired observability inequality, since $B_2 \subset \omega$, with N as in (54) and $K = C_5 + \exp\left(K_1 T/4 + K_2 T + C_6\right)$. $\square$

3 Proof of Theorems 1.1 and 1.2

We devote this section to prove Theorems 1.1 and 1.2. Both proofs, which are inspired in those of other known controllability results for nonlinear systems (see [12], [13], [3], [14],...), rely on controllability results for linear problems similar to the linearized system and appropriate fixed point arguments. The proof of Theorem 1.2 is similar to the one of Theorem 1.1 and it will be omitted here.

Proof of Theorem 1.1: We start with the existence of approximately insensitizing controls for a linearized version of (5), (6) for $y_0 = 0$. For given $a, c \in L^\infty(Q)$, $B, D \in L^\infty(Q)^N$ and $\xi \in L^2(Q)$, we consider the linear systems

$$\begin{cases} \partial_t y - \Delta y + ay + B \cdot \nabla y = \xi + v\mathbf{1}_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{56}$$

$$\begin{cases} -\partial_t q - \Delta q + cq - \nabla \cdot (Dq) = y\mathbf{1}_{\mathcal{O}} \ \text{ in } Q, \\ q = 0 \ \text{ on } \Sigma, \quad q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{57}$$

and the corresponding adjoint systems (12) and (13). The following result holds:

Proposition 3.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Let M and H be the positive constants provided by Theorem 2.1. For any $\varepsilon > 0$, there exists a control function $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the associated solution $(y_\varepsilon, q_\varepsilon)$ of* (56), (57) *satisfies*

$$|q_\varepsilon(0)|_{L^2(\Omega)} \leq \varepsilon. \tag{58}$$

In addition, if $\xi \in L^2(Q)$ satisfies

$$\iint_Q \exp\left(\frac{M}{t}\right) |\xi|^2 \, dx \, dt < \infty, \tag{59}$$

then the controls $\{v_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L^2(\omega \times (0,T))$. More precisely,

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq \sqrt{H} \left(\iint_Q \exp\left(\frac{M}{t}\right) |\xi|^2 \, dx \, dt \right)^{1/2}, \quad \forall \varepsilon > 0. \tag{60}$$

Proof: The structure of the proof being identical to the one in [2] and [3], we will not go into details. For fixed $\varepsilon > 0$, we introduce the functional defined on $L^2(\Omega)$

$$J(\varphi^0; a, c, B, D) = \frac{1}{2} \iint_{\omega\times(0,T)} |\psi|^2 + \varepsilon |\varphi^0|_{L^2(\Omega)} + \iint_Q \xi\psi, \tag{61}$$

where ψ solves (13), φ being the solution of (12) with initial data $\varphi^0 \in L^2(\Omega)$. In view of a unique continuation property for the adjoint systems (which follows, for instance, from Theorem 2.1), the continuous and convex functional $J(\cdot; a, c, B, D)$ is strictly convex and satisfies

$$\liminf_{|\varphi^0|_{L^2(\Omega)} \to +\infty} \frac{J(\varphi^0; a, c, B, D)}{|\varphi^0|_{L^2(\Omega)}} \geq \varepsilon. \tag{62}$$

Thus, $J(\cdot; a, c, B, D)$ is coercive and therefore it reaches its minimum at a unique $\varphi^0_\varepsilon \in L^2(\Omega)$. Set

$$v_\varepsilon = \psi_\varepsilon \mathbf{1}_\omega, \tag{63}$$

$(\varphi_\varepsilon, \psi_\varepsilon)$ solving (12), (13) with initial data φ^0_ε. Then, the solution $(y_\varepsilon, q_\varepsilon)$ of (56), (57) associated to v_ε satisfies (58). Indeed, v_ε is the unique control of minimal L^2–norm solving (56)–(58).

Now, assume that ξ satisfies (59). The optimality condition for φ^0_ε and Theorem 2.1 give

$$\begin{aligned} &\iint_{\omega\times(0,T)} |\psi_\varepsilon|^2 + \varepsilon|\varphi^0_\varepsilon|_{L^2(\Omega)} = -\iint_Q \xi\, \psi_\varepsilon \\ &\leq \left(H \iint_{\omega\times(0,T)} |\psi_\varepsilon|^2\right)^{1/2} \left(\iint_Q \exp\left(\frac{M}{t}\right) |\xi|^2\right)^{1/2}, \end{aligned}$$

which yields, together with (63), the uniform estimate (60). □

Remark 1 *In view of* (60)*, for any* $\xi \in L^2(Q)$ *verifying* (59)*, one can prove the existence of a control* $v \in L^2(\omega \times (0,T))$ *such that the associated solution* (y, q) *of* (56)*,* (57) *satisfies* (7)*. Moreover, this control* v *satisfies the estimate*

$$\|v\|_{L^2(\omega\times(0,T))} \leq \sqrt{H} \left(\iint_Q \exp\left(\frac{M}{t}\right)|\xi|^2 \, dx\, dt\right)^{1/2},$$

with M *and* H *as above. That is to say, an insensitivity result in the linear case can also be proved.*

We now apply a fixed point argument to prove an approximate insensitivity result in the nonlinear case.

Proposition 3.2 *For fixed* $\varepsilon > 0$*, under the assumptions in Theorem 1.1, there exist a positive constant* $\mathcal{M}$ *(depending on* Ω*,* ω*,* $\mathcal{O}$*,* T*, and* f*) such that for any* $\xi \in L^2(Q)$ *satisfying* (4)*, one can find a control* $v_\varepsilon \in L^2(\omega \times (0,T))$ *so that the associated solution* $(y_\varepsilon, q_\varepsilon)$ *of* (5)*,* (6) *satisfies* (58)*. Furthermore,*

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq \mathcal{H} \left(\iint_Q \exp\left(\frac{\mathcal{M}}{t}\right)|\xi|^2 \, dx\, dt\right)^{1/2}, \quad \forall \varepsilon > 0, \tag{64}$$

$\mathcal{H}$ *being a new positive constant depending on* Ω*,* ω*,* $\mathcal{O}$*,* T*, and* f*.*

PROOF: For a given function f as in Theorem 1.1, we can write

$$f(s,p) = g(s,p)s + G(s,p)\cdot p \quad \text{for all } (s,p) \in \mathbb{R}\times\mathbb{R}^N,$$

where $g : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ are the bounded continuous functions defined by

$$g(s,p) = \int_0^1 \partial_s f(\sigma s, \sigma p)\, d\sigma, \quad G(s,p) = \int_0^1 \partial_p f(\sigma s, \sigma p)\, d\sigma. \tag{65}$$

Since it is a fixed parameter, the dependence on ε will be omitted in this proof. For any $z \in L^2(0,T;H_0^1(\Omega))$, we consider the linear systems (56) and (57), with $a = a_z = g(z,\nabla z)$, $c = c_z = \partial_s f(z,\nabla z) \in L^\infty(Q)$ and $B = B_z = G(z,\nabla z)$, $D = D_z = \partial_p f(z,\nabla z) \in L^\infty(Q)^N$. Indeed, the hypothesis on f gives

$$\|a_z\|_\infty, \|c_z\|_\infty, \|B_z\|_\infty, \|D_z\|_\infty \le L, \quad \forall z \in L^2(0,T;H_0^1(\Omega)), \tag{66}$$

where $L > 0$ is a bound of $\partial_s f$ and $\partial_p f$ in $\mathbb{R}\times\mathbb{R}^N$. In view of Proposition 3.1, there exists a control $v_z \in L^2(\omega\times(0,T))$ such that the corresponding solution (y_z, q_z) of these systems satisfies

$$|q_z(0)|_{L^2(\Omega)} \le \varepsilon. \tag{67}$$

Let M_z and H_z be the positive constants provided by Theorem 2.1 for $a = a_z$, $c = c_z$, $B = B_z$, and $D = D_z$. Recalling the expressions of M_z and H_z, and using (66), there exist positive constants $\mathcal{M}$ and $\mathcal{H}$ of the form

$$\begin{cases} \mathcal{M} = C(\Omega,\omega,\mathcal{O})\,(1 + T\,(1+L^2))\,, \\ \mathcal{H} = \exp\left[C(\Omega,\omega,\mathcal{O})\left(1 + \dfrac{1}{T} + T + (1+T)L^2\right)\right], \end{cases} \tag{68}$$

such that, for all $z \in L^2(0,T;H_0^1(\Omega))$, $M_z \le \mathcal{M}$ and $\sqrt{H_z} \le \mathcal{H}$. Then, if ξ satisfies (4), using (60) we have the following estimate (uniform with respect to z and ε)

$$\|v_z\|_{L^2(\omega\times(0,T))} \le \mathcal{H}\left(\iint_Q \exp\left(\frac{\mathcal{M}}{t}\right)|\xi|^2\right)^{1/2}, \quad \forall z \in L^2(0,T;H_0^1(\Omega)). \tag{69}$$

We now consider the mapping $\Lambda_\varepsilon : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;H_0^1(\Omega))$ defined by $\Lambda_\varepsilon(z) = y_z$, y_z being the solution of (56) associated to the potentials $a = a_z$ and $B = B_z$ and the control v_z provided by Proposition 3.1. We will apply the Schauder fixed point theorem to prove that Λ_ε possesses at least one fixed point. First, by classical regularity results on the heat equation, y_z lies in the space $Y = \{u : u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\ \partial_t u \in L^2(Q)\}$, with

$$\|y_z\|_Y \le \exp\left[C\left(1 + \left(T + T^{1/2}\right)\|a_z\|_\infty + T\|B_z\|_\infty^2\right)\right]\left(\|\xi + v_z \mathbf{1}_\omega\|_{L^2(Q)}\right)$$

(here $\|y_z\|_Y = \|y_z\|_{L^2(H^2\cap H_0^1)} + \|\partial_t y_z\|_{L^2(Q)}$ and $\|\cdot\|_{L^2(H^2\cap H_0^1)}$ denotes the norm in $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)))$. Taking into account (66) and (69), one deduces that Λ_ε maps $L^2(0,T;H_0^1(\Omega))$ into a bounded set of Y. This space being compactly embedded in $L^2(0,T;H_0^1(\Omega))$, there exists a fixed compact set K in $L^2(0,T;H_0^1(\Omega))$ such that

$$\Lambda_\varepsilon(L^2(0,T;H_0^1(\Omega))) \subset K. \tag{70}$$

Thus, Λ_ε is a compact mapping.

Now, let $\{z_j\} \subset L^2(0,T;H_0^1(\Omega))$ be such that $z_j \to z$ in $L^2(0,T;H_0^1(\Omega))$. From (66) and the regularity assumptions on f, one has

$$\begin{aligned} &a_{z_j} = g(z_j,\nabla z_j) \rightharpoonup a_z, \quad c_{z_j} = \partial_s f(z_j,\nabla z_j) \rightharpoonup c_z \text{ weak-}\star \text{ in } L^\infty(Q),\\ &B_{z_j} = G(z_j,\nabla z_j) \rightharpoonup B_z, \quad D_{z_j} = \partial_p f(z_j,\nabla z_j) \rightharpoonup D_z \text{ weak-}\star \text{ in } L^\infty(Q)^N. \end{aligned} \tag{71}$$

Let $\hat\varphi^0$ (resp. $\hat\varphi_j^0$, $j \geq 1$) be the unique minimizer in $L^2(\Omega)$ of the functional J defined by (61) with $a = a_z$, $c = c_z$, $B = B_z$ and $D = D_z$ (resp. with $a = a_{z_j}$, $c = c_{z_j}$, $B = B_{z_j}$ and $D = D_{z_j}$). Reasoning as in [12] and [15], the coercivity property (62) is proved to be hold uniformly on potentials a, c, B and D uniformly bounded. Then, one can see that the sequence $\{\hat\varphi_j^0\}$ is bounded in $L^2(\Omega)$ and, finally, one proves that

$$\hat\varphi_j^0 \to \hat\varphi^0 \quad \text{in } L^2(\Omega). \tag{72}$$

Let now $(\hat\varphi,\hat\psi)$ (resp. $(\hat\varphi_j,\hat\psi_j)$, $j \geq 1$) be the solution of (12), (13) with $a = a_z$, $c = c_z$, $B = B_z$, $D = D_z$ (resp. $a = a_{z_j}$, $c = c_{z_j}$, $B = B_{z_j}$, $D = D_{z_j}$) and the initial condition $\hat\varphi^0$ (resp. $\hat\varphi_j^0$). From (71) and (72), we have

$$\hat\varphi_j \to \hat\varphi, \quad \hat\psi_j \to \hat\psi \quad \text{in } L^2(Q). \tag{73}$$

By definition of Λ_ε, $\Lambda_\varepsilon(z)$ (resp. $\Lambda_\varepsilon(z_j)$, $j \geq 1$) is the solution of (56) associated to the control $\hat v = \hat\psi \mathbf{1}_\omega$ (resp. $\hat v_j = \hat\psi_j \mathbf{1}_\omega$) with $a = a_z$, $c = c_z$, $B = B_z$ and $D = D_z$ (resp. $a = a_{z_j}$, $c = c_{z_j}$, $B = B_{z_j}$, and $D = D_{z_j}$). From (73) one has $\hat v_j \to \hat v$ in $L^2(Q)$, so that from (71) one gets that $\Lambda_\varepsilon(z_j) \to \Lambda_\varepsilon(z)$ in $L^2(Q)$ and also in $L^2(0,T;H_0^1(\Omega))$, due to (70). This proves the continuity of Λ_ε.

All the assumptions of the Schauder theorem being fulfilled, Λ_ε possesses at least one fixed point $y_\varepsilon \in L^2(0,T;H_0^1(\Omega))$. Then, the control $v_\varepsilon = v_{y_\varepsilon}$ is such that y_ε solves

$$\begin{cases} \partial_t y_\varepsilon - \Delta y_\varepsilon + g(y_\varepsilon,\nabla y_\varepsilon) y_\varepsilon + G(y_\varepsilon,\nabla y_\varepsilon)\cdot\nabla y_\varepsilon = \xi + v_\varepsilon \mathbf{1}_\omega \text{ in } Q,\\ y_\varepsilon = 0 \text{ on } \Sigma, \quad y_\varepsilon(x,0) = 0 \text{ in } \Omega, \end{cases} \tag{74}$$

and the solution q_ε of

$$\begin{cases} -\partial_t q_\varepsilon - \Delta q_\varepsilon + \partial_s f(y_\varepsilon, \nabla y_\varepsilon) q_\varepsilon - \nabla \cdot (\partial_p f(y_\varepsilon, \nabla y_\varepsilon) q_\varepsilon) = y_\varepsilon \mathbf{1}_{\mathcal{O}} \text{ in } Q, \\ q_\varepsilon = 0 \text{ on } \Sigma, \quad q_\varepsilon(x,T) = 0 \text{ in } \Omega, \end{cases} \tag{75}$$

satisfies (58). In other words, we have found a control function $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the associated solution of (5), (6) (with $y_0 = 0$) verifies (58). Finally, estimate (64) follows readily from (69), which ends the proof of Proposition 3.2. $\Box$

We will end the proof of Theorem 1.1 by passing to the limit in (74), (75), and (58). Since the controls v_ε provided by Proposition 3.2 are uniformly bounded in $L^2(\omega \times (0,T))$ and (66) holds, due to the regularizing effect of the heat equation, $\{(y_\varepsilon, q_\varepsilon)\}$ lies in a bounded set of $Y \times W(0,T)$ (Y defined in page 20 and $W(0,T) := \{u : u \in L^2(0,T; H_0^1(\Omega)),\ \partial_t u \in L^2(0,T; H^{-1}(\Omega))\}$) and accordingly, in a compact set of $L^2(0,T; H_0^1(\Omega)) \times L^2(Q)$. Then, up to a subsequence, one has

$$v_\varepsilon \rightharpoonup v \text{ weakly in } L^2(\omega \times (0,T)),$$
$$(y_\varepsilon, q_\varepsilon) \to (y, q) \text{ in } L^2(0,T; H_0^1(\Omega)) \times L^2(Q), \quad q_\varepsilon(0) \to q(0) \text{ in } L^2(\Omega),$$

for some $v \in L^2(\omega \times (0,T))$, $y \in Y$, $q \in W(0,T)$. Due to the continuity of g and G, one can pass to the limit in (74) and (75), deducing that (y,q) solves (5), (6) with control term v and initial datum $y_0 = 0$. Moreover, from (58), the function q satisfies (7). Thus, the function v is an insensitizing control for the functional Φ given by (2). Finally, (64) and the convergences above allow one to estimate

$$\|v\|_{L^2(\omega \times (0,T))} \le \mathcal{H} \left(\iint_Q \exp\left(\frac{\mathcal{M}}{t}\right) |\xi|^2 \, dx\, dt \right)^{1/2}, \tag{76}$$

with $\mathcal{M}$ and $\mathcal{H}$ given by (68) and the proof is complete. $\Box$

Remark 2 *The method used in Theorem 1.1 to obtain such a control v provides an upper bound of the cost of insensitizing the functional Φ. Indeed, in the proof of the theorem it is shown that the control function v can be chosen satisfying estimate* (76)*, with $\mathcal{M}$ and $\mathcal{H}$ given by* (68)*. Inspired in* [6]*, denote by $\mathcal{U}_{ad}$ the nonempty set*

$$\mathcal{U}_{ad} = \{v \in L^2(\omega \times (0,T)) : (y,q) \text{ satisfies (5)–(7) with } y_0 = 0\}.$$

Thus, the quantity $\mathcal{C}_{ins} = \inf\{\|v\|_{L^2(\omega \times (0,T))} : v \in \mathcal{U}_{ad}\}$, which measures the cost of insensitizing the functional Φ, can be estimated as follows

$$\mathcal{C}_{ins} \le \mathcal{H} \left(\iint_Q \exp\left(\frac{\mathcal{M}}{t}\right) |\xi|^2 \, dx\, dt \right)^{1/2}.$$

4 Comments and conclusions

Boundary Fourier conditions. Proving an insensitivity result for the system:

$$\begin{cases} \partial_t y - \Delta y + f(y, \nabla y) = \xi + v 1_\omega \text{ in } Q, \\ \partial_n y + h y = 0 \text{ on } \Sigma, \quad y(x,0) = \tau \hat{y}_0(x) \text{ in } \Omega, \end{cases}$$

with a C^1 globally Lipschitz-continuous function f is a much more difficult problem. Let us observe that such an insensitivity result is equivalent to the following null controllability problem:

$$\begin{cases} \partial_t y - \Delta y + f(y, \nabla y) = \xi + v 1_\omega \text{ in } Q, \\ \partial_n y + h y = 0 \text{ on } \Sigma, \quad y(x,0) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -\partial_t q - \Delta q - \nabla \cdot (\partial_p f(y, \nabla y) q) + \partial_s f(y, \nabla y) q = y 1_{\mathcal{O}} \text{ in } Q, \\ \partial_n q + h q + (\partial_p f(y, \nabla y) \cdot n)\, q = 0 \text{ on } \Sigma, \quad q(x,T) = 0 \text{ in } \Omega, \end{cases}$$

$$q(x,0) = 0 \quad \text{in } \Omega.$$

This leads us to analyze the null controllability problem for the cascade linear system

$$\begin{cases} \partial_t y - \Delta y + a y + B \cdot \nabla y = \xi + v 1_\omega \text{ in } Q, \\ \partial_n y + h y = 0 \text{ on } \Sigma, \quad y(x,0) = 0 \text{ in } \Omega, \end{cases} \tag{77}$$

$$\begin{cases} -\partial_t q - \Delta q - \nabla \cdot (D q) + c q = y 1_{\mathcal{O}} \text{ in } Q, \\ \partial_n q + (h + D \cdot n)\, q = 0 \text{ on } \Sigma, \quad q(x,T) = 0 \text{ in } \Omega, \end{cases} \tag{78}$$

under the hypothesis $a, c \in L^\infty(Q)$ and $B, D \in L^\infty(Q)^N$ (which are the natural assumptions on these potentials for the given function f). For this, an observability inequality for the corresponding adjoint problem should be proved. This adjoint problem is:

$$\begin{cases} \partial_t \varphi - \Delta \varphi + c \varphi + D \cdot \nabla y = 0 \text{ in } Q, \\ \partial_n \varphi + h \varphi = 0 \text{ on } \Sigma, \quad \varphi(x,0) = \varphi^0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -\partial_t \psi - \Delta \psi - \nabla \cdot (B \psi) + a \psi = \varphi 1_{\mathcal{O}} \text{ in } Q, \\ \partial_n \psi + (h + B \cdot n)\, \psi = 0 \text{ on } \Sigma, \quad \psi(x,T) = 0 \text{ in } \Omega. \end{cases}$$

In order to obtain such an observability inequality, we need a Carleman inequality for these adjoint problems. The presence of the term $(B \cdot n)\psi$ in the boundary condition for ψ and the unique hypothesis $B \in L^\infty(Q)^N$ makes it quite difficult (even in the case of a null controllability problem for a unique linear heat equation with the same kind of boundary conditions) and this is out of the scope of this paper.

Superlinear nonlinearities. The observability results proved in this paper are of wider use than the scope of this article. First, Theorem 2.1 is used in [4] and [5] for a semilinear heat equation with a superlinear nonlinearity $f(y)$. Theorem 2.6 is also used in [7] for the case of nonlinear Fourier boundary conditions. It is of interest to notice that in Theorem 2.6, the dependency of the constants with respect to the boundary data h and k is not explicit. This comes from the proof of Lemma 2.5 (see Lemma 1.2 in [9]).

In view of known null controllability results, it is natural to think of extending Theorem 1.1 to C^1 locally Lipschitz-continuous functions f such that $f(0,0)=0$ and

$$\lim_{|(s,p)|\to\infty} \frac{|g(s,p)|}{\log^{3/2}(1+|s|+|p|)}=0, \qquad \lim_{|(s,p)|\to\infty} \frac{|G(s,p)|}{\log^{1/2}(1+|s|+|p|)}=0,$$

with g and G the functions given by (65) (see Theorem 1.1 in [11]). Observe that such nonlinearities may lead to blow-up phenomena. However, the idea in [11] of taking short control times to avoid blow-up to occur fails here (even if $G\equiv 0$), since the initial and final times are fixed in insensitivity problems.

In [4] and [5] the authors introduce a new technique and prove an insensitivity result for nonlinearities $f=f(y)$ with certain superlinear growth at infinity, e.g. for f such as $|f(s)|=|p_1(s)|\log^{\alpha}(1+|p_2(s)|)$ for all $|s|\geq s_0>0$, with $\alpha\in[0,1)$, p_1 and p_2 being first order real polynomial functions. The crucial point in these works is the construction, in the linear case, of regular controls starting from insensitizing controls in L^2. The idea is as follows. Let us consider two open sets $\mathcal{B}_0$ and $\mathcal{B}$ such that $\mathcal{B}_0 \subset\subset \mathcal{B}\subset \omega\cap\mathcal{O}$. Let $\hat{v}$ be an L^2–control, with $\operatorname{supp}\hat{v}\subset \overline{\mathcal{B}}_0\times[0,T]$, such that the corresponding solution $(\hat{y},\hat{q})$ of (56), (57) for $B=D=0$ satisfies (7). Then, setting $q=(1-\theta)\hat{q},\ y=(1-\theta)\,\hat{y}+2\nabla\theta\cdot\nabla\hat{q}+(\Delta\theta)\hat{q}$, with $\theta\in\mathcal{D}(\mathcal{B})$ such that $\theta\equiv 1$ in a neighborhood of $\mathcal{B}_0$, it is possible to furnish a regular insensitizing control v supported on $\overline{\mathcal{B}}\times[0,T]$. This construction uses local regularization properties of the heat equation. This technique does not apply to the case involving gradient terms because of the lack of regularity introduced by the term $-\nabla\cdot(Dq)$ in (57). Indeed, in this case, the expression of the regular control we wish to build contains some terms, which are not regular enough to make the state y lie in a suitable space to apply a fixed point argument. This is why in Theorem 1.1 we cannot consider nonlinearities of higher order.

In [7], a local result on the existence of insensitizing controls for a semilinear heat equation with nonlinear boundary conditions of Fourier type is proved. Such boundary conditions lead to seek a fixed point, thus also control functions, in certain Hölder spaces. A construction similar to that used in [4] and [5], allows one to build, in the linear case, controls with hölderian regularity starting from L^2–controls. Again, this is one of the essential points in the referenced work.

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