Controllability results for some nonlinear coupled parabolic systems by one control force

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Abstract

In this paper, we present new controllability results for some nonlinear coupled parabolic systems considered in a bounded domain Ω of $\mathbb{R}^N$ (with $N \geq 1$ being arbitrary) when the control force acts on a unique equation of the system through an arbitrarily small open set $\omega \subset \Omega$. As a model example, we consider a nonlinear phase field system with certain superlinear nonlinearities and prove the null controllability, the exact controllability to the trajectories and the approximate controllability of the model. The crucial point in this paper is the new strategy developed to deal with the null controllability of linear coupled parabolic systems by a unique control force. Global Carleman estimates and the parabolic regularizing effect of the problem are used.

Key words. Controllability; Nonlinear coupled systems of parabolic type.

1 Introduction

The aim of the present paper is the description of a new approach that allows one to prove new controllability results for some (linear and nonlinear) coupled parabolic systems considered in a bounded domain $\Omega \subset \mathbb{R}^N$ (for arbitrary $N \geq 1$) when they are controlled by a unique control force that acts on an arbitrarily small open set $\omega \subset \Omega$. In order to describe our strategy, we will consider (as a model example) a system of two nonlinear coupled parabolic PDEs that generalizes the *phase field model* introduced by Caginalp in its enthalpy formulation (cf. [1]).

Suppose that a material that may be in either of two phases (e.g. solid and liquid) occupies a bounded region Ω in the space $\mathbb{R}^N$ ($N \geq 1$ arbitrary), with boundary $\partial\Omega$ of class C^2. For a given $T > 0$, the nonlinear phase field system

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reads as follows:

$$\begin{cases} \partial_t u - \Delta u + f(u, \nabla u, \phi, \nabla \phi) = -\Delta \phi + v \mathbf{1}_\omega \quad \text{in } Q = \Omega \times (0,T), \\ \partial_t \phi - \Delta \phi + h(\phi) = u \quad \text{in } Q, \\ u = 0, \ \ \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x), \ \ \phi(x,0) = \phi_0(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz-continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function, the initial datum (u_0, ϕ_0) is given in a suitable space, and $v \in L^2(Q)$ is a control function to be determined. Here, $\omega \subset \Omega$ is an arbitrarily small open control set, $\Sigma = \partial\Omega \times (0,T)$, and $\mathbf{1}_\omega$ denotes the characteristic function of ω.

This model provides a mathematical description of free boundary problems arising from physical phenomena that occurs with a phase transition, such as the phenomenon of solidification of a liquid. The variable ϕ is the so-called *phase field* function and is used to distinguish between the two phases of the material involved in the solidification process. The *enthalpy* of the system, u, is given by $u = \theta + \phi$, where θ is the temperature of the material. From a physical point of view, observe that θ (which satisfies a nonlinear heat equation such as $\partial_t \theta + \partial_t \phi - \Delta\theta + F(\theta, \nabla\theta) = v\mathbf{1}_\omega$ in Q) is the variable to be controlled. Thus, in system (1) we control on the equation satisfied by u.

We are interested in analyzing the controllability properties of system (1) when nonlinearities with a slight superlinear growth at infinity are considered. This analysis is more intricate than the study of the controllability properties for a scalar superlinear heat equation (cf. [2] and [3], for instance) since we want a coupled parabolic system to be controlled by a unique distributed control and, even in the linear case, additional technical difficulties arise owing to the coupling of the equations.

The first controllability results for a nonlinear phase field system by one control force are proved in [4] under certain restriction on the dimension N. For $1 \le N \le 5$, Ammar Khodja et al. prove the exact controllability to the trajectories when $f \equiv 0$ and h satisfies $h(0) = 0$ and

$$\lim_{|\sigma| \to \infty} \frac{|h(\sigma)|}{|\sigma| \log^{3/2}(1 + |\sigma|)} = 0. \tag{2}$$

In [5], the authors introduce a new approach to deal with the null controllability of some linear coupled parabolic systems that makes it possible to generalize the results in [4] to a more general phase field system such as (1), where both f and h are allowed to have a slight superlinear growth at infinity. The proof of these results as well as the above-mentioned strategy, sketched in [5], are developed in the present paper. To be precise, assume that $f(0,0,0,0) = 0$ and

$$f(s,p,\sigma,\pi) = g_1(s,p,\sigma,\pi)s + G_1(s,p,\sigma,\pi) \cdot p + g_2(s,p,\sigma,\pi)\sigma + G_2(s,p,\sigma,\pi) \cdot \pi$$

for any $(s,p,\sigma,\pi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, where $g_i : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $G_i : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, 2$, are some L^∞_{loc} functions. First, we prove a null controllability result for system (1) (see Theorem 2.1) when the nonlinearities satisfy, together with other assumptions, hypothesis (2) and

$$\begin{cases} \displaystyle\lim_{|s|,|\sigma|\to\infty} \frac{|g_1(s,p,\sigma,\pi)|}{\log^{3/2}(1+|s|+|\sigma|)} = 0, \quad \lim_{|s|,|\sigma|\to\infty} \frac{|G_1(s,p,\sigma,\pi)|}{\log^{1/2}(1+|s|+|\sigma|)} = 0, \\ \displaystyle\lim_{|s|,|\sigma|\to\infty} \frac{|g_2(s,p,\sigma,\pi)|}{\log^{2}(1+|s|+|\sigma|)} = 0, \quad \text{and} \quad \lim_{|s|,|\sigma|\to\infty} \frac{|G_2(s,p,\sigma,\pi)|}{\log(1+|s|+|\sigma|)} = 0 \\ \text{uniformly in } p,\pi \in \mathbb{R}^N. \end{cases}$$

Under slightly different hypothesis on the nonlinearities, we also show a result on the exact controllability to the trajectories for system (1), Theorem 2.2, that extends the results given in [4]. As a consequence, an approximate controllability result for (1) is obtained (see Theorem 2.3).

Theorem 2.1 in the present paper and the main results in [4] can be proved, as is already usual (see [6], [3], and [7], for instance), by combining a similar controllability result for the corresponding linearized system with and appropriate fixed-point argument. The presence of superlinear nonlinearities leads to obtain, in the linear case, a 'good' control to the effect that a fixed point in an appropriate space can be obtained. One of the main goals of the present paper is to develop the strategy sketched in [5] to obtain such a good control in the linear case. It is worth pointing out that this approach, which is completely different from the one used in [4], makes it possible to obtain controllability results which are valid for arbitrary $N \geq 1$.

In Section 4, we develop in detail the above-mentioned strategy to prove the null controllability of a linear phase field system by one control force. For the sake of clarity, let us describe in this introduction the main ideas in our method to deal with a simpler linear null controllability problem.

The strategy of fictitious control functions: We consider the linear null controllability problem

$$\begin{cases} \partial_t u - \Delta u = a_1 u + b_1 \phi + v\mathbf{1}_\omega \quad \text{in } Q, \\ \partial_t \phi - \nu\Delta\phi = a_2 u + b_2\phi \quad \text{in } Q, \\ u = 0, \;\; \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x), \;\; \phi(x,0) = \phi_0(x) \quad \text{in } \Omega, \end{cases} \tag{3}$$

$$u(x,T) = 0, \quad \phi(x,T) = 0 \quad \text{in } \Omega, \tag{4}$$

where $a_i, b_i \in L^\infty(Q)$ $(i = 1,2)$, $u_0, \phi_0 \in L^2(\Omega)$ (at least), and $v \in L^2(Q)$ is a control function to be determined ($\omega \subset \Omega$ is the control open set and $\nu > 0$ is the diffusion coefficient of the second PDE). Assume that there exist a nonempty open set $\mathcal{B}$ and a constant $a_0 > 0$ such that

$$\mathcal{B} \subset\subset \omega, \qquad |a_2| \geq a_0 > 0 \;\text{ in } \mathcal{B}\times(0,T).$$

We are interested in obtaining a control v such that the corresponding solution (u,ϕ) to (3) not only satisfies (4) but also lies in $L^\infty(Q)^2$. Appropriate estimates of the control v and the solution (u, ϕ) with respect to the size of the data must also be obtained.

We proceed in two steps. Let $\mathcal{B}_0$ be a nonempty open set such that $\mathcal{B}_0 \subset\subset \mathcal{B}$. In the first place, we introduce a fictitious control in the second PDE in (3) and prove the null controllability of the linear system

$$\begin{cases} \partial_t u - \Delta u = a_1 u + b_1 \phi + \hat{v}_1 \mathbf{1}_{\mathcal{B}_0} \quad \text{in } Q, \\ \partial_t \phi - \nu \Delta \phi = a_2 u + b_2 \phi + \hat{v}_2 \mathbf{1}_{\mathcal{B}_0} \quad \text{in } Q, \\ u = 0, \ \ \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x), \ \ \phi(x,0) = \phi_0(x) \quad \text{in } \Omega. \end{cases} \tag{5}$$

It is proved that there exist two control functions $\hat{v}_1, \hat{v}_2 \in L^2(Q)$, with $\operatorname{supp} \hat{v}_1, \operatorname{supp} \hat{v}_2 \subset \overline{\mathcal{B}}_0 \times [0,T]$, such that the corresponding solution $(\hat{u}, \hat{\phi})$ to (5) satisfies (4). Moreover, $\hat{v}_1$ and $\hat{v}_2$ can be chosen so that

$$||\hat{v}_1||^2_{L^2(Q)} + ||\hat{v}_2||^2_{L^2(Q)} \leq \exp(CH_0) \left(||u_0||^2_{L^2(\Omega)} + ||\phi_0||^2_{L^2(\Omega)} \right),$$

with $C = C(\Omega, \mathcal{B}_0, \nu) > 0$ and $H_0 = H_0(T, \|a_1\|_\infty, \|a_2\|_\infty, \|b_1\|_\infty, \|b_2\|_\infty) > 0$ (the explicit dependence of the constant H_0 with respect to T and the size of the potentials can be given). This controllability result is a consequence of the observability inequality

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)} \leq \exp(CH_0) \iint_{\mathcal{B}_0 \times (0,T)} \left(|\varphi|^2 + |\psi|^2 \right) \, dx \, dt$$

(C and H_0 as above) for the solutions to the adjoint system

$$\begin{cases} -\partial_t \psi - \Delta \psi = a_1 \psi + a_2 \varphi \quad \text{in } Q, \\ -\partial_t \varphi - \nu \Delta \varphi = b_1 \psi + b_2 \varphi \quad \text{in } Q, \\ \varphi = 0, \ \ \psi = 0 \quad \text{on } \Sigma, \\ \varphi(x,T) = \varphi^0(x), \ \ \psi(x,T) = \psi^0(x) \quad \text{in } \Omega \quad (\varphi^0, \psi^0 \in L^2(\Omega)), \end{cases}$$

which is deduced by combining an appropriate Carleman inequality and the energy estimates for these solutions.

In a second step, we eliminate $\hat{v}_2$ and construct a control $v \in L^r(Q)$ ($r \in [2, \infty)$ being arbitrary) that gives the null controllability of system (3), with associated solution (u, ϕ) in $L^\infty(Q)^2$. This can be carried out by adapting to the present situation the technique of construction of regular controls (from L^2-controls) introduced in [8] (also see [9]). We proceed as follows. Let $\eta \in C^\infty([0,T])$ be such that $\eta \equiv 1$ in $[0, T/3]$, $\eta \equiv 0$ in $[2T/3, T]$, and $0 \leq \eta \leq 1$, $|\eta'(t)| \leq C/T$ in $[0,T]$. We introduce the change of variables $u = U + \eta \overline{u}$, $\phi = \Phi + \eta \overline{\phi}$, where $(\overline{u}, \overline{\phi})$ is the weak solution to

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} = a_1 \overline{u} + b_1 \overline{\phi} \quad \text{in } Q, \\ \partial_t \overline{\phi} - \nu \Delta \overline{\phi} = a_2 \overline{u} + b_2 \overline{\phi} \quad \text{in } Q, \\ \overline{u} = 0, \ \ \overline{\phi} = 0 \quad \text{on } \Sigma, \quad \overline{u}(x,0) = u_0(x), \ \ \overline{\phi}(x,0) = \phi_0(x) \quad \text{in } \Omega. \end{cases}$$

Observe that a control v solves the null controllability problem (3), (4) if and only if v solves:

$$\begin{cases} \partial_t U - \Delta U = a_1 U + b_1 \Phi - \eta' \overline{u} + v \mathbf{1}_\omega \ \text{ in } Q, \\ \partial_t \Phi - \nu \Delta \Phi = a_2 U + b_2 \Phi - \eta' \overline{\phi} \ \text{ in } Q, \\ U = 0, \ \ \Phi = 0 \quad \text{on } \Sigma, \\ U(x,0) = 0, \ \ \Phi(x,0) = 0, \ \ U(x,T) = 0, \ \ \Phi(x,T) = 0 \quad \text{in } \Omega. \end{cases} \tag{6}$$

Thus, we are reduced to obtaining a control force in $L^r(Q)$ ($r \in [2, \infty)$) that solves (6). To this end, let $(\hat{u}, \hat{\phi})$ be the solution to system (5) associated to two arbitrary L^2-controls $\hat{v}_1$ and $\hat{v}_2$ that give the null controllability of (5). We can write $\hat{u} = \hat{U} + \eta \overline{u}$ and $\hat{\phi} = \hat{\Phi} + \eta \overline{\phi}$, with η and $(\overline{u}, \overline{\phi})$ as above and $(\hat{U}, \hat{\Phi})$ being the solution to

$$\begin{cases} \partial_t \hat{U} - \Delta \hat{U} = a_1 \hat{U} + b_1 \hat{\Phi} - \eta' \overline{u} + \hat{v}_1 \mathbf{1}_{\mathcal{B}_0} \ \text{ in } Q, \\ \partial_t \hat{\Phi} - \nu \Delta \hat{\Phi} = a_2 \hat{U} + b_2 \hat{\Phi} - \eta' \overline{\phi} + \hat{v}_2 \mathbf{1}_{\mathcal{B}_0} \ \text{ in } Q, \\ \hat{U} = 0, \ \ \hat{\Phi} = 0 \quad \text{on } \Sigma, \quad \hat{U}(x,0) = 0, \ \ \hat{\Phi}(x,0) = 0 \quad \text{in } \Omega, \end{cases}$$

which also satisfies $\hat{U}(x,T) = 0$ and $\hat{\Phi}(x,T) = 0$ in Ω. We now consider a function $\theta \in \mathcal{D}(\mathcal{B})$ satisfying $\theta \equiv 1$ in a neighborhood of $\mathcal{B}_0$. We set

$$\Phi = (1-\theta)\, \hat{\Phi}, \quad U = (1-\theta)\, \hat{U} + \frac{1}{a_2} \left(\theta \eta' \overline{\phi} + 2\nu \nabla\theta \cdot \nabla \hat{\Phi} + \nu (\Delta\theta) \hat{\Phi} \right),$$

and

$$v = \theta \eta' \overline{u} + 2 \nabla\theta \cdot \nabla \hat{U} + (\Delta\theta) \hat{U} + (\partial_t - \Delta - a_1) \left[\frac{1}{a_2} \left(\theta \eta' \overline{\phi} + 2\nu \nabla\theta \cdot \nabla \hat{\Phi} + \nu (\Delta\theta) \hat{\Phi} \right) \right].$$

By local parabolic regularity, under appropriate assumptions on the potentials a_2 and b_2, the functions U, Φ, and v are regular enough. In addition, for regular initial data, the functions $u = U + \eta \overline{u}$ and $\phi = \Phi + \eta \overline{\phi}$ are also regular. Suitable estimates for (u, ϕ) and v can be obtained. Furthermore, v (together with (U, Φ)) solves (6), thus v (together with (u, ϕ)) solves the null controllability problem (3), (4). Notice that, in fact, it suffices to assume that $|a_2| \geq a_0 > 0$ in $\mathcal{B} \times (0, T_0)$ for certain $T_0 \in (0, T)$, since we can drive, in this case, system (3) to zero at time T_0 and set $v \equiv 0$ for the rest of the time interval.

Let us end this introduction with a brief comment on the last main result in this paper. We consider the linear coupled system:

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + c\varphi - \nabla \cdot (F\psi) + e\psi = -\Delta \psi \quad \text{in } Q, \\ -\partial_t \psi - \Delta \psi - \nabla \cdot (B\psi) + a\psi = \varphi \quad \text{in } Q, \\ \varphi = 0, \ \ \psi = 0 \quad \text{on } \Sigma, \quad \varphi(x,T) = \varphi^0(x), \ \ \psi(x,T) = \psi^0(x) \quad \text{in } \Omega, \end{cases} \tag{7}$$

where $a, c, e \in L^\infty(Q)$, $B, F \in L^\infty(Q)^N$, and $\varphi^0, \psi^0 \in L^2(\Omega)$. For a given open set $\mathcal{B}_0$, Theorem 2.4 provides a Carleman inequality for the solutions (φ, ψ)

to (7), by means of which, some global terms of φ and ψ are bounded just in terms of ψ 'localized' in $\mathcal{B}_0$ (see (12)). Although we have opted in this paper for the strategy of introducing a fictitious control, Theorem 2.4 leads to an observability result for the solutions to (7) that allows one to prove the null controllability of a linear phase field system such as (16) by a control $\hat{v} \in L^2(Q)$ supported in $\overline{\mathcal{B}}_0 \times [0,T]$, and the L^2-norm of $\hat{v}$ can be estimated.

The rest of the present work is organized as follows. Our main results are stated in the following Section. In Section 3, we compile some technical results which will be used later. Section 4 provides an exhaustive study of the null controllability property for a linear phase field system. In the next three Sections, we prove the controllability results for system (1) stated in Section 2, and Section 8 is devoted to proving Theorem 2.4. We end the paper with further results and comments.

2 Main results

We devote this Section to stating the relevant results in this paper. Assume that $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz-continuous function, with $f(0,0,0,0) = 0$, and $h \in C^1(\mathbb{R})$, both with certain superlinear growth at infinity (which is specified below). Notice that, under these assumptions, one can write

$$f(s,p,\sigma,\pi) = g_1(s,p,\sigma,\pi)s + G_1(s,p,\sigma,\pi) \cdot p + g_2(s,p,\sigma,\pi)\sigma + G_2(s,p,\sigma,\pi) \cdot \pi$$

for any $(s,p,\sigma,\pi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, where $g_i : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $G_i : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, 2$, are the L^∞_{loc}-functions defined, respectively, by

$$\begin{aligned} g_1(s,p,\sigma,\pi) &= \int_0^1 \partial_s f(\lambda s, \lambda p, \lambda\sigma, \lambda\pi)\, d\lambda, \\ G_1(s,p,\sigma,\pi) &= \int_0^1 \partial_p f(\lambda s, \lambda p, \lambda\sigma, \lambda\pi)\, d\lambda, \\ g_2(s,p,\sigma,\pi) &= \int_0^1 \partial_\sigma f(\lambda s, \lambda p, \lambda\sigma, \lambda\pi)\, d\lambda, \quad \text{and} \\ G_2(s,p,\sigma,\pi) &= \int_0^1 \partial_\pi f(\lambda s, \lambda p, \lambda\sigma, \lambda\pi)\, d\lambda, \end{aligned}$$

for any $(s,p,\sigma,\pi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$. Here we denoted by $\partial_s f$ (resp. $\partial_\sigma f$) the derivative of f with respect to s (resp. σ) and by $\partial_p f$ (resp. $\partial_\pi f$) the gradient of f with respect to p (resp. π). Our first main result establishes the null controllability of system (1).

Theorem 2.1 *Let $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz-continuous function such that $f(0,0,0,0) = 0$ and let $h \in C^1(\mathbb{R})$ satisfy $h'' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $h(0) = 0$. Let us assume that:*

i) For any $R > 0$*, there exists* $M_R > 0$ *such that*

$$|g_1(s,p,\sigma,\pi)| + |G_1(s,p,\sigma,\pi)| + |g_2(s,p,\sigma,\pi)| + |G_2(s,p,\sigma,\pi)| \le M_R$$

for every $s, \sigma \in [-R, R]$ *and* $p, \pi \in \mathbb{R}^N$*;*

ii)

$$\begin{cases} \lim_{|s|,|\sigma|\to\infty} \dfrac{|g_1(s,p,\sigma,\pi)|}{\log^{3/2}(1+|s|+|\sigma|)} = 0, \quad \lim_{|s|,|\sigma|\to\infty} \dfrac{|G_1(s,p,\sigma,\pi)|}{\log^{1/2}(1+|s|+|\sigma|)} = 0, \\ \lim_{|s|,|\sigma|\to\infty} \dfrac{|g_2(s,p,\sigma,\pi)|}{\log^{2}(1+|s|+|\sigma|)} = 0, \ \ \textit{and} \ \lim_{|s|,|\sigma|\to\infty} \dfrac{|G_2(s,p,\sigma,\pi)|}{\log(1+|s|+|\sigma|)} = 0 \\ \textit{uniformly in } p, \pi \in \mathbb{R}^N; \end{cases}$$

$$\lim_{|\sigma|\to\infty} \frac{|h(\sigma)|}{|\sigma| \log^{3/2}(1+|\sigma|)} = 0. \tag{8}$$

Then, for any $T > 0$ *and* $(u_0, \phi_0) \in (W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega))^2$*, with* $s_1 \in (N/2+1, \infty)$*, there exists a control function* $v \in L^2(Q)$ *such that system* (1) *possesses a solution* $(u, \phi) \in L^\infty(Q)^2$ *that satisfies*

$$u(x,T) = 0, \quad \phi(x,T) = 0 \quad \textit{in } \Omega.$$

The proof of this Theorem combines a similar null controllability result for the corresponding linearized system with and appropriate fixed-point argument, and will be given in Section 5.

Remark 2.1 *In particular, under the hypothesis in Theorem 2.1, for any* $(u_0, \phi_0) \in (W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega))^2$*, with* $s_1 \in (N/2+1, \infty)$*, there exists a control* v *such that system* (1) *admits a solution* (u, ϕ) *which is globally defined in* $[0, T]$*. Observe that this assertion does not remain valid for any control term* v *and any initial datum* (u_0, ϕ_0) *since we are in the range of nonlinearities for which blow-up phenomena may occur at an instant* $T^* < T$*.*

Under hypothesis (on the nonlinearities) slightly different from the ones in Theorem 2.1, one is able to show a result on the exact controllability to the trajectories of system (1). To be precise, for arbitrary $(s_0, p_0, \sigma_0, \pi_0)$ and (s, p, σ, π) in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, let us now write

$$\begin{aligned} f(s_0+s, p_0+p, \sigma_0+\sigma, \pi_0+\pi) &= f(s_0, p_0, \sigma_0, \pi_0) \\ + g_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)s &+ G_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) \cdot p \\ + g_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)\sigma &+ G_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) \cdot \pi, \end{aligned}$$

where

$$g_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) = \int_0^1 \partial_s f(s_0+\lambda s, p_0+\lambda p, \sigma_0+\lambda\sigma, \pi_0+\lambda\pi)\, d\lambda,$$

$$G_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) = \int_0^1 \partial_p f(s_0+\lambda s, p_0+\lambda p, \sigma_0+\lambda\sigma, \pi_0+\lambda\pi)\, d\lambda,$$

$$g_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) = \int_0^1 \partial_\sigma f(s_0 + \lambda s, p_0 + \lambda p, \sigma_0 + \lambda\sigma, \pi_0 + \lambda\pi)\, d\lambda,$$

and

$$G_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi) = \int_0^1 \partial_\pi f(s_0 + \lambda s, p_0 + \lambda p, \sigma_0 + \lambda\sigma, \pi_0 + \lambda\pi)\, d\lambda.$$

The following result holds, which generalizes the main results in [4] and [10].

Theorem 2.2 *Let $h \in C^1(\mathbb{R})$ satisfy $h'' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and hypothesis (8) and let $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz-continuous function such that*

$$\begin{aligned}
&\lim_{|s|,|\sigma|\to\infty} \frac{|g_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)|}{\log^{3/2}(1+|s|+|\sigma|)} = 0,\\
&\lim_{|s|,|\sigma|\to\infty} \frac{|G_1(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)|}{\log^{1/2}(1+|s|+|\sigma|)} = 0,\\
&\lim_{|s|,|\sigma|\to\infty} \frac{|g_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)|}{\log^{2}(1+|s|+|\sigma|)} = 0,\\
&\lim_{|s|,|\sigma|\to\infty} \frac{|G_2(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)|}{\log(1+|s|+|\sigma|)} = 0
\end{aligned} \tag{9}$$

uniformly in $(s_0, p_0, p, \sigma_0, \pi_0, \pi) \in K \times \mathbb{R}^N \times \mathbb{R}^N \times K \times \mathbb{R}^N \times \mathbb{R}^N$, with $K \subset \mathbb{R}$ being compact. Assume, in addition, that for any $R > 0$ and $k^* > 0$, there exists $M_{R,k^*} > 0$ such that

$$\begin{aligned}
|g_i(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)| \le M_{R,k^*}, \quad i = 1, 2,\\
|G_i(s_0, p_0, \sigma_0, \pi_0; s, p, \sigma, \pi)| \le M_{R,k^*}, \quad i = 1, 2,
\end{aligned} \tag{10}$$

for every $s_0, \sigma_0 \in [-k^*, k^*]$, $s, \sigma \in [-R, R]$, and $p_0, p, \pi_0, \pi \in \mathbb{R}^N$. For an arbitrary $T > 0$, let (u^*, ϕ^*) be a weak solution to (1) in $L^\infty(Q)^2$ associated to $v^* \in L^2(Q)$ and $(u_0^*, \phi_0^*) \in (W^{2-2/s_1, s_1}(\Omega) \cap H_0^1(\Omega))^2$, with $s_1 > N/2 + 1$. Then, for any $(u_0, \phi_0) \in (W^{2-2/s_1, s_1}(\Omega) \cap H_0^1(\Omega))^2$, there exists a control $v \in L^2(Q)$ and a state $(u, \phi) \in L^\infty(Q)^2$ associated to v and (u_0, ϕ_0) such that $u(x, T) = u^*(x, T)$ and $\phi(x, T) = \phi^*(x, T)$ in Ω.

Remark 2.2 *Notice that, when $f(0, 0, 0, 0) = h(0) = 0$, the null controllability can be read as the exact controllability to the trajectory $(u^*, \phi^*) \equiv (0, 0)$ (associated to $v^* = 0$ and $(u_0^*, \phi_0^*) = (0, 0)$). Thus, the hypothesis (9) and (10) required to prove Theorem 2.2 are, as natural, slightly stronger than those in Theorem 2.1. On the other hand, assumption (8) on the growth of h, which is not other that*

$$\lim_{|\sigma|\to\infty} \frac{1}{\log^{3/2}(1+|\sigma|)} \int_0^1 h'(\lambda\sigma)\, d\lambda = 0,$$

is equivalent to

$$\begin{cases}
\displaystyle\lim_{|\sigma|\to\infty} \frac{1}{\log^{3/2}(1+|\sigma|)} \left| \int_0^1 h'(\sigma_0 + \lambda\sigma)\, d\lambda \right| = 0,\\
\text{uniformly in } \sigma_0 \in K \text{ for any compact set } K \subset \mathbb{R},
\end{cases}$$

as already observed in [11].

A consequence of Theorem 2.2 is the following approximate controllability result for system (1).

Theorem 2.3 *Let f and h be two functions as in Theorem 2.2. Then, for any $T > 0$, $u_0, \phi_0 \in W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega))$, with $s_1 > N/2 + 1$, $u_d, \phi_d \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $v \in L^2(Q)$ and a state (u, ϕ) in $L^\infty(Q)^2$ associated to v and (u_0, ϕ_0) such that*

$$\|u(\cdot, T) - u_d\|_{L^2(\Omega)} \le \varepsilon \quad \textit{and} \quad \|\phi(\cdot, T) - \phi_d\|_{L^2(\Omega)} \le \varepsilon. \tag{11}$$

Our last main result is concerned with a Carleman estimate for the solutions to the linear (adjoint) system (7).

Theorem 2.4 *Let $\mathcal{B}_0 \subset \omega$ be a nonempty open set. Then, there exists a function $\alpha_0 \in C^2(\overline{\Omega})$ and there exist two constants $C, \hat{\sigma} > 0$ (that only depend on Ω and $\mathcal{B}_0$) such that the solution (φ, ψ) to (7) associated to any $(\varphi^0, \psi^0) \in L^2(\Omega)^2$ satisfies*

$$\begin{aligned} & s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla \varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\ & + s^4 \iint_Q e^{-2s\alpha} t^{-4}(T-t)^{-4} |\nabla \psi|^2 + s^6 \iint_Q e^{-2s\alpha} t^{-6}(T-t)^{-6} |\psi|^2 \\ & \qquad \le C s^7 \iint_{\mathcal{B}_0 \times (0,T)} e^{-2s\alpha} t^{-7}(T-t)^{-7} |\psi|^2, \end{aligned} \tag{12}$$

for every $s \ge \hat{s} = \hat{\sigma}(\Omega, \mathcal{B}_0)(T + T^2 M)$, where

$$\begin{aligned} M &= 1 + \|a+c\|_\infty^{1/2} + \|a\|_\infty^{1/2} + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|e\|_\infty^{1/4} + \|e\|_\infty^{1/3} \\ &+ \|B\|_\infty + \|B\|_\infty^2 + \|F\|_\infty^{1/4} + \|F\|_\infty^{1/3} + \|F\|_\infty^{2/5} + \|F\|_\infty^{1/2}. \end{aligned}$$

In (12), the function α is given by $\alpha(x,t) = \dfrac{\alpha_0(x)}{t(T-t)}$, $x \in \Omega$, $t \in (0,T)$.

Remark 2.3 *Theorem 2.4 improves the Carleman estimate established in [4]. To be precise, Ammar Khodja et al. consider the adjoint system (7) with $a \equiv e \equiv 0$ and $B \equiv F \equiv 0$ and, by using a technique completely different from ours, are able to bound some global terms of φ and ψ by*

$$C(\Omega, T, \|c\|_\infty) \iint_{\mathcal{B}_0 \times (0,T)} e^{-rs\alpha} |\psi|^2 \, dx \, dt \qquad (\textit{with } r \in (0,2)).$$

Observe that, owing to the structure of the function $\alpha(x,t)$, the weight function $e^{-rs\alpha}$ is worse than the weight $e^{-2s\alpha} t^{-7}(T-t)^{-7}$ in the right-hand side of (12). Such a Carleman inequality enables the authors to obtain controls in $L^{q_N}(Q)$, together with appropriate estimates, that give the exact controllability to the *trajectories (thus also controls that give the null controllability) of the linear system, with $q_N \in (2,\infty)$ if $N = 1$ or 2, and $N/2+1 < q_N \leq 2(N+2)/(N-2)$ if $3 \leq N \leq 5$. It is worthy of remark that their strategy uses the global terms of $|\Delta\psi|^2$ and $|\partial_t\psi|^2$ appearing in the left-hand side of the Carleman inequality, thus it cannot be applied in our case because of the presence of the term $-\nabla\cdot(B\psi)$ in (7).*

Remark 2.4 *Let us consider a nonempty open set $\mathcal{B}_0$ such that $\mathcal{B}_0 \subset\subset \omega$ ($\omega \subset \Omega$ being the control set). By combining Theorem 2.4 with the corresponding energy estimates, the following observability inequality for the solutions to (7) is easily derived:*

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)} \leq \exp(CH) \iint_{\mathcal{B}_0\times(0,T)} |\psi|^2\,dx\,dt \quad \forall \varphi^0,\psi^0 \in L^2(\Omega),$$

where $C = C(\Omega,\mathcal{B}_0) > 0$ and $H = H(T,\|a\|_\infty,\|c\|_\infty,\|e\|_\infty,\|B\|_\infty,\|F\|_\infty)$ is given by

$$H = M + \frac{1}{T} + T\left(1 + \|a\|_\infty + \|c\|_\infty + \|e\|_\infty + \|B\|^2_\infty + \|F\|^2_\infty\right),$$

with $M > 0$ as in Theorem 2.4. By a standard argument, the previous observability result enables us to prove the existence of a control $\hat{v} \in L^2(Q)$ supported in $\overline{\mathcal{B}}_0 \times [0,T]$ that gives the null controllability of a linear phase field system such as (16), for initial data in $L^2(\Omega)$ and potentials in L^∞. Moreover, we can estimate

$$||\hat{v}||^2_{L^2(Q)} \leq \exp(CH)\left(||u_0||^2_{L^2(\Omega)} + ||\phi_0||^2_{L^2(\Omega)}\right),$$

with $C = C(\Omega,\mathcal{B}_0) > 0$ and $H > 0$ being the constants above.

Remark 2.5 *Theorem 2.4 would imply, in particular, the following unique continuation property for the solutions to the adjoint system (7): "If $\varphi^0,\psi^0 \in L^2(\Omega)$, (φ,ψ) is the associated solution to (7) and $\psi = 0$ in $\omega\times(0,T)$, then $\varphi \equiv \psi \equiv 0$ in Q". Nevertheless, such a unique continuation property cannot be obtained as a direct consequence of the classical unique continuation property for the heat equation, owing to the coupling of the PDEs in (7).*

3 Some technical results

The strategy developed in this paper to deal with the null controllability of some linear coupled parabolic systems uses the parabolic regularizing effect. For this reason, we devote this Section to stating some technical results on parabolic regularity of a linear phase field system which will be used later. Although these results are naturally expected, we include them here so as to obtain the explicit dependence of the constants in the corresponding estimates with respect to the size of the potentials, which is essential in our analysis.

We previously introduce the following notation, which is used all along the paper. For $p \in [1,\infty]$ and a given Banach space X, $\|\cdot\|_{L^p(X)}$ will denote the norm in the space $L^p(0,T;X)$ (for simplicity, $\|\cdot\|_\infty$ will stand for the norm in $L^\infty(Q)$). For arbitrary $r \in [2,\infty)$, $\delta \in [0,T)$, and any open set $\mathcal{V} \subset \mathbb{R}^N$, we introduce the Banach space

$$X^r(\delta,T;\mathcal{V}) = \left\{u : u \in L^r(\delta,T;W^{2,r}(\mathcal{V})),\, \partial_t u \in L^r(\delta,T;L^r(\mathcal{V}))\right\}$$

and its natural norm $\|u\|_{X^r(\delta,T;\mathcal{V})} = \|u\|_{L^r(\delta,T;W^{2,r}(\mathcal{V}))} + \|\partial_t u\|_{L^r(\delta,T;L^r(\mathcal{V}))}$. We denote by X^r the Banach space

$$X^r = \{u : u \in L^r(0,T;W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)),\ \partial_t u \in L^r(Q)\}$$

and by $\|\cdot\|_{X^r}$ its natural norm $\|u\|_{X^r} = \|u\|_{L^r(W^{2,r}(\Omega))} + \|\partial_t u\|_{L^r(Q)}$.

On the other hand, for $\beta \in (0,1)$ and $u \in C^0(\overline{Q})$, we define the quantity

$$[u]_{\beta,\frac{\beta}{2}} = \sup_{\substack{\overline{Q} \\ x \neq x'}} \frac{|u(x,t) - u(x',t)|}{|x-x'|^\beta} + \sup_{\substack{\overline{Q} \\ t \neq t'}} \frac{|u(x,t) - u(x,t')|}{|t-t'|^{\frac{\beta}{2}}}.$$

We will consider the space $C^{\beta,\frac{\beta}{2}}(\overline{Q}) = \left\{u \in C^0(\overline{Q}) : [u]_{\beta,\frac{\beta}{2}} < \infty\right\}$, which is a Banach space with its natural norm $|u|_{\beta,\frac{\beta}{2};\overline{Q}} = \|u\|_\infty + [u]_{\beta,\frac{\beta}{2}}$. We will also consider the Banach space defined by

$$C^{1+\beta,\frac{1+\beta}{2}}(\overline{Q}) = \left\{u \in C^0(\overline{Q}) : \nabla u \in C^{\beta,\frac{\beta}{2}}(\overline{Q})^N,\ \sup_{\overline{Q}} \frac{|u(x,t) - u(x,t')|}{|t-t'|^{\frac{1+\beta}{2}}} < \infty\right\},$$

with norm denoted by $|\cdot|_{1+\beta,\frac{1+\beta}{2};\overline{Q}}$. The norm in the space $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ will be denoted by $\|\cdot\|_{L^2(H^1)\cap C(L^2)}$ and $W(0,T)$ will stand for the space

$$W(0,T) = \left\{y : y \in L^2(0,T;H_0^1(\Omega)),\, \partial_t y \in L^2(0,T;H^{-1}(\Omega))\right\}.$$

Finally, throughout the paper C will stand for a generic positive constant that only depends on the geometric data (Ω, the control set ω, and/or other open sets which will be considered further) and, eventually, on the dimension N (and/or a given number $r \in [2,\infty)$), whose value may change from one line to another. We will denote by $\mathcal{K}$ another generic positive constant that only depends on N, whose value may also change from line to line. From now on, we will only specify the dependence of the constants with respect to the arguments which will be relevant in our analysis (thus, for instance, the dependence on N and r will usually be omitted).

We start with a result on existence, uniqueness, and regularity of solution for linear phase field systems such as

$$\begin{cases} \partial_t u - \Delta u + B\cdot\nabla u + au + F\cdot\nabla\phi + e\phi = -\Delta\phi + g_1 & \text{in } Q, \\ \partial_t \phi - \Delta\phi + D\cdot\nabla\phi + c\phi = u + g_2 & \text{in } Q, \\ u = 0,\ \ \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x),\ \ \phi(x,0) = \phi_0(x) & \text{in } \Omega, \end{cases} \tag{13}$$

where $a, c, e \in L^\infty(Q)$, $B, D, F \in L^\infty(Q)^N$, $u_0, \phi_0 \in L^2(\Omega)$, and $g_1, g_2 \in L^2(0,T; H^{-1}(\Omega))$ (at least).

Proposition 3.1 *Let $a, c, e \in L^\infty(Q)$ and $B, D, F \in L^\infty(Q)^N$ be given. It holds that:*

a) If $u_0, \phi_0 \in L^2(\Omega)$ and $g_1, g_2 \in L^2(0,T; H^{-1}(\Omega))$, there exists a unique weak solution (u, ϕ) to (13) satisfying $(u, \phi) \in W(0,T)^2$ together with the estimate

$$\|(u,\phi)\|_{W(0,T)^2} \le \exp(CH_1)\left(\|(u_0,\phi_0)\|_{L^2(\Omega)^2} + \|(g_1,g_2)\|_{L^2(H^{-1}(\Omega))^2}\right),$$

where $C = C(\Omega) > 0$ and $H_1 = H_1(T, \|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|D\|_\infty, \|F\|_\infty)$ is given by

$$H_1 = 1 + T\left(1 + \|a\|_\infty + \|c\|_\infty + \|e\|_\infty + \|B\|_\infty^2 + \|D\|_\infty^2 + \|F\|_\infty^2\right) \tag{14}$$

(here, $\|y\|_{W(0,T)} = \|y\|_{L^2(H_0^1(\Omega))} + \|\partial_t u\|_{L^2(H^{-1}(\Omega))}$, $y \in W(0,T)$).

b) If $u_0, \phi_0 \in W^{2-2/r,r}(\Omega) \cap H_0^1(\Omega)$ and $g_1, g_2 \in L^r(Q)$, with $r \in [2,\infty)$ being arbitrary, the weak solution (u,ϕ) to (13) lies in $X^r \times X^r$ and there exist positive constants $C = C(\Omega, N, r)$ and $\mathcal{K} = \mathcal{K}(N)$ such that

$$\|(u,\phi)\|_{X^r \times X^r} \le \exp(CH_1) H_2^{\mathcal{K}} \left(\|(u_0,\phi_0)\|_{W^{2-2/r,r}(\Omega)^2} + \|(g_1,g_2)\|_{L^r(Q)^2}\right),$$

with $H_1 > 0$ as above and $H_2 = H_2(\|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|D\|_\infty, \|F\|_\infty)$ given by

$$H_2 = 1 + \|a\|_\infty + \|c\|_\infty + \|e\|_\infty + \|B\|_\infty + \|D\|_\infty + \|F\|_\infty. \tag{15}$$

The proof of this Proposition is sketched in Appendix A.

One can also obtain the following result on local parabolic regularity of the linear phase field system (13):

Proposition 3.2 *For given $a, c, e \in L^\infty(Q)$, $B, D, F \in L^\infty(Q)^N$, $g_1, g_2 \in L^2(0,T; H^{-1}(\Omega))$, and $u_0, \phi_0 \in L^2(\Omega)$, let us consider the weak solution $(u,\phi) \in W(0,T)^2$ to (13). Let $\mathcal{V} \subset \Omega$ and $\mathcal{O} \subset\subset \Omega$ be two open sets and let $r \in [2,\infty)$ be given. The following holds:*

a) If $g_1, g_2 \in L^r(\delta, T; L^r(\Omega))$, with $\delta \in [0,T)$ being arbitrary, then (u,ϕ) lies in $X^r(\delta', T; \Omega)^2$ for any $\delta' \in (\delta, T)$ and there exists a positive constant C independent of T such that

$$\|(u,\phi)\|_{X^r(\delta',T;\Omega)^2} \le \exp\left[C(1+T)\right]\left(1 + \frac{1}{\delta' - \delta}\right)^{\mathcal{K}} H_2^{\mathcal{K}}$$
$$\times \left(\|(g_1,g_2)\|_{L^r(\delta,T;L^r(\Omega))^2} + \|(u,\phi)\|_{W(0,T)^2}\right),$$

with $\mathcal{K} = \mathcal{K}(N) > 0$ and $H_2 = H_2(\|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|D\|_\infty, \|F\|_\infty) > 0$ as in Proposition 3.1.

b) Suppose, in addition, that $u(x,0) = 0$ and $\phi(x,0) = 0$ in Ω and $g_1, g_2 \in L^r(0,T;L^r(\mathcal{V}))$ (resp. $g_1, g_2 \in L^r(0,T;L^r(\Omega \setminus \overline{\mathcal{O}}))$). Then, for any open set $\mathcal{V}' \subset\subset \mathcal{V}$ (resp. $\mathcal{O}'$ such that $\mathcal{O} \subset\subset \mathcal{O}' \subset\subset \Omega$), it holds that

$$(u,\phi) \in X^r(0,T;\mathcal{V}')^2 \qquad (\textit{resp. } (u,\phi) \in X^r(0,T;\Omega \setminus \overline{\mathcal{O}'})^2).$$

Moreover, there exists a new positive constant C independent of T such that

$$\begin{aligned} &\|(u,\phi)\|_{X^r(0,T;\mathcal{V}')^2} \\ &\le \exp\left[C(1+T)\right] H_2^{\mathcal{K}} \left(\|(g_1,g_2)\|_{L^r(L^r(\mathcal{V}))^2} + \|(u,\phi)\|_{W(0,T)^2}\right) \end{aligned}$$

(resp.

$$\begin{aligned} &\|(u,\phi)\|_{X^r(0,T;\Omega\setminus\overline{\mathcal{O}'})^2} \\ &\le \exp\left[C(1+T)\right] H_2^{\mathcal{K}} \left(\|(g_1,g_2)\|_{L^r(L^r(\Omega\setminus\overline{\mathcal{O}}))^2} + \|(u,\phi)\|_{W(0,T)^2}\right)\Big), \end{aligned}$$

with $\mathcal{K} = \mathcal{K}(N) > 0$ and $H_2 = H_2(\|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|D\|_\infty, \|F\|_\infty) > 0$ as above.

c) Assume the hypothesis in the previous point together with $g_2, u \in L^r(0,T;W^{1,r}(\mathcal{V}))$, $\nabla c \in L^\gamma(Q)^N$, with γ given by (24), and $D \equiv 0$. Then, for any open set $\mathcal{V}' \subset\subset \mathcal{V}$, one has

$$\phi \in L^r(0,T;W^{3,r}(\mathcal{V}')), \quad \partial_t \phi \in L^r(0,T;W^{1,r}(\mathcal{V}')).$$

Furthermore, for a new positive constant C independent of T, the following estimate holds

$$\begin{aligned} \|\phi\|_{L^r(W^{3,r}(\mathcal{V}'))} + \|\partial_t\phi\|_{L^r(W^{1,r}(\mathcal{V}'))} &\le \exp\left[C(1+T)\right]\left(1+\frac{1}{T}\right) H_2^{\mathcal{K}}(1+\|c\|_\infty) \\ \times \left(1+\|\nabla c\|_{L^\gamma(Q)}\right) &\left(\|g_1\|_{L^r(L^r(\mathcal{V}))} + \|(g_2,u)\|_{L^r(W^{1,r}(\mathcal{V}))^2} + \|\phi\|_{W(0,T)}\right), \end{aligned}$$

with $\mathcal{K} = \mathcal{K}(N) > 0$.

The proof of this Proposition combines the local regularity of the heat equation with an argument of 'bootstrap' type and, far from being the aim of this paper, it will be omitted.

We end this Section by recalling the following result, which is readily obtained by rewriting Lemma 3.3 in [12] with our notation (also see Lemma 2.2 in [9]):

Lemma 3.1 *Let $\mathcal{V} \subset \mathbb{R}^N$ be a bounded open set with $\partial\mathcal{V} \in C^2$ ($N \ge 1$ being arbitrary) and let $r \in [1,\infty)$ be given. The following continuous embeddings hold:*

i) If $r < \dfrac{N}{2}+1$, then $X^r(0,T;\mathcal{V}) \hookrightarrow L^p(\mathcal{V}\times(0,T))$, where $\dfrac{1}{p} = \dfrac{1}{r} - \dfrac{2}{N+2}$.

ii) If $r = \dfrac{N}{2}+1$, then $X^r(0,T;\mathcal{V}) \hookrightarrow L^q(\mathcal{V}\times(0,T))$ for any $q < \infty$.

iii) If $\dfrac{N}{2}+1 < r < N+2$, *then* $X^r(0,T;\mathcal{V}) \hookrightarrow C^{\beta,\frac{\beta}{2}}(\overline{\mathcal{V}}\times[0,T])$, *where* $\beta = 2-(N+2)/r$.

iv) If $r = N+2$, *then* $X^r(0,T;\mathcal{V}) \hookrightarrow C^{l,\frac{l}{2}}(\overline{\mathcal{V}}\times[0,T])$ *for any* $l\in(0,1)$.

v) If $r > N+2$, *then* $X^r(0,T;\mathcal{V}) \hookrightarrow C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\mathcal{V}}\times[0,T])$, *where* $\alpha = 1-(N+2)/r$.

Moreover, the constant in each embedding can be written as $C\,(1+1/T)$, *with* $C = C(\mathcal{V},N,r) > 0$.

4 Null controllability of a linear phase field system

The purpose of this Section is to prove the null controllability of the linear phase field system

$$\begin{cases} \partial_t u - \Delta u + B\cdot\nabla u + au + F\cdot\nabla\phi + e\phi = -\Delta\phi + v\mathbf{1}_\omega \quad \text{in } Q,\\ \partial_t\phi - \Delta\phi + c\phi = u \quad \text{in } Q,\\ u=0,\ \ \phi=0 \quad \text{on } \Sigma, \quad u(x,0)=u_0(x),\ \ \phi(x,0)=\phi_0(x) \quad \text{in } \Omega, \end{cases} \tag{16}$$

where $a,c,e \in L^\infty(Q)$, $B,F\in L^\infty(Q)^N$, $u_0,\phi_0 \in L^2(\Omega)$ (at least), and $v\in L^2(Q)$ is a control function to be determined ($\omega\subset\Omega$ is the control open set). Since the present analysis is directed towards the study of the interesting nonlinear case when certain superlinear nonlinearities are considered, we are indeed interested in obtaining a null control v for system (16) so that the associated solution (u,ϕ) lies in $L^\infty(Q)^2$. Furthermore, appropriate estimates of v and (u,ϕ) with respect to the size of the data must be obtained. As in the example described in the introduction to the paper, we proceed in two steps. First, we introduce a fictitious control in the second PDE in (16) and give a null controllability result with two controls $\hat{v}_1$ and $\hat{v}_2$ in $L^2(Q)$ (see Theorem 4.1). In Subsection 4.2, we develop the strategy to remove $\hat{v}_2$ and construct a control $v\in L^r(Q)$ ($r\in[2,\infty)$ arbitrary) that gives the null controllability of system (16), with associated solution (u,ϕ) in $L^\infty(Q)^2$. We again adapt to the present case the technique of construction of regular controls (from L^2-controls) introduced in [8].

4.1 The linear null controllability problem with two controls

Let $\mathcal{B}_0$ be a regular nonempty open set such that $\mathcal{B}_0\subset\subset\omega$. Let us consider the linear system

$$\begin{cases} \partial_t u - \Delta u + B\cdot\nabla u + au + F\cdot\nabla\phi + e\phi = -\Delta\phi + \hat{v}_1\mathbf{1}_{\mathcal{B}_0} \quad \text{in } Q,\\ \partial_t\phi - \Delta\phi + c\phi = u + \hat{v}_2\mathbf{1}_{\mathcal{B}_0} \quad \text{in } Q,\\ u=0,\ \ \phi=0 \quad \text{on } \Sigma, \quad u(x,0)=u_0(x),\ \ \phi(x,0)=\phi_0(x) \quad \text{in } \Omega, \end{cases} \tag{17}$$

where $\hat{v}_1, \hat{v}_2 \in L^2(Q)$ are two control functions to be determined. The following null controllability result for system (17) can be proved:

Theorem 4.1 *Let $a, c, e \in L^\infty(Q)$, $B, F \in L^\infty(Q)^N$, and $u_0, \phi_0 \in L^2(\Omega)$ be given. Then, there exist two control functions $\hat{v}_1, \hat{v}_2 \in L^2(Q)$, with $\operatorname{supp} \hat{v}_1$, $\operatorname{supp} \hat{v}_2 \subset \overline{\mathcal{B}}_0 \times [0,T]$, such that the corresponding solution $(\hat{u}, \hat{\phi})$ to (17) satisfies $\hat{u}(x,T) = 0$, $\hat{\phi}(x,T) = 0$ in Ω. Moreover, $\hat{v}_1$ and $\hat{v}_2$ can be chosen so that*

$$||\hat{v}_1||^2_{L^2(Q)} + ||\hat{v}_2||^2_{L^2(Q)} \leq \exp(CH_0) \left(||u_0||^2_{L^2(\Omega)} + ||\phi_0||^2_{L^2(\Omega)} \right), \tag{18}$$

with $C = C(\Omega, \mathcal{B}_0) > 0$ and $H_0 = H_0(T, \|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|F\|_\infty) > 0$ given by

$$\begin{aligned} H_0 &= 1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|e\|_\infty^{1/3} + \|B\|^2_\infty + \|F\|^{1/2}_\infty \\ &+ T \left(1 + ||a||_\infty + ||c||_\infty + ||e||_\infty + ||B||^2_\infty + ||F||^2_\infty \right). \end{aligned} \tag{19}$$

The proof of this Theorem is a standard consequence of the observability result for the solutions to the adjoint system (7) established in the following Theorem, and will be omitted.

Theorem 4.2 *There exist positive constants $C = C(\Omega, \mathcal{B}_0)$ and*

$$H_0 = H_0(T, \|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|F\|_\infty)$$

such that, for every $\varphi^0, \psi^0 \in L^2(\Omega)$, the corresponding solution (φ, ψ) to (7) satisfies

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)} \leq \exp(CH_0) \iint_{\mathcal{B}_0 \times (0,T)} \left(|\varphi|^2 + |\psi|^2 \right) \, dx \, dt.$$

To be precise, H_0 is given by (19).

The proof of this observability result combines a suitable Carleman estimate for the solutions to (7) with the corresponding energy estimates for these solutions. The basic tool used in the proof is a global Carleman inequality (which we recall in Lemma 4.1) for linear systems such as

$$\begin{cases} \partial_t z - \Delta z = \widetilde{f} \ \text{ in } Q, \\ z = 0 \ \text{ on } \Sigma, \quad z(x,0) = z^0(x) \ \text{ in } \Omega, \end{cases} \tag{20}$$

with $z^0 \in L^2(\Omega)$ and $\widetilde{f} \in L^2(0,T; H^{-1}(\Omega))$.

Lemma 4.1 *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open set. Let us assume that $\widetilde{f} = f_0 + \sum_{i=1}^N \dfrac{\partial f_i}{\partial x_i}$, with $f_i \in L^2(Q)$, $i = 0, 1, \ldots, N$. Then, there exist a function*

$\alpha_0 \in C^2(\overline{\Omega})$ and two positive constants C_0 and σ_0 (that only depend on Ω and $\mathcal{B}$) such that, for any $z^0 \in L^2(\Omega)$, the associated solution z to (20) *satisfies*

$$
\begin{aligned}
s^{1+l} \iint_Q e^{-2s\alpha} t^{-1-l}(T-t)^{-1-l} |\nabla z|^2 + s^{3+l} \iint_Q e^{-2s\alpha} t^{-3-l}(T-t)^{-3-l}|z|^2 \\
\leq C_0 \Bigg(s^{3+l} \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha} t^{-3-l}(T-t)^{-3-l}|z|^2 \\
+ s^l \iint_Q e^{-2s\alpha} t^{-l}(T-t)^{-l}|f_0|^2 + s^{2+l} \sum_{i=1}^{N} \iint_Q e^{-2s\alpha} t^{-2-l}(T-t)^{-2-l}|f_i|^2 \Bigg),
\end{aligned}
$$

for any $s \geq s_0 = \sigma_0(\Omega, \mathcal{B})\,(T+T^2)$ and $l \geq 0$. The function α is given by $\alpha(x,t) = \dfrac{\alpha_0(x)}{t(T-t)}$, $x \in \Omega$, $t \in (0,T)$.

The proof of this Lemma can be found in [13] although the authors do not precise the way the constant s_0 depends on T. This explicit dependence can be obtained arguing as in [14] (also see [11] and [15]).

The following Carleman estimate for the solutions to (7) is proved:

Proposition 4.1 *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open set. There exist a function $\alpha_0 \in C^2(\overline{\Omega})$ and two positive constants C and $\overline{\sigma}$ (that only depend on Ω and $\mathcal{B}$) such that the solution (φ, ψ) to* (7) *associated to any $(\varphi^0, \psi^0) \in L^2(\Omega)^2$ satisfies*

$$
\begin{aligned}
s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1}|\nabla\varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \\
+ s^4 \iint_Q e^{-2s\alpha} t^{-4}(T-t)^{-4}|\nabla\psi|^2 + s^6 \iint_Q e^{-2s\alpha} t^{-6}(T-t)^{-6}|\psi|^2 \\
+ \leq C \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha} \left(s^3 t^{-3}(T-t)^{-3}|\varphi|^2 + s^6 t^{-6}(T-t)^{-6}|\psi|^2 \right)
\end{aligned}
\tag{21}
$$

for every $s \geq \overline{s}$, where

$$
\overline{s} = \overline{\sigma}(\Omega, \mathcal{B}) \left(T + T^2 \left(1 + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|e\|_\infty^{1/3} + \|B\|_\infty^2 + \|F\|_\infty^{1/2} \right) \right). \tag{22}
$$

In (21)*, the function α is given by $\alpha(x,t) = \dfrac{\alpha_0(x)}{t(T-t)}$, $x \in \Omega$, $t \in (0,T)$.*

Proof. Let (φ, ψ) be the solution to (7) associated to $(\varphi^0, \psi^0) \in L^2(\Omega)^2$ and let α_0 and α be the functions in Lemma 4.1 associated to $\mathcal{B}$.

Firstly, we apply Lemma 4.1 to the function φ, with $l = 0$ and $\widetilde{f} = -c\varphi - e\psi + \nabla \cdot (F\psi - \nabla\psi)$. Secondly, we apply the same Lemma to ψ, by taking this time $l = 3$ and $\widetilde{f} = \varphi - a\psi + \nabla \cdot (B\psi)$. By combining both Carleman estimates for φ and ψ, one infers the existence of two positive constants $C_1 = C_1(\Omega, \mathcal{B})$ and $\sigma_1 = \sigma_1(\Omega, \mathcal{B})$ such that, for any $s \geq s_1 =$

$\sigma_1(\Omega,\mathcal{B})\left(T+T^2\left(1+\|a\|_\infty^{2/3}+\|c\|_\infty^{2/3}+\|B\|_\infty^2\right)\right)$, it holds that

$$\begin{aligned}
& s\iint_Q e^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\varphi|^2+s^3\iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2 \\
&+s^4\iint_Q e^{-2s\alpha}t^{-4}(T-t)^{-4}|\nabla\psi|^2+s^6\iint_Q e^{-2s\alpha}t^{-6}(T-t)^{-6}|\psi|^2 \\
&\le C_1\left(s^3\iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2+\|e\|_\infty^2\iint_Q e^{-2s\alpha}|\psi|^2\right. \\
&\left.+s^6\iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}t^{-6}(T-t)^{-6}|\psi|^2+s^2\|F\|_\infty^2\iint_Q e^{-2s\alpha}t^{-2}(T-t)^{-2}|\psi|^2\right).
\end{aligned}$$

This immediately yields estimate (21) for every $s\ge\overline{s}$, with $\overline{s}$ given by (22), which was our claim. □

In the proof of Theorem 4.2, we also use the following Lemma whose proof, being an easy matter, will be omitted.

Lemma 4.2 *Let $\gamma_0\in C^0(\overline{\Omega})$ be such that $\gamma_0(x)\ge m>0$ $\forall x\in\overline{\Omega}$. Let us set $\gamma(x,t)=\dfrac{\gamma_0(x)}{t(T-t)}$ for $(x,t)\in Q$, and $m_0=\min_{\overline{\Omega}}\gamma_0$. Then,*

$$s^6e^{-2s\gamma}t^{-6}(T-t)^{-6}\le\left(\frac{3}{em_0}\right)^6 \quad \textit{for any } s>\frac{3T^2}{4m_0} \textit{ and } (x,t)\in Q.$$

We end this Subsection by proving Theorem 4.2.

Proof of Theorem 4.2. Let (φ,ψ) be the solution to (7) associated to $\varphi^0,\psi^0\in L^2(\Omega)$. The proof uses Proposition 4.1 and the energy estimates for φ and ψ.

For t almost everywhere (a.e.) in $(0,T)$, it is seen that

$$-\frac{d}{dt}\left[\|\varphi(t)\|_{L^2(\Omega)}^2+\|\psi(t)\|_{L^2(\Omega)}^2\right]\le H_*\left(\|\varphi(t)\|_{L^2(\Omega)}^2+\|\psi(t)\|_{L^2(\Omega)}^2\right),$$

with $H_*=1+2\left(\|a\|_\infty+\|c\|_\infty+\|e\|_\infty+\|B\|_\infty^2+\|F\|_\infty^2\right)>0$, whence

$$\|\varphi(0)\|_{L^2(\Omega)}^2+\|\psi(0)\|_{L^2(\Omega)}^2\le\frac{4}{T}\exp\left(H_*T\right)\iint_{\Omega\times(T/2,3T/4)}\left(|\varphi|^2+|\psi|^2\right). \quad (23)$$

The proof will be completed by combining (21) and (23). Let α_0 and α be the functions in Proposition 4.1 associated to $\mathcal{B}_0$ and consider an arbitrary $s\ge\overline{s}$, with $\overline{s}$ given by (22). Observe that

$$e^{-2s\alpha}t^{-3}(T-t)^{-3}\ge 2^6T^{-6}\exp\left(-2^5\mathcal{M}_0 s/(3T^2)\right) \quad \forall(x,t)\in\Omega\times(T/2,3T/4),$$

with $\mathcal{M}_0 = \max_{\overline{\Omega}} \alpha_0$. On the other hand, since $s \geq CT^2$, one has $s^3 t^{-6}(T-t)^{-6} \geq Ct^{-3}(T-t)^{-3}$ for all $t \in (0,T)$. Then,

$$\begin{aligned}&\iint_{\Omega\times(T/2,3T/4)} e^{-2s\alpha}\left(t^{-3}(T-t)^{-3}|\varphi|^2 + s^3t^{-6}(T-t)^{-6}|\psi|^2\right)\\ &\qquad \geq CT^{-6}\exp\left(-\frac{Cs}{T^2}\right)\iint_{\Omega\times(T/2,3T/4)}\left(|\varphi|^2+|\psi|^2\right)\,dx\,dt.\end{aligned}$$

Now, by applying Proposition 4.1 (with $\mathcal{B} = \mathcal{B}_0$) and Lemma 4.2, one can estimate

$$\begin{aligned}&\iint_{\Omega\times(T/2,3T/4)}\left(|\varphi|^2+|\psi|^2\right) \leq CT^6\exp\left(\frac{Cs}{T^2}\right)\times\\ &\left(\iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2 + s^3\iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-6}(T-t)^{-6}|\psi|^2\right)\\ &\qquad \leq CT^6\exp\left(\frac{Cs}{T^2}\right)\iint_{\mathcal{B}_0\times(0,T)}\left(|\varphi|^2+|\psi|^2\right)\end{aligned}$$

for every $s \geq \hat{s}$, where

$$\hat{s} = \hat{\sigma}(\Omega,\mathcal{B}_0)\left(T + T^2\left(1 + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|e\|_\infty^{1/3} + \|B\|_\infty^2 + \|F\|_\infty^{1/2}\right)\right),$$

with $\hat{\sigma}(\Omega,\mathcal{B}_0) = \max\{\overline{\sigma}(\Omega,\mathcal{B}_0), 3/(4m_0)\}$ and $m_0 = \min_{\overline{\Omega}} \alpha_0$. Finally, by setting $s = \hat{s}$ in the previous estimate and by recalling (23), we end the proof. □

4.2 The linear null controllability problem with one control

For a given $s_1 \in [2,\infty)$, we set

$$Z^{s_1} = \begin{cases} L^{s_1}(0,T;W_0^{1,s_1}(\Omega)) & \text{if} \quad s_1 \in [2, N/2+1],\\ L^{s_1}(0,T;W_0^{1,s_1}(\Omega)) \cap C^0(\overline{Q}) & \text{if} \quad s_1 > N/2+1,\end{cases}$$

and $X^{s_1} = \{\phi : \phi \in L^{s_1}(0,T;W^{2,s_1}(\Omega)\cap W_0^{1,s_1}(\Omega)),\ \partial_t\phi \in L^{s_1}(Q)\}$. In this Subsection, we prove the following null controllability result for system (16).

Theorem 4.3 *Let $r, s_1 \in [2,\infty)$ and $T > 0$ be given. Assume that $u_0, \phi_0 \in W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega)$, $a, e \in L^\infty(Q)$, $B, F \in L^\infty(Q)^N$, and $c \in L^\infty(Q) \cap L^\gamma(0,T;W^{1,\gamma}(\Omega))$, with γ defined by*

$$\gamma = \begin{cases} \max\{r, N/2+1\} & \textit{if} \quad r \neq N/2+1,\\ N/2+1+\varepsilon' \ \ (\varepsilon' > 0 \textit{ arbitrarily small}) & \textit{if} \quad r = N/2+1.\end{cases} \tag{24}$$

Then, there exists a control function $v \in L^r(Q)$ *supported in* $\omega \times [0,T]$ *such that the corresponding solution* (u, ϕ) *to* (16) *lies in* $Z^{s_1} \times X^{s_1}$ *and satisfies* $u(x,T) = 0$ *and* $\phi(x,T) = 0$ *in* Ω*. Moreover,*

$$\|u\|_{Z^{s_1}} + \|\phi\|_{X^{s_1}} \leq \exp\left(CH_0\right) \|(u_0, \phi_0)\|_{W^{2-2/s_1, s_1}(\Omega)^2} \tag{25}$$

and

$$\|v\|_{L^r(Q)} \leq \exp\left(CH_0\right)(1 + \|\nabla c\|_{L^\gamma(Q)})\|(u_0, \phi_0)\|_{L^2(\Omega)^2}, \tag{26}$$

with $C = C(\Omega, \omega) > 0$ *and* $H_0 = H_0(T, \|a\|_\infty, \|c\|_\infty, \|e\|_\infty, \|B\|_\infty, \|F\|_\infty) > 0$ *given by* (19)*.*

Proof. Let $\mathcal{B}_0$ be a regular nonempty open set such that $\mathcal{B}_0 \subset\subset \omega$. Let $\hat{v}_1, \hat{v}_2 \in L^2(Q)$ be two controls provided by Theorem 4.1 (associated to $\mathcal{B}_0$) and denote by $(\hat{u}, \hat{\phi})$ the corresponding solution to (17). We will eliminate the control $\hat{v}_2$ and construct a new control v as in the statement.

We proceed as follows. First, we consider a function $\eta \in C^\infty([0,T])$ such that $\eta \equiv 1$ in $[0, T/3]$, $\eta \equiv 0$ in $[2T/3, T]$, and $0 \leq \eta \leq 1$, $|\eta'(t)| \leq C/T$ in $[0,T]$. Then, we introduce the change of variables

$$u = U + \eta \overline{u}, \quad \phi = \Phi + \eta \overline{\phi}, \tag{27}$$

where $(\overline{u}, \overline{\phi})$ is the weak solution to

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + B \cdot \nabla \overline{u} + a \overline{u} + F \cdot \nabla \overline{\phi} + e \overline{\phi} = -\Delta \overline{\phi} \quad \text{in } Q, \\ \partial_t \overline{\phi} - \Delta \overline{\phi} + c \overline{\phi} = \overline{u} \quad \text{in } Q, \\ \overline{u} = 0, \;\; \overline{\phi} = 0 \quad \text{on } \Sigma, \quad \overline{u}(x,0) = u_0(x), \;\; \overline{\phi}(x,0) = \phi_0(x) \quad \text{in } \Omega. \end{cases}$$

Observe that the proof is reduced to obtaining a control v as in the statement that solves the null controllability problem

$$\begin{cases} \partial_t U - \Delta U + B \cdot \nabla U + aU + F \cdot \nabla \Phi + e\Phi = -\Delta \Phi - \eta' \overline{u} + v \mathbf{1}_\omega \;\; \text{in } Q, \\ \partial_t \Phi - \Delta \Phi + c\Phi = U - \eta' \overline{\phi} \;\; \text{in } Q, \\ U = 0, \;\; \Phi = 0 \quad \text{on } \Sigma, \\ U(x,0) = 0, \;\; \Phi(x,0) = 0, \;\; U(x,T) = 0, \;\; \Phi(x,T) = 0 \quad \text{in } \Omega. \end{cases} \tag{28}$$

We can also write $\hat{u} = \hat{U} + \eta \overline{u}$ and $\hat{\phi} = \hat{\Phi} + \eta \overline{\phi}$, where $(\hat{U}, \hat{\Phi})$ solves

$$\begin{cases} \partial_t \hat{U} - \Delta \hat{U} + B \cdot \nabla \hat{U} + a\hat{U} + F \cdot \nabla \hat{\Phi} + e\hat{\Phi} = -\Delta \hat{\Phi} - \eta' \overline{u} + \hat{v}_1 \mathbf{1}_{\mathcal{B}_0} \;\; \text{in } Q, \\ \partial_t \hat{\Phi} - \Delta \hat{\Phi} + c\hat{\Phi} = \hat{U} - \eta' \overline{\phi} + \hat{v}_2 \mathbf{1}_{\mathcal{B}_0} \;\; \text{in } Q, \\ \hat{U} = 0, \;\; \hat{\Phi} = 0 \quad \text{on } \Sigma, \\ \hat{U}(x,0) = 0, \;\; \hat{\Phi}(x,0) = 0, \;\; \hat{U}(x,T) = 0, \;\; \hat{\Phi}(x,T) = 0 \quad \text{in } \Omega. \end{cases} \tag{29}$$

We now consider three new regular open sets $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}$ such that

$$\mathcal{B}_0 \subset\subset \mathcal{B}_1 \subset\subset \mathcal{B}_2 \subset\subset \mathcal{B} \subset\subset \omega,$$

and a function $\theta \in \mathcal{D}(\mathcal{B})$ satisfying $\theta \equiv 1$ in $\mathcal{B}_2$. We set

$$\Phi = (1-\theta)\,\hat{\Phi}, \quad U = (1-\theta)\,\hat{U} + \theta\eta'\overline{\phi} + 2\nabla\theta\cdot\nabla\hat{\Phi} + (\Delta\theta)\hat{\Phi}, \tag{30}$$

and

$$\begin{aligned} v = {} & \theta\eta'\overline{u} - 2\nabla\theta\cdot\nabla\hat{\Phi} - (\Delta\theta)\hat{\Phi} + 2\nabla\theta\cdot\nabla\hat{U} + (\Delta\theta)\hat{U} - \nabla\theta\cdot(B\hat{U}) \\ & -\nabla\theta\cdot(F\hat{\Phi}) + (\partial_t - \Delta + B\cdot\nabla + a)\left[\theta\eta'\overline{\phi} + 2\nabla\theta\cdot\nabla\hat{\Phi} + (\Delta\theta)\hat{\Phi}\right]. \end{aligned} \tag{31}$$

The rest of the proof is a consequence of the parabolic regularity results in Section 3. For clarity, it will be divided into three steps.

Step 1. Let $r \in [2,\infty)$ be given. First, the above-introduced function v is supported in $\overline{\mathcal{B}} \times [0,T]$ (by the choice of θ). On the other hand, Proposition 3.2 yields $(\overline{u},\overline{\phi}) \in X^r(\delta,T;\Omega)^2$ for all $\delta > 0$. By observing the right-hand side of the PDEs in (29), Proposition 3.2 now gives $(\hat{U},\hat{\Phi}) \in X^r(0,T;\Omega\setminus\overline{\mathcal{B}}_1)^2$ and (by the first point of Proposition 3.1 and the properties on η) we have

$$\begin{aligned} & \|(\hat{U},\hat{\Phi})\|_{X^r(0,T;\Omega\setminus\overline{\mathcal{B}}_1)^2} \\ & \le \exp[C(1+T)]M_2^{\mathcal{K}}\left(\frac{1}{T}\|(\overline{u},\overline{\phi})\|_{X^r(T/3,T;\Omega)^2} + \|(\hat{U},\hat{\Phi})\|_{W(0,T)^2}\right) \\ & \le \exp(CM_1)M_2^{\mathcal{K}}\left(\|(u_0,\phi_0)\|_{L^2(\Omega)^2} + \|(\hat{v}_1,\hat{v}_2)\|_{L^2(Q)^2}\right), \end{aligned} \tag{32}$$

with $\mathcal{K} = \mathcal{K}(N) > 0$ and $M_1, M_2 > 0$ given by

$$M_1 = 1 + \frac{1}{T} + T\left(1 + \|a\|_\infty + \|c\|_\infty + \|e\|_\infty + \|B\|_\infty^2 + \|F\|_\infty^2\right),$$

$$M_2 = 1 + \|a\|_\infty + \|c\|_\infty + \|e\|_\infty + \|B\|_\infty + \|F\|_\infty.$$

Since $\nabla c \in L^\gamma(Q)^N$, with γ given by (24), the third point of Proposition 3.2 can be applied to $\hat{\Phi}$ inferring that

$$\hat{\Phi} \in L^r(0,T;W^{3,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2)), \quad \partial_t\hat{\Phi} \in L^r(0,T;W^{1,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2))$$

and

$$\begin{aligned} & \|\hat{\Phi}\|_{L^r(W^{3,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2))} + \|\partial_t\hat{\Phi}\|_{L^r(W^{1,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2))} \\ & \le \exp[C(1+T)]\,M_3\left(1 + \|\nabla c\|_{L^\gamma(Q)}\right)\Big(\|-\eta'\overline{\phi} + \hat{v}_2\mathbf{1}_{\mathcal{B}_0}\|_{L^r(W^{1,r}(\Omega\setminus\overline{\mathcal{B}}_1))} \\ & \quad + \|\hat{U}\|_{L^r(W^{1,r}(\Omega\setminus\overline{\mathcal{B}}_1))} + \|\hat{\Phi}\|_{W(0,T)}\Big) \end{aligned}$$

(here, $M_3 = (1+1/T)\,H_2^{\mathcal{K}}(1+\|c\|_\infty)$, with $\mathcal{K} = \mathcal{K}(N) > 0$) which, combined with (32) and (18), gives

$$\begin{aligned} & \|\hat{\Phi}\|_{L^r(W^{3,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2))} + \|\partial_t\hat{\Phi}\|_{L^r(W^{1,r}(\mathcal{B}\setminus\overline{\mathcal{B}}_2))} \\ & \le \exp(CH_0)\left(1 + \|\nabla c\|_{L^\gamma(Q)}\right)\|(u_0,\phi_0)\|_{L^2(\Omega)^2}. \end{aligned}$$

Then, the term $(\partial_t - \Delta)[2\nabla\theta \cdot \nabla\hat{\Phi}]$ in (31) lies in $L^r(Q)$, thus $v \in L^r(Q)$ and estimate (26) holds.

Step 2. First, assume that $u_0, \phi_0 \in L^2(\Omega)$. Let us have in mind that $(\hat{U}, \hat{\Phi}) \in X^p(0,T;\Omega \setminus \overline{\mathcal{B}}_1)^2$ for all $p \in [2,\infty)$ together with estimates (32) and (18). Then, (U,Φ) defined in (30) lies in $L^p(0,T;W_0^{1,p}(\Omega)) \times X^p$. By taking any $p > N+2$, Lemma 3.1 gives

$$(\hat{U}, \hat{\Phi}) \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega \setminus \mathcal{B}_1} \times [0,T])^2 \quad \forall \alpha \in (0,1).$$

In particular, $(U,\Phi) \in (L^p(0,T;W_0^{1,p}(\Omega)) \cap C^0(\overline{Q})) \times X^p$ for any $p \in [2,\infty)$ and

$$\|U\|_{L^p(W_0^{1,p}(\Omega)) \cap C^0(\overline{Q})} + \|\Phi\|_{X^p} \leq \exp(CH_0)\|(u_0,\phi_0)\|_{L^2(\Omega)^2} \quad (H_0 > 0 \text{ as above}).$$

Now, suppose that $u_0, \phi_0 \in W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega)$, for a given $s_1 \in [2,\infty)$. Proposition 3.1 immediately yields

$$(\overline{u}, \overline{\phi}) \in X^{s_1} \times X^{s_1}, \quad \|(\overline{u}, \overline{\phi})\|_{X^{s_1} \times X^{s_1}} \leq \exp(CH_1)M_2^{\mathcal{K}}\|(u_0,\phi_0)\|_{W^{2-2/s_1,s_1}(\Omega)^2},$$

with $\mathcal{K} = \mathcal{K}(N) > 0$ and $H_1 > 0$ given by (14) (here, $D \equiv 0$). Just by recalling the change of variables (27), we have $(u,\phi) = (U + \eta\overline{u}, \Phi + \eta\overline{\phi}) \in Z^{s_1} \times X^{s_1}$ and estimate (25) holds (we again use here Lemma 3.1 and, in particular, the continuous embedding $X^{s_1} \hookrightarrow C^0(\overline{Q})$ if $s_1 > N/2 + 1$).

Step 3. Notice that the functions U and Φ introduced in (30) satisfy

$$U = \Phi = 0 \text{ on } \Sigma, \quad U(x,0) = \Phi(x,0) = U(x,T) = \Phi(x,T) = 0 \text{ in } \Omega.$$

In particular, we use that $\operatorname{supp}(\nabla\theta)$, $\operatorname{supp}(\Delta\theta) \subset\subset \Omega$, $\hat{\Phi} \in C([0,T];H_0^1(\Omega))$, $\hat{U} \in C([0,T];L^2(\Omega))$, and

$$\hat{U} = \hat{\Phi} = 0 \text{ on } \Sigma, \quad \hat{U}(x,0) = \hat{\Phi}(x,0) = \hat{U}(x,T) = \hat{\Phi}(x,T) = 0 \text{ in } \Omega.$$

It is already a simple matter to check that the control function v given by (31) (together with (U,Φ)) solves (28). Hence, v (together with $u = U + \eta\overline{u}$ and $\phi = \Phi + \eta\overline{\phi}$) gives the null controllability of system (16). This completes the proof of Theorem 4.3. □

Remark 4.1 It is worthy of remark that the regularity of the control v (resp. estimate (26)) is obtained independently of the regularity of u and ϕ (resp. estimate (25)). Indeed, the regularity of v only depends on the local parabolic regularizing effect (thus on the regularity of the term ∇c), while the regularity of (u,ϕ) just depends on the regularity of the initial condition (u_0,ϕ_0).

5 Proof of the null controllability of a nonlinear phase field system

This Section is devoted to proving Theorem 2.1. The proof will be divided into two steps. Firstly, we will prove the result in the case when g_i and G_i $(i = 1,2)$ are continuous functions and $h \in C^2(\mathbb{R})$, by applying an appropriate fixed-point argument. A suitable regularizing argument will give us the result in the general case.

5.1 The case when the functions g_i, G_i $(i = 1,2)$, and h'' are continuous

Assume that the hypothesis in the statement hold and that g_i, G_i $(i = 1,2)$, and h'' are continuous functions. Let $H : \mathbb{R} \to \mathbb{R}$ be the C^1-function defined by

$$H(\sigma) = \begin{cases} \dfrac{h(\sigma)}{\sigma} & \text{if } \sigma \neq 0, \\ h'(0) & \text{if } \sigma = 0. \end{cases}$$

Then, for each $\varepsilon > 0$, there exists a positive constant C_ε (which only depends on ε, f, and h) such that

$$\begin{aligned} |g_1(s,p,\sigma,\pi)|^{2/3} + |G_1(s,p,\sigma,\pi)|^2 + |g_2(s,p,\sigma,\pi)|^{1/2} + |G_2(s,p,\sigma,\pi)| \\ + |H(\sigma)|^{2/3} \leq C_\varepsilon + \varepsilon \log(1 + |s| + |\sigma|), \end{aligned} \tag{33}$$

for any $(s,p,\sigma,\pi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$. Let $s_1 \in (N/2+1, \infty)$ be given and set

$$q = \begin{cases} 2 & \text{if } \ N = 1, \\ q \in (2, s_1) & \text{if } \ N = 2, \\ \dfrac{N}{2} + 1 & \text{if } \ N \geq 3, \end{cases} \qquad \mathbf{T}_R(s) = \begin{cases} s & \text{si } |s| \leq R, \\ R\,\mathrm{sgn}(s) & \text{si } |s| > R, \end{cases} \tag{34}$$

with $R > 0$ to be determined later.

For any (z,ζ) in $L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$, we consider the linear coupled system (for simplicity, we omit the dependence on R):

$$\begin{cases} \partial_t u - \Delta u + B_{z,\zeta} \cdot \nabla u + a_{z,\zeta} u + F_{z,\zeta} \cdot \nabla \phi + e_{z,\zeta} \phi = -\Delta \phi + v \mathbf{1}_\omega \quad \text{in } Q, \\ \partial_t \phi - \Delta \phi + c_\zeta \phi = u \quad \text{in } Q, \\ u = 0, \ \ \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x), \ \ \phi(x,0) = \phi_0(x) \quad \text{in } \Omega, \end{cases} \tag{35}$$

with potentials given by

$$\begin{aligned} a_{z,\zeta} &= g_1(\mathbf{T}_R(z), \nabla z, \mathbf{T}_R(\zeta), \nabla \zeta), \quad & e_{z,\zeta} &= g_2(\mathbf{T}_R(z), \nabla z, \mathbf{T}_R(\zeta), \nabla \zeta), \\ B_{z,\zeta} &= G_1(\mathbf{T}_R(z), \nabla z, \mathbf{T}_R(\zeta), \nabla \zeta), \quad & F_{z,\zeta} &= G_2(\mathbf{T}_R(z), \nabla z, \mathbf{T}_R(\zeta), \nabla \zeta), \end{aligned}$$

and $c_\zeta = H(\mathbf{T}_R(\zeta))$, which satisfy

$$a_{z,\zeta}, e_{z,\zeta} \in L^\infty(Q), \ B_{z,\zeta}, F_{z,\zeta} \in L^\infty(Q)^N, \text{ and } c_\zeta \in L^\infty(Q) \cap L^q(0,T;W_0^{1,q}(\Omega)).$$

Moreover, for any $(z,\zeta) \in L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$, one can see that

$$\begin{aligned} \|a_{z,\zeta}\|_\infty \leq \alpha_{1,R}, \quad \|e_{z,\zeta}\|_\infty \leq \alpha_{2,R}, \quad \|B_{z,\zeta}\|_\infty \leq \beta_{1,R}, \\ \|F_{z,\zeta}\|_\infty \leq \beta_{2,R}, \ \text{ and } \ \|c_\zeta\|_\infty \leq \kappa_R, \end{aligned} \tag{36}$$

where

$$\alpha_{i,R} = \sup_{\substack{|s|,|\sigma| \leq R \\ p,\pi \in \mathbb{R}^N}} |g_i(s,p,\sigma,\pi)|, \quad \beta_{i,R} = \sup_{\substack{|s|,|\sigma| \leq R \\ p,\pi \in \mathbb{R}^N}} |G_i(s,p,\sigma,\pi)|, \quad i = 1,2, \tag{37}$$

and

$$\kappa_R = \max_{|\sigma| \le R} |H(\sigma)|. \tag{38}$$

In particular, we have used hypothesis i) in the statement and the continuity of H', together with Stampacchia's Theorem (see Theorem A.4.2., p. 256 of [16]) and the fact that $\mathbf{T}_R$ is a globally Lipschitz-continuous function.

Let us now reason as in [3] (also see [11]). Let us associate, to each pair (z, ζ) in $L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$, a family of controls that lead system (35) to zero at time T. The idea is to applying Theorem 4.3 to (35) in an appropriate time interval (eventually smaller than $[0,T]$). To be precise, let us set

$$T_R = \min\left\{T, \alpha_{1,R}^{-1/3}, \alpha_{2,R}^{-1/2}, \kappa_R^{-1/3}, \beta_{2,R}^{-1}\right\}.$$

For any $(z, \zeta) \in L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$, one applies Theorem 4.3 in the time interval $(0, T_R)$, with $s_1 > N/2 + 1$ and $r = 2$ (observe that, in this case, γ given by (24) is equal to q). If $u_0, \phi_0 \in W^{2-2/s_1, s_1}(\Omega) \cap H_0^1(\Omega)$, one deduces the existence of a control $v_{z,\zeta} \in L^2(\Omega \times (0, T_R))$ such that the associated solution, $(u_{z,\zeta}, \phi_{z,\zeta})$, to (35) in the cylinder $\Omega \times (0, T_R)$ satisfies

$$u_{z,\zeta} \in L^{s_1}(0, T_R; W_0^{1,s_1}(\Omega)) \cap C^0(\overline{\Omega} \times [0, T_R]), \quad \phi_{z,\zeta} \in X^{s_1}(0, T_R; \Omega),$$

and

$$u_{z,\zeta}(x, T_R) = 0, \quad \phi_{z,\zeta}(x, T_R) = 0 \quad \text{in } \Omega.$$

Moreover, the following estimates hold

$$\begin{gathered} \|u_{z,\zeta}\|_{L^{s_1}(0,T_R;W_0^{1,s_1}(\Omega)) \cap C^0(\overline{\Omega} \times [0,T_R])} + \|\phi_{z,\zeta}\|_{X^{s_1}(0,T_R;\Omega)} \\ \le C_1(\Omega, \omega, f, h, T_R, z, \zeta) \|(u_0, \phi_0)\|_{W^{2-2/s_1, s_1}(\Omega)^2}, \end{gathered} \tag{39}$$

$$\|v_{z,\zeta}\|_{L^2(\Omega \times (0,T_R))} \le C_2(\Omega, \omega, f, h, T_R, z, \zeta) \|(u_0, \phi_0)\|_{L^2(\Omega)}, \tag{40}$$

where

$$C_1 \equiv \exp[C(\Omega, \omega)\, \mathcal{H}_{R,z,\zeta}],$$

$$C_2 \equiv C_1(\Omega, \omega, f, h, T_R, z, \zeta)(1 + \|H'(\mathbf{T}_R(\zeta)) \nabla(\mathbf{T}_R(\zeta))\|_{L^q(Q)}),$$

and $\mathcal{H}_{R,z,\zeta} > 0$ is given by

$$\begin{aligned} \mathcal{H}_{R,z,\zeta} &= 1 + \frac{1}{T_R} + \|a_{z,\zeta}\|_\infty^{2/3} + \|c_\zeta\|_\infty^{2/3} + \|e_{z,\zeta}\|_\infty^{1/3} + \|B_{z,\zeta}\|_\infty^2 + \|F_{z,\zeta}\|_\infty^{1/2} \\ &+ T_R \left(1 + \|a_{z,\zeta}\|_\infty + \|c_\zeta\|_\infty + \|e_{z,\zeta}\|_\infty + \|B_{z,\zeta}\|_\infty^2 + \|F_{z,\zeta}\|_\infty^2\right). \end{aligned}$$

We now extend the functions $v_{z,\zeta}$, $u_{z,\zeta}$, and $\phi_{z,\zeta}$ by zero to the whole cylinder Q. Denote such extensions by $\widetilde{v}_{z,\zeta}$, $\widetilde{u}_{z,\zeta}$, and $\widetilde{\phi}_{z,\zeta}$, respectively. Then, $(\widetilde{u}_{z,\zeta}, \widetilde{\phi}_{z,\zeta})$ lies in $Z^{s_1} \times X^{s_1}$ (here $Z^{s_1} = L^{s_1}(0,T;W_0^{1,s_1}(\Omega)) \cap C^0(\overline{Q})$, since $s_1 > N/2 + 1$), solves the linearized system (35) in Q with control term $v = \widetilde{v}_{z,\zeta} \in L^2(Q)$, and satisfies

$$\widetilde{u}_{z,\zeta}(x, T) = 0, \quad \widetilde{\phi}_{z,\zeta}(x, T) = 0 \quad \text{in } \Omega.$$

Furthermore, by recalling the definitions of $\mathcal{H}_{R,z,\zeta}$ and T_R and (36), from estimates (39) and (40), one infers

$$\|\widetilde{u}_{z,\zeta}\|_{Z^{s_1}} + \|\widetilde{\phi}_{z,\zeta}\|_{X^{s_1}} \leq C_3(\Omega, \omega, f, h, T, R)\|(u_0, \phi_0)\|_{W^{2-2/s_1,s_1}(\Omega)^2}, \tag{41}$$

$$\|\widetilde{v}_{z,\zeta}\|_{L^2(Q)} \leq C_4(\Omega, \omega, f, h, T, R, \zeta)\|(u_0, \phi_0)\|_{L^2(\Omega)^2}, \tag{42}$$

with

$$C_3(\Omega, \omega, f, h, T, R) = \exp\left[C(\Omega, \omega, T)\left(1 + \alpha_{1,R}^{2/3} + \alpha_{2,R}^{1/2} + \kappa_R^{2/3} + \beta_{1,R}^2 + \beta_{2,R}\right)\right],$$

$$C_4(\Omega, \omega, f, h, T, R, \zeta) = C_3(\Omega, \omega, f, h, T, R)\left(1 + \|\nabla\zeta\|_{L^q(Q)} \max_{|\sigma|\leq R} |H'(\sigma)|\right),$$

for a new positive constant C which now depends on Ω, ω, and also on T.

For a fixed control $v \in L^2(Q)$, we now denote by (u_v, ϕ_v) the solution to (35) associated to v and the potentials $a_{z,\zeta}$, c_ζ, $e_{z,\zeta}$, $B_{z,\zeta}$, and $F_{z,\zeta}$ (we have omitted the dependence on (z,ζ) to simplify the notation). For any $(z,\zeta) \in L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$, one defines the family of controls

$$\mathcal{A}_R(z,\zeta) = \{v \in L^2(Q) : (u_v, \phi_v) \in Z^{s_1} \times X^{s_1},\ u_v(x,T) = \phi_v(x,T) = 0 \text{ in } \Omega, \\ \text{and } v \text{ satisfies (42)}\}.$$

Thus, one can introduce the multi-valued mapping

$$\Lambda_R : (z,\zeta) \in Y := L^2(0,T;H_0^1(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega)) \longmapsto \Lambda_R(z,\zeta) \subset Y,$$

where $\Lambda_R(z,\zeta)$ is the family of functions (u_v, ϕ_v) in $Z^{s_1} \times X^{s_1}$ such that $v \in \mathcal{A}_R(z,\zeta)$ and (u_v, ϕ_v) satisfies (41).

It can be seen that, for any $R > 0$, the Kakutani fixed-point Theorem can be applied to Λ_R, thus ensuring the existence of (at least) one fixed point of Λ_R in Y. First, it is easy to check that $\Lambda_R(z,\zeta)$ is a nonempty closed convex subset of Y, for any $(z,\zeta) \in Y$. Moreover, for any $(z,\zeta) \in Y$, each $(u,\phi) \in \Lambda_R(z,\zeta)$ satisfies the uniform estimate (41). Hence, Λ_R maps the whole space Y in a bounded subset of Y. On the other hand, let $\mathcal{C} \subset Y$ be a bounded set. From estimate (42), $\mathcal{A}(\mathcal{C}) = \bigcup\{\mathcal{A}(z,\zeta) : (z,\zeta) \in \mathcal{C}\}$ is uniformly bounded in $L^2(Q)$. By using Proposition 3.1 (with $r = 2$) and estimate (41), $\Lambda_R(\mathcal{C})$ is a bounded set in $X^2 \times X^{s_1}$, whence each $\Lambda_R(z,\zeta)$ is compact in Y and $\Lambda_R(\mathcal{C})$ is relatively compact in Y (we use here the compact embeddings $X^2 \rightrightarrows L^2(0,T;H_0^1(\Omega))$ and $X^{s_1} \rightrightarrows L^q(0,T;W_0^{1,q}(\Omega))$). Finally, one is able to prove that Λ_R is upper hemicontinuous, that is, for any linear and continuous form μ defined on Y, the real-valued function

$$(z,\zeta) \in Y \longmapsto \sup_{(u,\phi)\in\Lambda_R(z,\zeta)} \langle \mu, (u,\phi)\rangle$$

is upper semicontinuous (see [9] and [11], for instance, for similar proofs). Thus, the Kakutani fixed-point Theorem can be applied to Λ_R and one infers the existence of (at least) a fixed point (u_R, ϕ_R) of Λ_R in Y.

To conclude the proof in this case, it is sufficient to find $R > 0$ such that $\mathbf{T}_R(u_R) = u_R$ and $\mathbf{T}_R(\phi_R) = \phi_R$. Let us see that there exists $R > 0$ (large enough) such that

$$\|u_R\|_\infty \leq R, \quad \|\phi_R\|_\infty \leq R. \tag{43}$$

Indeed, it will be seen that any fixed point of Λ_R satisfies (43). Let (u,ϕ) be a fixed point of Λ_R. From (41), (37), (38) and (33), it is deduced that

$$\begin{aligned}\|u\|_\infty + \|\phi\|_\infty &\leq \exp[C\,(1 + C_\varepsilon + \varepsilon \log(1+R))]\,\|(u_0,\phi_0)\|_{W^{2-2/s_1,s_1}(\Omega)^2}\\ &= \exp\left[C(1+C_\varepsilon)\right](1+R)^{C\varepsilon}\|(u_0,\phi_0)\|_{W^{2-2/s_1,s_1}(\Omega)^2},\end{aligned}$$

with $C = C(\Omega,\omega,T) > 0$ (observe that, for $s_1 > N/2+1$, $X^{s_1} \hookrightarrow L^\infty(Q)$, with continuous embedding). Then, by taking (for instance) $\varepsilon = (2C)^{-1}$, one infers that

$$\|u\|_\infty + \|\phi\|_\infty \leq C(\Omega,\omega,T)(1+R)^{1/2}\|(u_0,\phi_0)\|_{W^{2-2/s_1,s_1}(\Omega)^2},$$

whence $\|u\|_\infty + \|\phi\|_\infty \leq R$, for $R > 0$ large enough. The proof of Theorem 2.1 is thus complete when g_i, G_i $(i = 1,2)$ are continuous and $h \in C^2(\mathbb{R})$.

5.2 The general case

Assume the hypothesis in the statement. We consider two functions $\rho \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ and $\widetilde{\rho} \in \mathcal{D}(\mathbb{R})$ such that $\rho \geq 0$ in $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$, $\widetilde{\rho} \geq 0$ in $\mathbb{R}$, $\operatorname{supp}\rho \subset \overline{B}((0,0,0,0);1)$, $\operatorname{supp}\widetilde{\rho} \subset [-1,1]$, and

$$\iiiint_{\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N} \rho(s,p,\sigma,\pi)\,ds\,dp\,d\sigma d\pi = \int_{\mathbb{R}} \widetilde{\rho}(\sigma)\,d\sigma = 1.$$

For every $n \geq 1$, we introduce the following functions

$$\rho_n(s,p,\sigma,\pi) = n^{2N+2}\rho(ns,np,n\sigma,n\pi) \quad \forall (s,p,\sigma,\pi) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N,$$

$$\widetilde{\rho}_n(\sigma) = n\widetilde{\rho}(n\sigma) \quad \forall \sigma \in \mathbb{R},$$

$$g_{1,n} = \rho_n * g_1,\ G_{1,n} = \rho_n * G_1,\ g_{2,n} = \rho_n * g_2,\ G_{2,n} = \rho_n * G_2,\ H_n = \widetilde{\rho}_n * H,$$

with H defined in page 22. Finally, for any $n \geq 1$, we set

$$\begin{aligned}f_n(s,p,\sigma,\pi) &= g_{1,n}(s,p,\sigma,\pi)s + G_{1,n}(s,p,\sigma,\pi)\cdot p\\ &\quad + g_{2,n}(s,p,\sigma,\pi)\sigma + G_{2,n}(s,p,\sigma,\pi)\cdot\pi\end{aligned}$$

for every $(s,p,\sigma,\pi) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$ and

$$h_n(\sigma) = H_n(\sigma)\sigma \quad \forall \sigma \in \mathbb{R}.$$

The functions that we have just introduced satisfy the following properties:

i) $g_{i,n} \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$, $G_{i,n} \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)^N$ $(i = 1,2)$, $h_n \in C^2(\mathbb{R})$ (indeed, they are C^∞-functions), $f_n(0,0,0,0) = 0$, and $h_n(0) = 0$, for any $n \geq 1$.

ii) $f_n \to f$ uniformly in the compact sets of $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.

iii) $H_n \to H$ uniformly in the compact sets of $\mathbb{R}$.

iv) For any $M > 0$, there exists $C(M) > 0$ such that, for every $n \geq 1$, it holds that

$$\sup_{\substack{|s|,|\sigma|\leq M\\ p,\pi\in\mathbb{R}^N}} (|g_{i,n}(s,p,\sigma,\pi)| + |G_{i,n}(s,p,\sigma,\pi)|) \leq C(M), \quad i = 1,2,$$

$$\sup_{|\sigma|\leq M} (|H_n(\sigma)| + |H_n'(\sigma)|) \leq C(M).$$

v) The functions $g_{i,n}$, $G_{i,n}$ $(i = 1,2)$, and h_n satisfy hypothesis ii) in Theorem 2.1 uniformly in n, that is to say, for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that, whenever $|s|, |\sigma| > M_\varepsilon$, $p, \pi \in \mathbb{R}^N$, and $n \geq 1$, one has

$$\begin{aligned} &|g_{1,n}(s,p,\sigma,\pi)|^{2/3} + |G_{1,n}(s,p,\sigma,\pi)|^2 + |g_{2,n}(s,p,\sigma,\pi)|^{1/2} \\ &\quad + |G_{2,n}(s,p,\sigma,\pi)| + |H_n(\sigma)|^{2/3} \leq \varepsilon \log(1 + |s| + |\sigma|). \end{aligned}$$

For any $n \geq 1$, we consider the linear system

$$\begin{cases} \partial_t u_n - \Delta u_n + f_n(u_n, \nabla u_n, \phi_n, \nabla\phi_n) = -\Delta\phi_n + v_n \mathbf{1}_\omega & \text{in } Q, \\ \partial_t \phi_n - \Delta\phi_n + H_n(\phi_n)\phi_n = u_n & \text{in } Q, \\ u_n = 0, \;\; \phi_n = 0 & \text{on } \Sigma, \\ u_n(x,0) = u_0(x), \;\; \phi_n(x,0) = \phi_0(x) & \text{in } \Omega, \end{cases} \tag{44}$$

where $u_0, \phi_0 \in W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega)$, with $s_1 > N/2 + 1$. As a consequence of the properties above, by proceeding as in Subsection 5.1, for any $n \geq 1$ there exists a control $v_n \in L^2(Q)$ such that system (44) admits at least one solution $(u_n, \phi_n) \in Z^{s_1} \times X^{s_1}$ satisfying

$$u_n(x,T) = 0, \quad \phi_n(x,T) = 0 \quad \text{in } \Omega. \tag{45}$$

Moreover, $\{v_n\}_{n\geq 1}$ (resp. (u_n, ϕ_n)) is uniformly bounded in $L^2(Q)$ (resp. in $Z^{s_1} \times X^{s_1}$). From Proposition 3.1 and the estimates obtained in Subsection 5.1, there exist subsequences (still denoted by $\{v_n\}_{n\geq 1}$ and $\{(u_n, \phi_n)\}_{n\geq 1}$) such that $v_n \rightharpoonup v$ weakly in $L^2(Q)$ and $(u_n, \phi_n) \to (u, \phi)$ strongly in $L^2(0,T; H_0^1(\Omega)) \times L^q(0,T; W_0^{1,q}(\Omega))$, for some $v \in L^2(Q)$ and $(u, \phi) \in Z^{s_1} \times X^{s_1}$. Finally, one is able to pass to the limit in (44) and (45) to infer that v (together with (u, ϕ)) gives the null controllability of system (1), which proves Theorem 2.1.

Remark 5.1 *By inspection of the proof of Theorem 2.1, it is observed that the same result remains valid for more general nonlinear terms such as* $f(x,t; u(x,t), \nabla u(x,t), \phi(x,t), \nabla\phi(x,t))$ *and* $h(x,t; \phi(x,t))$, *with* $(x,t) \in Q$. *It is not difficult to see that the result holds if* f *and* h *satisfy the following properties:*

1. $f(x,t; 0,0,0,0) = 0$ *and* $h(x,t;0) = 0$ *for* (x,t) *a.e. in* Q.

2. $h(x,t;\cdot) \in C^1(\mathbb{R})$ *and* $\dfrac{\partial^2 h}{\partial \sigma^2}(x,t;\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ *for* (x,t) *a.e. in* Q.

3. f *can be written as*

$$f(\cdot;s,p,\sigma,\pi) = g_1(\cdot;s,p,\sigma,\pi)s + G_1(\cdot;s,p,\sigma,\pi)\cdot p + g_2(\cdot;s,p,\sigma,\pi)\sigma + G_2(\cdot;s,p,\sigma,\pi)\cdot\pi$$

for any (s,p,σ,π) *in* $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$, *with*

$$\lim_{|s|,|\sigma|\to\infty} \frac{|g_1(x,t;s,p,\sigma,\pi)|}{\log^{3/2}(1+|s|+|\sigma|)} = 0, \qquad \lim_{|s|,|\sigma|\to\infty} \frac{|G_1(x,t;s,p,\sigma,\pi)|}{\log^{1/2}(1+|s|+|\sigma|)} = 0,$$

$$\lim_{|s|,|\sigma|\to\infty} \frac{|g_2(x,t;s,p,\sigma,\pi)|}{\log^{2}(1+|s|+|\sigma|)} = 0, \text{ and } \quad \lim_{|s|,|\sigma|\to\infty} \frac{|G_2(x,t;s,p,\sigma,\pi)|}{\log(1+|s|+|\sigma|)} = 0$$

uniformly in $(p,\pi)\in\mathbb{R}^N\times\mathbb{R}^N$ *and in* (x,t) *a.e. in* Q.

4.
$$\lim_{|\sigma|\to\infty} \frac{|h(x,t;\sigma)|}{|\sigma|\log^{3/2}(1+|\sigma|)} = 0 \quad \text{uniformly in } (x,t) \text{ a.e. in } Q.$$

5. *For any* $R>0$, *there exists* $M_R>0$ *such that*

$$|g_1(x,t;s,p,\sigma,\pi)| + |G_1(x,t;s,p,\sigma,\pi)| + |g_2(x,t;s,p,\sigma,\pi)| + |G_2(x,t;s,p,\sigma,\pi)| + \left|\frac{\partial h}{\partial\sigma}(x,t;\sigma)\right| + \left|\frac{\partial^2 h}{\partial\sigma^2}(x,t;\sigma)\right| \le M_R,$$

for any $s,\sigma\in[-R,R]$, $p,\pi\in\mathbb{R}^N$, *and* (x,t) *a.e. in* Q.

6. $h(\cdot;\sigma) = H(\cdot;\sigma)\sigma$ *for all* $\sigma\in\mathbb{R}$, *where* H *has the property that for any* $R>0$ *there exists* $w_R\in L^q(Q)$ *(*q *given in* (34)*) such that*

$$\left|\frac{\partial H}{\partial x_i}(x,t;\sigma)\right| \le w_R(x,t),\ 1\le i\le N, \quad \forall\sigma\in[-R,R],\ (x,t) \text{ a.e. in } Q.$$

We will omit the proof in this case.

6 Proof of the exact controllability to the trajectories

In this Section, we prove Theorem 2.2. We proceed as follows. Let us set $w = u - u^*$ and $q = \phi - \phi^*$, with (u^*,ϕ^*) as in the statement. Observe that (u,ϕ) solves (1) with control term v if and only if (w,q) satisfies

$$\begin{cases} \partial_t w - \Delta w + \widetilde{f}(x,t;w,\nabla w,q,\nabla q) = -\Delta q + \nu\mathbf{1}_\omega & \text{in } Q, \\ \partial_t q - \Delta q + \widetilde{h}(x,t;q) = w & \text{in } Q, \\ w = 0,\ \ q = 0 \quad \text{on } \Sigma, \quad w(0) = u_0 - u_0^*,\ \ \phi(0) = \phi_0 - \phi_0^* & \text{in } \Omega, \end{cases} \tag{46}$$

where $\nu = v - v^*$ and $\widetilde{f}$, $\widetilde{h}$ are given by

$$\begin{aligned}\widetilde{f}(x,t;s,p,\sigma,\pi) &= f(u^*(x,t)+s, \nabla u^*(x,t)+p, \phi^*(x,t)+\sigma, \nabla\phi^*(x,t)+\pi)\\ &\quad - f(u^*(x,t), \nabla u^*(x,t), \phi^*(x,t), \nabla\phi^*(x,t)),\\ \widetilde{h}(x,t;\sigma) &= h(\phi^*(x,t)+\sigma) - h(\phi^*(x,t)),\end{aligned}$$

for any $(x,t) \in Q$, $s, \sigma \in \mathbb{R}$, and $p, \pi \in \mathbb{R}^N$. Thus, the proof of Theorem 2.2 is reduced to proving the existence of a control $\nu \in L^2(Q)$ such that system (46) possesses a solution $(w,q) \in L^\infty(Q)^2$ satisfying

$$w(x,T) = 0, \quad q(x,T) = 0 \quad \text{in } \Omega.$$

So as to ensure the existence of such a control, it suffices to check that $\widetilde{f}$ and $\widetilde{h}$ satisfy the properties in Remark 5.1. Notice that $\widetilde{f}$ can be written as

$$\begin{aligned}\widetilde{f}(x,t;s,p,\sigma,\pi) &= g_1(u^*(x,t), \nabla u^*(x,t), \phi^*(x,t), \nabla\phi^*(x,t); s,p,\sigma,\pi)s\\ &\quad + G_1(u^*(x,t), \nabla u^*(x,t), \phi^*(x,t), \nabla\phi^*(x,t); s,p,\sigma,\pi)\cdot p\\ &\quad + g_2(u^*(x,t), \nabla u^*(x,t), \phi^*(x,t), \nabla\phi^*(x,t); s,p,\sigma,\pi)\sigma\\ &\quad + G_2(u^*(x,t), \nabla u^*(x,t), \phi^*(x,t), \nabla\phi^*(x,t); s,p,\sigma,\pi)\cdot \pi\end{aligned} \tag{47}$$

for any $(x,t) \in Q$, $s, \sigma \in \mathbb{R}$, and $p, \pi \in \mathbb{R}^N$, with g_i and G_i $(i = 1,2)$ defined in page 7, and

$$\widetilde{h}(x,t;\sigma) = \widetilde{H}(x,t;\sigma)\sigma \quad \text{for any } (x,t) \in Q \text{ and } \sigma \in \mathbb{R},$$

where

$$\widetilde{H}(x,t;\sigma) = \int_0^1 h'(\phi^*(x,t)+\lambda\sigma)\,d\lambda.$$

Now, by taking into account the regularity of h and (u^*, ϕ^*), the expressions above and hypothesis (8), (9), and (10) (also see Remark 2.2), it is an easy exercise to check that $\widetilde{f}$ and $\widetilde{h}$ satisfy the conditions in Remark 5.1, which ends the proof.

7 Proof of the approximate controllability result

The goal of this Section is to prove Theorem 2.3. Let f and h be as in the statement. Let $T > 0$ and $u_0, \phi_0 \in W^{2-2/s_1, s_1}(\Omega) \cap H_0^1(\Omega)$ (with $s_1 > N/2 + 1$) be given. Observe that we only need to prove the result for final data $u_d, \phi_d \in W^{2-2/s_1, s_1}(\Omega) \cap H_0^1(\Omega)$, since this space is dense in $L^2(\Omega)$. The proof will be divided into some steps.

Firstly, there exists $\delta_0 > 0$ (that just depends on Ω, u_d, ϕ_d, f, and h) such that the system

$$\begin{cases} \partial_t w - \Delta w + f(w, \nabla w, q, \nabla q) = -\Delta q \quad \text{in } \Omega \times (0, \delta_0),\\ \partial_t q - \Delta q + h(q) = w \quad \text{in } \Omega \times (0, \delta_0),\\ w = 0, \;\; q = 0 \quad \text{on } \partial\Omega \times (0, \delta_0), \quad w(x,0) = u_d(x), \;\; q(x,0) = \phi_d(x) \quad \text{in } \Omega, \end{cases}$$

possesses a solution $(w,q) \in L^\infty(\Omega \times (0,\delta_0))^2$ satisfying

$$w(\cdot,t),\ q(\cdot,t) \in W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega) \quad \forall t \in [0,\delta_0].$$

For a given $\varepsilon > 0$, one can choose $\delta_1 \in (0,\delta_0]$ small enough so that

$$\|w(\cdot,t) - u_d\|_{L^2(\Omega)} \le \varepsilon, \quad \|q(\cdot,t) - \phi_d\|_{L^2(\Omega)} \le \varepsilon \quad \forall t \in [0,\delta_1]. \tag{48}$$

Secondly, for a given δ_1 for which (48) is satisfied, let us fix $m \in \boldsymbol{N}$ such that $\delta := T/m \le \delta_1$. In view of Theorem 2.2, there exists $v_1 \in L^2(\Omega \times (0,\delta))$ such that the system

$$\begin{cases} \partial_t u - \Delta u + f(u, \nabla u, \phi, \nabla\phi) = -\Delta\phi + v_1 \mathbf{1}_\omega \quad \text{in } \Omega \times (0,\delta), \\ \partial_t \phi - \Delta\phi + h(\phi) = u \quad \text{in } \Omega \times (0,\delta), \\ u = 0,\ \ \phi = 0 \quad \text{on } \partial\Omega \times (0,\delta), \quad u(x,0) = u_0(x),\ \ \phi(x,0) = \phi_0(x) \quad \text{in } \Omega, \end{cases}$$

admits a solution $(u_1,\phi_1) \in L^\infty(\Omega \times (0,\delta))^2$ such that

$$u_1(x,\delta) = w(x,\delta), \quad \phi_1(x,\delta) = q(x,\delta) \ \text{ in } \Omega.$$

In the third place, again from Theorem 2.2, there exists a control $\widetilde{v} \in L^2(\Omega \times (0,\delta))$ such that the system

$$\begin{cases} \partial_t u - \Delta u + f(u, \nabla u, \phi, \nabla\phi) = -\Delta\phi + \widetilde{v} \mathbf{1}_\omega \text{ in } \Omega \times (0,\delta), \\ \partial_t \phi - \Delta\phi + h(\phi) = u \text{ in } \Omega \times (0,\delta), \\ u = 0, \quad \phi = 0 \text{ on } \partial\Omega \times (0,\delta), \quad u(x,0) = w(x,\delta), \quad \phi(x,0) = q(x,\delta) \text{ in } \Omega, \end{cases}$$

has a solution $(\widetilde{u},\widetilde{\phi}) \in L^\infty(\Omega \times (0,\delta))^2$ that satisfies

$$\widetilde{u}(x,\delta) = w(x,\delta), \quad \widetilde{\phi}(x,\delta) = q(x,\delta) \quad \text{in } \Omega.$$

The required control v is now constructed as follows. Let us set $I_k = [(k-1)\delta, k\delta]$, for $1 \le k \le m$. For (x,t) a.e. in Q, we define

$$v(x,t) = \begin{cases} v_1(x,t) & \text{if } (x,t) \in \Omega \times I_1, \\ \widetilde{v}(x, t-(k-1)\delta) & \text{if } (x,t) \in \Omega \times I_k, \quad 2 \le k \le m. \end{cases}$$

By the construction above, the control v lies in $L^2(Q)$ and the corresponding system (1) admits a solution $(u,\phi) \in L^\infty(Q)^2$ such that

$$u(x,T) = w(x,\delta), \quad \phi(x,T) = q(x,\delta) \quad \text{in } \Omega.$$

Finally, by recalling (48) (and that $\delta \le \delta_1$), (11) holds, which was our claim.

8 Proof of Theorem 2.4

In this Section, we prove the Carleman inequality stated in Theorem 2.4. The structure of the proof is similar to that of Lemma 1 in [17] (also see [18]). Here, we adapt the method exhibited in [17] to the lack of regularity in the terms $-\nabla \cdot (B\psi)$ and $-\nabla \cdot (F\psi)$ in (7).

For a given $\mathcal{B}_0$ as in the statement, let us consider an auxiliary nonempty open set $\mathcal{B}_1$ such that $\mathcal{B}_1 \subset\subset \mathcal{B}_0$. Let (φ, ψ) be the solution to (7) associated to an arbitrary $(\varphi^0, \psi^0) \in L^2(\Omega)^2$. Firstly, by applying Proposition 4.1 (with $\mathcal{B} = \mathcal{B}_1$), there exist two positive constants $C_1 = C_1(\Omega, \mathcal{B}_0)$ and $\sigma_1 = \sigma_1(\Omega, \mathcal{B}_0)$ such that

$$\begin{aligned}
&s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1}|\nabla\varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \\
&+ s^4 \iint_Q e^{-2s\alpha} t^{-4}(T-t)^{-4}|\nabla\psi|^2 + s^6 \iint_Q e^{-2s\alpha} t^{-6}(T-t)^{-6}|\psi|^2 \\
&\qquad \leq C_1 \left(s^3 \iint_{\mathcal{B}_1\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \right. \\
&\qquad\qquad \left. + s^6 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha} t^{-6}(T-t)^{-6}|\psi|^2 \right),
\end{aligned} \tag{49}$$

for any $s \geq s_1$, with

$$s_1 = \sigma_1(\Omega, \mathcal{B}_0)\left(T + T^2\left(1 + \|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|e\|_\infty^{1/3} + \|B\|_\infty^2 + \|F\|_\infty^{1/2}\right)\right).$$

In (49), α_0 is the function associated to $\mathcal{B} = \mathcal{B}_1$ provided by Proposition 4.1 and $\alpha(x,t) = \dfrac{\alpha_0(x)}{t(T-t)}$, $x \in \Omega$, $t \in (0,T)$.

Let us now consider a function $\xi_1 \in C_0^\infty(\Omega)$ satisfying

$$0 \leq \xi_1 \leq 1 \text{ in } \Omega, \quad \xi_1 \equiv 1 \text{ in } \mathcal{B}_1, \quad \operatorname{supp}\xi_1 \subset \mathcal{B}_0, \tag{50}$$

$$\Delta\xi_1/\xi_1^{1/2} \in L^\infty(\Omega), \quad \text{and} \quad \nabla\xi_1/\xi_1^{1/2} \in L^\infty(\Omega)^N. \tag{51}$$

This is achieved by setting $\xi_1 = \zeta^4$, with $\zeta \in C_0^\infty(\Omega)$ satisfying (50). Let $s \geq s_1$ be, with s_1 as above. We set $u = e^{-2s\alpha}s^3t^{-3}(T-t)^{-3}$. Multiply by $\varphi\xi_1 u$ the PDE satisfied by ψ and integrate in Q. We get

$$\begin{aligned}
s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 &= \iint_Q \left[-\partial_t\psi - \Delta\psi - \nabla\cdot(B\psi) + a\psi\right]\varphi\xi_1 u \\
= \iint_Q \varphi\psi\xi_1\partial_t u &+ \iint_Q \psi\xi_1 u[-\Delta\varphi + c\varphi + \Delta\psi - \nabla\cdot(F\psi) + e\psi] \\
+ \iint_Q (\nabla\psi\cdot\nabla(\xi_1 u))\varphi &+ \iint_Q (\nabla\psi\cdot\nabla\varphi)\xi_1 u + \iint_Q ((B\psi)\cdot\nabla\varphi)\xi_1 u \\
&+ \iint_Q ((B\psi)\cdot\nabla(\xi_1 u))\varphi + \iint_Q a\varphi\psi\xi_1 u,
\end{aligned}$$

whence (just by integrating by parts)

$$
\begin{aligned}
s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \xi_1 &= \iint_Q (a+c)\varphi\psi\xi_1 u + \iint_Q (B\cdot\nabla\varphi)\psi\xi_1 u \\
&+ \iint_Q \varphi\psi\big[\xi_1\partial_t u - \Delta(\xi_1 u) + B\cdot\nabla(\xi_1 u)\big] + 2\iint_Q (\nabla\psi\cdot\nabla\varphi)\xi_1 u \\
&- \iint_Q (\nabla(\xi_1 u)\cdot\nabla\psi)\psi + \iint_Q ((F\psi)\cdot\nabla\psi)\xi_1 u + \iint_Q ((F\psi)\cdot\nabla(\xi_1 u))\psi \\
&+ \iint_Q e\xi_1 u|\psi|^2 - \iint_Q |\nabla\psi|^2 \xi_1 u := \sum_{k=1}^{11} J_k.
\end{aligned} \tag{52}
$$

Let us estimate each J_k, $1 \le k \le 10$ (notice that $J_{11} = -\iint_Q |\nabla\psi|^2\xi_1 u \le 0$). By using Hölder and Young inequalities, it holds that

$$
J_1 = \iint_Q (a+c)\varphi\psi\xi_1 u \le \delta_1 \iint_Q \xi_1 u|\varphi|^2 + \frac{1}{4\delta_1}\|a+c\|_\infty^2 \iint_Q \xi_1 u|\psi|^2,
$$

$$
\begin{aligned}
J_2 = \iint_Q (B\cdot\nabla\varphi)\,\psi\xi_1 u &\le \gamma_1 s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1}|\nabla\varphi|^2\xi_1 \\
&+ \frac{1}{4\gamma_1}\|B\|_\infty^2 s^5 \iint_Q e^{-2s\alpha} t^{-5}(T-t)^{-5}|\psi|^2\xi_1,
\end{aligned}
$$

for any $\delta_1, \gamma_1 > 0$ to be determined later. Observe that

$$
|\partial_t u| \le T s^3 e^{-2s\alpha} t^{-5}(T-t)^{-5}\left(Cs + 3T^2/4\right) \le CTs^4 e^{-2s\alpha} t^{-5}(T-t)^{-5},
$$

since $s \ge \sigma_1(\Omega, \mathcal{B}_0)T^2$. Thus, we can estimate

$$
\begin{aligned}
J_3 = \iint_Q \varphi\psi\xi_1\partial_t u &\le CTs^4 \iint_Q e^{-2s\alpha} t^{-5}(T-t)^{-5}|\varphi||\psi|\xi_1 \\
&\le \delta_2 s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 \\
&+ \frac{CT^2}{\delta_2} s^5 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2\xi_1 \\
&\le \delta_2 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_2} s^7 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2\xi_1
\end{aligned} \tag{53}
$$

for any $\delta_2 > 0$, since $s \ge \sigma_1(\Omega, \mathcal{B}_0)T$.

In order to estimate $J_4 = -\iint_Q \varphi\psi\Delta(\xi_1 u)$, notice that

$$
\Delta(\xi_1 u) = s^3 t^{-3}(T-t)^{-3}\left((\Delta\xi_1)e^{-2s\alpha} + 2\nabla\xi_1\cdot\nabla(e^{-2s\alpha}) + \xi_1\Delta(e^{-2s\alpha})\right),
$$

with

$$
|\nabla(e^{-2s\alpha})| = 2se^{-2s\alpha} t^{-1}(T-t)^{-1}|\nabla\alpha_0| \le Cse^{-2s\alpha} t^{-1}(T-t)^{-1}, \tag{54}
$$

$$
\begin{aligned}
|\Delta(e^{-2s\alpha})| &\le 2se^{-2s\alpha}t^{-2}(T-t)^{-2}\left(2s|\nabla\alpha_0|^2 + t(T-t)|\Delta\alpha_0|\right)\\
&\le Cse^{-2s\alpha}t^{-2}(T-t)^{-2}(s+T^2) \le Cs^2e^{-2s\alpha}t^{-2}(T-t)^{-2}.
\end{aligned}
$$

These considerations together with (51) give

$$
\begin{aligned}
J_4 \le C\Bigg(& s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi||\psi|\xi_1^{1/2}\\
&+ s^4 \iint_Q e^{-2s\alpha}t^{-4}(T-t)^{-4}|\varphi||\psi|\xi_1^{1/2} + s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\varphi||\psi|\xi_1\Bigg).
\end{aligned}
$$

We now use Hölder and Young inequalities and (50) to get

$$
\begin{aligned}
J_4 \le \delta_3 &\iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_3}s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2\mathbf{1}_{\mathcal{B}_0}\\
&+ \frac{C}{\delta_3}s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{\mathcal{B}_0} + \frac{C}{\delta_3}s^7 \iint_Q e^{-2s\alpha}t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{\mathcal{B}_0},
\end{aligned}
$$

with $\delta_3 > 0$ to be fixed later. Notice that, for any $n, m \in \mathbb{N}$ with $n \ge m$, we have

$$
\begin{aligned}
s^m t^{-m}(T-t)^{-m} &= s^m t^{-n}(T-t)^{-n}\left(t(T-t)\right)^{n-m}\\
&\le s^m t^{-n}(T-t)^{-n}\left(\frac{T^2}{4}\right)^{n-m} \le Cs^n t^{-n}(T-t)^{-n},
\end{aligned}
\tag{55}
$$

since $s \ge \sigma_1(\Omega, \mathcal{B}_0)T^2$. Then

$$
J_4 \le \delta_3 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_3}s^7 \iint_Q e^{-2s\alpha}t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{\mathcal{B}_0}.
$$

In a similar way, we can estimate

$$
\begin{aligned}
J_5 &= \iint_Q \varphi\psi\left(B\cdot\nabla(\xi_1 u)\right)\\
&\le \delta_4 \iint_Q \xi_1 u|\varphi|^2 + \frac{C}{\delta_4}\|B\|_\infty^2 s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{\mathcal{B}_0},
\end{aligned}
$$

$$
\begin{aligned}
J_6 &= 2\iint_Q (\nabla\psi\cdot\nabla\varphi)\xi_1 u\\
&\le \gamma_2 s \iint_Q e^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\varphi|^2\xi_1 + \frac{1}{\gamma_2}s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\nabla\psi|^2\xi_1,
\end{aligned}
$$

for any $\delta_4, \gamma_2 > 0$. On the other hand, we have

$$
\begin{aligned}
J_7 &= -\iint_Q (\nabla(\xi_1 u)\cdot\nabla\psi)\psi = -\frac{1}{2}\iint_Q \nabla(\xi_1 u)\cdot\nabla|\psi|^2 = \frac{1}{2}\iint_Q \Delta(\xi_1 u)|\psi|^2\\
&\le Cs^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2\xi_1^{1/2}\left[1 + s(t(T-t))^{-1} + s^2(t(T-t))^{-2}\xi_1^{1/2}\right]\\
&\le Cs^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{\mathcal{B}_0}.
\end{aligned}
$$

Finally, we can bound

$$J_8 = \iint_Q ((F\psi)\cdot\nabla\psi)\xi_1 u \le \delta_5 s^4 \iint_Q e^{-2s\alpha} t^{-4}(T-t)^{-4}|\nabla\psi|^2\xi_1 + \frac{1}{4\delta_5}\|F\|_\infty^2 s^2 \iint_Q e^{-2s\alpha}t^{-2}(T-t)^{-2}|\psi|^2\xi_1$$

for any $\delta_5 > 0$,

$$J_9 = \iint_Q ((F\psi)\cdot\nabla(\xi_1 u))\psi \le C\|F\|_\infty s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2\xi_1^{1/2}[1+st^{-1}(T-t)^{-1}\xi_1^{1/2}],$$

and

$$J_{10} = \iint_Q e\xi_1 u|\psi|^2 \le \|e\|_\infty s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2\xi_1.$$

Let us now deal with the terms in which $|\nabla\psi|^2$ appears. As it will be seen later, the constants γ_2 and δ_5 only depend on Ω and $\mathcal{B}_0$. We use this fact to deduce that, for a new constant $C = C(\Omega,\mathcal{B}_0) > 0$, it holds that

$$\frac{1}{\gamma_2}s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\nabla\psi|^2\xi_1 + \delta_5 s^4 \iint_Q e^{-2s\alpha}t^{-4}(T-t)^{-4}|\nabla\psi|^2\xi_1 \le \frac{2}{\gamma_2}\iint_Q |\nabla\psi|^2\xi_1\overline{u} \tag{56}$$

for any $s \ge CT^2$, where $\overline{u} = e^{-2s\alpha}s^5t^{-5}(T-t)^{-5}$. Thus, to estimate J_6 and J_8, it suffices to bound the right-hand side of (56). To this end, we multiply by $\psi\xi_1\overline{u}$ the PDE satisfied by ψ and integrate in Q, obtaining

$$\begin{aligned}\iint_Q \varphi\psi\xi_1\overline{u} &= \iint_Q [-\partial_t\psi - \Delta\psi - \nabla\cdot(B\psi) + a\psi]\,\psi\xi_1\overline{u}\\ &= \frac{1}{2}\iint_Q |\psi|^2\xi_1\partial_t\overline{u} + \iint_Q |\nabla\psi|^2\xi_1\overline{u} - \frac{1}{2}\iint_Q |\psi|^2\Delta(\xi_1\overline{u})\\ &+ \iint_Q a|\psi|^2\xi_1\overline{u} + \iint_Q (B\cdot\nabla\psi)\psi\xi_1\overline{u} + \iint_Q (B\cdot\nabla(\xi_1\overline{u}))|\psi|^2.\end{aligned}$$

Since

$$-\iint_Q (B\cdot\nabla\psi)\psi\xi_1\overline{u} \le \frac{1}{2}\iint_Q |\nabla\psi|^2\xi_1\overline{u} + \frac{1}{2}\|B\|_\infty^2 \iint_Q |\psi|^2\xi_1\overline{u},$$

the term $\iint_Q |\nabla\psi|^2\xi_1\overline{u}$ is bounded as follows:

$$\begin{aligned}\iint_Q |\nabla\psi|^2\xi_1\overline{u} &\le 2\iint_Q |\varphi||\psi|\xi_1\overline{u} + \iint_Q |\psi|^2\xi_1|\partial_t\overline{u}| + \iint_Q |\psi|^2|\Delta(\xi_1\overline{u})|\\ &+ 2\|B\|_\infty \iint_Q |\psi|^2|\nabla(\xi_1\overline{u})| + (2\|a\|_\infty + \|B\|_\infty^2)\iint_Q |\psi|^2\xi_1\overline{u} := \sum_{i=1}^5 A_i.\end{aligned} \tag{57}$$

We now estimate A_i, $1 \le i \le 4$. Firstly,

$$A_1 = 2 \iint_Q |\varphi||\psi|\xi_1 \overline{u} \le \delta \iint_Q \xi_1 u |\varphi|^2 + \frac{1}{\delta} s^7 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2 \mathbf{1}_{\mathcal{B}_0}$$

for any $\delta > 0$ (to be chosen further). Observe that

$$|\partial_t \overline{u}| \le T s^5 e^{-2s\alpha} t^{-7}(T-t)^{-7}\left(Cs + 5T^2/4\right) \le CTs^6 e^{-2s\alpha} t^{-7}(T-t)^{-7},$$

since $s \ge \sigma_1(\Omega, \mathcal{B}_0)T^2$. Then

$$A_2 = \iint_Q |\psi|^2 \xi_1 |\partial_t \overline{u}| \le Cs^7 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2 \mathbf{1}_{\mathcal{B}_0},$$

since $s \ge \sigma_1(\Omega, \mathcal{B}_0)T$. To estimate $A_3 = \iint_Q |\psi|^2 |\Delta(\xi_1 \overline{u})|$, notice that

$$\Delta(\xi_1 \overline{u}) = s^5 t^{-5}(T-t)^{-5}\left((\Delta\xi_1)e^{-2s\alpha} + 2\nabla\xi_1 \cdot \nabla(e^{-2s\alpha}) + \xi_1 \Delta(e^{-2s\alpha})\right),$$

which (together with (51), (54), and (55)) yields

$$A_3 \le Cs^7 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2 \mathbf{1}_{\mathcal{B}_0}.$$

In a similar way, we estimate

$$A_4 = 2\|B\|_\infty \iint_Q |\psi|^2 |\nabla(\xi_1 \overline{u})| \le C\|B\|_\infty s^6 \iint_Q e^{-2s\alpha} t^{-6}(T-t)^{-6}|\psi|^2 \mathbf{1}_{\mathcal{B}_0}$$

(here, we use that $\nabla(\xi_1 \overline{u}) = (\nabla\xi_1)\overline{u} - 2\xi_1 e^{-2s\alpha} s^6 t^{-6}(T-t)^{-6}\nabla\alpha_0$, together with (51) and (55)). By combining the above-obtained estimates for A_i, $1 \le i \le 4$, from (57) we then deduce

$$\begin{aligned} \iint_Q |\nabla\psi|^2 \xi_1 \overline{u} &\le \delta \iint_Q \xi_1 u |\varphi|^2 \\ &+ \left(\frac{1}{\delta} + C\right) s^7 \iint_Q e^{-2s\alpha} t^{-7}(T-t)^{-7}|\psi|^2 \mathbf{1}_{\mathcal{B}_0} \\ &+ C\|B\|_\infty s^6 \iint_Q e^{-2s\alpha} t^{-6}(T-t)^{-6}|\psi|^2 \mathbf{1}_{\mathcal{B}_0} \\ &+ (2\|a\|_\infty + \|B\|_\infty^2) s^5 \iint_Q e^{-2s\alpha} t^{-5}(T-t)^{-5}|\psi|^2 \mathbf{1}_{\mathcal{B}_0}, \end{aligned} \tag{58}$$

which is valid for any $s \ge C(T + T^2)$ and $\delta > 0$.

We now take the corresponding estimates of J_k ($1 \le k \le 10$) to (52). By using (56), (58), and (50), we get

$$\iint_Q \xi_1 u|\varphi|^2 \le \left(\sum_{i=1}^4 \delta_i + \frac{2\delta}{\gamma_2}\right) \iint_Q \xi_1 u|\varphi|^2$$
$$+(\gamma_1+\gamma_2)s \iint_Q e^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\varphi|^2\xi_1$$
$$+\frac{1}{4\delta_5}\|F\|_\infty^2 s^2 \iint_Q e^{-2s\alpha}t^{-2}(T-t)^{-2}|\psi|^2\xi_1$$
$$+\left(\frac{1}{4\delta_1}\|a+c\|_\infty^2 + \|e\|_\infty + C\|F\|_\infty\right) s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2\mathbf{1}_{\mathcal{B}_0}$$
$$+C\|F\|_\infty s^4 \iint_Q e^{-2s\alpha}t^{-4}(T-t)^{-4}|\psi|^2\xi_1$$
$$+\left(C+\frac{4}{\gamma_2}\|a\|_\infty + \left(\frac{1}{4\gamma_1}+\frac{2}{\gamma_2}+\frac{C}{\delta_4}\right)\|B\|_\infty^2\right) s^5 \iint_Q e^{-2s\alpha}t^{-5}(T-t)^{-5}|\psi|^2\mathbf{1}_{\mathcal{B}_0}$$
$$+\frac{2C}{\gamma_2}\|B\|_\infty s^6 \iint_Q e^{-2s\alpha}t^{-6}(T-t)^{-6}|\psi|^2\mathbf{1}_{\mathcal{B}_0}$$
$$+\left(\frac{C}{\delta_2}+\frac{C}{\delta_3}+\frac{2}{\gamma_2\delta}+\frac{2C}{\gamma_2}\right) s^7 \iint_Q e^{-2s\alpha}t^{-7}(T-t)^{-7}|\psi|^2\mathbf{1}_{\mathcal{B}_0},$$

for any $s \ge s_2 = \sigma_2(\Omega,\mathcal{B}_0)(T+T^2)$. We now set $\delta_i = 1/10$ (for $1 \le i \le 4$ and, for instance, $\delta = \delta_5 = 1$), $\gamma_1 = \gamma_2 = 1/(8C_1)$, with $C_1 = C_1(\Omega,\mathcal{B}_0)$ as in (49), and $\delta = \gamma_2/20$. For any $s \ge s_2$, it holds that

$$s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 \le \frac{1}{2C_1}s \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\varphi|^2$$
$$+C\|F\|_\infty^2 s^2 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-2}(T-t)^{-2}|\psi|^2$$
$$+C\left(\|a+c\|_\infty^2 + \|e\|_\infty + \|F\|_\infty\right) s^3 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-3}(T-t)^{-3}|\psi|^2$$
$$+C\|F\|_\infty s^4 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-4}(T-t)^{-4}|\psi|^2$$
$$+C\left(1+\|a\|_\infty + \|B\|_\infty^2\right) s^5 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-5}(T-t)^{-5}|\psi|^2$$
$$+C\|B\|_\infty s^6 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-6}(T-t)^{-6}|\psi|^2$$
$$+Cs^7 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-7}(T-t)^{-7}|\psi|^2,$$

whence it is immediately inferred that

$$s^3 \iint_Q e^{-2s\alpha}t^{-3}(T-t)^{-3}|\varphi|^2\xi_1 \le \frac{1}{2C_1}s \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-1}(T-t)^{-1}|\nabla\varphi|^2$$
$$+Cs^7 \iint_{\mathcal{B}_0\times(0,T)} e^{-2s\alpha}t^{-7}(T-t)^{-7}|\psi|^2$$

for any $s \geq s_3 = \sigma_3(\Omega, \mathcal{B}_0)\left(T + T^2 M'\right)$, with

$$M' = 1 + \|a + c\|_\infty^{1/2} + \|a\|_\infty^{1/2} + \|e\|_\infty^{1/4} + \|B\|_\infty + \|F\|_\infty^{1/4} + \|F\|_\infty^{1/3} + \|F\|_\infty^{2/5}.$$

Finally, by combining the previous estimate with the Carleman inequality (49), we infer the desired Carleman estimate (12), valid for any $s \geq \hat{s} = \hat{\sigma}(T + T^2 M)$, with $\hat{\sigma} = \hat{\sigma}(\Omega, \mathcal{B}_0)$ and $M > 0$ as in the statement.

9 Further results and comments

1. The proof of Theorems 2.1 and 2.2 can be adapted to give other controllability results for system (1):

a) The null controllability property of system (1) remains valid if hypothesis ii) in Theorem 2.1 is replaced by this other slightly weaker one:

$$\begin{cases} \limsup\limits_{|s|+|\sigma|\to\infty} \dfrac{|g_1(s,p,\sigma,\pi)|}{\log^{3/2}(1+|s|+|\sigma|)} \leq l_1, \quad \limsup\limits_{|s|+|\sigma|\to\infty} \dfrac{|G_1(s,p,\sigma,\pi)|}{\log^{1/2}(1+|s|+|\sigma|)} \leq l_2, \\ \limsup\limits_{|s|+|\sigma|\to\infty} \dfrac{|g_2(s,p,\sigma,\pi)|}{\log^{2}(1+|s|+|\sigma|)} \leq l_3, \quad \text{and} \quad \limsup\limits_{|s|+|\sigma|\to\infty} \dfrac{|G_2(s,p,\sigma,\pi)|}{\log(1+|s|+|\sigma|)} \leq l_4 \\ \text{uniformly in } p, \pi \in \mathbb{R}^N; \end{cases}$$

$$\limsup_{|\sigma|\to\infty} \frac{|h(\sigma)|}{|\sigma| \log^{3/2}(1+|\sigma|)} \leq l_5,$$

where l_i, $1 \leq i \leq 5$, are positive constants small enough that only depend on Ω, ω, and T.

b) A local null controllability result for system (1) under no restrictions on the growth of f nor h can also be obtained. More precisely, if $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz-continuous function such that $f(0,0,0,0) = 0$ and $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function, with $h'' \in L^\infty_{\text{loc}}(\mathbb{R})$ and $h(0) = 0$, there exists a positive $\rho = \rho(\Omega, \omega, T, f, h)$ with the following property: *" For any initial data (u_0, ϕ_0) in $(W^{2-2/s_1,s_1}(\Omega) \cap H_0^1(\Omega))^2$ ($s_1 > N/2 + 1$) satisfying $\|u_0\|_{W^{2-2/s_1,s_1}(\Omega)} + \|\phi_0\|_{W^{2-2/s_1,s_1}(\Omega)} \leq \rho$, one is able to find a control $v \in L^2(Q)$ such that system (1) admits a unique solution $(u, \phi) \in \left(L^{s_1}(0,T; W_0^{1,s_1}(\Omega)) \cap C^0(\overline{Q})\right) \times X^{s_1}$ satisfying $u(x,T) = 0$ and $\phi(x,T) = 0$ in Ω".* This result generalizes the main result in [10], which establishes the local null controllability, by two control functions, of a nonlinear phase field system such as (1), when $f \equiv 0$ and Ω is a bounded domain in $\mathbb{R}^N$, with $1 \leq N \leq 3$.

c) By adapting the proof of Theorem 2.2, a result on the local exact controllability to the trajectories for system (1) is immediately deduced.

2. The strategy developed in this paper to deal with the null controllability of linear coupled parabolic systems by a single control force (by introducing a

fictitious control) can be applied in the case of cascade systems such as (for instance):

$$\begin{cases} \partial_t u - \nu_1 \Delta u + B \cdot \nabla u + au + F \cdot \nabla \phi + e\phi = v \mathbf{1}_\omega \quad \text{in } Q, \\ \partial_t \phi - \nu_2 \Delta \phi + D \cdot \nabla \phi + c\phi = gu \quad \text{in } Q, \\ u = 0, \;\; \phi = 0 \quad \text{on } \Sigma, \quad u(x,0) = u_0(x), \;\; \phi(x,0) = \phi_0(x) \quad \text{in } \Omega, \end{cases}$$

when regular potentials are considered and there exist a constant $g_0 > 0$ and an open set ω_0 such that

$$\omega_0 \subset\subset \omega, \qquad |g| \geq g_0 > 0 \text{ in } \omega_0 \times (0, T_0) \;\text{ for some } T_0 > 0.$$

(in particular, when the system has constant potentials, with $g \neq 0$). Observe that different diffusion coefficients $\nu_1, \nu_2 > 0$ can be considered. Indeed, in such a case, a result similar to Proposition 4.1 can be obtained for the corresponding adjoint system, with constants C and $\overline{\sigma}$ that depend on $\mathcal{B}$, Ω, and ν_i, $i = 1, 2$. This yields the existence of two L^2-controls that give the null controllability of the system, and our technique allows us to get rid of the fictitious control.

We would like to remark that all the known null controllability results for coupled systems by one control force are proved when cascade systems are considered (cf. [4], [9], [19], [20],...). The case when the system is not written in a cascade form, which is a much more complicated situation, cannot be dealt with our technique and is at present open.

3. The above-mentioned strategy can also be applied to control to zero a linear heat-wave cascade system such as

$$\begin{cases} \partial_t y - \Delta y = v\zeta \quad \text{in } Q, \\ \partial_{tt}^2 q - \Delta q = y \mathbf{1}_{\mathcal{O}} \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \;\; y(x,0) = y_0(x) \quad \text{in } \Omega, \\ q = 0 \quad \text{on } \Sigma, \;\; q(x,0) = q_0(x), \;\; \partial_t q(x,0) = q_1(x) \quad \text{in } \Omega, \end{cases} \tag{59}$$

where v is a control function supported in an arbitrarily small open control set $\omega \subset \Omega$, $\mathcal{O} \subset \Omega$ is an open set such that $\omega \cap \mathcal{O} \neq \emptyset$, and $\zeta \in \mathcal{D}(\omega)$ is a function such that $\zeta \equiv 1$ in a nonempty open set $\mathcal{B}_0 \subset\subset \omega \cap \mathcal{O}$. More precisely, suppose that $\hat{v}_1$ and $\hat{v}_2$ are two control functions supported in $\overline{\mathcal{B}}_0 \times [0, T]$ that give the null controllability of the heat-wave cascade system ($\hat{v}_2$ being a fictitious distributed control introduced in the wave equation). Observe that the existence of such controls $\hat{v}_1$ and $\hat{v}_2$ is guaranteed if certain geometrical condition on $\Omega \cap \mathcal{O}$ is satisfied (see, for instance, [19] and the references therein). By adapting our strategy to the present situation, one is able to eliminate $\hat{v}_2$ and to construct a control $v \in L^2(0, T; D(-\Delta)')$ that drives the cascade system (59) to zero. It is worthy of mention that the irregularity of this control v is not related to the technique used to obtain it but to the fact that we are dealing with the wave equation, as already observed in [20].

4. Our strategy cannot be applied to infer the null controllability of (1) when nonlinearities $h(\phi, \nabla \phi)$ with certain superlinear growth at infinity are

considered. Indeed, we ought to be able to construct a control v that solves the linear null controllability problem

$$\begin{cases} \partial_t U - \Delta U + B \cdot \nabla U + aU + F \cdot \nabla \Phi + e\Phi = -\Delta \Phi - \eta' \overline{u} + v \mathbf{1}_\omega & \text{in } Q, \\ \partial_t \Phi - \Delta \Phi + D \cdot \nabla \Phi + c\Phi = U - \eta' \overline{\phi} & \text{in } Q, \\ U = 0, \quad \Phi = 0 & \text{on } \Sigma, \\ U(x,0) = 0, \quad \Phi(x,0) = 0, \quad U(x,T) = 0, \quad \Phi(x,T) = 0 & \text{in } \Omega, \end{cases}$$

with associated (U, Φ) in $L^\infty(Q)^2$. Here, $(\overline{u}, \overline{\phi})$ is now the weak solution to

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + B \cdot \nabla \overline{u} + a\overline{u} + F \cdot \nabla \overline{\phi} + e\overline{\phi} = -\Delta \overline{\phi} & \text{in } Q, \\ \partial_t \overline{\phi} - \Delta \overline{\phi} + D \cdot \nabla \overline{\phi} + c\overline{\phi} = \overline{u} & \text{in } Q, \\ \overline{u} = 0, \quad \overline{\phi} = 0 \ \text{ on } \Sigma, \quad \overline{u}(x,0) = u_0(x), \quad \overline{\phi}(x,0) = \phi_0(x) & \text{in } \Omega. \end{cases}$$

Let $\mathcal{B}_0 \subset \omega$ be a nonempty open set. Suppose that we have already obtained two controls $\hat{v}_1, \hat{v}_2 \in L^2(Q)$, with $\operatorname{sop} \hat{v}_1, \operatorname{sop} \hat{v}_2 \subset \overline{\mathcal{B}}_0 \times [0,T]$, that give the null controllability of the linear coupled system

$$\begin{cases} \partial_t \hat{U} - \Delta \hat{U} + B \cdot \nabla \hat{U} + a\hat{U} + F \cdot \nabla \hat{\Phi} + e\hat{\Phi} = -\Delta \hat{\Phi} - \eta' \overline{u} + \hat{v}_1 \mathbf{1}_{\mathcal{B}_0} & \text{in } Q, \\ \partial_t \hat{\Phi} - \Delta \hat{\Phi} + D \cdot \nabla \hat{\Phi} + c\hat{\Phi} = \hat{U} - \eta' \overline{\phi} + \hat{v}_2 \mathbf{1}_{\mathcal{B}_0} & \text{in } Q, \\ \hat{U} = 0, \quad \hat{\Phi} = 0 \quad \text{on } \Sigma, \quad \hat{U}(x,0) = 0, \quad \hat{\Phi}(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{60}$$

The expression of a new control obtained (from $\hat{v}_1$, $\hat{v}_2$, and the corresponding solution $(\hat{U}, \hat{\Phi})$ to (60)) by means of our strategy would be:

$$\begin{gathered} v = \theta \eta' \overline{u} - 2\nabla\theta \cdot \nabla \hat{\Phi} - (\Delta\theta)\hat{\Phi} + 2\nabla\theta \cdot \nabla \hat{U} + (\Delta\theta)\hat{U} - \nabla\theta \cdot (B\hat{U}) \\ -\nabla\theta \cdot (F\hat{\Phi}) + (\partial_t - \Delta + B \cdot \nabla + a)[\theta\eta'\overline{\phi} + 2\nabla\theta \cdot \nabla\hat{\Phi} + (\Delta\theta)\hat{\Phi} - \nabla\theta \cdot (D\hat{\Phi})], \end{gathered}$$

where $\theta \in \mathcal{D}(\omega)$ satisfies $\theta \equiv 1$ in a neighborhood of $\mathcal{B}_0$. Nevertheless, observe that, if $D \in L^\infty(Q)^N$, some terms in this formula are not regular enough to make the state (U, Φ) lie in a suitable space to apply an appropriate fixed-point argument. Thus, our technique cannot be applied in this case.

5. *Controllability results for some coupled systems of m parabolic PDEs by one control force.* In [21], the authors are able to control to zero some cascade systems of m linear parabolic PDEs by a single distributed control. The crucial point in the referenced work is a Carleman inequality for the solutions (φ_i, $1 \le i \le m$) to the corresponding adjoint system by means of which some global terms of these solutions are bounded in terms of only one of them "localized" in a nonempty open subset $\mathcal{B}_0$ of the control set ω. The proof of such a Carleman estimate is a generalization of that of Theorem 2.4. An appropriate fixed-point argument enables M. González-Burgos and L. de Teresa to show the null controllability of some cascade systems of m nonlinear parabolic PDEs by one control force when certain superlinear nonlinearities are considered. Both the exact controllability to the trajectories and the approximate controllability for certain superlinear cascade systems can also be obtained.

6. *Other comments.* All along this paper, other kind of boundary conditions, such as Fourier (or Robin) boundary conditions, could have also been considered. In such a case, one is able to obtain a Carleman estimate analogous to inequality (21) (with the same weight functions) for the solutions to the corresponding adjoint system. The existence of two L^2-controls that give the null controllability of the system is then guaranteed. Finally, our strategy, being local in time and space, enables one to remove the second control.

A null controllability result for system (1) analogous to Theorem 2.1, as well as the exact controllability to the trajectories and the approximate controllability under slightly different hypothesis, can also be obtained for an unbounded domain Ω such that $\Omega \setminus \overline{\omega}$ is bounded (cf. [22]).

In view of known controllability results for a semilinear heat equation, it would be natural to wonder wether the main results in this paper remain valid when one considers boundary controls. Nevertheless, there exist negative results for some 1-d linear coupled parabolic systems (cf. [23]), which reveals the different nature of the controllability properties for a single heat equation and for coupled parabolic systems.

A Proof of Proposition 3.1

The proof of the first point of Proposition 3.1 and that of the second one for $r = 2$ uses the Galerkin method and the energy estimates of the corresponding approximate solutions, and being a standard proof, it will be omitted. We sketch here the proof of the second point for $r > 2$, which combines Theorem 2.3 in [24] with an argument of 'bootstrap' type. We will restrict our attention to the case when $N > 2$, the discussion being similar but more direct when $N = 1$ or 2.

Firstly, the weak solution (u, ϕ) to (13) lies in $X^2 \times X^2$ and satisfies the estimate

$$\|(u, \phi)\|_{X^2 \times X^2} \leq \exp(CH_1) H_2^{\mathcal{K}} \left(\|(\nabla u_0, \nabla \phi_0)\|_{L^2(\Omega)^2} + \|(g_1, g_2)\|_{L^2(Q)^2} \right),$$

with $H_1, H_2 > 0$ as in the statement (recall that $\mathcal{K}$ stands for a generic positive constant that only depends on N, whose value may also change from one line to another). Notice that ϕ solves

$$\begin{cases} \partial_t \phi - \Delta \phi = f_2 \quad \text{in } Q, \\ \phi = 0 \quad \text{on } \Sigma, \quad \phi(x, 0) = \phi_0(x) \quad \text{in } \Omega, \end{cases}$$

with $f_2 = -D \cdot \nabla \phi - c\phi + u + g_2$. From usual Sobolev embeddings, we have

$$D \cdot \nabla \phi \in L^2(0, T; L^{2^\star}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),$$

with $\dfrac{1}{2^\star} = \dfrac{1}{2} - \dfrac{1}{N}$ (recall that $N > 2$), whence f_2 lies in $L^r(0, T; L^{p_0}(\Omega))$, with $p_0 = \min\{r, 2Nr/(Nr - 4)\}$, and

$$\|f_2\|_{L^r(L^{p_0}(\Omega))} \leq C \left(1 + \|c\|_\infty + \|D\|_\infty\right) \left(\|(u, \phi)\|_{X^2 \times X^2} + \|g_2\|_{L^r(Q)} \right), \quad (61)$$

by classical interpolation estimates. Then, in view of Theorem 2.3 in [24], it holds that

$$\begin{gathered}\phi \in L^r(0,T;W^{2,p_0}(\Omega)), \quad \partial_t\phi \in L^r(0,T;L^{p_0}(\Omega)),\\ \|\phi\|_{L^r(W^{2,p_0}(\Omega))} + \|\partial_t\phi\|_{L^r(L^{p_0}(\Omega))} \le C\left(\|f_2\|_{L^r(L^{p_0}(\Omega))} + \|\phi_0\|_{W^{2-2/r,r}(\Omega)}\right),\end{gathered}$$

where C is a positive constant independent of T. By combining the previous estimate with (61), we get

$$\begin{aligned}\|\phi\|_{L^r(W^{2,p_0}(\Omega))} + \|\partial_t\phi\|_{L^r(L^{p_0}(\Omega))} &\le C\left(1+\|c\|_\infty+\|D\|_\infty\right)\\ &\quad\times\left(\|(u,\phi)\|_{X^2\times X^2} + \|g_2\|_{L^r(Q)} + \|\phi_0\|_{W^{2-2/r,r}(\Omega)}\right).\end{aligned} \tag{62}$$

On the other hand, u solves

$$\begin{cases}\partial_t u - \Delta u = f_1 & \text{in } Q,\\ u = 0 \quad \text{on } \Sigma, \quad u(x,0)=u_0(x) & \text{in } \Omega,\end{cases}$$

where $f_1 = -B\cdot\nabla u - au - \Delta\phi - F\cdot\nabla\phi - e\phi + g_1$. Reasoning as above and using (62), one deduces that $f_1 \in L^r(0,T;L^{p_0}(\Omega))$ and

$$\|f_1\|_{L^r(L^{p_0}(\Omega))} \le CH_2\left(\|(u,\phi)\|_{X^2\times X^2} + \|(g_1,g_2)\|_{L^r(Q)^2} + \|\phi_0\|_{W^{2-2/r,r}(\Omega)}\right),$$

where $H_2 = 1+\|a\|_\infty+\|c\|_\infty+\|e\|_\infty+\|B\|_\infty+\|D\|_\infty+\|F\|_\infty$. We now apply the above-mentioned Theorem 2.3 in [24] to the function u, inferring

$$\begin{gathered}u \in L^r(0,T;W^{2,p_0}(\Omega)), \quad \partial_t u \in L^r(0,T;L^{p_0}(\Omega)),\\ \|u\|_{L^r(W^{2,p_0}(\Omega))} + \|\partial_t u\|_{L^r(L^{p_0}(\Omega))} \le C\left(\|f_1\|_{L^r(L^{p_0}(\Omega))} + \|u_0\|_{W^{2-2/r,r}(\Omega)}\right)\end{gathered}$$

for a new positive constant C independent of T, whence

$$\begin{gathered}\|u\|_{L^r(W^{2,p_0}(\Omega))} + \|\partial_t u\|_{L^r(L^{p_0}(\Omega))}\\ \le CH_2\left(\|(u,\phi)\|_{X^2\times X^2} + \|(u_0,\phi_0)\|_{W^{2-2/r,r}(\Omega)^2} + \|(g_1,g_2)\|_{L^r(Q)^2}\right),\end{gathered}$$

by using the estimate of $\|f_1\|_{L^r(L^{p_0}(\Omega))}$ obtained above.

Finally, one can apply an argument of "bootstrap" type which yields the result in a finite number of steps.

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