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006

A RESULT CONCERNING CONTROLLABILITY FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. The main goal of this paper is to present a new result concerning controllability of the time-dependent Navier–Stokes equations. Here, the control variable is the trace of the velocity field on a "small" part of the boundary. The main result states that the linear space spanned by final states is dense in the L^2 space of admissible fields. For the proof, one uses a duality argument that is suggested by the linear theory. This reduces the task to an existence/regularity result for a nonlinear problem.

Key words. approximate controllability, Navier–Stokes equations, nonlinear parabolic partial differential equations

AMS subject classifications. 93C20, 93B05, 76D05, 35D05

1. Statement of the problem: The main result. In what follows it will be assumed that $\Omega \subset \mathbb{R}^N$ is a bounded open set ($N = 2$ or 3) whose boundary $\partial\Omega$ is of class $C^{1,1}$. We denote by γ a component of $\partial\Omega$ and we assume that $\partial\Omega\backslash\gamma$ has positive measure. We consider the following spaces:

$$\tilde{\mathcal{V}}(\Omega) = \{\mathbf{v}; \mathbf{v} \in \mathcal{D}(\bar{\Omega})^N,\ \nabla\cdot\mathbf{v} = 0,\ \operatorname{Supp}\mathbf{v} \subset \Omega\cup\gamma\},$$

$$\tilde{H}(\Omega) = \text{the closure of } \tilde{\mathcal{V}}(\Omega) \text{ in the space } L^2(\Omega)^N,$$

$$\tilde{V}(\Omega) = \text{the closure of } \tilde{\mathcal{V}}(\Omega) \text{ in the space } H^1(\Omega)^N.$$

Obviously, $\tilde{V}(\Omega)$ and $\tilde{H}(\Omega)$ are Hilbert spaces for the usual scalar products in $H^1(\Omega)^N$ and $L^2(\Omega)^N$, respectively. Furthermore, in $\tilde{V}(\Omega)$, the seminorm

$$\mathbf{u} \to \|\nabla\mathbf{u}\|_{L^2}$$

is in fact a norm, equivalent to the norm in $H^1(\Omega)^N$. For simplicity, we put $\tilde{V}$ and $\tilde{H}$ instead of $\tilde{V}(\Omega)$ and $\tilde{H}(\Omega)$, resp.

Let $T > 0$ be given. Consider the following Navier–Stokes problem in $Q_T = \Omega\times(0,T)$, where we impose nonzero Dirichlet data:

$$(1)\qquad \begin{cases} \dfrac{\partial\mathbf{y}}{\partial t} + (\mathbf{y}\cdot\nabla)\mathbf{y} - \nu\Delta\mathbf{y} + \nabla\pi = 0, & \nabla\cdot\mathbf{y} = 0 \text{ in } Q_T,\\ \mathbf{y} = \mathbf{v} & \text{on } \Lambda_T = \gamma\times(0,T),\\ \mathbf{y} = 0 & \text{on } S_T = (\partial\Omega\backslash\gamma)\times(0,T),\\ \mathbf{y}(0) = 0 & \text{in } \Omega. \end{cases}$$

Here, ν is the kinematic viscosity ($\nu > 0$) and $\mathbf{v} \in L^2(0,T;H^{-1/2}(\gamma)^N)$.

THEOREM 1.1. (a) *Assume* $\mathbf{v} = \mathbf{curl}\,\zeta\,|_\gamma$, *with*

$$(2)\qquad \begin{array}{ll} \boldsymbol{\zeta} \in L^2(0,T;H^2(\Omega)^M), & \dfrac{\partial\boldsymbol{\zeta}}{\partial t} \in L^2(0,T;H^1(\Omega)^M),\\ \boldsymbol{\zeta} \in L^\infty(0,T;W^{1,p}(\Omega)^3) & \textit{for some } p > 3 \textit{ if } N = 3,\\ \boldsymbol{\zeta} = \zeta \in L^\infty(0,T;W^{1,p}(\Omega)) & \textit{for some } p > 2 \textit{ if } N = 2,\\ \mathbf{v}(0)\cdot\mathbf{n} = 0 & \textit{in } H^{-1/2}(\gamma)^N \end{array}$$

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(here, $\mathbf{n}$ *is the unit outward normal vector on* $\partial\Omega$*;* $M = 1$ *if* $N = 2$ *and* $M = 3$ *if* $N = 3$*). Then,* (1) *possesses at least one weak solution* $(\mathbf{y}_\mathbf{v}, \pi_\mathbf{v})$*. One has*

$$\mathbf{y}_\mathbf{v} \in L^2(0,T;\tilde{V}) \cap L^\infty(0,T;\tilde{H}),$$

$$\frac{\partial \mathbf{y}_\mathbf{v}}{\partial t} \in L^\sigma(0,T;H^{-1}(\Omega)^N) \qquad (\sigma = 2 \text{ if } N = 2 \text{ and } \sigma = 4/3 \text{ if } N = 3),$$

$$\mathbf{y}_\mathbf{v} \in C^0([0,T];L^2(\Omega)^N) \quad \text{if } N = 2,$$

$$\pi_\mathbf{v} \in L^2(Q_T).$$

(b) *If* $N = 2$*, there exists at most one weak solution to* (1) *(of course,* $\pi_\mathbf{v}$ *is unique up to a constant).*

The proof of this result can be easily obtained arguing as in [8], [9], [12]. Now, for each $\mathbf{v} \in L^2(0,T;H^{1/2}(\gamma)^N)$, let us set

$$\tilde{Y}_\mathbf{v}(T) = \{\mathbf{y}_\mathbf{v}(T); \mathbf{y}_\mathbf{v} \text{ solves, together with } \pi_\mathbf{v}, \text{ problem } (1)\}.$$

In this paper, we are concerned with the following problems.

PROBLEM (P). *Prove that the set*

$$\left(\bigcup_\mathbf{v} \tilde{Y}_\mathbf{v}(T)\right) \cap \tilde{H}$$

is dense in $\tilde{H}$.

PROBLEM (Q). *Let* $\tilde{Z}$ *be the subspace of* $\tilde{H}$ *spanned by*

$$\left(\bigcup_\mathbf{v} \tilde{Y}_\mathbf{v}(T)\right) \cap \tilde{H}.$$

Prove that $\tilde{Z}$ *is dense in* $\tilde{H}$.

Problem (P) is an approximate controllability problem in the sense of [10]. It admits the following physical interpretation: assume (for instance) that $\Omega = \mathcal{O}\backslash\bar{\Delta}$, where $\mathcal{O}$ and Δ are bounded and simply connected open sets. Also, assume that $\gamma = \partial\Delta$. If Problem (P) is solved, then a viscous incompressible fluid in $\mathcal{O}\backslash\bar{\Delta}$ that is initially at rest can be conduced to a mechanical state arbitrarily close to a given desired field acting exclusively on $\partial\Delta$.

Unfortunately, we are not able to solve Problem (P); instead, we solve Problem (Q) in this paper (see Theorem 1.2 below). Of course, the former is a much more interesting question. However, it must be noticed that in a similar linear situation Problems (P) and (Q) are equivalent. This happens, for instance, with (1) being replaced by the Stokes problem; thus, arguing as in the proof of Theorem 1.2, we obtain approximate controllability in this case (and this no matter how small γ is!).

On the other hand, recall that in the Navier–Stokes case not much is known on the nature of the set formed by all final states $\mathbf{y}_\mathbf{v}(T)$. In particular, it is not clear at all whether this set is very different from its linear span $\tilde{Z}$. In our opinion, this suffices to justify an analysis of Problem (Q).

Let us denote by U_{ad} the family of all admissible control functions:

$$U_{\text{ad}} = \{\mathbf{v};\ \mathbf{v} \in L^2(0,T;H^{1/2}(\gamma)^N),\ \exists\, \text{solution } (\mathbf{y}_\mathbf{v}, \pi_\mathbf{v}) \text{ to } (1)\}.$$

The main result in this paper is as follows.

THEOREM 1.2. (a) *Assume* $N = 2$ *and let* $\tilde{Y}$ *be the subspace of* $\tilde{H}$ *spanned by the set*

$$\{\mathbf{y}_{\mathbf{v}}(T); \mathbf{v} \in U_{\text{ad}}\}.$$

Then $\tilde{Y}$ *is dense in* $\tilde{H}$.

(b) *Assume* $N = 3$ *and let* $\tilde{Z}$ *be the subspace of* $\tilde{H}$ *spanned by*

$$\left(\bigcup_{\mathbf{v}} \tilde{Y}_{\mathbf{v}}(T)\right) \cap \tilde{H}.$$

Then $\tilde{Z}$ *is dense in* $\tilde{H}$.

Theorem 1.2 is related to a conjecture formulated by Lions in [11]. In this reference, one is also concerned with approximate controllability, but there one imposes vanishing Dirichlet conditions on the whole $\partial\Omega \times (0, T)$ and one introduces L^2 control functions in the right side of the Navier–Stokes equations. In what follows this will be referred to as the distributed control variant of Problem (P). Bardos and Tartar [1] have considered in their paper a similar question; this time, the control is exerted on the initial condition and boundary data and second members vanish. Our result is similar to that in [1] (for $N = 2$ and initial data control) and also to those in [4] and [5] (for distributed control). See also [6] and the references therein for some related questions.

2. Some technical lemmas. Before we give the proof of Theorem 1.2, we present some technical results. First, we establish existence and regularity for the stationary Stokes problem with boundary conditions of different kinds on γ and on $\partial\Omega\backslash\gamma$ (recall that $\partial\Omega$ is a $C^{1,1}$ boundary and γ is a component of $\partial\Omega$). Let $\mathbf{f} \in L^2(\Omega)^N$, $g \in L^2(\Omega)$, and $\mathbf{b} \in H^{-1/2}(\gamma)^N$ be given and consider the following problem:

$$-\nu\Delta\mathbf{y} + \nabla\pi = \mathbf{f}, \qquad \nabla\cdot\mathbf{y} = g \quad \text{in } \Omega, \tag{3}$$

$$(-\pi\mathbf{Id} + \nu\nabla\mathbf{y})\cdot\mathbf{n} = \mathbf{b} \quad \text{on } \gamma, \tag{4}$$

$$\mathbf{y} = 0 \quad \text{on } \partial\Omega\backslash\gamma. \tag{5}$$

LEMMA 2.1. *There exists one and only one solution to* (3)–(5), $(\mathbf{y}, \pi) \in \tilde{V} \times L^2(\Omega)$. *For this couple*, (3) *is satisfied almost everywhere* (*a.e.*) *in* Ω, (4) *is satisfied as an equality in* $H^{-1/2}(\gamma)^N$, *and* (5) *is satisfied in the sense of the trace on* $\partial\Omega\backslash\gamma$. *Finally, there exists a constant* $C > 0$, *only depending on* Ω *and* γ, *such that*

$$\|\mathbf{y}\|_{H^1} + \|\pi\|_{L^2} \le C(\|\mathbf{f}\|_{L^2} + \|g\|_{L^2} + \|\mathbf{b}\|_{H^{-1/2}}).$$

The proof of this lemma can be achieved by means of well-known arguments. One also has the following.

LEMMA 2.2. *Let* $m \ge 0$ *be an integer. If* $\partial\Omega$ *is* $C^{m+1,1}$, $\mathbf{f} \in H^m(\Omega)^N$, $g \in H^{m+1}(\Omega)$, *and* $\mathbf{b} \in H^{m+1/2}(\gamma)^N$, *then* $(\mathbf{y}, \pi) \in H^{m+2}(\Omega)^N \times H^{m+1}(\Omega)$. *Furthermore, there exists a constant* $C > 0$, *only depending on* Ω, γ, *and* m, *such that*

$$\|\mathbf{y}\|_{H^{m+2}} + \|\pi\|_{H^{m+1}} \le C(\|\mathbf{f}\|_{H^m} + \|g\|_{H^{m+1}} + \|\mathbf{b}\|_{H^{m+1/2}}).$$

The proof of this result is rather technical. For instance, when $m = 0$, it relies on adequate uniform bounds for the finite difference quotients

$$\frac{1}{h}(\mathbf{y}(x + he_i) - \mathbf{y}(x)) \quad \text{and} \quad \frac{1}{h}(\pi(x + he_i) - \pi(x))$$

in $H^1(\Omega)^N$ and $L^2(\Omega)$, resp. The details are given in [7] (see also [2] and the references therein for other related results).

LEMMA 2.3. *There exists a sequence* $\{\lambda_j\}$, *with*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \dots, \qquad \lambda_j \nearrow \infty,$$

and an orthonormal basis of $\tilde{H}$, *denoted* $\{\mathbf{w}_j\}$, *such that, for all* j, *one has*

$$\mathbf{w}_j \in C^\infty(\Omega)^N \cap H^2(\Omega)^N \cap \tilde{V}$$

and

$$\int_\Omega \nabla \mathbf{w}_j : \nabla \mathbf{v}\, dx = \lambda_j \int_\Omega \mathbf{w}_j \cdot \mathbf{v}\, dx \quad \forall \mathbf{v} \in \tilde{V}.$$

The function $\mathbf{w}_j$ *is, together with some* $q_j \in C^\infty(\Omega) \cap H^1(\Omega)$, *the unique solution to*

$$\begin{cases} -\Delta \mathbf{w}_j + \nabla q_j = \lambda_j \mathbf{w}_j, \qquad \nabla \cdot \mathbf{w}_j = 0 \quad \text{in}\, \Omega, \\ (-q_j\, \mathbf{Id} + \nabla \mathbf{w}_j) \cdot \mathbf{n} = 0 \quad \text{on}\, \gamma, \\ \mathbf{w}_j = 0 \quad \text{on}\, \partial\Omega \backslash \gamma, \\ \|\mathbf{w}_j\|_{L^2} = 1. \end{cases}$$

Of course, the proof of Lemma 2.3 relies on the fact that the embedding $\tilde{V} \hookrightarrow \tilde{H}$ is dense and compact (see [7] for the details).

DEFINITION 2.4. *We introduce the trilineal form* $\tilde{b}$ *on* $H^1(\Omega)^N$ *by putting*

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \equiv \frac{1}{2}[((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - ((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v})].$$

Here, $(\cdot, \cdot)$ *stands for the usual scalar product in* $L^2(\Omega)^N$. *We also introduce the bilinear operator* $\tilde{B} : \tilde{V} \times \tilde{V} \to \tilde{V}'$ *by putting*

$$\langle \tilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = \tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \tilde{V}.$$

Now, $\langle \cdot, \cdot \rangle$ *stands for the duality pairing between* $\tilde{V}'$ *and* $\tilde{V}$.

Assume that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^N$ and $\nabla \cdot \mathbf{u} = 0$ in Ω. Then

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - \frac{1}{2} \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) \mathbf{v} \cdot \mathbf{w}\, dS.$$

On the other hand,

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^N.$$

Finally, notice that if $\mathbf{u}$ and $\mathbf{v}$ belong to $L^2(0, T; \tilde{V}) \cap L^\infty(0, T; \tilde{H})$, then

$$\tilde{B}(\mathbf{u}, \mathbf{v}) \in L^\sigma(0, T; \tilde{V}'),$$

where σ is arbitrary in $[1, 2)$ if $N = 2$ and $\sigma = 4/3$ if $N = 3$.

3. The existence of a solution to a coupled nonlinear problem. In order to prove Theorem 1.1, it will be convenient to demonstrate an existence result for a certain nonlinear problem. More precisely, for each $\mathbf{w} \in \tilde{H}$, let us introduce the system

$$
(6) \qquad \begin{cases}
\dfrac{\partial \mathbf{y}}{\partial t} + (\mathbf{y}\cdot\nabla)\mathbf{y} - \nu\Delta\mathbf{y} + \nabla\pi = 0, \qquad \nabla\cdot\mathbf{y} = 0 \quad \text{in } Q_T,\\
-\dfrac{\partial \mathbf{q}}{\partial t} - (\mathbf{y}\cdot\nabla)\mathbf{q} - \nu\Delta\mathbf{q} + \nabla Q = 0, \qquad \nabla\cdot\mathbf{q} = 0 \quad \text{in } Q_T,\\
(-\pi\,\mathbf{Id} + \nu\nabla\mathbf{y})\cdot\mathbf{n} - \dfrac{1}{2}(\mathbf{y}\cdot\mathbf{n})\mathbf{y} = \mathbf{q} \quad \text{on } \Lambda_T,\\
(-Q\,\mathbf{Id} + \nu\nabla\mathbf{q})\cdot\mathbf{n} + \dfrac{1}{2}(\mathbf{y}\cdot\mathbf{n})\mathbf{q} = 0 \quad \text{on } \Lambda_T,\\
\mathbf{y} = \mathbf{q} = 0 \quad \text{on } S_T,\\
\mathbf{y}(0) = 0, \quad \mathbf{q}(T) = \mathbf{w} \quad \text{in } \Omega.
\end{cases}
$$

Then one has the following theorem.

THEOREM 3.1. *If* $\mathbf{w} \in \tilde{H}$, *then the corresponding problem* (6) *possesses at least one weak solution* $(\mathbf{y}, \pi, \mathbf{q}, Q)$ *also satisfying*:

$$
(7) \qquad \begin{cases}
\mathbf{y},\mathbf{q} \in L^2(0,T;\tilde{V}) \cap L^\infty(0,T;\tilde{H}), \qquad \dfrac{\partial \mathbf{y}}{\partial t}, \dfrac{\partial \mathbf{q}}{\partial t} \in L^\sigma(0,T;\tilde{V}'),\\
\mathbf{y},\mathbf{q} \in C^0([0,T];\tilde{V}') \cap C^0_w([0,T];\tilde{H}), \qquad \pi, Q \in L^2(Q_T),
\end{cases}
$$

(*again*, σ *is arbitrary in* $[1,2)$ *if* $N = 2$ *and* $\sigma = 4/3$ *if* $N = 3$). *Moreover*, $\mathbf{y}$ *satisfies the energy inequalities*

$$
(8) \qquad \|\mathbf{y}(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla\mathbf{y}(s)\|^2_{L^2}\,ds \le 2\int_0^t\int_\gamma \mathbf{q}(s)\cdot\mathbf{y}(s)\,dS\,ds
$$

and one has

$$
(9) \qquad (\mathbf{y}(T),\mathbf{w}) = \int\!\!\int_{\Lambda_T} |\mathbf{q}|^2\,dS\,dt.
$$

Proof. Let us see that there exist functions

$$
\mathbf{y},\mathbf{q} \in L^2(0,T;\tilde{V}) \cap L^\infty(0,T;\tilde{H}),
$$

which solve the weak formulation of (6), i.e., such that

$$
(10) \qquad \begin{cases}
\left\langle \dfrac{\partial \mathbf{y}}{\partial t}, \mathbf{v}\right\rangle + \tilde{b}(\mathbf{y},\mathbf{y},\mathbf{v}) + \nu(\nabla\mathbf{y},\nabla\mathbf{v}) = \int_\gamma \mathbf{q}(t)\cdot\mathbf{v}\,dS \quad \forall\,\mathbf{v}\in\tilde{V},\\
-\left\langle \dfrac{\partial \mathbf{q}}{\partial t}, \mathbf{v}\right\rangle - \tilde{b}(\mathbf{y},\mathbf{q},\mathbf{v}) + \nu(\nabla\mathbf{q},\nabla\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in\tilde{V},\\
\mathbf{y}(0) = 0, \qquad \mathbf{q}(T) = \mathbf{w}.
\end{cases}
$$

The proof consists of three steps.

First step: The existence of approximate solutions. We use the orthonormal basis furnished by Lemma 2.3. We denote by $\tilde{V}_m$ the linear space spanned by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ and we put

$$\mathbf{w}_{0m} = \sum_{j=1}^{m} (\mathbf{w}, \mathbf{w}_j)\mathbf{w}_j,$$

i.e., $\mathbf{w}_{0m}$ is the orthogonal projection of $\mathbf{w}$ on $\tilde{V}_m$. For each $m \geq 1$, we search for functions

$$\mathbf{y}_m, \mathbf{q}_m \in C^0([0,T]; \tilde{V}_m)$$

such that

$$\text{(11)} \qquad \begin{cases} (\mathbf{y}'_m, \mathbf{w}_j) + \tilde{b}(\mathbf{y}_m, \mathbf{y}_m, \mathbf{w}_j) + \nu(\nabla \mathbf{y}_m, \nabla \mathbf{w}_j) = \displaystyle\int_\gamma \mathbf{q}_m \cdot \mathbf{w}_j \, dS \\ (1 \leq j \leq m), \qquad \mathbf{y}_m(0) = 0, \end{cases}$$

$$\text{(12)} \qquad \begin{cases} -(\mathbf{q}'_m, \mathbf{w}_j) - \tilde{b}(\mathbf{y}_m, \mathbf{q}_m, \mathbf{w}_j) + \nu(\nabla \mathbf{q}_m, \nabla \mathbf{w}_j) = 0 \\ (1 \leq j \leq m), \qquad \mathbf{q}_m(T) = \mathbf{w}_{0m}. \end{cases}$$

We argue as follows. If the function $\mathbf{p}_m$ is given in $C^0([0,T]; \tilde{V}_m)$, there exists exactly one maximal (in time) solution $\mathbf{y}_m = \mathbf{y}_m(\mathbf{p}_m)$ to the ordinary differential problem (11) with $\mathbf{q}_m = \mathbf{p}_m$. It is not difficult to check that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{y}_m(t)\|_{L^2}^2 + \nu\|\nabla \mathbf{y}_m(t)\|_{L^2}^2 = \int_\gamma \mathbf{p}_m(t) \cdot \mathbf{y}_m(t) dS.$$

Hence,

$$\|\mathbf{y}_m(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla \mathbf{y}_m(s)\|_{L^2}^2 \, ds \leq C \int\int_{\Lambda_T} |\mathbf{p}_m(t)|^2 \, dS \, dt$$

for some C only depending on Ω, γ, and ν. From this inequality, we deduce that $\mathbf{y}_m$ is defined for all $t \in [0,T]$. Now, let us denote by $\mathbf{q}_m = \mathbf{q}_m(\mathbf{y}_m)$ the unique maximal solution to (12). It is clear that $\mathbf{q}_m$ is also defined for all $t \in [0,T]$. Moreover,

$$-\frac{1}{2}\frac{d}{dt}\|\mathbf{q}_m(t)\|_{L^2}^2 + \nu\|\nabla \mathbf{q}_m(t)\|_{L^2}^2 \equiv 0,$$

whence

$$\|\mathbf{q}_m(t)\|_{L^2}^2 + 2\nu \int_t^T \|\nabla \mathbf{q}_m(s)\|_{L^2}^2 \, ds = \|\mathbf{w}_{0m}\|_{L^2}^2 \leq \|\mathbf{w}\|_{L^2}^2.$$

This proves that $\mathbf{q}_m$ is bounded in $C^0([0,T]; \tilde{V}_m)$ independently from $\mathbf{y}_m$. Let W be the ball $\bar{B}(0; \|\mathbf{w}\|_{L^2}^2)$ in $C^0([0,T]; \tilde{V}_m)$ and let Φ be given as follows:

$$\Phi(\mathbf{p}_m) = \mathbf{q}_m(\mathbf{y}_m(\mathbf{p}_m)) \quad \forall \mathbf{p}_m \in W.$$

Then $\Phi : W \to W$ is a continuous compact mapping (due to the fact that $\Phi(\mathbf{p}_m) \in C^1([0,T]; \tilde{V}_m)$ for each $\mathbf{p}_m$). Consequently, Schauders' theorem applies and Φ possesses a fixed point $\mathbf{q}_m \in W$. Obviously, $\mathbf{q}_m$ and $\mathbf{y}_m = \mathbf{y}_m(\mathbf{q}_m)$ satisfy (11) and (12).

Second step: "A priori" estimates. From (11) and (12), one easily obtains

$$\mathbf{y}_m, \mathbf{q}_m \in \text{bounded set in } L^2(0,T;\tilde{V}) \cap L^\infty(0,T;\tilde{H}). \tag{13}$$

Consequently,

$$\tilde{B}(\mathbf{y}_m, \mathbf{y}_m), \tilde{B}(\mathbf{y}_m, \mathbf{q}_m) \in \text{bounded set in } L^\sigma(0,T;\tilde{V}'),$$

with σ being as before. Now, the choice of the basis $\{\mathbf{w}_j\}$ yields

$$\mathbf{y}'_m, \mathbf{q}'_m \in \text{bounded set in } L^\sigma(0,T;\tilde{V}'). \tag{14}$$

On the other hand, from (11) and (12), one easily deduces that

$$(\mathbf{y}_m(T), \mathbf{w}_{0m}) = \int\int_{\Lambda_T} |\mathbf{q}_m|^2 \, dS \, dt. \tag{15}$$

Third step: The choice of a convergent sequence — conclusion. From (13) and (14), one deduces that functions $\mathbf{y}$ and $\mathbf{q}$ and subsequences $\{\mathbf{y}_\rho\}$ and $\{\mathbf{q}_\rho\}$ must exist with

$$\mathbf{y}, \mathbf{q} \in L^2(0,T;\tilde{V}) \cap L^\infty(0,T;\tilde{H}) \cap C^0([0,T];\tilde{V}'),$$

$$\frac{\partial \mathbf{y}}{\partial t}, \frac{\partial \mathbf{q}}{\partial t} \in L^\sigma(0,T;\tilde{V}'),$$

and

$$\begin{cases} \mathbf{y}_\rho \text{ (resp., } \mathbf{q}_\rho) \rightharpoonup \mathbf{y} \text{ (resp., } \mathbf{q}) \text{ weakly in } L^2(0,T;\tilde{V}), \\ \mathbf{y}_\rho \text{ (resp., } \mathbf{q}_\rho) \rightharpoonup \mathbf{y} \text{ (resp., } \mathbf{q}) \text{ weakly } * \text{ in } L^\infty(0,T;\tilde{H}), \\ \mathbf{y}_\rho \text{ (resp., } \mathbf{q}_\rho) \to \mathbf{y} \text{ (resp., } \mathbf{q}) \text{ strongly in } L^2(0,T;\tilde{V}_s), \\ \dfrac{\partial \mathbf{y}_\rho}{\partial t} \left(\text{resp., } \dfrac{\partial \mathbf{q}_\rho}{\partial t}\right) \rightharpoonup \dfrac{\partial \mathbf{y}}{\partial t} \left(\text{resp., } \dfrac{\partial \mathbf{q}}{\partial t}\right) \text{ weakly in } L^\sigma(0,T;\tilde{V}'). \end{cases}$$

Here, $1/2 < s < 1$ and $\tilde{V}_s$ stands for the closure of $\tilde{\mathcal{V}}$ with respect to the norm in $H^s(\Omega)^N$ (a new Hilbert space for the same norm). These convergence properties allow us to take limits in (11) and (12), which proves that $\mathbf{y}$ and $\mathbf{q}$ solve (10). Obviously, (8) is satisfied; on the other hand, from (15) and the previous properties, it is easy to deduce (9). This ends the proof of Theorem 3.1.

4. The proof of the main result. From a well-known consequence of the Hahn–Banach theorem (for instance, see [3, Cor. I.8]), we know that the following is a statement equivalent to Theorem 1.2.

THEOREM 4.1. *Assume* $\mathbf{w} \in \tilde{H}$ *satisfies*

$$\begin{aligned} &(\mathbf{y_v}(T), \mathbf{w}) = 0 \quad \forall \mathbf{v} \in U_{\text{ad}} \quad \textit{if } N = 2, \\ &(\mathbf{y_v}(T), \mathbf{w}) = 0 \quad \forall \mathbf{v} \in \left(\bigcup_{\mathbf{v} \in U_{\text{ad}}} \tilde{Y}_\mathbf{v}(T)\right) \cap \tilde{H} \quad \textit{if } N = 3. \end{aligned} \tag{16}$$

Then $\mathbf{w} = 0$.

Proof. Let $\mathbf{w} \in \tilde{H}$ be given and assume that (16) is satisfied. Let $(\mathbf{y}^*, \pi^*, \mathbf{q}^*, Q^*)$ be the weak solution to (6) furnished by Theorem 3.1. Recall that $(\mathbf{y}^*, \pi^*, \mathbf{q}^*, Q^*)$ satisfies (7)–(9).

Let $\mathbf{v}$ be the trace of $\mathbf{y}^*$ on $\Lambda_T = \gamma \times (0,T)$. Then $\mathbf{v} \in U_{\rm ad}$ and, moreover, the couple $(\mathbf{y}^*, \pi^*)$ is a state associated to $\mathbf{v}$. Accordingly, taking into account (9) and (16), one has

$$\mathbf{q}^* = 0 \quad \text{on } \Lambda_T. \tag{17}$$

From (8), we also deduce that $\mathbf{y}^* \equiv 0$. Thus, we have found a function $\mathbf{q}^*$ that vanishes on Λ_T and solves, together with Q^*, the following final value-boundary value problem:

$$-\frac{\partial \mathbf{q}}{\partial t} - \nu\Delta\mathbf{q} + \nabla Q = 0, \quad \nabla\cdot\mathbf{q} = 0 \quad \text{in } Q_T, \tag{18}$$

$$(-Q\,\mathbf{Id} + \nu\nabla\mathbf{q})\cdot\mathbf{n} = 0 \quad \text{on } \Lambda_T, \tag{19}$$

$$\mathbf{q} = 0 \quad \text{on } S_T, \tag{20}$$

$$\mathbf{q}(T) = \mathbf{w} \quad \text{in } \Omega. \tag{21}$$

It is not difficult to prove that (18)–(21) possesses exactly one solution pair $(\mathbf{q}, Q)$, with (at least)

$$\mathbf{q} \in L^2(0,T;\tilde{V}) \cap C^0([0,T];\tilde{H}),$$

$$\frac{\partial \mathbf{q}}{\partial t} \in L^2(0,T;\tilde{V}'), \qquad Q \in L^2(Q_T).$$

Necessarily, $(\mathbf{q}, Q) = (\mathbf{q}^*, Q^*)$. Consequently, Theorem 4.1 is implied by Proposition 4.2 (see below).

PROPOSITION 4.2. *Assume the couple* $(\mathbf{q}^*, Q^*)$ *satisfies*

$$\mathbf{q}^* \in L^2_{\rm loc}(0,T;\tilde{V}) \cap L^\infty_{\rm loc}(0,T;\tilde{H}),$$

$$\frac{\partial \mathbf{q}^*}{\partial t} \in L^2_{\rm loc}(0,T;\tilde{V}'), \qquad Q^* \in L^2_{\rm loc}(0,T;L^2(\Omega))$$

and (17)–(20). *Then* $\mathbf{q}^* \equiv 0$.

Proof. Let $x_0 \in \gamma$ be given. Choose $r > 0$ such that

$$B(x_0;r) \cap \partial\Omega \subset \gamma$$

and consider the open sets $\omega = B(x_0;r)$ and $\tilde{\Omega} = \Omega \cup \omega$. Let $(\tilde{\mathbf{q}}, \tilde{Q})$ be the extension by zero of $(\mathbf{q}^*, Q^*)$ to the whole cylinder $\tilde{\Omega} \times (0,T)$. From (17), we see that

$$\tilde{\mathbf{q}} \in L^2_{\rm loc}(0,T;V(\tilde{\Omega})) \cap L^\infty_{\rm loc}(0,T;H(\tilde{\Omega})),$$

$$\tilde{Q} \in L^2_{\rm loc}(\tilde{\Omega} \times (0,T)).$$

Here,

$$V(\tilde{\Omega}) = \{\mathbf{v}; \mathbf{v} \in H^1_0(\tilde{\Omega})^N,\ \nabla\cdot\mathbf{v} = 0 \text{ in } \tilde{\Omega}\},$$

$$H(\tilde{\Omega}) = \{\mathbf{v}; \mathbf{v} \in L^2(\tilde{\Omega})^N,\ \nabla \cdot \mathbf{v} = 0 \text{ in } \tilde{\Omega},\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\tilde{\Omega}\}.$$

$V(\tilde{\Omega})$ and $H(\tilde{\Omega})$ are endowed with the norms of $H^1(\tilde{\Omega})^N$ and $L^2(\tilde{\Omega})^N$, resp. It is easy to check that

$$\frac{\partial \tilde{\mathbf{q}}}{\partial t} \in L^2_{\text{loc}}(0, T; V(\tilde{\Omega})')$$

and

$$\begin{cases} -\dfrac{\partial \tilde{\mathbf{q}}}{\partial t} - \nu \Delta \tilde{\mathbf{q}} + \nabla \tilde{Q} = 0, \quad \nabla \cdot \tilde{\mathbf{q}} = 0 \quad \text{in } \tilde{\Omega} \times (0, T), \\ \qquad \tilde{\mathbf{q}} = 0 \quad \text{on } \partial\tilde{\Omega} \times (0, T). \end{cases}$$

In particular, we deduce that both $\tilde{\mathbf{q}}$ and $\tilde{Q}$ are analytical functions in the space in $\tilde{\Omega} \times (0, T)$ (cf. e.g. [8]). But $\tilde{\mathbf{q}} = 0$ in $(\tilde{\Omega} \backslash \bar{\Omega}) \times (0, T)$. Hence, necessarily $\tilde{\mathbf{q}} \equiv 0$.

For the sake of completeness, let us state (and prove) a regularity result for (18)–(21).

LEMMA 4.3. *Let $\mathbf{w} \in \tilde{H}$ and $\delta > 0$ be given. Then the unique solution $(\mathbf{q}, Q)$ to (18)–(21) satisfies*

$$\mathbf{q} \in L^2(0, T - \delta; H^2(\Omega)^N) \cap L^\infty(0, T - \delta; \tilde{V}) \cap L^2(0, T; \tilde{V}) \cap C^0([0, T]; \tilde{H}),$$

$$\frac{\partial \mathbf{q}}{\partial t} \in L^2(0, T - \delta; H^1(\Omega)^N) \cap L^\infty(0, T - \delta; L^2(\Omega)^N) \cap L^2(0, T; \tilde{V}'),$$

$$Q \in L^2(0, T - \delta; H^1(\Omega)) \cap L^\infty(0, T - \delta; L^2(\Omega)) \cap L^2(Q_T).$$

Sketch of the proof. Let $\theta = \theta_\delta$ be a real-valued C^∞ function on $[0, +\infty)$ such that

$$\theta \equiv 1 \text{ in } [0, T - \delta), \qquad \theta \equiv 0 \text{ in } \left[T - \frac{\delta}{2}, +\infty\right).$$

Using θ, we introduce

$$\hat{\mathbf{q}} = \theta \mathbf{q} \quad \text{and} \quad \hat{Q} = \theta Q.$$

Then $\hat{\mathbf{q}} \in L^2(0, T; \tilde{V}) \cap L^\infty(0, T; \tilde{H})$ and, also,

$$\begin{cases} -\left\langle \dfrac{\partial \hat{\mathbf{q}}}{\partial t}(t), \mathbf{v} \right\rangle + \nu (\nabla \hat{\mathbf{q}}(t), \nabla \mathbf{v})_{0;\Omega} = (\mathbf{f}(t), \mathbf{v})_{0;\Omega} \quad \forall \mathbf{v} \in \tilde{V},\ t \in (0, T) \text{ a.e.}, \\ \hat{\mathbf{q}}(T) = 0, \end{cases}$$

where $\mathbf{f} = -\theta' \mathbf{q}$. Notice that

$$\mathbf{f} \in L^2(0, T; \tilde{V}) \cap L^\infty(0, T; \tilde{H}) \quad \text{and} \quad \frac{\partial \mathbf{f}}{\partial t} \in L^2(0, T; \tilde{V}').$$

It is clear that $\hat{\mathbf{q}}$ is the limit of approximate solutions $\hat{\mathbf{q}}_m$, with

$$\hat{\mathbf{q}}_m(t) = \sum_{j=1}^{m} \hat{q}^j_m(t) \mathbf{w}_j,$$

$$\text{(22)} \qquad \begin{cases} -(\hat{\mathbf{q}}'_m(t), \mathbf{w}_j)_{0;\Omega} + \nu(\nabla\hat{\mathbf{q}}_m(t), \nabla\mathbf{w}_j)_{0;\Omega} = (\mathbf{f}(t), \mathbf{w}_j)_{0;\Omega}, \\ (1 \le j \le m), \qquad \hat{\mathbf{q}}_m(T) = 0. \end{cases}$$

Differentiation with respect to t leads to the equalities

$$\text{(23)} \qquad -(\hat{\mathbf{q}}''_m(t), \mathbf{w}_j)_{0;\Omega} + \nu(\nabla\hat{\mathbf{q}}'_m(t), \nabla\mathbf{w}_j)_{0;\Omega} = \langle \mathbf{f}', \mathbf{w}_j \rangle.$$

Now, multiplying the jth equation in (22) by $\lambda_j \hat{q}^j_m(t)$, adding for $1 \le j \le m$, and integrating with respect to t, we are led to the inequalities

$$\|\nabla\hat{\mathbf{q}}_m(t)\|^2_{0;\Omega} + \nu \int_t^T \|\Delta\hat{\mathbf{q}}'_m(s)\|^2_{0;\Omega}\, ds \le C \int_0^T \|\mathbf{f}(t)\|^2_{0;\Omega}\, dt,$$

where C is a constant. This proves that

$$\hat{\mathbf{q}} \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T; \tilde{V}).$$

On the other hand, multiplying (23) by $(\hat{q}^j_m)'(t)$ and adding for $1 \le j \le m$, we obtain

$$-\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{q}}'_m(t)\|^2_{0;\Omega} + \nu\|\nabla\hat{\mathbf{q}}'_m(t)\|^2_{0;\Omega} = -\left\langle \frac{\partial \mathbf{f}}{\partial t}(t), \hat{\mathbf{q}}'_m(t) \right\rangle \quad \forall t \in [0,T].$$

After integration with respect to t, one has

$$\|\hat{\mathbf{q}}'_m(t)\|^2_{0;\Omega} + \nu \int_t^T \|\delta\hat{\mathbf{q}}'_m(s)\|^2_{0;\Omega}\, ds \le C \left\| \frac{\partial \mathbf{f}}{\partial t} \right\|^2_*,$$

where $\|\cdot\|_*$ stands for the norm in $L^2(0,T;\tilde{V}')$. Hence,

$$\frac{\partial \hat{\mathbf{q}}}{\partial t} \in L^2(0,T; H^1(\Omega)^N) \cap L^\infty(0,T; L^2(\Omega)^N).$$

This proves the lemma.

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