SOME CONTROLLABILITY RESULTS FOR THE N-DIMENSIONAL NAVIER–STOKES AND BOUSSINESQ SYSTEMS WITH $N - 1$ SCALAR CONTROLS*

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Abstract. In this paper we deal with some controllability problems for systems of the Navier–Stokes and Boussinesq kind with distributed controls supported in small sets. Our main aim is to control N-dimensional systems ($N + 1$ scalar unknowns in the case of the Navier–Stokes equations) with $N - 1$ scalar control functions. In a first step, we present some global Carleman estimates for suitable adjoint problems of linearized Navier–Stokes and Boussinesq systems. In this way, we obtain null controllability properties for these systems. Then, we deduce results concerning the local exact controllability to the trajectories. We also present (global) null controllability results for some (truncated) approximations of the Navier–Stokes equations.

1. Introduction and examples. Let $\Omega \subset \mathbf{R}^N$ ($N = 2$ or 3) be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance of class C^2). Let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

On the other hand, we will denote by C, C_1, C_2, ... various positive constants (usually depending on Ω and $\mathcal{O}$).

We will be concerned with the following controlled Navier–Stokes and Boussinesq systems:

$$(1) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v 1_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega \end{cases}$$

and

$$(2) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v 1_{\mathcal{O}} + \theta\, e_N, \quad \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = h 1_{\mathcal{O}} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega \end{cases}$$

(in both dimensions $N = 2$ and $N = 3$).

For $N = 2$, we will also consider the following approximation of the Navier–Stokes system with boundary conditions of the Navier kind:

$$
(3) \quad
\begin{cases}
y_t - \Delta y + (y \cdot \nabla)\mathbf{T}_M(y) + \nabla p = v1_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{in } Q, \\
y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\
y(0) = y^0 & \text{in } \Omega,
\end{cases}
$$

where $M > 0$, $\mathbf{T}_M(y) = (T_M(y_1), T_M(y_2))$ and T_M is given by

$$
T_M(s) =
\begin{cases}
-M & \text{if } s \leq -M, \\
s & \text{if } -M \leq s \leq M, \\
M & \text{if } s \geq M.
\end{cases}
$$

In systems (1), (2) and (3), $v = v(x,t)$ and $h = h(x,t)$ stand for the control functions. They act during the whole time interval $(0,T)$ over the set $\mathcal{O}$. The symbol $1_{\mathcal{O}}$ stands for the characteristic function of $\mathcal{O}$ and e_N is the Nth vector of the canonical basis of $\mathbf{R}^N$.

The controllability of Navier–Stokes systems has been the objective of considerable work over the last years. Up to our knowledge, the strongest results have been given in [7], where a strategy based on the methods in [13] and [14] has been followed. Recently, the techniques in [7] have been adapted in [12] to cover Boussinesq systems (see also [3], [4], [8] and [10] for other results).

This paper can be viewed as a continuation of [7]. We will present some new results which show that the N-dimensional systems (1) and (2) can be controlled, at least under some geometrical assumptions, with only $N-1$ scalar controls in $L^2(\mathcal{O} \times (0,T))$. In particular, the Boussinesq system (2) in dimension $N = 2$ can be controlled by an action performed only on the temperature equation. We will also prove that the two-dimensional system (3) can be controlled with controls of the form $v1_{\mathcal{O}}$ where v is the curl of a function in $L^2(0,T;H^1(\mathcal{O}))$.

In this paper, we will have to impose some regularity assumptions on the initial data. To this purpose, we introduce the spaces H, E and V, with

$$
(4) \qquad H = \{w \in L^2(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega, \, w \cdot n = 0 \text{ on } \partial\Omega\},
$$

$$
E =
\begin{cases}
H & \text{if } N = 2, \\
L^4(\Omega)^3 \cap H & \text{if } N = 3
\end{cases}
$$

and

$$
V = \{w \in H_0^1(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega\}.
$$

For system (1), we will assume that the control region $\mathcal{O}$ is adjacent to the boundary $\partial\Omega$ (see assumption (11) below) and we will deal with the *local exact controllability to the trajectories*. More precisely, our task will be to prove that, for any bounded and sufficiently regular solution $(\overline{y}, \overline{p})$ of the uncontrolled Navier–Stokes equations

$$
(5) \quad
\begin{cases}
\overline{y}_t - \Delta\overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla\overline{p} = 0, \quad \nabla \cdot \overline{y} = 0 & \text{in } Q, \\
\overline{y} = 0 & \text{on } \Sigma, \\
\overline{y}(0) = \overline{y}^0 & \text{in } \Omega,
\end{cases}
$$

there exists $\delta > 0$ such that, whenever $y^0 \in E$ and

$$\|y^0 - \overline{y}^0\|_E \leq \delta,$$

we can find L^2 controls v with $v_k \equiv 0$ for at least one k and associated states (y, p) satisfying

(6) $y(T) = \overline{y}(T)$ in Ω.

Notice that, under these circumstances, after time $t = T$ we can switch off the control and let the system follow the "ideal" trajectory $(\overline{y}, \overline{p})$.

For the Boussinesq system (2), we will assume that $\mathcal{O}$ is adjacent to $\partial\Omega$ near a point x^0 such that $n_k(x^0) \neq 0$ for some $k < N$. We will also be concerned with the local exact controllability to the trajectories. Now, a trajectory is a bounded and sufficiently regular solution $(\overline{y}, \overline{p}, \overline{\theta})$ of

(7)
$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla\overline{p} = \overline{\theta}\, e_N, \quad \nabla \cdot \overline{y} = 0 & \text{in } Q, \\ \overline{\theta}_t - \Delta\overline{\theta} + \overline{y} \cdot \nabla\overline{\theta} = 0 & \text{in } Q, \\ \overline{y} = 0, \quad \overline{\theta} = 0 & \text{on } \Sigma, \\ \overline{y}(0) = \overline{y}^0, \quad \overline{\theta}(0) = \overline{\theta}^0 & \text{in } \Omega. \end{cases}$$

The goal will be to prove that there exists $\delta > 0$ such that, whenever $(y^0, \theta^0) \in E \times L^2(\Omega)$ and

$$\|(y^0, \theta^0) - (\overline{y}^0, \overline{\theta}^0)\|_{E \times L^2} \leq \delta,$$

we can find L^2 controls v and h with $v_k \equiv v_N \equiv 0$ and associated states (y, p, θ) satisfying

(8) $y(T) = \overline{y}(T)$ and $\theta(T) = \overline{\theta}(T)$ in Ω.

In this context, the results established in [12] will be fundamental.

Notice that, in particular, when $N = 2$, we try to control the whole system (2) with just one scalar control h.

As far as (3) is concerned, our goal will be to prove the *(global) null controllability*. That is to say, for each $y^0 \in H$, we will try to find controls of the form $v1_{\mathcal{O}}$, where v belongs to the Hilbert space

(9) $W = \{\nabla \times z = (\partial_2 z, -\partial_1 z) : z \in L^2(0, T; H^1(\mathcal{O}))\},$

such that the associated solutions (y, p) satisfy

(10) $y(T) = 0$ in Ω.

Approximate controllability results have been established for analogous systems in [4].

Observe that in this system the boundary conditions are of the Navier kind as in [3] (for their physical meaning, see, for instance, [11]). This and the fact that $N = 2$ will be essential in the arguments presented below.

Similarly to the previous situation, an extension by zero of the control after time $t = T$ will keep (y, p) at rest.

As mentioned above, some hypotheses will be imposed on the control domain and the trajectories. More precisely, we will frequently assume that

$$(11) \qquad \exists x^0 \in \partial\Omega, \ \exists \varepsilon > 0 \text{ such that } \overline{\mathcal{O}} \cap \partial\Omega \supset B(x^0; \varepsilon) \cap \partial\Omega$$

($B(x^0; \varepsilon)$ is the ball centered at x^0 of radius ε),

$$(12) \qquad \overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0, T; L^\sigma(\Omega)^N) \quad \begin{pmatrix} \sigma > 1 & \text{if } N = 2 \\ \sigma > 6/5 & \text{if } N = 3 \end{pmatrix}$$

and

$$(13) \qquad \overline{\theta} \in L^\infty(Q), \quad \overline{\theta}_t \in L^2(0, T; L^\sigma(\Omega)) \quad \begin{pmatrix} \sigma > 1 & \text{if } N = 2 \\ \sigma > 6/5 & \text{if } N = 3 \end{pmatrix}.$$

Let us now present our main results in a precise form. The first one concerns the local exact controllability to the trajectories of system (1).

THEOREM 1. *Assume that $\mathcal{O}$ satisfies (11). Then, for any $T > 0$, (1) is locally exactly controllable at time T to the trajectories $(\overline{y}, \overline{p})$ satisfying (12) with controls $v \in L^2(\mathcal{O} \times (0, T))^N$ having one component identically zero.*

The second main result concerns the controllability of (2).

THEOREM 2. *Assume that $\mathcal{O}$ satisfies (11) with $n_k(x^0) \neq 0$ for some $k < N$. Then, for any $T > 0$, (2) is locally exactly controllable at time T to the trajectories $(\overline{y}, \overline{p}, \overline{\theta})$ satisfying (12)–(13) with L^2 controls v and h such that $v_k \equiv v_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability to the trajectories with controls $v \equiv 0$ and $h \in L^2(\mathcal{O} \times (0, T))$.*

The last main result we present in this paper follows in Theorem 3.

THEOREM 3. *Let $N = 2$. Then, for any $T > 0$ and any $M > 0$, (3) is null controllable at time T with controls of the form $v1_\mathcal{O}$, where $v \in W$.*

For the proofs of these results, following a standard approach, we will first deduce null controllability results for suitable linearized versions of (1), (2) and (3), namely,

$$(14) \qquad \begin{cases} y_t - \Delta y + (\overline{y} \cdot \nabla)y + (y \cdot \nabla)\overline{y} + \nabla p = f + v1_\mathcal{O}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

$$(15) \qquad \begin{cases} y_t - \Delta y + (\overline{y} \cdot \nabla)y + (y \cdot \nabla)\overline{y} + \nabla p = f + v1_\mathcal{O} + \theta \, e_N & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + \overline{y} \cdot \nabla\theta + y \cdot \nabla\overline{\theta} = k + h1_\mathcal{O} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega \end{cases}$$

and

$$(16) \qquad \begin{cases} y_t - \Delta y + (y \cdot \nabla)\overline{y} + \nabla p = v1_\mathcal{O}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

Then, appropriate arguments will be used to deduce the controllability of the nonlinear systems (1)–(3).

Remark 1. When $N = 3$, it is very natural to ask whether a result similar to Theorem 1 holds with controls having two zero components. In general, the answer is no. In fact, it seems difficult to identify the open sets Ω and $\mathcal{O}$ such that one has null controllability for all $T > 0$ with controls of this kind. This is unknown even for the classical Stokes equations for which, up to now, the only known results concern *approximate controllability*; see [16].

Remark 2. Assume that $N = 2$. The arguments in [7] implicitly show that, under hypotheses (12), we can find controls $v1_{\mathcal{O}}$ with $v \in W$ such that the associated solutions to (1) satisfy $y(T) = \overline{y}(T)$. Observe that the assumption (11) on the control domain is not necessary here.

This paper is organized as follows. We will first establish all the technical results needed in this work in section 2. Section 3 will deal with null controllability results for the linear control systems (14)–(16). Finally, the proofs of Theorems 1, 2 and 3 will be given in section 4.

2. Some previous results. In this section we will establish all the technical results needed in this paper. More precisely, we will present and prove the required *Carleman estimates* for the backward systems (19), (20) and (21), given below.

To do this, let us first introduce some weight functions:

$$\alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4},$$

$$\xi(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4},$$

(17) $$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4},$$

$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4},$$

$$\widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, \quad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4},$$

where $m > 4$ is a fixed real number. Here, η^0 is a function verifying

(18) $\qquad \eta^0 \in C^2(\overline{\Omega}), \quad |\nabla\eta^0| > 0 \ \text{in} \ \overline{\Omega \setminus \mathcal{O}_0}, \quad \eta^0 > 0 \ \text{in} \ \Omega \ \text{and} \ \eta^0 \equiv 0 \ \text{on} \ \partial\Omega$

with $\mathcal{O}_0$ a nonempty open subset of $\mathcal{O}$ that will be determined below. For any $\mathcal{O}_0$, the existence of such a function η^0 is proved in [9]. Note that these weights have already been used in [7] and [12].

We will be dealing in this section with the adjoint systems to (14) and (15), that is to say,

(19) $$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g, \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^0 & \text{in } \Omega \end{cases}$$

and

$$
(20) \quad
\begin{cases}
-\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g + \overline{\theta}\,\nabla\psi, \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\[4pt]
-\psi_t - \Delta\psi - \overline{y}\cdot\nabla\psi = q + \varphi_N & \text{in } Q, \\[4pt]
\varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\[4pt]
\varphi(T) = \varphi^0, \quad \psi(T) = \psi^0 & \text{in } \Omega
\end{cases}
$$

(where $D\varphi = \nabla\varphi + \nabla\varphi^t$) as well as with the adjoint system of $\omega := \nabla \times y$ (where y is the solution of (16)), which is

$$
(21) \quad
\begin{cases}
-\rho_t - \Delta\rho - \nabla \times ((\overline{y}\cdot\nabla\times)\nabla\gamma) = 0, \quad \Delta\gamma = \rho & \text{in } Q, \\[4pt]
\gamma = 0, \quad \rho = 0 & \text{on } \Sigma, \\[4pt]
\rho(T) = \rho^0 & \text{in } \Omega.
\end{cases}
$$

Here, $g \in L^2(Q)^N$, $q \in L^2(Q)$, $\varphi^0 \in H$, $\psi^0 \in L^2(\Omega)$ and $\rho^0 \in H^{-1}(\Omega)$ (of course, φ_N stands for the last component of the vector field φ).

2.1. New Carleman estimates for system (19). We will establish some new Carleman estimates for the solutions of (19). We will assume that $\mathcal{O}$ and $\overline{y}$ satisfy (11)–(12). To fix ideas, we will also assume for the moment that $N = 3$ and $n_1(x^0) \neq 0$ (x^0 appears in assumption (11)).

The desired Carleman inequalities will have the form

$$
I(\varphi) \leq C\left(\iint_Q \rho_1^2\,|g|^2\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_2^2\left(|\varphi_2|^2 + |\varphi_3|^2\right)\,dx\,dt \right),
$$

where $I(\varphi)$ contains global weighted integrals of $|\varphi|^2$, $|\nabla\varphi|^2$, etc. and ρ_1 and ρ_2 are appropriate weights that vanish exponentially as $t \to T$. This will suffice to prove in section 3 the null controllability of (14) with controls $v1_{\mathcal{O}}$ satisfying $v_1 \equiv 0$.

LEMMA 1. *Assume that $N = 3$, $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $\overline{y}$ verify (11)–(12). Then there exists a positive constant C such that, for any $g \in L^2(Q)^3$ and any $\varphi^0 \in H$, the associated solution to* (19) *satisfies:*

$$
(22) \quad
\begin{cases}
I(\varphi) := \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-12}(T-t)^{-12}\,|\varphi|^2\,dx\,dt \\[10pt]
\qquad + \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-4}(T-t)^{-4}\,|\nabla\varphi|^2)\,dx\,dt \\[10pt]
\qquad + \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^4(T-t)^4\left(|\Delta\varphi|^2 + |\varphi_t|^2\right)\,dx\,dt \\[10pt]
\qquad \leq C\left(\displaystyle\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-30}(T-t)^{-30}\,|g|^2\,dx\,dt \right. \\[10pt]
\qquad\qquad \left. + \displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}\left(|\varphi_2|^2 + |\varphi_3|^2\right)\,dx\,dt \right).
\end{cases}
$$

Here, $\overline{\alpha}$ and $\widetilde{\alpha}$ are constants only depending on Ω, $\mathcal{O}$, T and $\overline{y}$ satisfying $0 < \widetilde{\alpha} < \overline{\alpha}$ and $8\widetilde{\alpha} - 7\overline{\alpha} > 0$.

Proof. Let us first recall a Carleman inequality for the solutions of (19) which has been proved in [7] whenever (12) is fulfilled:

$$
\begin{aligned}
&s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \\
&\quad + s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\left(|\varphi_t|^2 + |\Delta\varphi|^2\right)dx\,dt \\
&\quad \le C_0(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt\right. \\
&\quad\quad \left. + s^{16}\lambda^{40}\iint_{\mathcal{O}_0\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt\right).
\end{aligned}
\tag{23}
$$

Here, $s \ge s_0$ and $\lambda \ge \lambda_0$ are arbitrarily large and C_0, s_0 and λ_0 are suitable constants depending on Ω, $\mathcal{O}_0$, T and $\overline{y}$; see Theorem 1 in [7].

Recall that an inequality like (23) had already been proved in [13] using stronger properties on $\overline{y}$ than (12).

It is immediate from (23) that, for some C_1, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}_0$, T and $\overline{y}$, we have:

$$
\begin{cases}
\displaystyle \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\left(t^{-12}(T-t)^{-12}|\varphi|^2 + t^{-4}(T-t)^{-4}|\nabla\varphi|^2\right)dx\,dt \\
\displaystyle \quad + \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^4(T-t)^4\left(|\Delta\varphi|^2 + |\varphi_t|^2\right)dx\,dt \\
\displaystyle \quad \le C_1\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}t^{-30}(T-t)^{-30}|g|^2\,dx\,dt\right. \\
\displaystyle \quad\quad \left. + \iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}t^{-64}(T-t)^{-64}|\varphi|^2\,dx\,dt\right).
\end{cases}
\tag{24}
$$

Indeed, it suffices to choose

$$
\begin{cases}
\overline{\alpha} = s_0\left(e^{5/4\lambda_0 m\|\eta^0\|_\infty} - e^{\lambda_0 m\|\eta^0\|_\infty}\right), \\
\widetilde{\alpha} = s_0\left(e^{5/4\lambda_0 m\|\eta^0\|_\infty} - e^{\lambda_0(m+1)\|\eta^0\|_\infty}\right)
\end{cases}
\tag{25}
$$

and $C_1 = C_0(1+T^2)s_0^{17}\lambda_0^{40}e^{17\lambda_0(m+1)\|\eta^0\|_\infty}$. Notice that $0 < \widetilde{\alpha} < \overline{\alpha}$. Moreover, it can be assumed that $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ (it suffices to notice that λ_0 is large enough in (25)).

We will apply (24) for the open set $\mathcal{O}_0 \subset \mathcal{O}$ defined as follows. We choose $\kappa > 0$ such that

$$
n_1(x) \ne 0 \quad \forall x \in B(x^0;\kappa)\cap\partial\mathcal{O}\cap\partial\Omega
$$

and we denote this set by Γ_κ. Then, we define

$$
\mathcal{O}_0 = \{x \in \Omega : x = w + \tau e_1, \, w \in \Gamma_\kappa, \, |\tau| < \tau^0\},
\tag{26}
$$

with $\kappa, \tau^0 > 0$ small enough so that we still have

$$
\mathcal{O}_0 \subset \mathcal{O} \quad \text{and} \quad d_0 := \text{dist}(\overline{\mathcal{O}_0}, \partial\mathcal{O}\cap\Omega) > 0.
\tag{27}
$$

Observe that, with this choice, each $P \in \mathcal{O}_0$ verifies that one of the two points where the straight line $\{P + \mathbf{R}\,e_1\}$ intersects $\partial\Omega$ belongs to $\partial\mathcal{O}_0$.

Once $\mathcal{O}_0$ is defined, we apply inequality (24) in this open set and we try to bound the term

$$\iint_{\mathcal{O}_0 \times (0,T)} e^{\frac{-8\alpha + 6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\, |\varphi_1|^2 \, dx\, dt$$

in terms of local integrals of φ_2 and φ_3.

To this end, for each $(x,t) \in \mathcal{O}_0 \times (0,T)$ we denote by $l(x,t)$ (resp., $\widetilde{l}(x,t)$) the segment that starts from (x,t) with direction e_1 in the positive (resp. negative) sense and ends at $\partial\mathcal{O}_0$. Then, since φ is divergence-free, it is not difficult to see that

$$\varphi_1(x,t) = \int_{l(x,t)} (\partial_2\varphi_2 + \partial_3\varphi_3)(y_1, x_2, x_3, t)\, dy_1$$

for each $(x,t) \in \mathcal{O}_0 \times (0,T)$. For simplicity, let us introduce the notation

$$\beta(t) = e^{\frac{-8\alpha + 6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64} \quad \forall t \in (0,T).$$

Applying at this point Hölder's inequality and Fubini's formula, we obtain

$$
\begin{aligned}
&\iint_{\mathcal{O}_0 \times (0,T)} \beta(t)\, |\varphi_1|^2 \, dx\, dt \\
&\quad \leq C_2 \iint_{\mathcal{O}_0 \times (0,T)} \beta(t) \left(\int_{l(x,t)} (|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2)\, dy_1 \right) dx\, dt \\
&\quad = C_2 \iint_{\mathcal{O}_0 \times (0,T)} (|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2) \left(\int_{\widetilde{l}(y_1)} \beta(t)\, dx_1 \right) dy_1\, dx_2\, dx_3\, dt \\
&\quad \leq C_3 \iint_{\mathcal{O}_0 \times (0,T)} \beta(t)(|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2)\, dx\, dt,
\end{aligned}
$$
(28)

where $\widetilde{l}(y_1)$ stands for the segment $\widetilde{l}(y_1, x_2, x_3, t)$. Then, we introduce a function $\zeta \in C^2(\overline{\mathcal{O}})$ such that

$$\zeta \equiv 1 \text{ in } \mathcal{O}_0, \quad 0 \leq \zeta \leq 1$$

and $\zeta(x) = 0$ at any point $x \in \mathcal{O}$ satisfying $\mathrm{dist}(x, \partial\mathcal{O} \cap \Omega) \leq d_0/2$ (d_0 was defined in (27)). This and the fact that $\varphi|_\Sigma \equiv 0$ imply

$$
\begin{aligned}
&\iint_{\mathcal{O}_0 \times (0,T)} \beta(t)\, |\partial_i\varphi_i|^2 \, dx\, dt \leq \iint_{\mathcal{O} \times (0,T)} \zeta\, \beta(t)\, |\partial_i\varphi_i|^2 \, dx\, dt \\
&\quad = \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} \partial_{ii}^2 \zeta\, \beta(t)\, |\varphi_i|^2 \, dx\, dt - \iint_{\mathcal{O} \times (0,T)} \zeta\, \beta(t)\, \partial_{ii}^2\varphi_i\, \varphi_i \, dx\, dt
\end{aligned}
$$

for $i = 2, 3$. Finally, in view of Young's inequality and regularity estimates for φ_i in

Ω ($\varphi_i \in H^2(\Omega)$ and $\|\varphi_i\|_{H^2} \leq C\|\Delta\varphi_i\|_{L^2}$), we also have:

$$\iint_{\mathcal{O}_0 \times (0,T)} \beta(t)\,|\partial_i\varphi_i|^2\,dx\,dt$$

$$\leq C_4 \iint_{\mathcal{O} \times (0,T)} e^{\frac{-16\alpha+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}\,|\varphi_i|^2\,dx\,dt$$

$$+\frac{1}{2C_1C_3}\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4\,|\Delta\varphi_i|^2\,dx\,dt,$$

which, combined with (24) and (28), yields (22).   □

Let us now present another Carleman inequality for (19) with weight functions not vanishing at time $t = 0$.

LEMMA 2. *Assume that* $N = 3$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $\overline{y}$ *verify* (11)–(12). *Then there exist positive constants* C, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *with* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and* $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ *depending on* Ω, $\mathcal{O}$, T *and* $\overline{y}$ *such that, for any* $g \in L^2(Q)^3$ *and any* $\varphi^0 \in H$, *the associated solution to* (19) *satisfies:*

$$(29) \quad \begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}}\left(\ell(t)^{-12}\,|\varphi|^2 + \ell(t)^{-4}\,|\nabla\varphi|^2\right)dx\,dt \\[2ex] \displaystyle\leq C\left(\iint_Q e^{\frac{-4\alpha+2\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-30}\,|g|^2\,dx\,dt\right.\\[2ex] \displaystyle\left.+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\alpha+14\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-132}\,(|\varphi_2|^2 + |\varphi_3|^2)\,dx\,dt\right), \end{cases}$$

where ℓ *is the* C^1 *function given by*

$$(30) \qquad \ell(t) = \begin{cases} \dfrac{T^2}{4} & \text{for } 0 \leq t \leq T/2, \\[2ex] t(T-t) & \text{for } T/2 \leq t \leq T. \end{cases}$$

To prove (29), it suffices to use (22) and the classical parabolic estimates for the Stokes system satisfied by φ. The argument has already been used in [9], [13] and [7] in several similar situations, so we omit it for simplicity.

For completeness, let us state the similar result that can be established when $N = 2$. Here, we assume again that $n_1(x^0) \neq 0$.

LEMMA 3. *Assume that* $N = 2$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $\overline{y}$ *verify* (11)–(12). *Then there exist positive constants* C, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *with* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and* $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ *depending on* Ω, $\mathcal{O}$, T *and* $\overline{y}$ *such that, for any* $g \in L^2(Q)^2$ *and any* $\varphi^0 \in H$, *the associated solution to* (19) *satisfies:*

$$(31) \quad \begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}}\left(\ell(t)^{-12}\,|\varphi|^2 + \ell(t)^{-4}\,|\nabla\varphi|^2\right)dx\,dt \\[2ex] \displaystyle\leq C\left(\iint_Q e^{\frac{-4\alpha+2\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-30}\,|g|^2\,dx\,dt\right.\\[2ex] \displaystyle\left.+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\alpha+14\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-132}\,|\varphi_2|^2\,dx\,dt\right), \end{cases}$$

where ℓ *is the function given by* (30).

2.2. New Carleman estimates for system (20). We will establish suitable Carleman inequalities for the solutions of (20). To this end, our approach will be similar to the one in subsection 2.1.

Thus, we will assume again that $N = 3$ and $n_1(x^0) \neq 0$ and we will prove an estimate of the form

$$K(\varphi, \psi) \leq C \left(\iint_Q \rho_3^2 \left(|g|^2 + |q|^2\right) dx\,dt + \iint_{\mathcal{O} \times (0,T)} \rho_4^2 \left(|\varphi_2|^2 + |\psi|^2\right) dx\,dt \right),$$

where $K(\varphi, \psi) = I(\varphi) + I(\psi)$ ($I(\varphi)$ has been given in (22)) and ρ_3 and ρ_4 are appropriate weights. This will be used in section 3 to find controls $v1_{\mathcal{O}}$ and $h1_{\mathcal{O}}$ with $v_1 \equiv v_3 \equiv 0$ leading to the null controllability of (15).

LEMMA 4. *Assume that* $N = 3$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $(\overline{y}, \overline{\theta})$ *satisfy* (11)–(13). *Then, there exist positive constants* C, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *depending on* Ω, $\mathcal{O}$, T, $\overline{y}$ *and* $\overline{\theta}$ *with* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and* $16\widetilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $g \in L^2(Q)^3$, $q \in L^2(Q)$, $\varphi^0 \in H$ *and* $\psi^0 \in L^2(\Omega)$, *the associated solution to* (20) *satisfies:*

$$(32) \quad \begin{cases} I(\varphi) + I(\psi) \leq C \left(\iint_Q e^{\frac{-4\overline{\alpha} + 2\widetilde{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30} |g|^2 \, dx\,dt \right. \\[2mm] \quad + \iint_Q e^{\frac{-32\overline{\alpha} + 30\widetilde{\alpha}}{t^4(T-t)^4}} t^{-252}(T-t)^{-252} |q|^2 \, dx\,dt \\[2mm] \quad + \iint_{\mathcal{O} \times (0,T)} e^{\frac{-16\overline{\alpha} + 14\widetilde{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132} |\varphi_2|^2 \, dx\,dt \\[2mm] \quad \left. + \iint_{\mathcal{O} \times (0,T)} e^{\frac{-32\overline{\alpha} + 30\widetilde{\alpha}}{t^4(T-t)^4}} t^{-268}(T-t)^{-268} |\psi|^2 \, dx\,dt \right). \end{cases}$$

Proof. Let us first recall a Carleman inequality for the solutions of (20) which has recently been proved in [12] (Proposition 1) whenever (12)–(13) are fulfilled:

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 (|\varphi|^2 + |\psi|^2) \, dx\,dt$$

$$+ s\lambda^2 \iint_Q e^{-2s\alpha} \xi(|\nabla\varphi|^2 + |\nabla\psi|^2) \, dx\,dt$$

$$(33) \quad + s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} \left(|\varphi_t|^2 + |\psi_t|^2 + |\Delta\varphi|^2 + |\Delta\psi|^2\right) dx\,dt$$

$$\leq C_5 (1 + T^2) \left(s^{15/2} \lambda^{24} \iint_Q e^{-4s\overline{\alpha} + 2s\alpha^*} \widehat{\xi}^{15/2} (|g|^2 + |q|^2) \, dx\,dt \right.$$

$$\left. + s^{16} \lambda^{48} \iint_{\mathcal{O}_0 \times (0,T)} e^{-8s\overline{\alpha} + 6s\alpha^*} \widehat{\xi}^{16} (|\varphi|^2 + |\psi|^2) \, dx\,dt \right).$$

Here, $s \geq s_1$ and $\lambda \geq \lambda_1$ are arbitrarily large and C_5, s_1 and λ_1 are suitable constants depending on Ω, $\mathcal{O}_0$, T, $\overline{y}$ and $\overline{\theta}$; see Proposition 1 in [12]. The proof of this inequality follows the same arguments employed in [7] to prove (23) and can be achieved without any further regularity on $\overline{y}$ or $\overline{\theta}$.

It is clear from (33) that, for some C_6, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}_0$, T, $\overline{y}$ and $\overline{\theta}$, we have:

$$
\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}\left(|\varphi|^2+|\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{4}(T-t)^{4}\left(|\Delta\varphi|^2+|\Delta\psi|^2+|\varphi_t|^2+|\psi_t|^2\right) dx\,dt \\
&\leq C_6\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\left(|g|^2+|q|^2\right) dx\,dt \right.\\
&\left. + \iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}\left(|\varphi|^2+|\psi|^2\right) dx\,dt\right).
\end{aligned}
\tag{34}
$$

Indeed, it suffices to take $\overline{\alpha}$ and $\widetilde{\alpha}$ as in (25) and

$$
C_6 = C_5(1+T^2)s_1^{17}\lambda_1^{48}e^{17\lambda_1(m+1)\|\eta^0\|_\infty}.
$$

We thus obtain $0 < \widetilde{\alpha} < \overline{\alpha}$ and, noticing that λ_1 is large enough, $16\widetilde{\alpha}-15\overline{\alpha} > 0$.

We apply (34) for the open set $\mathcal{O}_0$ defined in (26). Then we can argue as in subsection 2.1 and deduce that

$$
\begin{aligned}
&\iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}\,|\varphi_1|^2 dx\,dt \\
&\leq C_7\iint_{\mathcal{O}_1\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}\left(|\varphi_2|^2+|\varphi_3|^2\right) dx\,dt \\
&+ \varepsilon\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{4}(T-t)^{4}\left(|\Delta\varphi_2|^2+|\Delta\varphi_3|^2\right) dx\,dt,
\end{aligned}
$$

where $\mathcal{O}_1$ is an appropriate nonempty open set verifying

$$
\mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}, \quad d_1 := \operatorname{dist}(\overline{\mathcal{O}_1}, \partial\mathcal{O}\cap\Omega) > 0.
$$

This inequality combined with (34) yields:

$$
\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}\left(|\varphi|^2+|\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{4}(T-t)^{4}\left(|\Delta\varphi|^2+|\Delta\psi|^2+|\varphi_t|^2+|\psi_t|^2\right) dx\,dt \\
&\leq C_8\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\left(|g|^2+|q|^2\right) dx\,dt \right.\\
&+ \iint_{\mathcal{O}_1\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}\left(|\varphi_2|^2+|\varphi_3|^2\right) dx\,dt \\
&\left. + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}\,|\psi|^2 dx\,dt\right).
\end{aligned}
\tag{35}
$$

Our last task will be to estimate the integral

$$\iint_{\mathcal{O}_1 \times (0,T)} e^{\frac{-16\boldsymbol{a}+14\overline{\alpha}}{t^4(T-t)^4}} \, t^{-132}(T-t)^{-132} \, |\varphi_3|^2 \, dx \, dt$$

in terms of $\varepsilon I(\varphi_3)$ and local integrals of ψ and q. To do this, we set

$$\beta_1(t) = e^{\frac{-16\boldsymbol{a}+14\overline{\alpha}}{t^4(T-t)^4}} \, t^{-132}(T-t)^{-132}$$

and we introduce a function $\zeta_0 \in C^2(\overline{\mathcal{O}})$ such that

$$\zeta_0 \equiv 1 \ \text{in} \ \mathcal{O}_1, \quad 0 \le \zeta \le 1$$

and $\zeta_0(x) = 0$ at any point $x \in \mathcal{O}$ satisfying $\operatorname{dist}(x, \partial\mathcal{O} \cap \Omega) \le d_1/2$. From the differential equation satisfied by ψ (see (20)), we have

(36)
$$\iint_{\mathcal{O}_1 \times (0,T)} \beta_1(t) \, |\varphi_3|^2 \, dx \, dt \le \iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, |\varphi_3|^2 \, dx \, dt$$
$$= \iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, \varphi_3(-\psi_t - \Delta\psi - \overline{y} \cdot \nabla\psi - q) \, dx \, dt.$$

To end the proof, we perform integrations by parts in the last integral and pass all the derivatives from ψ to φ_3.

First, we integrate by parts in time taking into account that $\beta_1(0) = \beta_1(T) = 0$:

(37)
$$-\iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, \varphi_3 \, \psi_t \, dx \, dt$$
$$= \iint_{\mathcal{O} \times (0,T)} \beta_{1,t}(t) \, \zeta_0 \, \varphi_3 \, \psi \, dx \, dt + \iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, \varphi_{3,t} \, \psi \, dx \, dt$$
$$\le \varepsilon I(\varphi_3) + C_9(\varepsilon) \iint_{\mathcal{O} \times (0,T)} e^{\frac{-32\boldsymbol{a}+30\overline{\alpha}}{t^4(T-t)^4}} \, t^{-268}(T-t)^{-268} \, |\psi|^2 \, dx \, dt.$$

Next, we integrate by parts twice in space. Here, we use the properties of the cut-off function ζ and the Dirichlet boundary conditions for φ_3 and ψ:

(38)
$$-\iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, \varphi_3 \, \Delta\psi \, dx \, dt$$
$$= \iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, (-\Delta\zeta_0 \, \varphi_3 - 2\nabla\zeta_0 \cdot \nabla\varphi_3 - \zeta_0 \, \Delta\varphi_3) \, \psi \, dx \, dt$$
$$\le \varepsilon I(\varphi_3) + C_{10}(\varepsilon) \iint_{\mathcal{O} \times (0,T)} e^{\frac{-32\boldsymbol{a}+30\overline{\alpha}}{t^4(T-t)^4}} \, t^{-268}(T-t)^{-268} \, |\psi|^2 \, dx \, dt.$$

We also integrate by parts in the third term with respect to x and we use the incompressibility condition on $\overline{y}$:

(39)
$$-\iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \zeta_0 \, \varphi_3 \, \overline{y} \cdot \nabla\psi \, dx \, dt$$
$$= \iint_{\mathcal{O} \times (0,T)} \beta_1(t) \, \overline{y} \cdot (\varphi_3 \, \nabla\zeta + \zeta \, \nabla\varphi_3)\psi \, dx \, dt$$
$$\le \varepsilon I(\varphi_3) + C_{11}(\varepsilon) \iint_{\mathcal{O} \times (0,T)} e^{\frac{-32\boldsymbol{a}+30\overline{\alpha}}{t^4(T-t)^4}} \, t^{-260}(T-t)^{-260} \, |\psi|^2 \, dx \, dt.$$

We finally apply Young's inequality in the last term and we have:

(40)
$$-\iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3\,q\,dx\,dt$$
$$\leq \varepsilon I(\varphi_3) + C_{12}(\varepsilon)\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\boldsymbol{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-252}(T-t)^{-252}\,|q|^2\,dx\,dt.$$

From (35), (36) and (37)–(40), it is easy to deduce the desired inequality (32). □

Arguing as in subsection 2.1, that is to say, combining the previous result and the classical energy estimates satisfied by φ and ψ, we can deduce the following Carleman inequality.

LEMMA 5. *Assume that* $N = 3$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $(\overline{y},\overline{\theta})$ *satisfy* (11)–(13). *Then, there exist positive constants* C, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *depending on* Ω, $\mathcal{O}$, T, $\overline{y}$ *and* $\overline{\theta}$ *with* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and* $16\widetilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $g \in L^2(Q)^3$, $q \in L^2(Q)$, $\varphi^0 \in H$ *and* $\psi^0 \in L^2(\Omega)$, *the associated solution to* (20) *satisfies:*

(41)
$$\begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}}\left(\ell(t)^{-12}\left(|\varphi|^2+|\psi|^2\right)+\ell(t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)\right)\,dx\,dt \\[2mm] \qquad\leq C\left(\displaystyle\iint_Q e^{\frac{-4\boldsymbol{\alpha}+2\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-30}\,|g|^2\,dx\,dt\right. \\[2mm] \qquad\quad+\displaystyle\iint_Q e^{\frac{-32\boldsymbol{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-252}\,|q|^2\,dx\,dt \\[2mm] \qquad\quad+\displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\boldsymbol{\alpha}+14\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-132}\,|\varphi_2|^2\,dx\,dt \\[2mm] \qquad\quad\left.+\displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\boldsymbol{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-268}\,|\psi|^2\,dx\,dt\right), \end{cases}$$

where the function ℓ *was defined in* (30).

The similar result that can be established when $N = 2$ follows.

LEMMA 6. *Assume that* $N = 2$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $(\overline{y},\overline{\theta})$ *satisfy* (11)–(13). *Then, there exist positive constants* C, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *depending on* Ω, $\mathcal{O}$, T, $\overline{y}$ *and* $\overline{\theta}$ *with* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and* $16\widetilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $g \in L^2(Q)^2$, $q \in L^2(Q)$, $\varphi^0 \in H$ *and* $\psi^0 \in L^2(\Omega)$, *the associated solution to* (20) *satisfies:*

(42)
$$\begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}}\left(\ell(t)^{-12}\left(|\varphi|^2+|\psi|^2\right)+\ell(t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)\right)\,dx\,dt \\[2mm] \qquad\leq C\left(\displaystyle\iint_Q e^{\frac{-4\boldsymbol{\alpha}+2\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-30}\,|g|^2\,dx\,dt\right. \\[2mm] \qquad\quad+\displaystyle\iint_Q e^{\frac{-32\boldsymbol{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-252}\,|q|^2\,dx\,dt \\[2mm] \qquad\quad\left.+\displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\boldsymbol{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-268}\,|\psi|^2\,dx\,dt\right). \end{cases}$$

2.3. An observability estimate for system (21). We will prove an observability estimate for the system

$$
(43) \quad
\begin{cases}
-\rho_t - \Delta\rho - \nabla \times ((\overline{y} \cdot \nabla \times)\nabla\gamma) = 0, \quad \Delta\gamma = \rho & \text{in } Q, \\
\gamma = 0, \quad \rho = 0 & \text{on } \Sigma, \\
\rho(T) = \rho^0 & \text{in } \Omega.
\end{cases}
$$

This estimate will be implied by a Carleman inequality of the form

$$
S(\nabla\gamma) \leq C \iint_{\mathcal{O}\times(0,T)} |\nabla\gamma|^2 \, dx \, dt,
$$

where $S(\nabla\gamma)$ contains several global weighted integrals involving $\nabla\gamma$ (see (44)).

LEMMA 7. *Assume that $N = 2$ and $\overline{y} \in L^\infty(Q)^2$. There exist three positive constants C, $\overline{s}$ and $\overline{\lambda}$ depending on Ω, $\mathcal{O}$, T and $\overline{y}$ such that, for any $\rho^0 \in H^{-1}(\Omega)$, the associated solution to (43) satisfies:*

$$
(44) \quad
\begin{cases}
S(\nabla\gamma) := s^4\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^4|\nabla\gamma|^2 \, dx \, dt \\
\qquad + s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2 \, dx \, dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2 \, dx \, dt \\
\qquad \leq C\,s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2 \, dx \, dt,
\end{cases}
$$

for any $s \geq \overline{s}$ and any $\lambda \geq \overline{\lambda}$. Recall that α and ξ were defined in (17).

Proof. For the proof, s_j and λ_j $(j \geq 2)$ will denote various positive constants that can eventually depend on Ω, $\mathcal{O}$, T and $\overline{y}$.

Let $\mathcal{O}_0$ be a nonempty open set satisfying $\mathcal{O}_0 \subset\subset \mathcal{O}$ and let us apply to ρ a Carleman inequality for parabolic systems with right-hand sides in $L^2(0,T;H^{-1}(\Omega))$, originally proved in [15] (this version can be found in Lemma 2.1 of [6]):

$$
(45) \quad
\begin{aligned}
& s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2 \, dx \, dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2 \, dx \, dt \\
& \qquad \leq C_{13}\left(s^2\lambda^2\|\overline{y}\|_\infty^2 \iint_Q e^{-2s\alpha}\xi^2|\nabla(\nabla\times\gamma)|^2 \, dx \, dt \right. \\
& \qquad\qquad \left. + s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\rho|^2 \, dx \, dt \right),
\end{aligned}
$$

for any $s \geq s_2$ and $\lambda \geq \overline{\lambda}_2$.

Observe that, here, the assumption $\rho^0 \in H^{-1}(\Omega)$ may seem too weak to apply this result. Indeed, (45) can be proved as in [15] whenever $\rho \in C^1(\overline{Q})$ and, by a continuity argument, also for the solutions of problem (43) for which the left-hand side of (45) is finite. This is our case, since one can ensure that $\rho \in L^2(Q)$ as soon as $\rho^0 \in H^{-1}(\Omega)$ (for instance, taking into account the definition of ρ as the solution by transposition of (43)).

Once (45) has been justified, let us first estimate the last integral in its right-hand side. Thus, let $\zeta \in C^2(\overline{\mathcal{O}})$ be a cut-off function satisfying

$$
\zeta \equiv 1 \text{ in } \mathcal{O}_0, \quad 0 \leq \zeta \leq 1 \quad \text{and} \quad \zeta = 0 \text{ on } \partial\mathcal{O}.
$$

We have:

$$s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt \leq s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt$$

$$= -s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(\nabla\zeta\cdot\nabla\gamma)\Delta\gamma\,dx\,dt$$

$$-3s^3\lambda^5 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3(\nabla\eta^0\cdot\nabla\gamma)\Delta\gamma\,dx\,dt$$

$$+2s^4\lambda^5 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^4(\nabla\eta^0\cdot\nabla\gamma)\Delta\gamma\,dx\,dt$$

$$-s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3(\nabla\Delta\gamma\cdot\nabla\gamma)\,dx\,dt.$$

Now, we apply Young's inequality several times and we obtain

$$s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt$$

$$\leq C_{14}(\varepsilon)\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt$$

$$+\varepsilon\left(s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\Delta\gamma|^2\,dx\,dt\right),$$

for $s \geq s_3$ and $\lambda \geq \lambda_3$ and for any small positive constant ε. Combining this, the fact that $\rho = \Delta\gamma$, and (45), we get

$$s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt$$

(46)
$$\leq C_{15}\left(s^2\lambda^2\|\overline{y}\|_\infty^2 \iint_Q e^{-2s\alpha}\xi^2|\nabla(\nabla\times\gamma)|^2\,dx\,dt\right.$$

$$\left.+s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt\right)$$

for any $s \geq s_4$ and $\lambda \geq \lambda_4$.

Finally, we are going to estimate the first integral in the right-hand side of (45). To this end, let us notice that, for $j = 1$ and 2 and almost every $t \in (0,T)$, the function $\partial_j\gamma(t)$ satisfies:

$$\Delta(\partial_j\gamma)(t) = \partial_j\rho(t) \ \text{ in } \Omega.$$

Let us apply the main result in [14] to $\partial_j\gamma$. This yields the existence of two numbers $\widetilde{\tau} > 1$ and $\widetilde{\lambda} > 1$ such that

$$\tau^4\lambda^4 \int_\Omega e^{2\tau\eta}\eta^4|\partial_j\gamma|^2(t)\,dx + \tau^2\lambda^2 \int_\Omega e^{2\tau\eta}\eta^2|\nabla(\partial_j\gamma)|^2(t)\,dx$$

(47)
$$\leq C_{16}\left(\tau \int_\Omega e^{2\tau\eta}\eta|\partial_j\rho|^2(t)\,dx + \tau^4\lambda^4 \int_{\mathcal{O}} e^{2\tau\eta}\eta^4|\partial_j\gamma|^2(t)\,dx\right.$$

$$\left.+\tau^{5/2}\lambda^2\,e^{2\tau}\|\partial_j\gamma(t)\|_{H^{1/2}(\partial\Omega)}^2\right)$$

for $\tau \geq \widetilde{\tau}$ and $\lambda \geq \widetilde{\lambda}$. Here, we have introduced the function η, with

$$\eta(x) = e^{\lambda \eta^0(x)}.$$

In fact, the inequality one can find in [14] contains local integrals of $|\partial_j \gamma|^2$ and $|\nabla(\partial_j \gamma)|^2$ in the right-hand side. But it can be written for a smaller set $\mathcal{O}' \subset\subset \mathcal{O}$. Using localizing arguments together with the fact that we actually have a global weighted integral of $|\Delta(\partial_j \gamma)|^2$ in the left-hand side, (47) is easily found.

Following the same steps of [7], we set

$$\tau = \frac{s}{t^4(T-t)^4} \, e^{\lambda \, m \|\eta^0\|_\infty},$$

we multiply (47) by

$$\exp\left\{ -2s \frac{e^{5/4\lambda m \|\eta^0\|_\infty}}{t^4(T-t)^4} \right\}$$

and we integrate in time over $(0, T)$. This gives

$$s^4 \lambda^4 \iint_Q e^{-2s\alpha} \xi^4 |\partial_j \gamma|^2 \, dx \, dt + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |\nabla(\partial_j \gamma)|^2 \, dx \, dt$$

$$\leq C_{17} \left(s \iint_Q e^{-2s\alpha} \xi |\partial_j \rho|^2 \, dx \, dt + s^4 \lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^4 |\partial_j \gamma|^2 \, dx \, dt \right.$$

$$\left. + s^{5/2} \lambda^2 \int_0^T e^{-2s\alpha^*} (\xi^*)^{5/2} \|\partial_j \gamma\|_{H^{1/2}(\partial\Omega)}^2 \right)$$

for $s \geq s_5$ and $\lambda \geq \widetilde{\lambda}$. Combining this estimate and (46), we have

$$s^4 \lambda^4 \iint_Q e^{-2s\alpha} \xi^4 |\partial_j \gamma|^2 \, dx \, dt + s \lambda^2 \iint_Q e^{-2s\alpha} \, \xi |\nabla \rho|^2 \, dx \, dt$$

$$+ s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\rho|^2 \, dx \, dt \leq C_{18} \left(s^{5/2} \lambda^2 \int_0^T e^{-2s\alpha^*} (\xi^*)^{5/2} \|\partial_j \gamma\|_{H^{1/2}(\partial\Omega)}^2 \right.$$

$$\left. + s^5 \lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^5 |\nabla\gamma|^2 \, dx \, dt \right)$$

for any $s \geq s_6$ and $\lambda \geq \lambda_5$. On the other hand, the boundary term can readily be bounded using the continuity of the trace operator:

$$\|\partial_j \gamma(t)\|_{H^{1/2}(\partial\Omega)}^2 \leq C_{19}(\|\partial_j \gamma(t)\|_{L^2}^2 + \|\nabla(\partial_j \gamma)(t)\|_{L^2}^2).$$

Furthermore, since $\gamma|_\Sigma \equiv 0$, we know that there exists a positive constant C_{20} such that

$$\|\nabla(\partial_j \gamma)(t)\|_{L^2} \leq C_{20} \|\Delta\gamma(t)\|_{L^2} \quad \text{a.e. in } (0, T) \text{ for } j = 1, 2.$$

Consequently,

$$s^4 \lambda^4 \iint_Q e^{-2s\alpha} \xi^4 |\partial_j \gamma|^2 \, dx \, dt + s \lambda^2 \iint_Q e^{-2s\alpha} \, \xi |\nabla \rho|^2 \, dx \, dt$$

$$+ s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\rho|^2 \, dx \, dt \leq C_{21} \, s^5 \lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^5 |\nabla\gamma|^2 \, dx \, dt$$

for $s \geq s_6$.

This implies (44) and ends the proof of Lemma 7. $\quad\square$

Remark 3. An almost immediate consequence of the Carleman estimate (44) is the following observability inequality:

$$(48) \qquad \|(\nabla\gamma)(0)\|_{L^2}^2 \leq C \iint_{\mathcal{O}\times(0,T)} |\nabla\gamma|^2 \, dx \, dt.$$

Proof. All comes to prove a dissipation result for the L^2 norm of $\nabla\gamma$. Indeed, if we can prove that

$$(49) \qquad \|\nabla\gamma(t_1)\|_{L^2}^2 \leq C\|\nabla\gamma(t_2)\|_{L^2}^2 \quad \forall 0 \leq t_1 < t_2 \leq T,$$

then using the properties of the weight function $e^{-2s\alpha}$ and estimate (44), we readily deduce (48).

Thus, we multiply the equation in (43) by $-\gamma$ and we integrate in Ω. Taking into account that γ and ρ vanish on $\partial\Omega$, this yields:

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\gamma|^2 \, dx + \int_\Omega |\Delta\gamma|^2 \, dx - \int_\Omega ((\overline{y}\cdot\nabla\times)\nabla\gamma)\cdot\nabla\times\gamma \, dx = 0,$$

from which the dissipation estimate (49) follows. $\quad\square$

In fact, this is what will be used in section 3 to prove the null controllability of system (16).

3. Null controllability of the linearized systems (14), (15) and (16).

3.1. Null controllability of (14). We are dealing here with the following system:

$$(50) \qquad \begin{cases} y_t - \Delta y + (\overline{y}\cdot\nabla)y + (y\cdot\nabla)\overline{y} + \nabla p = f + v1_{\mathcal{O}}, & \nabla\cdot y = 0 \quad \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{O}$ satisfies (11) and $\overline{y}$ satisfies (12). Our goal will be to find a control v such that $y(T) = 0$ in Ω.

Let us introduce some weight functions:

$$\beta_2(t) = \exp\left\{\frac{\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^6, \quad \beta_3(t) = \exp\left\{\frac{2\widetilde{\alpha} - \overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{15}$$

and

$$\beta_4(t) = \exp\left\{\frac{8\widetilde{\alpha} - 7\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{66}$$

(recall that ℓ was defined in (30)), where $\overline{\alpha}$ and $\widetilde{\alpha}$ are the constants provided by Lemma 2 when $N = 3$ and Lemma 3 when $N = 2$. Recall that, in particular, $0 < \widetilde{\alpha} < \overline{\alpha}$ and $8\widetilde{\alpha} - 7\overline{\alpha} > 0$.

Of course, we will need some specific conditions on f and y^0 to get the null controllability of (50). We will use the arguments in [7].

Thus, let us set

$$(51) \qquad Ly = y_t - \Delta y + (\overline{y}, \nabla)y + (y, \nabla)\overline{y}$$

and let us introduce the spaces

$$E_2 = \{(y,p,v) : (y,v) \in E_0, \, \ell^{-4}\beta_2(Ly + \nabla p - v1_{\mathcal{O}}) \in L^2(0,T;H^{-1}(\Omega)^2)\}$$

when $N = 2$ and

$$E_3 = \{(y,p,v) : (y,v) \in E_0, \, \ell^{-2}\beta_2^{1/2}y \in L^4(0,T;L^{12}(\Omega)^3),$$
$$\ell^{-4}\beta_2(Ly + \nabla p - v1_{\mathcal{O}}) \in L^2(0,T;W^{-1,6}(\Omega)^3)\}$$

when $N = 3$, where

$$E_0 = \{(y,v) : \beta_3\, y, \, \beta_4\, v1_{\mathcal{O}} \in L^2(Q)^N, \, v_1 \equiv 0,$$
$$\ell^{-2}\beta_2^{1/2}y \in L^2(0,T;V) \cap L^\infty(0,T;H)\}.$$

It is clear that E_N is a Banach space for the norm $\|\cdot\|_{E_N}$, where

$$\|(y,p,v)\|_{E_2} = \Big(\|\beta_3\,y\|_{L^2}^2 + \|\beta_4\,v1_{\mathcal{O}}\|_{L^2}^2 + \|\ell^{-2}\beta_2^{1/2}y\|_{L^2(0,T;V)}^2$$
$$+ \|\ell^{-2}\beta_2^{1/2}y\|_{L^\infty(0,T;H)}^2 + \|\ell^{-4}\beta_2(Ly + \nabla p - v1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2\Big)^{1/2}$$

and

$$\|(y,p,v)\|_{E_3} = \Big(\|\beta_3\,y\|_{L^2}^2 + \|\beta_4\,v1_{\mathcal{O}}\|_{L^2}^2 + \|\ell^{-2}\beta_2^{1/2}y\|_{L^2(0,T;V)}^2$$
$$+ \|\ell^{-2}\beta_2^{1/2}y\|_{L^\infty(0,T;H)}^2 + \|\ell^{-2}\beta_2^{1/2}\,y\|_{L^4(0,T;L^{12})}^2$$
$$+ \|\ell^{-4}\beta_2(Ly + \nabla p - v1_{\mathcal{O}})\|_{L^2(0,T;W^{-1,6})}^2\Big)^{1/2}.$$

Remark 4. The spaces E_j $(j = 0,2,3)$ are natural spaces where solutions of the null controllability of (50) must be found in order to preserve these properties for the nonlinear term $(y \cdot \nabla)y$. More details are provided in subsection 4.1.

PROPOSITION 1. *Assume that $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $\bar{y}$ verify (11)–(12). Let $y^0 \in E$ and let us assume that*

$$\ell^{-4}\beta_2 f \in \begin{cases} L^2(0,T;H^{-1}(\Omega)^2) & \text{if } N = 2, \\ L^2(0,T;W^{-1,6}(\Omega)^3) & \text{if } N = 3. \end{cases}$$

Then, we can find a control v such that the associated solution (y,p) to (50) satisfies $(y,p,v) \in E_N$. In particular, $v_1 \equiv 0$ and $y(T) = 0$.

Sketch of the proof. The proof of this proposition is very similar to the one of Proposition 2 in [7], so we will just give the main ideas. For simplicity, we will only consider the case $N = 3$. When $N = 2$, the proof is even easier.

Following the arguments in [9] and [13], let us introduce the auxiliary optimal control problem

$$(52) \quad \begin{cases} \inf \dfrac{1}{2}\left(\displaystyle\iint_Q |\beta_3\,y|^2 dxdt + \iint_{\mathcal{O}\times(0,T)} |\beta_4\,v|^2 dxdt\right) \\[2mm] \text{subject to } v \in L^2(Q)^3, \text{ supp } v \subset \mathcal{O}\times(0,T), \, v_1 \equiv 0 \text{ and} \\[2mm] \begin{cases} Ly + \nabla p = f + v1_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y(T) = 0 & \text{in } \Omega. \end{cases} \end{cases}$$

Notice that a solution $(\widehat{y},\widehat{p},\widehat{v})$ to (52) is a good candidate to satisfy $(\widehat{y},\widehat{p},\widehat{v}) \in E_3$.

For the moment, let us assume that (52) possesses a solution $(\widehat{y}, \widehat{p}, \widehat{v})$. Then, by virtue of Lagrange's principle, there must exist dual variables $\widehat{z}$ and $\widehat{q}$ such that

$$(53) \quad \begin{cases} \widehat{y} = \beta_3^{-2}(L^*\widehat{z} + \nabla\widehat{q}), \quad \nabla \cdot \widehat{z} = 0 & \text{in } Q, \\ \widehat{v}_1 \equiv 0, \quad \widehat{v}_i = -\beta_4^{-2}\,\widehat{z}_i \quad (i = 2, 3) & \text{in } \mathcal{O} \times (0, T), \\ \widehat{z} = 0 & \text{on } \Sigma, \end{cases}$$

where L^* is the adjoint operator of L, i.e.,

$$L^*z = -z_t - \Delta z - (Dz)\,\overline{y}.$$

At least formally, the couple $(\widehat{z}, \widehat{q})$ satisfies

$$(54) \quad a((\widehat{z}, \widehat{q}), (w, h)) = \langle G, (w, h) \rangle \quad \forall (w, h) \in P_0,$$

where P_0 is the space

$$P_0 = \{(w, h) \in C^2(\overline{Q})^4 : \nabla \cdot w = 0, \ w = 0 \text{ on } \Sigma, \ \int_{\mathcal{O}} h(x, t)\, dx = 0\}$$

and we have used the notation

$$a((\widehat{z}, \widehat{q}), (w, h)) = \iint_Q \beta_3^{-2}\,(L^*\widehat{z} + \nabla\widehat{q}) \cdot (L^*w + \nabla h)\, dx\, dt$$

$$+ \iint_{\mathcal{O} \times (0, T)} \beta_4^{-2}\,(\widehat{z}_2\, w_2 + \widehat{z}_3\, w_3)\, dx\, dt$$

and

$$\langle G, (w, h) \rangle = \int_0^T \langle f(t), w(t) \rangle_{H^{-1}, H_0^1}\, dt + \int_\Omega y^0 \cdot w(0)\, dx.$$

Conversely, if we are able to "solve" (54) and then use (53) to define $(\widehat{y}, \widehat{p}, \widehat{v})$, we will probably have found a solution to (52).

Thus, let us consider the linear space P_0. It is clear that $a(\cdot, \cdot) : P_0 \times P_0 \mapsto \mathbf{R}$ is a symmetric, definite positive bilinear form on P_0. We will denote by P the completion of P_0 for the norm induced by $a(\cdot, \cdot)$. Then $a(\cdot, \cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman estimate (29), the linear form $(w, h) \mapsto \langle G, (w, h) \rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$(55) \quad \begin{cases} a((\widehat{z}, \widehat{q}), (w, h)) = \langle G, (w, h) \rangle \\ \forall (w, h) \in P, \quad (\widehat{z}, \widehat{q}) \in P, \end{cases}$$

possesses exactly one solution $(\widehat{z}, \widehat{q})$.

Let $\widehat{y}$ and $\widehat{v}$ be given by (53). Then, it is readily seen that they verify

$$\iint_Q \beta_3^2 |\widehat{y}|^2 dx\, dt + \iint_{\mathcal{O} \times (0, T)} \beta_4^2 |\widehat{v}|^2 dx\, dt < +\infty$$

and, also, that $\widehat{y}$ is, together with some pressure $\widehat{p}$, the weak solution (belonging to $L^2(0, T; V) \cap L^\infty(0, T; H)$) of the Stokes system in (52) for $v = \widehat{v}$.

In order to prove that $(\widehat{y},\widehat{p},\widehat{v}) \in E_3$, it only remains to check that $\ell^{-2}\,\beta_2^{1/2}\,\widehat{y}$ is, together with $\ell^{-2}\,\beta_2^{1/2}\,\widehat{p}$, a weak solution of a Stokes problem of the kind (50) with a right-hand side in $L^2(0,T;W^{-1,6}(\Omega)^3)$ that belongs to $L^4(0,T;L^{12}(\Omega)^3)$. To this end, we define the functions $y^* = \ell^{-2}\,\beta_2^{1/2}\,\widehat{y}$, $p^* = \ell^{-2}\,\beta_2^{1/2}\,\widehat{p}$ and $f^* = \ell^{-2}\,\beta_2^{1/2}(f+\widehat{v}1_{\mathcal{O}})$. Then (y^*,p^*) satisfies

(56)
$$\begin{cases} Ly^* + \nabla p^* = f^* + (\ell^{-2}\beta_2^{-1/2})_t\,\widehat{y}, & \nabla \cdot y^* = 0 & \text{in } Q, \\ y^* = 0 & & \text{on } \Sigma, \\ y^*(0) = \ell^{-2}(0)\beta_2^{1/2}(0)y^0 & & \text{in } \Omega. \end{cases}$$

From the fact that $f^* \in L^2(0,T;H^{-1}(\Omega)^3)$ and $y^0 \in H$, we have indeed

$$y^* \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

Finally, we deduce that $y^* \in L^4(0,T;L^{12}(\Omega)^3)$ from Lemma 2 in [7]. This ends the sketch of the proof of Proposition 1.   □

3.2. Null controllability of system (15). We will establish the null controllability of the linear system

(57)
$$\begin{cases} y_t - \Delta y + (\overline{y} \cdot \nabla)y + (y \cdot \nabla)\overline{y} + \nabla p = f + v1_{\mathcal{O}} + \theta\,e_N & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + \overline{y} \cdot \nabla\theta + y \cdot \nabla\overline{\theta} = k + h1_{\mathcal{O}} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{O}$ satisfies (11) and $\overline{y}$ and $\overline{\theta}$ satisfy (12) and (13), for suitable right-hand sides f and k.

The arguments we present here are completely analogous to those in [12] and subsection 3.1 of this paper, so that we will only give a sketch. Thus, we restrict ourselves again to the three-dimensional case with $n_1(x^0) \neq 0$.

Let us introduce the weight functions

$$\beta_5(t) = \exp\left\{\frac{\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^6, \quad \beta_6(t) = \exp\left\{\frac{2\widetilde{\alpha}-\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{15},$$

$$\beta_7(t) = \exp\left\{\frac{16\widetilde{\alpha}-15\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{126}, \quad \beta_8(t) = \exp\left\{\frac{8\widetilde{\alpha}-7\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{66}$$

and

$$\beta_9(t) = \exp\left\{\frac{16\widetilde{\alpha}-15\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{134},$$

where the constants $\overline{\alpha}$ and $\widetilde{\alpha}$ are furnished by Lemma 5 when $N=3$ and Lemma 6 when $N=2$ (and, in particular, $0 < \widetilde{\alpha} < \overline{\alpha}$ and $16\widetilde{\alpha}-15\overline{\alpha} > 0$).

Let us set

(58)
$$P\theta = \theta_t - \Delta\theta + \overline{y} \cdot \nabla\theta$$

and let us introduce the spaces

$$\widetilde{E}_2 = \{(y, p, \theta, v, h) : (y, \theta, v, h) \in \widetilde{E}_0,$$
$$\ell^{-4}\beta_5(Ly + \nabla p - v 1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega)^2),$$
$$\ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h 1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega))\}$$

when $N = 2$ and

$$\widetilde{E}_3 = \{(y, p, \theta, v, h) : (y, \theta, v, h) \in \widetilde{E}_0,$$
$$\ell^{-2}\beta_5^{1/2}y \in L^4(0, T; L^{12}(\Omega)^3),$$
$$\ell^{-4}\beta_5(Ly + \nabla p - v 1_{\mathcal{O}}) \in L^2(0, T; W^{-1,6}(\Omega)^3),$$
$$\ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h 1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega))\}$$

when $N = 3$, where

$$\widetilde{E}_0 = \{(y, \theta, v, h) : (\beta_6 y)_i, \beta_7 \theta, (\beta_8 v 1_{\mathcal{O}})_i, \beta_9 h 1_{\mathcal{O}} \in L^2(Q) \ (1 \le i \le N),$$
$$v_1 \equiv v_N \equiv 0, \ \ell^{-2}\beta_5^{1/2}y \in L^2(0, T; V) \cap L^{\infty}(0, T; H),$$
$$\ell^{-2}\beta_5^{1/2}\theta \in L^2(0, T; H_0^1(\Omega)) \cap L^{\infty}(0, T; L^2(\Omega))\}.$$

It can be readily seen now that $\widetilde{E}_0$, $\widetilde{E}_2$ and $\widetilde{E}_3$ are Banach spaces for the norms

$$\|(y, \theta, v, h)\|_{\mathbf{E}_0} = \left(\|\beta_6 y\|_{L^2}^2 + \|\beta_7 \theta\|_{L^2}^2 + \|\beta_8 v\|_{L^2}^2 \right.$$
$$+ \|\beta_9 h\|_{L^2}^2 + \|\ell^{-2}\beta_5^{1/2}y\|_{L^2(0,T;V)}^2 + \|\ell^{-2}\beta_5^{1/2}y\|_{L^{\infty}(0,T;H)}^2$$
$$\left. + \|\ell^{-2}\beta_5^{1/2}\theta\|_{L^2(0,T;H_0^1)}^2 + \|\ell^{-2}\beta_5^{1/2}\theta\|_{L^{\infty}(0,T;L^2)}^2 \right)^{1/2},$$

$$\|(y, p, \theta, v, h)\|_{\mathbf{E}_2} = \left(\|(y, \theta, v, h)\|_{\mathbf{E}_0}^2 \right.$$
$$+ \|\ell^{-4}\beta_5(Ly + \nabla p - v 1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2$$
$$\left. + \|\ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h 1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2 \right)^{1/2}$$

and

$$\|(y, p, \theta, v, h)\|_{\mathbf{E}_3} = \left(\|(y, \theta, v, h)\|_{\mathbf{E}_0}^2 + \|\ell^{-2}\beta_5^{1/2}y\|_{L^4(0,T;L^{12})}^2 \right.$$
$$+ \|\ell^{-4}\beta_5(Ly + \nabla p - v 1_{\mathcal{O}})\|_{L^2(0,T;W^{-1,6})}^2$$
$$\left. + \|\ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h 1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2 \right)^{1/2}.$$

PROPOSITION 2. *Assume that* $n_1(x^0) \ne 0$ *and* $\mathcal{O}$ *and* $(\overline{y}, \overline{\theta})$ *satisfy* (11)–(13). *Let* $y^0 \in E$, $\theta^0 \in L^2(\Omega)$ *and let us assume that*

$$\ell^{-4}\beta_1(f, k) \in \begin{cases} L^2(0, T; H^{-1}(\Omega)^2) \times L^2(0, T; H^{-1}(\Omega)) & \text{if } N = 2, \\ L^2(0, T; W^{-1,6}(\Omega)^3) \times L^2(0, T; H^{-1}(\Omega)) & \text{if } N = 3. \end{cases}$$

Then, we can find controls v and h such that the associated solution to (57) *satisfies* $(y, p, \theta, v, h) \in \widetilde{E}_N$. *In particular,* $v_1 \equiv v_N \equiv 0$ *and* $y(T) = \theta(T) = 0$.

We omit the proof of this proposition, since it is essentially the same as the one of Proposition 2 in [12] and follows the steps of Proposition 1 above. As we have already indicated, the main ideas come from [13].

3.3. Null controllability of system (16). We will prove the null controllability of the linear system

(59)
$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)\overline{y} + \nabla p = v 1_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

where $N = 2$ and $\overline{y} \in L^\infty(Q)^2$.

For this purpose, we first rewrite this system using the streamline-vorticity formulation. Thus, setting $\omega = \nabla \times y$, we have

(60)
$$\begin{cases} \omega_t - \Delta\omega + \nabla \times ((\nabla \times \psi \cdot \nabla)\overline{y}) = \nabla \times (v 1_{\mathcal{O}}), \quad \Delta\psi = \omega & \text{in } Q, \\ \psi = 0, \quad \omega = 0 & \text{on } \Sigma, \\ \omega(0) = \nabla \times y^0 & \text{in } \Omega. \end{cases}$$

PROPOSITION 3. *Assume that $y^0 \in H$ and $\overline{y} \in L^\infty(Q)^2$. Then, there exists a constant $C(\Omega, \mathcal{O}, T) > 0$ and controls $v 1_{\mathcal{O}}$ with $v \in W$ (W was defined in (9)), such that*

(61)
$$\|v\|_{L^2} \le C\|y^0\|_H$$

and the associated solutions of (59) *satisfy*

(62) $y \in L^2(0, T; H^1(\Omega)^2) \cap C^0([0, T]; L^2(\Omega)^2), \quad y_t \in L^2(0, T; H^{-1}(\Omega)^2),$

and $y(T) = 0$, with

(63) $\|y\|_{L^2(0,T;H^1)} + \|y\|_{C^0([0,T];L^2)} + \|y_t\|_{L^2(0,T;H^{-1})} \le C\|y^0\|_H.$

Proof. We first establish the null controllability property for y. This can be done in several ways. One of them is the following. We first define for each $\varepsilon > 0$ the functional

$$\begin{cases} J_\varepsilon(\gamma^0) = \dfrac{1}{2} \iint_{\mathcal{O} \times (0,T)} |\nabla \times \gamma|^2 \, dx \, dt + \varepsilon \|\nabla\gamma^0\|_{L^2} + ((\nabla \times \gamma)(0), y^0)_{L^2} \\ \forall \gamma^0 \in H_0^1(\Omega), \end{cases}$$

where γ is given by (43) with $\rho^0 = \Delta\gamma^0 \in H^{-1}(\Omega)$.

It is not difficult to see from the observability inequality (48) that this functional possesses a unique minimizer $\gamma_\varepsilon^0 \in H_0^1(\Omega)$ (see Proposition 2.1 in [5]). Now, from the necessary conditions for J_ε to reach a minimum, we have

(64)
$$\iint_Q ((\nabla \times \gamma_\varepsilon) 1_{\mathcal{O}}) \cdot (\nabla \times \gamma) \, dx \, dt + \varepsilon \left(\frac{\nabla \times \gamma_\varepsilon^0}{\|\nabla \times \gamma_\varepsilon^0\|_{L^2}}, \nabla \times \gamma^0 \right)_{L^2}$$
$$+ ((\nabla \times \gamma)(0), y^0)_{L^2} = 0 \quad \forall \gamma^0 \in H_0^1(\Omega).$$

Thus, setting $v_\varepsilon = (\nabla \times \gamma_\varepsilon) 1_{\mathcal{O}}$ and putting $\gamma^0 = \gamma_\varepsilon^0$, we find from (48) and (64) that (61) holds for v_ε for some C independent of ε:

(65)
$$\|\nabla \times \gamma_\varepsilon\|_{L^2(\mathcal{O} \times (0,T))^2} = \|v_\varepsilon\|_{L^2(\mathcal{O} \times (0,T))^2} \le C.$$

Let us denote by $(\omega_\varepsilon, \psi_\varepsilon)$ the solution to (60) for $v = v_\varepsilon$. Then, taking into account the systems satisfied by (ρ, γ) and $(\omega_\varepsilon, \psi_\varepsilon)$, we deduce that

$$\iint_Q \nabla \times (v_\varepsilon 1_{\mathcal{O}}) \, \gamma \, dx \, dt + (\nabla \times \gamma^0, (\nabla \times \psi_\varepsilon)(T))_{L^2}$$
$$-((\nabla \times \gamma)(0), y^0)_{L^2} = 0 \quad \forall \gamma^0 \in H_0^1(\Omega).$$

Combining this and (64), we obtain

(66) $$\|(\nabla \times \psi_\varepsilon)(T)\|_{L^2} \leq \varepsilon.$$

From (65) and (66) written for each $\varepsilon > 0$, we deduce that, at least for a subsequence, $v_\varepsilon \to v$ weakly in $L^2(\mathcal{O} \times (0,T))^2$, where the control $v1_{\mathcal{O}}$ is such that the corresponding solution (ω, ψ) to (60) satisfies

$$(\nabla \times \psi)(T) = y(T) = 0 \quad \text{in } \Omega.$$

Since $v \in L^2(\mathcal{O} \times (0,T))^2$ and $\nabla \cdot v = 0$ in $\mathcal{O} \times (0,T)$, we necessarily have $v \in W$ (from De Rham's lemma applied to $(v_2, -v_1)$).

In order to obtain the desired regularity for y, we will consider again the equations satisfied by ψ and ω and we will check that

(67) $$\psi \in L^2(0,T; H^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega)) \quad \text{and} \quad \psi_t \in L^2(Q),$$

with appropriate estimates.

For simplicity, we will only present the estimates. The rigorous argument relies on introducing a standard Galerkin approximation of (60) with a "special" basis of $H_0^1(\Omega)$ (more precisely, the basis formed by the eigenfunctions of the Laplacian-Dirichlet operator in Ω) and deducing for the associated approximate solutions the estimates below.

Thus, let us multiply the first equation in (60) by ψ and let us integrate by parts. We find that

$$\frac{1}{2} \int_\Omega |\nabla \psi(t)|^2 \, dx + \int_0^t \int_\Omega |\Delta \psi|^2 \, dx \, d\tau = \int_0^t \int_{\mathcal{O}} v \cdot (\nabla \times \psi) \, dx \, d\tau$$
$$- \int_0^t \int_\Omega (((\nabla \times \psi), \nabla) \, \overline{y}) \cdot (\nabla \times \psi) \, dx \, d\tau + \frac{1}{2} \|(\nabla \psi)(0)\|_{L^2}^2$$

for all $t \in (0,T)$. If we integrate by parts in the last integral, we also have

$$- \int_0^t \int_\Omega (((\nabla \times \psi) \cdot \nabla) \, \overline{y}) \cdot (\nabla \times \psi) \, dx \, d\tau$$
$$= \int_0^t \int_\Omega ((\nabla \times \psi) \cdot \nabla)(\nabla \times \psi) \cdot \overline{y} \, dx \, d\tau.$$

Since $\psi|_\Sigma \equiv 0$, we deduce that

(68) $$\psi \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega))$$

and

(69) $$\|\psi\|_{L^2(0,T;H^2)} + \|\psi\|_{L^\infty(0,T;H_0^1)} \leq C\|y^0\|_{L^2}.$$

Now, let us introduce for each t the function $\psi^*(t) = \Delta^{-1}\psi_t(t)$, i.e., the solution to

$$
\begin{cases}
-\Delta\psi^*(t) = \psi_t(t) & \text{in } \Omega \\
\psi^*(t) = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Observe that, whenever $\psi_t(t) \in L^2(\Omega)$, this function satisfies $\psi^*(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$
(70) \qquad\qquad \|\psi^*(t)\|_{H^2} \le C\|\psi_t(t)\|_{L^2}.
$$

Then, we multiply the first equation of (60) by ψ^* and we integrate by parts. This gives

$$
\iint_Q |\psi_t|^2\,dx\,dt = \iint_Q (\Delta\psi)\,\psi_t\,dx\,dt - \iint_Q ((\nabla\times\psi)\cdot\nabla)(\nabla\times\psi^*)\cdot\overline{y}\,dx\,dt
$$
$$
+ \iint_{\mathcal{O}\times(0,T)} v\cdot(\nabla\times\psi^*)\,dx\,dt.
$$

Using that $v \in L^2(Q)^2$ and we already have (69) and (70), we conclude that $\psi_t \in L^2(Q)$ and

$$
(71) \qquad\qquad \|\psi_t\|_{L^2} \le C\|y^0\|_{L^2}.
$$

From (69) and (71), we immediately obtain (67), (62) and (63).

This ends the proof of Proposition 3.   □

4. Proofs of the controllability results for the nonlinear systems. In this last section, we will give the proofs of Theorems 1, 2 and 3. For the proofs of Theorems 1 and 2 we employ an inverse mapping theorem, while a fixed point argument is used for Theorem 3.

4.1. Proof of Theorem 1. We also follow here the steps in [7].

Thus, we set $y = \overline{y} + z$ and $p = \overline{p} + \chi$ and we use these identities in (1). Taking into account that $(\overline{y}, \overline{p})$ solves (5), we find:

$$
(72) \qquad
\begin{cases}
Lz + (z\cdot\nabla)z + \nabla\chi = v1_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{in } Q, \\
z = 0 & \text{on } \Sigma, \\
z(0) = y^0 - \overline{y}^0 & \text{in } \Omega
\end{cases}
$$

(recall that L was defined in (51)).

This way, we have reduced our problem to a local null controllability result for the solution (z, χ) to the *nonlinear* problem (72).

We will use the following inverse mapping theorem (see [1]).

THEOREM 4. *Let B_1 and B_2 be two Banach spaces and let $\mathcal{A}: B_1 \mapsto B_2$ satisfy $\mathcal{A} \in C^1(B_1; B_2)$. Assume that $b_0 \in B_1$, $\mathcal{A}(b_0) = d_0$ and also that $\mathcal{A}'(b_0): B_1 \mapsto B_2$ is surjective. Then there exists $\delta > 0$ such that, for every $d \in B_2$ satisfying $\|d - d_0\|_{B_2} < \delta$, there exists a solution of the equation*

$$
\mathcal{A}(b) = d, \quad b \in B_1.
$$

We will apply this result with $B_1 = E_N$,

$$B_2 = \begin{cases} L^2(\ell^{-4}\beta_2; 0, T; H^{-1}(\Omega)^2) \times H & \text{if } N = 2, \\ L^2(\ell^{-4}\beta_2; 0, T; W^{-1,6}(\Omega)^3) \times (H \cap L^4(\Omega)^3) & \text{if } N = 3 \end{cases}$$

and

$$\mathcal{A}(z, \chi, v) = (Lz + (z \cdot \nabla)z + \nabla\chi - v1_{\mathcal{O}}, z(0)) \quad \forall(z, \chi, v) \in E_N.$$

From the facts that $\ell^{-2}\beta_2^{1/2}y \in L^4(0, T; L^{12}(\Omega)^3)$ and $\mathcal{A}$ is bilinear, it is not difficult to check that $\mathcal{A} \in C^1(B_1; B_2)$; more details can be found in [13] or [7].

Let b_0 be the origin in B_1. Notice that $\mathcal{A}'(0, 0, 0) : B_1 \mapsto B_2$ is given by

$$\mathcal{A}'(0, 0, 0)(z, \chi, v) = (Lz + \nabla\chi - v1_{\mathcal{O}}, z(0)) \quad \forall(z, \chi, v) \in E_N$$

and is surjective, in view of the null controllability result for (14) given in Proposition 1.

Consequently, we can indeed apply theorem 4 with these data and there exists $\delta > 0$ such that, if $\|z(0)\|_E \le \delta$, then we find a control v satisfying $v_1 \equiv 0$ such that the associated solution to (72) verifies $z(T) = 0$ in Ω.

This concludes the proof of Theorem 1.

4.2. Proof of Theorem 2. Again, we follow here the ideas of [12].

Therefore, we set $y = \overline{y} + z$, $p = \overline{p} + \chi$ and $\theta = \overline{\theta} + \rho$, so from (2) and (7), we find:

$$(73) \quad \begin{cases} Lz + (z \cdot \nabla)z + \nabla\chi = v1_{\mathcal{O}} + \rho\, e_N, \quad \nabla \cdot z = 0 & \text{in } Q, \\ P\rho + (z \cdot \nabla)\rho + z \cdot \nabla\overline{\theta} = h1_{\mathcal{O}} & \text{in } Q, \\ z = 0, \quad \rho = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \overline{y}^0, \quad \rho(0) = \theta^0 - \overline{\theta}(0) & \text{in } \Omega \end{cases}$$

(L and P were respectively defined in (51) and (58)).

We are thus led to prove the local null controllability of (73). To this end, we will use again Theorem 4, which was presented in subsection 4.1. Using the same notation as there, we set $B_1 = \widetilde{E}_N$,

$$B_2 = L^2(\ell^{-4}\beta_5; 0, T; H^{-1}(\Omega)^3) \times H \times L^2(\Omega)$$

if $N = 2$ and

$$B_2 = L^2(\ell^{-4}\beta_5; 0, T; W^{-1,6}(\Omega)^3 \times H^{-1}(\Omega)) \times (L^4(\Omega)^3 \cap H) \times L^2(\Omega)$$

if $N = 3$.

Let us introduce $\mathcal{A}$, with

$$\mathcal{A}(z, \chi, \rho, v, h) = (\mathcal{A}_1(z, \chi, \rho, v), \mathcal{A}_2(z, \rho, h), z(0), \rho(0)),$$

$$\mathcal{A}_1(z, \chi, \rho, v) = Lz + (z \cdot \nabla)z + \nabla\chi - v1_{\mathcal{O}} - \rho e_N$$

and

$$\mathcal{A}_2(z, \rho, h) = P\rho + (z \cdot \nabla)\rho + z \cdot \nabla\overline{\theta} - h1_{\mathcal{O}}$$

for every $(z, \chi, \rho, v, h) \in \widetilde{E}_N$.

Using the fact that $\ell^{-2}\beta_5^{1/2} z \in L^4(0,T;L^{12}(\Omega)^3)$, it can be checked that $\mathcal{A}_1$ is C^1. Then, since $\ell^{-2}\beta_5^{1/2}\rho \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and this space is continuously embedded in $L^4(0,T;L^3(\Omega))$, we deduce that

$$\ell^{-4}\beta_5(z,\nabla)\rho = \nabla \cdot (z\,\rho) \in L^2(0,T;W^{-1,12/5}(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$$

and, consequently, $\mathcal{A}$ is well-defined and satisfies $\mathcal{A} \in C^1(B_1;B_2)$.

The fact that $\mathcal{A}'(0,0,0,0,0) : B_1 \longmapsto B_2$ is surjective is an immediate consequence of the result given in Proposition 2.

As a conclusion, we can apply Theorem 4 and the null controllability for system (73) holds.

4.3. Proof of Theorem 3. Let us recall the nonlinear system we are dealing with:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)\mathbf{T}_M(y) + \nabla p = v1_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

In this case, we are going to apply Kakutani's fixed point theorem (see, for instance, [2]).

THEOREM 5. *Let Z be a Banach space and let $\Lambda : Z \mapsto Z$ be a set-valued mapping satisfying the following assumptions:*

- *$\Lambda(z)$ is a nonempty closed convex set of Z for every $z \in Z$.*
- *There exists a convex compact set $K \subset Z$ such that $\Lambda(K) \subset K$.*
- *Λ is upper-hemicontinuous in Z, i.e., for each $\sigma \in Z'$ the single-valued mapping*

$$(74) \qquad z \mapsto \sup_{y \in \Lambda(z)} \langle \sigma, y \rangle_{Z',Z}$$

is upper-semicontinuous.

Then Λ possesses a fixed point in the set K, i.e., there exists $z \in K$ such that $z \in \Lambda(z)$.

In order to apply this result, we set $Z = L^2(Q)^2$ and, for each $z \in Z$, we consider the following system:

$$(75) \qquad \begin{cases} y_t - \Delta y + (y \cdot \nabla)\mathbf{T}_M(z) + \nabla p = v1_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

Then, for each $z \in Z$, we denote by $A(z)$ the set of controls $v1_{\mathcal{O}}$ with $v \in W$ that drive system (75) to zero and satisfy (61). Finally, our set-valued mapping is given as follows: for each $z \in Z$, $\Lambda(z)$ is the set of functions y that solve, together with some p, the linear system (75) corresponding to a control $v \in A(z)$.

Let us check that the assumptions of Theorem 5 are satisfied in this setting. The first one holds easily, so we omit the proof. Next, the estimates (62) and (63) tell us that the whole space Z is actually mapped into a compact set.

Let us finally see that Λ is upper-hemicontinuous in Z. Assume that $\sigma \in Z'$ and let $\{z_n\}$ be a sequence in Z such that $z_n \to z$ in Z. We have to prove that

$$(76) \qquad \limsup_{n \to \infty} \sup_{y \in \Lambda(z_n)} \langle \sigma, y \rangle_{Z',Z} \leq \sup_{y \in \Lambda(z)} \langle \sigma, y \rangle_{Z',Z}.$$

Let us choose a subsequence $\{z_{n'}\}$ such that

$$(77) \qquad \limsup_{n \to \infty} \sup_{y \in \Lambda(z_n)} \langle \sigma, y \rangle_{Z',Z} = \lim_{n' \to \infty} \sup_{y \in \Lambda(z_{n'})} \langle \sigma, y \rangle_{Z',Z}.$$

From the fact that $\Lambda(z_{n'})$ is a compact set of Z, for each n' we have

$$\sup_{y \in \Lambda(z_{n'})} \langle \sigma, y \rangle_{Z',Z} = \langle \sigma, y_{n'} \rangle_{Z',Z}$$

for some $y_{n'} \in \Lambda(z_{n'})$. Obviously, it can be assumed that

$$(78) \qquad z_{n'}(x,t) \to z(x,t) \text{ a.e. } (x,t) \in Q$$

and

$$(79) \qquad v_{n'} \rightharpoonup v \text{ weakly in } L^2(Q)^2$$

with $v \in A(z)$. Furthermore, since all the $y_{n'}$ belong to a fixed compact set, we can also assume that

$$y_{n'} \to y \text{ in } Z$$

(after extraction of a subsequence). This, together with (77)–(79) implies that $y \in \Lambda(z)$, since we have a Stokes system with a right-hand side weakly converging in L^2 and a coefficient converging almost everywhere. As a conclusion, (76) holds and the proof of Theorem 3 is achieved.

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