Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 23, 2004, 275–300

COMBINING FAST, LINEAR AND SLOW DIFFUSION

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Abstract. Although the pioneering studies of G. I. Barenblatt ([8]) and A. G. Aronson and L. A. Peletier ([7]) did result into a huge industry around the porous media equation, none further study analyzed the effect of combining fast, slow, and linear diffusion simultaneously, in a spatially heterogeneous porous medium. Actually, it might be this is the first work where such a problem has been addressed. Our main findings show how the heterogeneous model possesses two different regimes in the presence of a priori bounds. The minimal steady-state of the model exhibits a genuine *fast diffusion behavior*, whereas the remaining states are rather reminiscent of the purely *slow diffusion model*. The mathematical treatment of these heterogeneous problems should deserve a huge interest from the point of view of its applications in fluid dynamics and population evolution.

1. Introduction

In this paper we study the positive solutions of the boundary value problem

$$\begin{cases} -\Delta(w^{\mathfrak{m}(x)}) = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain of class $\mathcal{C}^2$, $\lambda \in \mathbb{R}$, and

$$\mathfrak{m} = 1 + p\chi_{\Omega_+} - q\chi_{\Omega_-}$$

2000 *Mathematics Subject Classification*. 35B32, 35J25, 35J60, 35K57.

Key words and phrases. Heterogeneous nonlinear diffusion, fast, slow and linear diffusion.

The research of the first named author supported by the Spanish Ministry of Science and Technology under Grants BFM2000-0797 and BFM2003-06466.

The research of the second named author was as well supported by REN2003-00707.

where Ω_+ and Ω_- are two subdomains of Ω of class $\mathcal{C}^2$ such that

$$\overline{\Omega}_+ \subset \Omega, \quad \overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset, \tag{1.2}$$

and $p \in L^\infty(\Omega_+) \cap \mathcal{C}(\Omega_+)$, $q \in L^\infty(\Omega_-) \cap \mathcal{C}(\Omega_-)$ satisfy

$$p(x) > 0 \quad \text{and} \quad 0 < q(y) < 1 \quad \text{for each } (x, y) \in \Omega_+ \times \Omega_-. \tag{1.3}$$

Throughout this paper, for any measurable set $M \subset \Omega$, we denote by χ_M the characteristic function of M, i.e. $\chi_M(x) = 1$ if $x \in M$, and $\chi_M(x) = 0$ for each $x \in \Omega \setminus M$. Also, we set

$$\Omega_1 := \Omega \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-),$$

the open set where $\mathfrak{m} = 1$, and suppose, by simplicity, that Ω_1 is connected. Though we allow Ω_1, Ω_+, or Ω_- to be empty, Figure 1.1 shows one of the admissible configurations dealt with in this work.

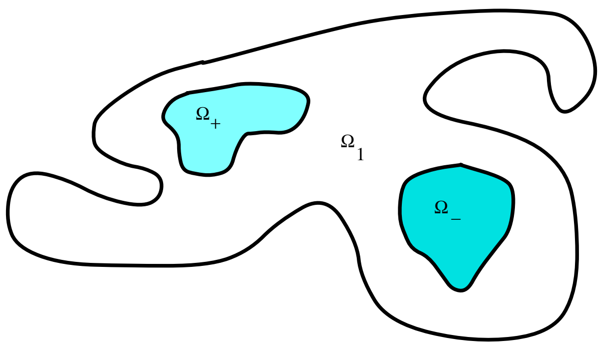

Figure 1.1. An admissible configuration

Throughout this paper we denote

$$m_+ := \mathfrak{m}|_{\Omega_+} = 1 + p\chi_{\Omega_+}, \quad m_- := \mathfrak{m}|_{\Omega_-} = 1 - q\chi_{\Omega_-}. \tag{1.4}$$

Then, $m_+(x) > 1$ for each $x \in \Omega_+$ and $0 < m_-(x) < 1$ for each $x \in \Omega_-$, and hence (1.1) provides us with the steady states of a porous medium equation where diffusion is linear in Ω_1 and nonlinear in $\Omega_+ \cup \Omega_-$ (slow in Ω_+ and fast in Ω_-). The analysis of these kind of boundary value problems generated a huge industry since the pioneering studies of G. I. Barenblatt ([8]) and A. G. Aronson, L. A. Peletier ([7]), although most of the literature treated the very special case when $\mathfrak{m}$ is constant.

Up to the best of our knowledge, the first work where $\mathfrak{m}$ has been allowed to vary is M. Delgado et al. in [12], where the special case when $m_- = 0$ was treated. The present paper seems to be the first work where the general problem of analyzing the interplay between slow, fast and linear diffusion, simultaneously, has been addressed. Therefore, most of the results found in this paper are completely new and, undoubtedly, open new research directions that might be of great relevance from the point of view of the applications of the underlying abstract mathematical theory to population dynamics and porous media dynamics. To summarize our main results we need to introduce some basic concepts and notations.

As the change of variable $u = w^{\mathfrak{m}}$ transforms (1.1) into

$$\begin{cases} -\Delta u = \lambda u^{1/\mathfrak{m}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

our efforts will be focused into the problem of analyzing the existence and multiplicity of positive solutions of (1.5). A function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is said to be a solution of (1.5) if $u^{1/\mathfrak{m}} \in L^{2N/(N+2)}(\Omega)$ and it satisfies the equation in the classical weak sense. By elliptic regularity, any weak non-negative solution $u \neq 0$ provides us with an strong solution almost everywhere twice differentiable in Ω and, as a result of the strong maximum principle, $u(x) > 0$ for each $x \in \Omega$ and $\partial u(x)/\partial n < 0$ for each $x \in \partial\Omega$, where n stands for the outward normal vector-field of Ω. In the remaining of this paper, it should be kept in mind that, as a result of the maximum principle, (1.5) cannot admit a positive solution if $\lambda \leq 0$.

Throughout the rest of this paper, for any potential $V \in L^\infty(\Omega)$ we denote by $\sigma[-\Delta + V; \Omega]$ the principal eigenvalue of $-\Delta + V$ in Ω under homogeneous Dirichlet boundary conditions. Note that if

$$\Omega_+ = \Omega_- = \emptyset,$$

then (1.5) becomes linear and, hence, it possesses a positive solution if, and only if, $\lambda = \sigma[-\Delta; \Omega]$. Therefore, we subsequently assume

$$\Omega_+ \cup \Omega_- \neq \emptyset. \tag{1.6}$$

Although most of our findings are completely new even in the special case when $\mathfrak{m} - 1$ does not change of sign, the most interesting results of this paper are those found for the general case when $\mathfrak{m} - 1$ changes sign, where one must assume m_+ and m_- to be constant to get optimal results. Under these assumptions our main result is Theorem 4.1, which can be rewritten as follows.

THEOREM 1.1. *Suppose Ω_+ and Ω_- are non-empty and m_+, m_- are constant. Then, there exist $\lambda^* > 0$ and an unbounded component, $\mathfrak{C}$, of the set of positive solutions (λ, u) of (1.5) such that:*

(a) $(\lambda, u) = (0, 0) \in \overline{\mathfrak{C}}$, *and* $\Lambda := \mathcal{P}_\lambda \mathfrak{C} \in \{(0, \lambda^*], (0, \lambda^*)\}$, *for* $\mathcal{P}_\lambda(\lambda, u) := \lambda$.

(b) *Problem* (1.5) *does not admit a positive solution if* $\lambda \in (-\infty, 0] \cup (\lambda^*, \infty)$.

(c) *For each* $\lambda \in \Lambda$, (1.5) *possesses a minimal positive solution, denoted by* θ_λ, *and the map* $\lambda \mapsto \theta_\lambda$ *is smooth and increasing. Moreover,*

$$\sigma[-\Delta - (\lambda/\mathfrak{m})\theta_\lambda^{1/\mathfrak{m}-1};\Omega] > 0 \quad \text{if } \lambda \in \Lambda \setminus \{\lambda^*\},$$

i.e. θ_λ *is linearly asymptotically stable with respect to the parabolic counterpart of* (1.5), *while*

$$\sigma[-\Delta - (\lambda^*/\mathfrak{m})\theta_{\lambda^*}^{1/\mathfrak{m}-1};\Omega] = 0 \quad \text{if } \lambda^* \in \Lambda,$$

i.e. θ_{λ^*} *is linearly neutrally stable.*

(d) *The component* $\mathfrak{C}$ *contains the arc of differentiable curve* $\Gamma := \{(\lambda,\theta_\lambda) : \lambda \in \Lambda \setminus \{\lambda^*\}\}$, $\lim_{\lambda\downarrow 0}\|\theta_\lambda\|_{\mathcal{C}_0(\overline{\Omega})} = 0$, *and* $\lim_{\lambda\uparrow\lambda^*}\theta_\lambda = \theta_{\lambda^*}$ *if* $\Lambda = (0,\lambda^*]$, *while* $\lim_{\lambda\uparrow\lambda^*}\|\theta_\lambda\|_{\mathcal{C}_0(\overline{\Omega})} = \infty$ *if* $\Lambda = (0,\lambda^*)$. *Actually,* $\mathfrak{C} = \Gamma$ *if* $\Lambda = (0,\lambda^*)$.

(e) *If* $\Lambda = (0,\lambda^*]$, *then there exists* $\lambda_\omega \in [0,\lambda^*)$ *such that* (1.5) *has two positive solutions, at least, for each* $\lambda \in (\lambda_\omega,\lambda^*)$. *Actually, if either* $N \in \{1,2\}$, *or* $N \ge 3$ *and* $m_- > (N-2)/(N+2)$, *then* $\Lambda = (0,\lambda^*]$ *and* $\lambda_\omega = 0$.

(f) *For each* $\lambda \in \Lambda$, θ_λ *provides us with the unique linearly stable positive solution of* (1.5).

The distribution of this paper is the following: Section 2 analyzes the case when $\Omega_+ = \emptyset$, Section 3 analyzes the case when $\Omega_- = \emptyset$, and, then, in Section 4, we prove Theorem 1.1. Throughout the manuscript we shortly describe some special perturbation results connecting each of these cases with the remaining one, though we have refrained to include the details of all their proofs to keep the length of the manuscript within a reasonable level. All those results will be deeply discussed and collected elsewhere.

2. The case $\Omega_+ = \emptyset$

As we are assuming (1.6), we have $\Omega_- \neq \emptyset$ and, hence, (1.5) is superlinear within Ω_-. The following result holds in the special case when $\Omega_1 = \emptyset$.

Theorem 2.1. *Suppose* $\Omega_+ = \Omega_1 = \emptyset$. *Then, the following assertions are true:*

(a) *Under the following condition*

$$\inf_{\Omega_-} m_- > \frac{N-2}{N+2} \quad \text{if } N \ge 3, \tag{2.1}$$

problem (1.5) *possesses a positive solution for each* $\lambda > 0$. *Moreover, if* (λ_n,u_n), $n \ge 1$, *is a sequence of positive solutions of* (1.5) *such that*

$\lim_{n\to\infty} \lambda_n = 0$, *then*

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{C}_0(\overline{\Omega})} = \infty. \tag{2.2}$$

(b) *If m_- is constant, then u is a positive solution of* (1.5) *if, and only if,*

$$u = \lambda^{-m_-/(1-m_-)} v$$

for some positive solution v of

$$\begin{cases} -\Delta v = v^{1/m_-} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In particular, the number of positive solutions of (1.5)*, for each $\lambda > 0$, equals the number of positive solutions of* (2.3) *and, therefore, the following holds:*

(b1) *Suppose $m_- > (N-2)/(N+2)$ if $N \geq 3$. Then* (2.3) *possesses a positive solution, at least, and, actually, each positive solution v of* (2.3) *provides us with a curve*

$$\lambda \mapsto u_\lambda := \lambda^{-m_-/(1-m_-)} v, \quad \lambda > 0,$$

of positive solutions of (1.5)*. Moreover,*

$$\lim_{\lambda\downarrow 0} u_\lambda = \infty \quad \text{and} \quad \lim_{\lambda\uparrow\infty} u_\lambda = 0$$

uniformly in compact subsets of Ω.

(b2) *Suppose $N \geq 3$, $m_- \leq (N-2)/(N+2)$, and Ω is star-shaped. Then,* (1.5) *cannot admit a positive solution.*

Subsequently, we shall denote by $\mathcal{P}_\rho : \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}) \to \mathcal{C}_0(\overline{\Omega})$ the ρ-projection operator, i.e.

$$\mathcal{P}_\rho(\rho, u) = \rho \quad \text{for each } (\rho, u) \in \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}).$$

Proof of Theorem 2.1. Suppose (2.1) and consider, for each $\lambda > 0$, the auxiliary problem

$$\begin{cases} -\Delta u = \mu u + \lambda u^{1/m_-} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\mu \in \mathbb{R}$ is regarded as a bifurcation parameter. Thanks to (2.1), the blowing-up argument of B. Gidas and J. Sprück (see [14]) can be easily adapted to show that the positive solutions of (2.4) possess $L^\infty(\Omega)$ a priori bounds uniform in compact intervals of $\mu \in \mathbb{R}$. Moreover, thanks to local bifurcation result of M. G. Crandall and P. H. Rabinowitz ([10]), $\mu := \sigma[-\Delta; \Omega]$ is a bifurcation value to positive solutions of (2.4) from the trivial state $(\mu, u) = (\mu, 0)$. Actually, by the global unilateral theorem of P. H. Rabinowitz ([22]), the component of positive solutions of (2.4) emanating from $(\mu, 0)$ at $\mu = \sigma[-\Delta; \Omega]$, subsequently denoted by $\mathfrak{C}$, must be unbounded in $\mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$ (cf. E. N. Dancer [11], as well as [19, Chapters 6, 7], for a complete development of the necessary abstract theory, as the original paper of P. H. Rabinowitz [22] contains some serious gaps). Suppose (2.4) possesses a positive solution. Then,

$$(-\Delta - \lambda u^{1/m_- -1})u = \mu u$$

and, hence, by the uniqueness of the principal eigenvalue,

$$\mu = \sigma[-\Delta - \lambda u^{1/m_- -1}; \Omega].$$

Thus, since $\lambda > 0$, it is apparent, from the monotonicity of the principal eigenvalue with respect to the potential, that $\mu < \sigma[-\Delta; \Omega]$, and, hence,

$$\mathcal{P}_\mu \mathfrak{C} \subset (-\infty, \sigma[-\Delta; \Omega]).$$

Actually, thanks to the existence of uniform a priori bounds,

$$\mathcal{P}_\mu \mathfrak{C} = (-\infty, \sigma[-\Delta; \Omega])$$

and, therefore, $0 \in \mathcal{P}_\mu \mathfrak{C}$. In particular, (1.5) possesses a positive solution. Now, let (λ_n, u_n), $n \geq 1$, be a sequence of positive solutions of (1.5) with $\lim_{n\to\infty} \lambda_n = 0$. If there exists a constant $M > 0$ such that

$$\|u_n\|_{\mathcal{C}_0(\overline{\Omega})} \leq M, \quad n \geq 1,$$

then, by the compactness $(-\Delta)^{-1}$ (the inverse of the operator $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions), along some subsequence of (λ_n, u_n), labeled again by n,

$$\lim_{n\to\infty} \|u_n - u_\infty\|_{\mathcal{C}_0(\overline{\Omega})} = 0,$$

for some strong solution u_∞ of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Necessarily $u_\infty = 0$ and, hence,

$$\lim_{n\to\infty} \|u_n\|_{\mathcal{C}_0(\overline{\Omega})} = 0.$$

Now, set

$$v_n := \frac{u_n}{\|u_n\|_{\mathcal{C}_0(\overline{\Omega})}}, \quad n \geq 1.$$

Then, for each $n \geq 1$, we have that

$$v_n = (-\Delta)^{-1}(\lambda_n v_n u_n^{1/m_- -1}),$$

and, hence, along some subsequence, labeled again by n, we have that

$$\lim_{n\to\infty} \|v_n - v_\infty\|_{\mathcal{C}_0(\overline{\Omega})} = 0.$$

Necessarily, $\|v_\infty\|_{\mathcal{C}_0(\overline{\Omega})} = 1$, $v_\infty > 0$, and v_∞ solves (2.5). This is impossible, since $u = 0$ is the unique solution of (2.5). This contradiction shows (2.2) and concludes the proof of (a).

(b1) is an easy consequence from (a), and (b2) follows readily from a celebrated identity by S. I. Pohozaev ([21]). $\square$

Even in the case when m_- is a constant satisfying (2.1), it is well known that the number of positive solutions of (1.5) is strongly dependent upon the geometry of the domain Ω. Indeed, if Ω consists of $n \geq 2$ separated balls joined by $n-1$ narrow corridors, then (2.3) has $2^n - 1$ positive solutions and, therefore, (1.5) possesses $2^n - 1$ global curves of positive solutions. Eventually, even for the simplest domain geometries, the number of solutions of (1.5) might be strongly dependent upon the local oscillation properties of the function $m_-(x)$ (cf. [15], as well as the references there in, for similar closely related discussions).

In the general case when $N \geq 3$ and the auxiliary function

$$s(x) := m_-(x) - \frac{N-2}{N+2}, \quad x \in \Omega,$$

changes of sign, the problem of characterizing the existence of positive solutions for (1.5) increases in complexity. The corresponding results will be given elsewhere, as they are still in progress.

In the most general case when $\Omega_1 \neq \emptyset$ the following result is satisfied.

Theorem 2.2. *Suppose $\Omega_+ = \emptyset$, $\Omega_1 \neq \emptyset$, and consider the function*

$$\sigma(\lambda) := \sigma[-\Delta - \lambda\chi_{\Omega_1}; \Omega], \quad \lambda \geq 0.$$

Then, the exists a unique $\lambda_0 = \lambda_0(\Omega_1) > 0$ satisfying

$$\sigma^{-1}(0) \cap [0, \infty) = \{\lambda_0\}.$$

Moreover, (1.5) *cannot admit a positive solution if $\lambda \geq \lambda_0$. Suppose, in addition, that*

$$\sup_{\Omega_-} m_- < 1$$

and regard to λ as a bifurcation parameter. Then $\lambda = \lambda_0$ is a bifurcation value from $(\lambda, u) = (\lambda, 0)$ to an unbounded continuum $\mathfrak{C} \subset (0, \lambda_0) \times \mathcal{C}_0(\overline{\Omega})$ of positive solutions of (1.5)*. Moreover, $\mathcal{P}_\lambda \mathfrak{C} = (0, \lambda_0)$ if condition* (2.1) *is satisfied, though, in general, $\mathcal{P}_\lambda \mathfrak{C}$ might be a proper subinterval of* $(0, \lambda_0)$.

Figure 2.1 shows three admissible situations within the setting of Theorem 2.2.

Figure 2.1(a) represents $\mathfrak{C}$ under assumption (2.1), while Figure 2.1(b), (c) represent two admissible $\mathfrak{C}$'s where (2.1) fails. In case (b), $\mathcal{P}_\lambda \mathfrak{C} = [\lambda_*, \lambda_0)$, for some $\lambda_* \in (0, \lambda_0)$, while, in case (c), $\mathcal{P}_\lambda \mathfrak{C} = (\lambda_*, \lambda_0)$. In all cases the problem

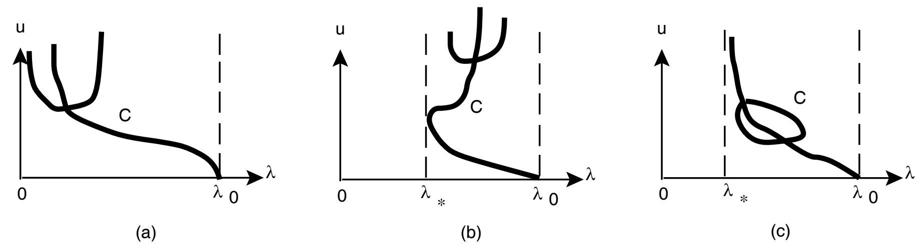

Figure 2.1. Three admissible bifurcation diagrams

might have an arbitrary number of solutions as a result of the geometry of Ω and the local properties of m_-.

A crucial feature, differentiating the case when $\Omega_1 = \emptyset$ from the case described by Theorem 2.2, is the fact there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0) \subset \mathcal{P}_\lambda \mathfrak{C}$ if $\Omega_1 \neq \emptyset$, and, therefore, (1.5) always possesses a positive solutions for each $\lambda < \lambda_0$ sufficiently close to λ_0, independently of the size of m_-; in strong contrast with the situation described by Theorem 2.1, where (1.5) cannot admit a positive solution if Ω is star-shaped, $N \geq 3$ and $m_- \leq (N-2)/(N+2)$.

If Ω_1^δ, $\delta \in [0,1]$, stands for an increasing family of smooth domains such that $\Omega_1^1 = \Omega_1$ and $\lim_{\delta \downarrow 0} \Omega_1^\delta = \emptyset$, then, $\lim_{\delta \downarrow 0} \lambda_0(\Omega_1^\delta) = \infty$ (cf. the details of the proof of [13, Theorem 12]). Actually, the corresponding bifurcation diagrams approximate, as $\delta \downarrow 0$, to the bifurcation diagram of the problem in case $\Omega_1 = \emptyset$, though, being outside the general scope of this work, this sharper analysis will appear elsewhere.

Proof of Theorem 2.2. By the monotonicity of the principal eigenvalue with respect to the potential, the function $\sigma(\lambda)$ is decreasing with λ. Moreover, $\sigma(0) = \sigma[-\Delta; \Omega] > 0$, and, for any ball $B \subset \Omega_1$ and $\lambda > 0$, we have that

$$\sigma(\lambda) < \sigma[-\Delta - \lambda; B] = \sigma[-\Delta; B] - \lambda,$$

and, hence, $\lim_{\lambda \uparrow \infty} \sigma(\lambda) = -\infty$. This shows the existence and the uniqueness of λ_0.

Suppose (1.5) possesses a positive solution u. Then,

$$(-\Delta - \lambda \chi_{\Omega_1}) u = \lambda \chi_{\Omega_-} u^{1/m_-} > 0$$

and, hence, u is a strict positive supersolution of $-\Delta - \lambda \chi_{\Omega_1}$ in Ω under homogeneous Dirichlet boundary conditions. Thus, thanks to [17, Theorem 2.5],

$$\sigma[-\Delta - \lambda \chi_{\Omega_1}; \Omega] > 0$$

and, therefore, $\lambda < \lambda_0$. Now, we regard to λ as the main bifurcation parameter and consider the nonlinear operator $\mathfrak{F}\colon \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}) \to \mathcal{C}_0(\overline{\Omega})$ defined by

$$\mathfrak{F}(\lambda, u) := u - (-\Delta)^{-1}(\lambda \chi_{\Omega_1} u + \lambda \chi_{\Omega_-} |u|^{1/m_-}), \tag{2.6}$$

whose positive fixed points provide us with the positive solutions of (1.5). For each $\lambda \in \mathbb{R}$, $\mathfrak{F}(\lambda, 0) = 0$. Moreover, $\mathfrak{F}$ is continuous and admits the decomposition

$$\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u - \lambda(-\Delta)^{-1}(\chi_{\Omega_-}|u|^{1/m_-}),$$

where

$$\mathfrak{L}(\lambda)u := u - \lambda(-\Delta)^{-1}(\chi_{\Omega_1} u), \quad u \in \mathcal{C}_0(\overline{\Omega}).$$

Therefore, it adjusts to the abstract setting of [19, Chapter 6]. It should be noted that condition $\sup_{\Omega_-} m_- < 1$ cannot be relaxed, because otherwise the nonlinearity would not be $o(\|u\|_{\mathcal{C}_0(\overline{\Omega})})$.

Let $\varphi_0 > 0$ denote a principal eigenfunction associated to $\sigma[-\Delta - \lambda_0\chi_{\Omega_1}; \Omega]$. Then,

$$N[\mathfrak{L}(\lambda_0)] = \mathrm{span}[\varphi_0] \quad \text{and} \quad \frac{d}{d\lambda}\mathfrak{L}(\lambda_0)\varphi_0 \notin R[\mathfrak{L}(\lambda_0)], \tag{2.7}$$

where, given any linear continuous operator L, $N[L]$ and $R[L]$ stand for the null space and the range of L, respectively. Indeed, the first identity of (2.7) is true by construction. For the second, suppose

$$-(-\Delta)^{-1}(\chi_{\Omega_1}\varphi_0) = u - \lambda_0(-\Delta)^{-1}(\chi_{\Omega_1} u) \tag{2.8}$$

for some $u \in \mathcal{C}_0(\overline{\Omega})$. Then,

$$(-\Delta - \lambda_0\chi_{\Omega_1})u = -\chi_{\Omega_1}\varphi_0$$

and multiplying this identity by φ_0 and integrating by parts in Ω gives

$$\int_{\Omega_1} \varphi_0^2 = 0,$$

which is impossible, since $\varphi_0(x) > 0$ for each $x \in \Omega$. Therefore, since $\mathfrak{L}(\lambda)$ is a Fredholm operator of index zero, λ_0 is a 1-transversal eigenvalue of the family $\mathfrak{L}(\lambda)$ and, hence, the generalized algebraic multiplicity $\chi[\mathfrak{L}; \lambda_0]$ introduced in [19, Chapter 4] equals 1. Therefore, thanks to [19, Theorem 4.2.4], λ_0 is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$. Actually, this fact is a direct consequence from the main local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz ([10]). It should be noted that the main theorem of [10] does not apply in order to get the existence of a curve of positive solutions of (1.5) emanating from $u = 0$ at $\lambda = \lambda_0$, because our nonlinearity does not have the required regularity. But this is far from being a trouble, since, due to [19, Theorem 5.6.2], the index – local topological degree – of $\mathfrak{L}(\lambda)$ at zero, $\mathrm{Ind}(\mathfrak{L}(\lambda), 0)$, $\lambda \sim \lambda_0$, $\lambda \neq \lambda_0$, must change as λ crosses λ_0, because $\chi[\mathfrak{L}; \lambda_0] = 1$. Therefore, thanks to [19, Theorem 6.2.1] there is a component, $\mathcal{C}$, of the set of nontrivial solutions of (1.5) such that $(\lambda_0, 0) \in \mathcal{C}$. Finally, the proof of [19, Theorem 6.5.5] carries over *mutatis mutandis* to show the existence of an unbounded subcomponent of $\mathcal{C}$, $\mathfrak{C}$, entirely consisting of positive solutions of (1.5) and such that $(\lambda_0, 0) \in \mathfrak{C}$. It should be noted that, although Assumption B of [19, Section 6.5] is not satisfied, because $(-\Delta)^{-1}(\chi_{\Omega_1} u) = 0$ if supp$u \subset \Omega_-$, Assumption B is not really needed in the proof of [19, Theorem 6.5.5], because all necessary features to complete all technical details of the proof can be obtained straight away from the properties of the elliptic operator $-\Delta - \lambda_0 \chi_{\Omega_1}$. Indeed, if we choose $Y = R[\mathfrak{L}(\lambda_0)]$ and there exist $y \in Y$, $y \gg 0$, and $u \in \mathcal{C}_0(\overline{\Omega})$ such that

$$u - \lambda_0(-\Delta)^{-1}(\chi_{\Omega_1} u) = y,$$

then

$$(-\Delta - \lambda_0 \chi_{\Omega_1})u = -\Delta y$$

and, multiplying by φ_0 and integrating by parts, we find that

$$\lambda_0 \int_{\Omega_1} y \varphi_0 = 0,$$

which is a contradiction. As λ_0 is the unique bifurcation value to positive solutions from $u = 0$, the proof of the first part of the theorem is completed.

Now, suppose condition (2.1) is satisfied, fix $\widehat{\lambda} \in (0, \lambda_0)$, and set $J := [\widehat{\lambda}, \lambda_0)$. Thanks to (2.1), the blowing-up argument of B. Gidas and J. Sprück ([14]) carries over *mutatis mutandis* to show the existence of a positive constant $M > 0$ such that $\|\theta\|_{\mathcal{C}(\overline{\Omega}_-)} \leq M$ for any $\lambda \in J$ and any positive solution θ of (1.5). Thus, $\theta|_{\Omega_1}$ must be a subsolution of the linear problem

$$\begin{cases} -\Delta u = \lambda_0 u & \text{in } \Omega_1, \\ u = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ u = M & \text{on } \partial\Omega_1 \cap \Omega. \end{cases} \tag{2.9}$$

By the monotonicity of the principal eigenvalue with respect to the domain,

$$\sigma[-\Delta - \lambda_0; \Omega_1] = \sigma[-\Delta - \lambda_0 \chi_{\Omega_1}; \Omega_1] > \sigma[-\Delta - \lambda_0 \chi_{\Omega_1}; \Omega] = 0,$$

and, hence, (2.9) possesses a unique solution, necessarily positive. Moreover, thanks to the strong maximum principle, $\theta|_{\Omega_1}$ is bounded above by the unique solution of (2.9). Therefore, there exists a constant $\widetilde{M} > M$ such that $\|\theta\|_{\mathcal{C}_0(\overline{\Omega})} \leq \widetilde{M}$. This concludes the proof of the theorem. □

3. The case when $\Omega_+ \neq \emptyset$ and $\Omega_- = \emptyset$

In this case our main result reads as follows.

Theorem 3.1. *Suppose $\Omega_+ \neq \emptyset$ and $\Omega_- = \emptyset$. Then, the following assertions are true:*

(a) *In case $\Omega_1 = \emptyset$, problem* (1.5) *possesses a positive solution if, and only if, $\lambda > 0$. Moreover, it is unique if it exists and if we denote it by θ_λ,* *then, for each* $\alpha \in (0,1)$*, the map* $\lambda \mapsto \theta_\lambda$ *is increasing and of class* $\mathcal{C}^1((0,\infty);\mathcal{C}_0^\alpha(\overline{\Omega}))$*. Furthermore,*

$$\lim_{\lambda\downarrow 0} \|\theta_\lambda\|_{\mathcal{C}^{1+\alpha}(\overline{\Omega})} = 0 \quad \textit{and} \quad \lim_{\lambda\uparrow\infty} \|\theta_\lambda\|_{\mathcal{C}(K)} = \infty, \tag{3.1}$$

for any compact subset $K \subset \Omega$*.*

(b) *In case* $\Omega_1 \neq \emptyset$*, problem* (1.5) *possesses a positive solution if and only if*

$$0 < \lambda < \lambda_0, \tag{3.2}$$

where λ_0 *is the unique positive zero of* $\lambda \mapsto \sigma[-\Delta - \lambda\chi_{\Omega_1};\Omega]$*. Moreover, it is unique if it exists and if we denote it by* θ_λ*, then, for each* $\alpha \in (0,1)$*, the map* $\lambda \mapsto \theta_\lambda$ *is increasing and of class* $\mathcal{C}^1((0,\lambda_0);\mathcal{C}_0^\alpha(\overline{\Omega}))$*. Furthermore,*

$$\lim_{\lambda\downarrow 0} \|\theta_\lambda\|_{\mathcal{C}^{1+\alpha}(\overline{\Omega})} = 0 \quad \textit{and} \quad \lim_{\lambda\uparrow\lambda_0} \|\theta_\lambda\|_{\mathcal{C}(K)} = \infty, \tag{3.3}$$

for any compact subset $K \subset \Omega$*.*

The proof of this theorem relies on some comparison techniques based on the strong maximum principle. In [12, Section 3] are given the details of the proof of (a) in the special case when m_+ is constant. The proof of this special case easily adapts to cover the general case we are dealing with, and so we will omit it here. Part (b) is the main theorem of [13]. In Figure 3.1 we have represented the corresponding diagrams of positive solutions of (1.5).

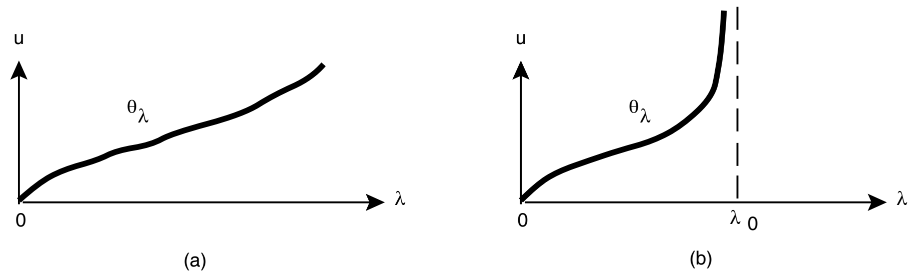

Figure 3.1. Bifurcation diagrams in cases (a) and (b) of Theorem 3.1

The bifurcation diagram consists of an increasing differentiable curve emanating from $u = 0$ at $\lambda = 0$ and blowing-up to infinity, everywhere in Ω, as $\lambda \uparrow \infty$ (resp. $\lambda \uparrow \lambda_0$) in case (a) (resp. (b)).

As in the setting of Theorem 2.2, if Ω_1^δ, $\delta \in [0,1]$, stands for an increasing family of smooth domains such that $\Omega_1^1 = \Omega_1$ and $\lim_{\delta\downarrow 0}\Omega_1^\delta = \emptyset$, then, $\lim_{\delta\downarrow 0}\lambda_0(\Omega_1^\delta) = \infty$ and the corresponding bifurcation diagrams approximate, as $\delta \downarrow 0$, to the bifurcation diagram of the problem in case $\Omega_1 = \emptyset$, though, being outside the general scope of this work, this sharper analysis will appear elsewhere.

4. The general case when $\Omega_+ \neq \emptyset$ and $\Omega_- \neq \emptyset$

Although some of the results found in this section are valid for general m_+ and m_-, our main global theorem needs assuming that m_+ and m_- are constant. The main result of this section reads as follows.

THEOREM 4.1. *Suppose Ω_+ and Ω_- are non-empty and m_+, m_- are constant. Then, there exist $\lambda^* > 0$ and an unbounded component, $\mathfrak{C} \subset (0,\infty) \times \mathcal{C}_0(\overline{\Omega})$, of the set of positive solutions of* (1.5) *such that:*

(a) $(\lambda, u) = (0,0) \in \overline{\mathfrak{C}}$, *and* $\Lambda := \mathcal{P}_\lambda \mathfrak{C} \in \{(0,\lambda^*], (0,\lambda^*)\}$.

(b) *Problem* (1.5) *does not admit a positive solution if* $\lambda \in (-\infty, 0] \cup (\lambda^*, \infty)$.

(c) *For each* $\lambda \in \Lambda$, (1.5) *possesses a minimal positive solution, denoted by* θ_λ, *and, for each* $\alpha \in (0,1)$, *the map* $\lambda \mapsto \theta_\lambda$ *is of class* $\mathcal{C}^1(\Lambda; \mathcal{C}_0^{1+\alpha}(\overline{\Omega}))$. *Moreover, for each* $\lambda \in \Lambda \setminus \{\lambda^*\}$,

$$\sigma\bigg[-\Delta - \frac{\lambda}{\mathfrak{m}}\theta_\lambda^{1/\mathfrak{m}-1};\Omega\bigg] > 0, \tag{4.1}$$

i.e. θ_λ *is linearly asymptotically stable with respect to the parabolic counterpart of* (1.5), *while*

$$\sigma\bigg[-\Delta - \frac{\lambda^*}{\mathfrak{m}}\theta_{\lambda^*}^{1/\mathfrak{m}-1};\Omega\bigg] = 0 \tag{4.2}$$

if $\lambda^* \in \Lambda$, *i.e.* θ_{λ^*} *is linearly neutrally stable.*

(d) *The component* $\mathfrak{C}$ *contains the arc of differentiable curve*

$$\Gamma := \{(\lambda, \theta_\lambda) : \lambda \in \Lambda \setminus \{\lambda^*\}\}. \tag{4.3}$$

Moreover, $\lim_{\lambda \downarrow 0} \|\theta_\lambda\|_{\mathcal{C}_0(\overline{\Omega})} = 0$, *and*

$$\lim_{\lambda \uparrow \lambda^*} \theta_\lambda = \theta_{\lambda^*} \quad \text{if } \Lambda = (0,\lambda^*], \tag{4.4}$$

while

$$\lim_{\lambda \uparrow \lambda^*} \|\theta_\lambda\|_{\mathcal{C}_0(\overline{\Omega})} = \infty \quad \text{if } \Lambda = (0,\lambda^*). \tag{4.5}$$

Actually, $\mathfrak{C} = \Gamma$ *if* $\Lambda = (0,\lambda^*)$.

(e) *If* $\Lambda = (0,\lambda^*]$, *then there exists* $\lambda_\omega \in [0,\lambda^*)$ *such that* (1.5) *has two positive solutions, at least, for each* $\lambda \in (\lambda_\omega, \lambda^*)$. *Actually, if either* $N \in \{1,2\}$, *or* $N \geq 3$ *and* $m_- > (N-2)/(N+2)$, *then* $\Lambda = (0,\lambda^*]$ *and* $\lambda_\omega = 0$.

(f) *For each* $\lambda \in \Lambda$, θ_λ *provides us with the unique linearly stable positive solution of* (1.5).

Figure 4.1 shows some of the possible bifurcation diagrams of positive solutions that, according to Theorem 4.1, problem (1.5) might exhibit. Continuous lines are filled in by linearly asymptotically stable solutions, necessarily the minimal ones, while dotted lines represent sub-continua, eventually curves, consisting of linearly unstable positive solutions.

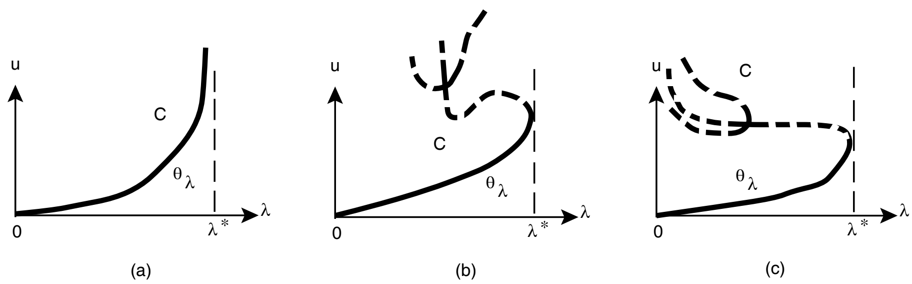

Figure 4.1. Three admissible components $\mathfrak{C}$.

We already know that the case illustrated by Figure 4.1(a) occurs when $\Omega_- = \emptyset$ (cf. Figure 3.1(b)), but it remains an open problem to ascertain if it can occur when $\Omega_- \neq \emptyset$, or not. In the affirmative case, the positive solutions of the problem should not exhibit a priori bounds in compact intervals of $(0, \lambda^*)$, of course. Thus, it cannot occur unless $N \geq 3$ and $m_- \leq (N-2)/(N+2)$. Figure 4.1(c) represents a typical bifurcation diagram in the presence of a priori bounds for all positive solutions, e.g. in the case when $N \in \{1,2\}$, or $N \geq 3$ with m_- sufficiently close to unity. As $m_- \uparrow 1$ and Ω_- approaches the empty set, the stable sub-continuum approaches the one exhibited by Figure 3.1(b), while the unstable sub-continuum blows up to infinity; otherwise, the associated problem with $\Omega_- = \emptyset$ would have two solutions somewhere, which is impossible. Therefore, adding m_-, or Ω_-, basically provokes the bifurcation diagram of the case when $\Omega_- = \emptyset$ to fold backwards; this is precisely what's going on in the context of classical superlinear indefinite problems with linear diffusion (cf. e.g. [3]). Figure 4.1(b) represents a situation case where the a priori bounds are lost somewhere within $(0, \lambda^*)$. It cannot happen if $N \in \{1,2\}$, or $N \geq 3$ but m_- is sufficiently close to unity. It must be remarked that, in spite of the drastic change of behaviour exhibited by the model when $\Omega_+ = \emptyset$, if either Ω_+ approximates the empty set, or $m_+ \downarrow 1$, then the minimal solution curve $(\lambda, \theta_\lambda)$ approaches zero, whereas the unstable sub-continuum must approximate some of the continua shown in Figure 2.1, though the technical details of this sharp perturbation analysis will appear elsewhere.

The remaining of this section is devoted to the proof of Theorem 4.1. The section itself will be divided into several subsections where we shall obtain all necessary analytical and/or topological properties of the positive solutions of (1.5) before giving the proof of the theorem in the last one.

4.1. Some non-existence results. The main result of this section is the following.

PROPOSITION 4.2. *There exists* $\lambda_1 > 0$ *such that* (1.5) *cannot have a positive solution if*

$$\lambda \in (-\infty, 0] \cup [\lambda_1, \infty).$$

Moreover, the unique possible bifurcation value to positive solutions of (1.5) *from* $u = 0$ *is* $\lambda = 0$. *Also,* $\lambda_1 \le \lambda_0^+$ *if* $\Omega_1 \ne \emptyset$, *where* $\lambda_0^+ > 0$ *stands for the unique zero of the map*

$$\lambda \mapsto \sigma[-\Delta - \lambda\chi_{\Omega_1}; \Omega_1 \cup \Omega_+].$$

It should be noted that λ_0^+ approaches the value λ_0 of the statement of Theorem 3.1 as $\Omega_- \downarrow \emptyset$, since, in this case, $\Omega_1 \cup \Omega_+ \uparrow \Omega$. In this sense, the estimate $\lambda_1 \le \lambda_0^+$ is optimal.

PROOF OF PROPOSITION 4.2. We already know that $\lambda > 0$ if (1.5) admits a positive solution. So, suppose $\lambda > 0$ and (1.5) has a positive solution u_λ. Then, u_λ provides us with a positive supersolution of the auxiliary problem

$$\begin{cases} -\Delta v = \lambda\chi_{\Omega_+} v^{1/m_+} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Now, adapting the proof of Theorem 3.2 in [12], it is easily to see that (4.6) possesses a unique positive solution and that the unique positive solution of (4.6), denoted by v_λ, must satisfy

$$v_\lambda \le u_\lambda \quad \text{in } \Omega_+.$$

As the change of variable $v = \lambda^{m_+/(m_+-1)} w$ transforms (4.6) into

$$\begin{cases} -\Delta w = \chi_{\Omega_+} w^{1/m_+} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

it is apparent that

$$\lambda^{m_+/(m_+-1)} v_1 \le u_\lambda \quad \text{in } \Omega, \tag{4.7}$$

and, hence,

$$\begin{aligned} 0 &= \sigma[-\Delta - \lambda u_\lambda^{1/\mathfrak{m}-1}; \Omega] < \sigma[-\Delta - \lambda u_\lambda^{1/m_- -1}; \Omega_-] \\ &< \sigma[-\Delta - \lambda^{(m_+-m_-)/(m_-(m_+-1))} v_1^{1/m_- -1}; \Omega_-]. \end{aligned}$$

Therefore, there exists $\lambda_1 > 0$ such that $\lambda \le \lambda_1$, because

$$\lim_{\lambda\uparrow\infty} \sigma[-\Delta - \lambda^{(m_+-m_-)/(m_-(m_+-1))} v_1^{1/m_- -1}; \Omega_-] = -\infty.$$

Moreover, thanks to (4.7), $\lambda = 0$ is the unique possible bifurcation value to positive solutions from $u = 0$.

Now, suppose $\Omega_1 \neq \emptyset$. Then, u_λ is a positive supersolution of the problem

$$\begin{cases} -\Delta v = \lambda v^{1/\mathfrak{m}} & \text{in } \Omega_1 \cup \Omega_+, \\ v = 0 & \text{on } \partial(\Omega_1 \cup \Omega_+), \end{cases} \tag{4.8}$$

and, since (4.8) possesses arbitrarily small subsolutions (cf. the proof of Theorem 3.1(b) in [13]), (4.8) has a positive solution and, therefore, thanks to Theorem 3.1(b), $\lambda < \lambda_0^+$. This concludes the proof. □

4.2. The existence of the component $\mathfrak{C}$. This section shows the existence of a component of the set of positive solutions, $\mathfrak{C} \subset (0,\infty) \times \mathcal{C}_0(\overline{\Omega})$, such that $(0,0) \in \overline{\mathfrak{C}}$.

Consider the auxiliary function

$$f(\lambda, x, s) := \begin{cases} \lambda s^{1/\mathfrak{m}(x)} & \text{if } (\lambda, x, s) \in \mathbb{R} \times \overline{\Omega} \times [0,\infty), \\ 0 & \text{if } (\lambda, x, s) \in \mathbb{R} \times \overline{\Omega} \times (-\infty, 0), \end{cases}$$

and the nonlinear operator $\mathcal{K}\colon \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}) \to \mathcal{C}_0(\overline{\Omega})$ defined by

$$\mathcal{K}(\lambda, u) := u - (-\Delta)^{-1}(f(\lambda, \cdot\,, u)).$$

For each $\lambda \in \mathbb{R}$, $\mathcal{K}(\lambda, \cdot\,)$ is a compact perturbation of the identity map such that $\mathcal{K}(\lambda, 0) = 0$. Moreover, $\mathcal{K}(\lambda, u) = 0$ for some $(\lambda, u) \in \mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$ with $u \neq 0$ if, and only if, $\lambda > 0$ and u is a positive solution of (1.5). Indeed, it is rather obvious that any positive solution of (1.5) provides us with a zero of $\mathcal{K}$. Moreover, if $(\lambda, u) \in \mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$ satisfies $\mathcal{K}(\lambda, u) = 0$, then multiplying (1.5) by u^- gives $\int_\Omega |\nabla u^-|^2 = 0$ and, hence, $u \geq 0$. Therefore, if $u \neq 0$, necessarily u is a positive solution of (1.5) and $\lambda > 0$. Consequently, the non-trivial zeroes of $\mathcal{K}$ are the positive solutions of (1.5). By a non-trivial zero it is meant a solution pair (λ, u) with $u \neq 0$. In the remaining of this paper, by a solution of (1.5) it is meant a pair $(\lambda, u) \in \mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$ such that

$$\mathcal{K}(\lambda, u) = 0.$$

Subsequently, we shall denote by B_ρ the ball of radius $\rho > 0$ centered at $u = 0$ in $\mathcal{C}_0(\overline{\Omega})$. The existence of $\mathfrak{C}$ is based upon the following result.

Theorem 4.3. *For each $\lambda \in \mathbb{R} \setminus \{0\}$, $u = 0$ is an isolated zero of $\mathcal{K}(\lambda, \cdot\,)$. Moreover,*

$$\operatorname{Ind}(\mathcal{K}(\lambda, \cdot\,), 0) = 1 \quad \text{if } \lambda < 0 \tag{4.9}$$

and

$$\operatorname{Ind}(\mathcal{K}(\lambda, \cdot\,), 0) = 0 \quad \text{if } \lambda > 0. \tag{4.10}$$

Thus, there exists a continuum of positive solutions of (1.5) *emanating from* $u = 0$ *at* $\lambda = 0$*. The maximal continuum, for the inclusion, provides us with the component* $\mathfrak{C}$.

Proof. The proof of (4.9) and (4.10) is based upon some homotopies coming from A. Ambrosetti and P. Hess ([5]), and D. Arcoya et al. ([6]).

Fix $\lambda < 0$ and consider the map $H_1\colon [0,1] \times \mathcal{C}_0(\overline{\Omega}) \to \mathcal{C}_0(\overline{\Omega})$ defined by

$$H_1(t,u) := u - (-\Delta)^{-1}(tf(\lambda,\,\cdot\,,u)).$$

Since the nontrivial zeroes of $H_1(t,\,\cdot\,)$ are the positive solutions of

$$\tag{4.11} \begin{cases} -\Delta u = t\lambda u^{1/\mathfrak{m}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $t\lambda \le 0$, we obtain that $H_1(t,u) \ne 0$ if $t \in [0,1]$ and $u \ne 0$. Thus, for each $R > 0$, the homotopy invariance of the topological degree gives

$$\begin{aligned} \operatorname{Ind}(\mathcal{K}(\lambda,\,\cdot\,),0) &= \operatorname{Deg}(\mathcal{K}(\lambda,\,\cdot\,),B_R) = \operatorname{Deg}(H_1(1,\,\cdot\,),B_R) \\ &= \operatorname{Deg}(H_1(0,\,\cdot\,),B_R) = \operatorname{Deg}(I,B_R) = 1, \end{aligned}$$

which concludes the proof of (4.9).

Now, fix $\lambda > 0$, $\phi \in \mathcal{C}_0(\overline{\Omega})$, $\phi > 0$, and consider the map $H_2\colon [0,1] \times \mathcal{C}_0(\overline{\Omega}) \to \mathcal{C}_0(\overline{\Omega})$ defined by

$$H_2(t,u) := u - (-\Delta)^{-1}(f(\lambda,\,\cdot\,,u) + t\phi).$$

We claim that there exists $\delta > 0$ such that $H_2(t,u) \ne 0$ for each $t \in [0,1]$ and $u \in \overline{B}_\delta \setminus \{0\}$. Note that, in particular, this shows that $u = 0$ is an isolated solution of (1.5). We shall proceed by contradiction. First, note that if $H_2(t,u) = 0$ for some $t \in [0,1]$ and $u \ne 0$, then

$$-\Delta u = \lambda f(\lambda,\,\cdot\,,u) + t\phi$$

and, hence, $\int_\Omega |\nabla u^-|^2 = 0$, since $t\phi \ge 0$. Consequently, $u > 0$. Now, suppose there is a sequence

$$(t_n,u_n) \in [0,1] \times (\mathcal{C}_0(\overline{\Omega}) \setminus \{0\}), \quad n \ge 1,$$

such that $\lim_{n\to\infty} u_n = 0$ and $H_2(t_n,u_n) = 0$ for each $n \ge 1$. Then, for each $n \ge 1$, we have that $u_n > 0$ and

$$-\Delta u_n = \lambda u_n^{1/m_+} + t_n\phi \ge \lambda \|u_n\|_{\mathcal{C}(\overline{\Omega}_+)}^{1/m_+ - 1} u_n + t_n\phi \quad \text{in } \Omega_+.$$

Moreover, $u_n > 0$ on $\partial\Omega_+$. Thus, $u_n|_{\Omega_+}$ provides us with a strict positive supersolution of

$$-\Delta - \lambda \|u_n\|_{\mathcal{C}(\overline{\Omega}_+)}^{1/m_+ - 1}$$

in Ω_+, under homogeneous Dirichlet boundary conditions. Thus, thanks to [17, Theorem 3.2],

$$\sigma[-\Delta - \lambda\|u_n\|^{1/m_+-1}_{\mathcal{C}(\overline{\Omega}_+)}; \Omega_+] = \sigma[-\Delta; \Omega_+] - \lambda\|u_n\|^{1/m_+-1}_{\mathcal{C}(\overline{\Omega}_+)} > 0.$$

This is impossible, since

$$\lim_{n\to\infty} \lambda\|u_n\|^{1/m_+-1}_{\mathcal{C}(\overline{\Omega}_+)} = \infty.$$

This contradiction shows the claim above. Now, thanks to the homotopy invariance of the topological degree, we obtain that

$$\begin{aligned}\operatorname{Ind}(\mathcal{K}(\lambda, \cdot\,), 0) &= \operatorname{Deg}(\mathcal{K}(\lambda, \cdot\,), B_\delta)\\ &= \operatorname{Deg}(H_2(0, \cdot\,), B_\delta) = \operatorname{Deg}(H_2(1, \cdot\,), B_\delta) = 0,\end{aligned}$$

since $H_2(1, 0) = -(-\Delta)^{-1}\phi < 0$, and, hence, $H_2(1, u) \neq 0$ for each $u \in \overline{B}_\delta$. This concludes the proof of (4.10).

Now, fix $\lambda_1 < 0 < \lambda_2$, pick $\varepsilon > 0$ such that $\mathcal{K}(\lambda_j, u) \neq 0$ for each $j \in \{1, 2\}$ and $u \in \overline{B}_\varepsilon \setminus \{0\}$, and consider the cylinders

$$Q_\eta := [\lambda_1, \lambda_2] \times \overline{B}_\eta \subset \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}), \quad \eta \in (0, \varepsilon].$$

Fix $\eta \in (0, \varepsilon]$. We claim that there exist $\lambda_\eta \in [\lambda_1, \lambda_2]$ and $u_\eta \in \partial B_\eta$ such that

$$\mathcal{K}(\lambda_\eta, u_\eta) = 0.$$

Note that, necessarily, $\lambda_\eta > 0$. Indeed, thanks to (4.9) and (4.10), if this were not true, then, by the homotopy invariance of the degree, we would get

$$1 = \operatorname{Deg}(\mathcal{K}(\lambda_1, \cdot\,), B_\eta) = \operatorname{Deg}(\mathcal{K}(\lambda_2, \cdot\,), B_\eta) = 0,$$

which is a contradiction. By the compactness of $\mathcal{K}$, it follows that there exists a sequence $\eta_n \in (0, \varepsilon)$, $n \geq 1$, such that

$$\lim_{n\to\infty} \eta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} (\lambda_{\eta_n}, u_{\eta_n}) = (0, 0).$$

Actually, thanks to a celebrated result by G. T. Whyburn ([24]), there is a continuum of non-trivial zeroes of $\mathcal{K}$ connecting $(0, 0)$ with $\|u\|_{\mathcal{C}_0(\overline{\Omega})} = \eta$. As the technical details of the proof have been already given in the proof of [19, Theorem 6.2.1], we will omit them here in (cf. [1, Theorem 3.1] and [6, Theorem 4.4] as well). This concludes the proof. □

4.3. The existence and linear stability of the minimal solution. The main result of this section is the following.

Proposition 4.4. *Suppose* (1.5) *possesses a positive solution. Then, it possesses a minimal positive solution, denoted by* θ_λ. *By minimal it is meant that* $\theta_\lambda < u$ *for any other positive solution* u *of* (1.5). *Moreover,* θ_λ *is linearly stable, i.e.*

$$\sigma\bigg[-\Delta - \frac{\lambda}{\mathfrak{m}}\theta_\lambda^{1/\mathfrak{m}-1}\bigg] \geq 0. \tag{4.12}$$

Proof. Suppose (1.5) has a positive solution, say u. Necessarily, $\lambda > 0$. Let B be any ball such that $\overline{B} \subset \Omega_+$, denote by ψ the unique positive eigenfunction associated to $\sigma[-\Delta; B]$, normalized so that $\|\psi\|_{\mathcal{C}_0(\overline{B})} = 1$, and set

$$\Psi := \begin{cases} \psi & \text{in } \overline{B}, \\ 0 & \text{in } \Omega \setminus B. \end{cases}$$

Then, for sufficiently small $\varepsilon > 0$, the function $\varepsilon\Psi$ provides us with a subsolution of (1.5) such that $\varepsilon\Psi < u$. As a consequence, (1.5) possesses a minimal positive solution in the order interval $[\varepsilon\Psi, u]$ of $\mathcal{C}_0(\overline{\Omega})$. Thus, it possesses a minimal positive solution in the order interval $[0, u]$, since λ cannot be a bifurcation value to positive solutions from $u = 0$, because of Proposition 4.2. Let θ_λ^u denote the minimal positive solution in $[0, u]$ and let $u(x, t; \varepsilon\Psi)$ be the unique solution of the parabolic counterpart of (1.5) starting at $\varepsilon\Psi < \theta_\lambda^u \leq u$. Thanks to the theory of D. Sattinger [23], $u(\,\cdot\,, t; \varepsilon\Psi)$ is increasing in time and it approaches θ_λ^u as $t \uparrow \infty$. Suppose v is another positive solution of (1.5) and shorten ε, if necessary, so that $\varepsilon\Psi < v$. Then, by the uniqueness of the limit $\lim_{t\uparrow\infty} u(\,\cdot\,, t; \varepsilon\Psi)$, we find that $\theta_\lambda^u = \theta_\lambda^v$ and, therefore, θ_λ^u is independent of the positive solution u. Thus, it provides us with the minimal positive solution θ_λ of (1.5). Relation (4.12) follows from [2, Proposition 20.4] (cf. [4, Lemma 3.5] as well). □

4.4. Solution curves through linearly stable solutions. The main result of this section reads as follows. Note that, thanks to Proposition 4.4, it reveals some crucial properties satisfied by all minimal solutions θ_λ of (1.5).

Theorem 4.5. *Suppose* (λ_0, u_0) *is a positive solution of* (1.5).

(a) *If*

$$\sigma\bigg[-\Delta - \frac{\lambda_0}{\mathfrak{m}}u_0^{1/\mathfrak{m}-1}; \Omega\bigg] > 0, \tag{4.13}$$

then, there exist $\varepsilon > 0$ *and a real analytic map* $U\colon (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathcal{C}_0^{1+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, *such that* $U(\lambda_0) = u_0$ *and* $(\lambda, U(\lambda))$ *is a positive solution of* (1.5) *for each* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. *Moreover, the map* $\lambda \mapsto U(\lambda)$ *is point-wise increasing and there exists a neighbourhood* $\mathcal{N}$ *of* (λ_0, u_0) *in* $(0, \infty) \times \mathcal{C}_0(\overline{\Omega})$ *such that if* $(\lambda, u) \in \mathcal{N}$ *solves* (1.5), *then* $u = U(\lambda)$.

(b) *If*

$$\sigma\Big[-\Delta-\frac{\lambda_0}{\mathfrak{m}}u_0^{1/\mathfrak{m}-1};\Omega\Big]=0, \tag{4.14}$$

then, there exist $\varepsilon>0$ *and a real analytic map* $(\Lambda,U)\colon(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{C}_0^{1+\alpha}(\overline{\Omega})$, $0<\alpha<1$, *such that* $(\Lambda(0),U(0))=(\lambda_0,u_0)$ *and for each* $s\in(-\varepsilon,\varepsilon)$, $(\Lambda(s),U(s))$ *is a positive solution of* (1.5). *Moreover, there exists a neighbourhood* $\mathcal{N}$ *of* (λ_0,u_0) *in* $(0,\infty)\times\mathcal{C}_0(\overline{\Omega})$ *such that if* $(\lambda,u)\in\mathcal{N}$ *solves* (1.5), *then* $(\lambda,u)=(\Lambda(s),U(s))$ *for some* $s\in(-\varepsilon,\varepsilon)$. *Furthermore, if* $\Phi>0$ *denotes a principal eigenfunction associated with the principal eigenvalue* (4.14), *then the function* $U(s)$ *can be chosen so that the auxiliary map* $s\mapsto V(s)$ *defined by*

$$V(s):=U(s)-u_0-s\Phi,\quad |s|<\varepsilon, \tag{4.15}$$

satisfy $\int_\Omega V(s)\Psi=0$ *and* $V(s)=O(s^2)$, *as* $s\to 0$. *Also, for this choice,*

$$\begin{gathered}\Lambda(s)=\lambda_0+s^2\lambda_2+O(s^3),\\ \lambda_2:=\frac{\lambda_0}{2}\int_\Omega\Big[\Phi^3u_0^{1/\mathfrak{m}-2}\frac{1}{\mathfrak{m}}\Big(1-\frac{1}{\mathfrak{m}}\Big)\Big]\Big/\int_\Omega u_0^{1/\mathfrak{m}}\Phi<0,\end{gathered} \tag{4.16}$$

and, for each $s\in(-\varepsilon,\varepsilon)$,

$$\operatorname{sign}\frac{d\Lambda}{ds}(s)=\operatorname{sign}\sigma\Big[-\Delta-\frac{\Lambda(s)}{\mathfrak{m}}U(s)^{1/\mathfrak{m}-1};\Omega\Big]. \tag{4.17}$$

Summarizing, around any linearly asymptotically stable positive solution the set of solutions of (1.5) consists of a smooth curve of linearly asymptotically stable solutions, while around any linearly neutrally stable positive solution the set of solutions consists of a second order sub-critical turning point whose *upper curve* is filled in by linearly unstable positive solutions, whereas its *lower curve* is filled in by linearly asymptotically stable positive solutions. For a more detailed discussion we send to the interested reader to [15] and [16], where the linear diffusion case was treated.

Proof of Theorem 4.5. Part (a) is an easy consequence from the implicit function theorem applied to the operator $\mathcal{K}$ defined in Section 4.2. As any non-trivial solution pair (λ,u) must have the second component, u, in the interior of the cone of positive functions of $\mathcal{C}_0(\overline{\Omega})$ and we are assuming that $\overline{\Omega}_+\subset\Omega$, the map $u\mapsto\mathcal{K}(\lambda,u)$ is analytic for each $\lambda>0$. Thus, the implicit function theorem provides us with an analytic solution curve.

The existence and the uniqueness of the curve $(\Lambda(s),U(s))$ in Part (b), as well as (4.17), have been already shown in [2, Proposition 20.8]. Actually, they can be obtained by applying the implicit function theorem to a certain operator related to $\mathcal{K}$ through a Lyapunov–Schmidt decomposition parallel to span$[\Phi]$. It should be noted that, thanks to (4.14), $\Lambda'(0) = 0$, where $'$ stands for differentiation with respect to the *pseudo-length* of arc of curve s. Consequently, the proof will be completed if we show that $\lambda_2 = \Lambda''(0)/2$ satisfies (4.16). Indeed, for each $s \in (-\varepsilon, \varepsilon)$ we have that

$$-\Delta[u_0 + s\Phi + V(s)] = [\lambda_0 + s^2\lambda_2 + O(s^3)][u_0 + s\Phi + V(s)]^{1/\mathfrak{m}}, \tag{4.18}$$

and, hence, differentiating (4.18) twice with respect s, particularizing the resulting expression at $s = 0$ and rearranging terms gives

$$\left(-\Delta - \frac{\lambda_0}{\mathfrak{m}} u_0^{1/\mathfrak{m}-1} \right) V''(0) = 2\lambda_2 u_0^{1/\mathfrak{m}} + \frac{\lambda_0}{\mathfrak{m}} \left(\frac{1}{\mathfrak{m}} - 1 \right) u_0^{1/\mathfrak{m}-2} \Phi^2. \tag{4.19}$$

It should be noted that the second term in the right hand side of (4.19) makes sense since $u_0^{-2}\Phi^2 \in \mathcal{C}(\overline{\Omega})$. Now, multiplying (4.19) by Φ, integrating in Ω and applying the formula of integration by parts gives

$$\lambda_2 = \frac{\lambda_0}{2} \int_\Omega \left[\Phi^3 u_0^{1/\mathfrak{m}-2} \frac{1}{\mathfrak{m}} \left(1 - \frac{1}{\mathfrak{m}} \right) \right] \bigg/ \int_\Omega u_0^{1/\mathfrak{m}} \Phi.$$

Thus, to conclude the proof, it remains to show that

$$\int_\Omega \left[\Phi^3 u_0^{1/\mathfrak{m}-2} \frac{1}{\mathfrak{m}} \left(1 - \frac{1}{\mathfrak{m}} \right) \right] < 0. \tag{4.20}$$

As in [15] and [16], this inequality will be obtained from a celebrated variational identity attributed to M. Picone [20] (cf. e.g. [9, Section 4] and [18, Lemma 4.1]). For any u, $v \in \mathcal{C}_0^1(\overline{\Omega})$ twice differentiable a.e. in Ω and such that $v/u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$, and every $\Upsilon \in \mathcal{C}^1([0,\infty); \mathbb{R})$, the following identity, usually referred to as Picone's identity, holds

$$\int_\Omega \Upsilon\left(\frac{v}{u} \right) (-v\Delta u + u\Delta v) = -\int_\Omega \Upsilon'\left(\frac{v}{u} \right) u^2 \left| \nabla \left(\frac{v}{u} \right) \right|^2. \tag{4.21}$$

Choosing

$$\Upsilon(t) = t^2, \quad v = \Phi, \quad u = u_0,$$

identity (4.21) gives

$$\int_\Omega \left[\Phi^3 u_0^{1/\mathfrak{m}-2} \left(1 - \frac{1}{\mathfrak{m}} \right) \right] = \int_\Omega \left[\left(\frac{\Phi}{u_0} \right)^2 (-\Phi\Delta u_0 + u_0\Delta\Phi) \right] < 0, \tag{4.22}$$

since Φ cannot be a multiple of u_0. Clearly, (4.22) implies

$$\int_\Omega \left[\Phi^3 u_0^{1/\mathfrak{m}-2} \left(1 - \frac{1}{\mathfrak{m}} \right) \frac{1}{\mathfrak{m}} \right] \le \int_\Omega \left[\Phi^3 u_0^{1/\mathfrak{m}-2} \left(1 - \frac{1}{\mathfrak{m}} \right) \right] < 0,$$

since $(1-x)x \le 1-x$ for each $x \in \mathbb{R}$. This shows (4.20) and concludes the proof of the theorem. $\square$

As an immediate consequence from Theorem 4.5, the following result holds.

Corollary 4.6. *Let* (λ_0, u_0) *be a positive solution of* (1.5) *satisfying* (4.14)*. Then, there exists* $\varepsilon > 0$ *such that for each* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0)$*,* (1.5) *has, at least, two positive solutions; one of them linearly asymptotically stable and the other linearly unstable. Moreover, there exists a neighbourhood* $\mathcal{N}$ *of* (λ_0, u_0) *in* $\mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$ *such that* (1.5) *cannot admit a positive solution in* $\mathcal{N}$ *if* $\lambda > \lambda_0$*.*

4.5. Local structure of $\mathfrak{C}$ at $(\lambda, u) = (0,0)$**.** The main result of this section reads as follows.

Proposition 4.7. *There exist* $\varepsilon > 0$ *and* $\beta > 0$ *such that, for each* $\lambda \in (0, \varepsilon]$*, the minimal positive solution* θ_λ *is the unique positive solution of* (1.5) *in* $\overline{B}_\beta$*. In particular,*

$$\mathfrak{C} \cap [(0,\varepsilon] \times \overline{B}_\beta] = \{(\lambda, \theta_\lambda) : 0 < \lambda \le \varepsilon\}.$$

Actually, thanks to Corollary 4.6*, for each* $\lambda \in (0, \varepsilon]$*, the following holds*

$$\sigma\bigg[-\Delta - \frac{\lambda}{\mathfrak{m}}\theta_\lambda^{1/\mathfrak{m}-1}; \Omega\bigg] > 0$$

and, therefore, thanks to Theorem 4.5(a)*,* $\mathfrak{C} \cap [(0,\varepsilon] \times \overline{B}_\beta]$ *is a compact arc of analytic curve.*

Proof. Thanks to Proposition 4.2 and Theorem 4.3, there exists $R > 0$ such that (1.5) has a positive solution, at least, for each $\lambda \in (0, R]$, because $\mathcal{P}_\lambda \mathfrak{C}$ is a connected interval of $(0, \infty)$. Actually, due to Proposition 4.4, (1.5) possesses a minimal solution, θ_λ, for each $\lambda \in (0, R]$. Thus, θ_λ is well defined for any sufficiently small $\lambda > 0$.

Suppose (1.5) possesses, for some $\lambda \in (0, R]$, a further solution u_λ. Then, $u_\lambda > \theta_\lambda$ and, hence,

$$\begin{aligned}(-\Delta - \lambda\chi_{\Omega_1})(u_\lambda - \theta_\lambda) &= \lambda\chi_{\Omega_+}(u_\lambda^{1/m_+} - \theta_\lambda^{1/m_+}) + \lambda\chi_{\Omega_-}(u_\lambda^{1/m_-} - \theta_\lambda^{1/m_-}) \\ &\le \frac{\lambda}{m_+}\chi_{\Omega_+}\theta_\lambda^{1/m_+-1}(u_\lambda - \theta_\lambda) + \frac{\lambda}{m_-}\chi_{\Omega_-}u_\lambda^{1/m_--1}(u_\lambda - \theta_\lambda).\end{aligned}$$

Thus,

$$\bigg(-\Delta - \lambda\chi_{\Omega_1} - \frac{\lambda}{m_+}\chi_{\Omega_+}\theta_\lambda^{1/m_+-1} - \frac{\lambda}{m_-}\chi_{\Omega_-}u_\lambda^{1/m_--1}\bigg)(u_\lambda - \theta_\lambda) \le 0,$$

and, therefore, thanks to the strong maximum principle,

$$\sigma\bigg[-\Delta - \lambda\chi_{\Omega_1} - \frac{\lambda}{m_+}\chi_{\Omega_+}\theta_\lambda^{1/m_+-1} - \frac{\lambda}{m_-}\chi_{\Omega_-}u_\lambda^{1/m_--1}; \Omega\bigg] \le 0. \tag{4.23}$$

The proof of the proposition will follow from (4.23), arguing by contradiction. Suppose there exists a sequence $(\lambda_n, u_{\lambda_n})$, $n \ge 1$, of positive solutions of (1.5) such that

$$\lim_{n\to\infty}(\lambda_n, u_{\lambda_n}) = (0,0), \quad u_{\lambda_n} > \theta_{\lambda_n} > 0, \quad n \ge 1.$$

Then, thanks to (4.23),

$$\sigma\Big[-\Delta-\lambda_n\chi_{\Omega_1}-\frac{\lambda_n}{m_+}\chi_{\Omega_+}\theta_{\lambda_n}^{1/m_+-1}-\frac{\lambda_n}{m_-}\chi_{\Omega_-}u_{\lambda_n}^{1/m_--1};\Omega\Big]\le 0,\quad n\ge 1. \tag{4.24}$$

Since $m_- < 1$,

$$\lim_{n\to\infty}\frac{\lambda_n}{m_-}\chi_{\Omega_-}u_{\lambda_n}^{1/m_--1}=0. \tag{4.25}$$

Moreover, thanks to the estimate (4.7), we have that

$$\theta_{\lambda_n}\ge\lambda_n^{m_+/(m_+-1)}v_1,\quad n\ge 1,$$

and, hence,

$$-\frac{\lambda_n}{m_+}\chi_{\Omega_+}\theta_{\lambda_n}^{1/m_+-1}\ge-\frac{1}{m_+}\chi_{\Omega_+}v_1^{1/m_+-1},\quad n\ge 1.$$

Thus, thanks to (4.24) and (4.25), passing to the limit as $n\to\infty$ gives

$$\sigma\Big[-\Delta-\frac{1}{m_+}\chi_{\Omega_+}v_1^{1/m_+-1};\Omega\Big]\le 0,$$

which is impossible, since v_1 is a non-degenerate solution of (4.6) with $\lambda=1$. This contradiction concludes the proof of the proposition. □

4.6. The component $\mathfrak{C}$ is unbounded. The main result of this section is the following.

Proposition 4.8. *The component $\mathfrak{C}$ is unbounded in $\mathbb{R}\times\mathcal{C}_0(\overline{\Omega})$.*

Proof. We will argue by contradiction. Suppose $\mathfrak{C}$ is bounded. Then, the extended component

$$\mathfrak{C}_0:=\mathfrak{C}\cup\{(0,0)\}$$

is bounded in $X:=\mathbb{R}\times\mathcal{C}_0(\overline{\Omega})$, and, hence, it is compact, since it consists of fixed points of the compact operator $\mathcal{K}$ defined in Section 4.2. Thus, since

$$\mathcal{K}^{-1}(0)\cap(\{0\}\times\mathcal{C}_0(\overline{\Omega}))=\{(0,0)\},$$

it is apparent, from Proposition 4.7, that there exists $\eta\in(0,\varepsilon]$ such that

$$\mathfrak{C}_0\cap([0,\eta]\times\mathcal{C}_0(\overline{\Omega}))=\{(\lambda,\theta_\lambda):0\le\lambda\le\eta\}. \tag{4.26}$$

Subsequently, we use the notations introduced in the statement of Proposition 4.7. Set $\delta:=\beta/2$ and consider the open neighborhood of $\mathfrak{C}_0$ defined by

$$U:=\mathfrak{C}_0+[(-\eta/2,\eta/2)\times B_\delta],$$

as well as the set of non-trivial zeroes of $\mathcal{K}$

$$\mathfrak{S}:=\{(\lambda,u)\in X:\mathcal{K}(\lambda,u)=0,\ u\ne 0\}\cup\{(0,0)\}.$$

Subsequently, a bounded open set $\mathcal{O} \subset X$ is said to be an open isolating neighborhood of $\mathfrak{C}_0$ in X if $\mathfrak{C}_0 \subset \mathcal{O}$ and

$$\partial\mathcal{O} \cap \mathfrak{S} = \emptyset. \tag{4.27}$$

If $\partial U \cap \mathfrak{S} = \emptyset$, then U provides us with an open isolating neighborhood of the component $\mathfrak{C}_0$, but, in general, $\partial U \cap \mathfrak{S} \neq \emptyset$. When this is the case, Whyburn's Lemma [24] uses the fact that $\mathfrak{C}_0$ is a maximal compact and connected subset of $\mathfrak{S}$ to show the existence of an open isolating neighbourhood $\mathcal{O}$ of $\mathfrak{C}_0$ such that

$$\mathfrak{C}_0 \subset \mathcal{O} \subset U$$

(cf. e.g. the proof of [19, Theorem 6.3.1]). Now, for each $\lambda > 0$ we set

$$\mathcal{O}_\lambda := \{u \in X : (\lambda, u) \in \mathcal{O}\}.$$

By construction,

$$\overline{\mathcal{O}}_{\eta/3} \cap \mathfrak{S} = \{\theta_{\eta/3}\}.$$

Thus, combining Leray–Schauder's formula with Proposition 4.7 gives

$$\mathrm{Deg}(\mathcal{K}(\eta/3,\,\cdot\,), \mathcal{O}_{\eta/3}) = \mathrm{Ind}(\mathcal{K}(\eta/3,\,\cdot\,), \theta_{\eta/3}) = 1$$

and, hence, by homotopy invariance, $\mathrm{Deg}(\mathcal{K}(\lambda,\,\cdot\,), \mathcal{O}_\lambda) = 1$ for all $\lambda > 0$. On the other hand, for sufficiently large λ we have that $\mathcal{O}_\lambda = \emptyset$ and, hence, $\mathrm{Deg}(\mathcal{K}(\lambda,\,\cdot\,), \mathcal{O}_\lambda) = 0$. This contradiction concludes the proof. □

4.7. Proof of Theorem 4.1. Suppose there exist $\widehat{\lambda} > 0$ and $u_{\widehat{\lambda}} \neq \theta_{\widehat{\lambda}}$ such that $(\widehat{\lambda}, u_{\widehat{\lambda}})$ is linearly stable (either neutrally stable, or asymptotically stable). Then, thanks to Proposition 4.4 and Theorem 4.5 (cf. Corollary 4.6), by global continuation to the left of $\widehat{\lambda}$, (1.5) must admit two linearly asymptotically stable solutions for each $\lambda \in (0, \widehat{\lambda})$. As the solutions in each of the corresponding curves are increasing with λ, thanks to Proposition 4.7, (1.5) must admit a positive solution for $\lambda = 0$. This is impossible. Therefore, for each $\lambda > 0$, θ_λ is the unique linearly stable positive solution of (1.5) if it admits a solution. This shows (f). It should be noted that, thanks to Proposition 4.2, $\lambda = 0$ is the unique bifurcation value to positive solutions from $u = 0$.

Let λ^* be the maximal $\lambda > 0$ satisfying the following condition

$$\sigma\Big[-\Delta - \frac{\lambda}{\mathfrak{m}}\theta_\lambda^{1/\mathfrak{m}-1}; \Omega\Big] > 0, \quad \lambda \in (0, \lambda^*). \tag{4.28}$$

Thanks to Propositions 4.2, Proposition 4.7, λ^* is well defined. Moreover, since $\mathfrak{C}$ is the maximal connected set such that $(0,0) \in \overline{\mathfrak{C}}$,

$$\gamma := \{(\lambda, \theta_\lambda) : \lambda \in (0, \lambda^*)\} \subset \mathfrak{C}, \tag{4.29}$$

because γ is connected.

Either γ is bounded in $\mathbb{R}\times\mathcal{C}_0(\overline{\Omega})$, or it is unbounded. Suppose γ is bounded. Then,

$$u_{\lambda^*} := \lim_{\lambda\uparrow\lambda^*}\theta_\lambda$$

provides us with a solution of (1.5) for $\lambda=\lambda^*$. Moreover, by the continuous dependence of the principal eigenvalue with respect to the potential, (4.28) implies

$$\sigma\bigg[-\Delta-\frac{\lambda^*}{\mathfrak{m}}u_{\lambda^*}^{1/\mathfrak{m}-1};\Omega\bigg]=0,$$

because of the maximality of λ^*. As θ_{λ^*} is the unique linearly stable solution, necessarily

$$u_{\lambda^*}=\theta_{\lambda^*}.$$

Actually, thanks to Corrollary 4.6, around $(\lambda^*,\theta_{\lambda^*})$, $\mathfrak{C}$ consists of a second order sub-critical turning point. In particular, there exists $\lambda_\omega\in[0,\lambda^*)$ such that $\mathfrak{C}$ possesses two solutions, at least, for each $\lambda\in(\lambda_\omega,\lambda^*)$; this shows the first claim of Part (e). Note that there exists an open set $\mathcal{O}$ such that:

(1) $\{(\lambda,\theta_\lambda):\lambda\in(0,\lambda^*]\}\subset\mathcal{O}$.
(2) Any solution of (1.5) in $\overline{\mathcal{O}}$ lies in $\mathfrak{C}$.
(3) Any positive solution of (1.5) in $\partial\mathcal{O}$ is linearly unstable.

Clearly,

$$(0,\lambda^*]\subset\Lambda:=\mathcal{P}_\lambda\mathfrak{C}.$$

We claim that $\Lambda=(0,\lambda^*]$. Indeed, suppose there exists $\widehat{\lambda}>\lambda^*$ such that $\widehat{\lambda}\in\Lambda$. Then, by global continuation from $(\widehat{\lambda},\theta_{\widehat{\lambda}})$ to the left of $\widehat{\lambda}$ one can construct a linearly stable positive solution of (1.5), outside $\mathcal{O}$, e.g. for $\lambda=\lambda^*$. This contradicts the uniqueness of the stable solution, and, therefore,

$$\Lambda=(0,\lambda^*].$$

To complete the proof of the theorem when γ is bounded it remains to show that $\mathfrak{C}$ possesses two positive solutions for each $\lambda\in(0,\lambda^*)$ if either $N\in\{1,2\}$, or $N\geq 3$ and $m_->(N-2)/(N+2)$. It suffices to show that, under these conditions, the component $\mathfrak{C}$ is bounded in $[\varepsilon,\lambda^*]\times\mathcal{C}_0(\overline{\Omega})$ for any $\varepsilon\in(0,\lambda^*)$. Pick one of those ε's. Then, the blowing-up argument of B. Gidas and J. Sprück ([14]) carries over *mutatis mutandis* to show the existence of a positive constant $M>0$ such that

$$\|u_\lambda\|_{\mathcal{C}(\overline{\Omega}_-)}\leq M$$

for any positive solution (λ,u_λ) of (1.5) with $\lambda\in[\varepsilon,\lambda^*]$. Thus, $u_\lambda|_{\Omega_1\cup\Omega_+}$ is a subsolution of

$$\begin{cases}-\Delta u=\lambda u^{1/\mathfrak{m}} & \text{in } \Omega_1\cup\Omega_+,\\ u=0 & \text{on } \partial\Omega,\\ u=M & \text{on } \partial\Omega_-.\end{cases}\tag{4.30}$$

Now, we have to distinguish two different cases. Assume $\Omega_1 = \emptyset$. Then (4.30) possesses a unique positive solution for each $\lambda > 0$, say v_λ, and, as an easy consequence from the strong maximum principle,

$$u_\lambda|_{\Omega_+} \leq v_\lambda \quad \text{in } \Omega_+,$$

for each $\lambda \in [\varepsilon, \lambda^*]$, which provides us with the desired a priori bounds. If $\Omega_1 \neq \emptyset$, then, thanks to Proposition 4.2, $\lambda^* < \lambda_0^+$, and, similarly, $u_\lambda|_{\Omega_1 \cup \Omega_+}$ is bounded above by the unique positive solution of (4.30). The existence and the uniqueness of the positive solution of (4.30) follows with the same argument used in [13] to treat the case of homogeneous Dirichlet boundary conditions. This concludes the proof of the theorem when γ is bounded.

Now, suppose γ is unbounded (cf. (4.29)). Then, necessarily, (4.5) holds. Indeed, if (1.5) possesses a positive solution (λ^*, u^*), then it possesses a minimal solution $(\lambda^*, \theta_{\lambda^*})$ and, consequently, it possesses two stable positive solutions for some range $\lambda < \lambda^*$, which is impossible. Actually, in this case $\mathfrak{C} = \gamma$. This concludes the proof. □

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Manuscript received December 20, 2003

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