Houston Journal of Mathematics

Volume 29, No. 3, 2003

POSITIVE SOLUTIONS FOR THE DEGENERATE LOGISTIC INDEFINITE SUPERLINEAR PROBLEM: THE SLOW DIFFUSION CASE

MANUEL DELGADO AND ANTONIO SUÁREZ

Communicated by Haïm Brezis

Abstract. In this work we study the existence, stability and multiplicity of the positive steady-states solutions of the degenerate logistic indefinite superlinear problem. By an adequate change of variable, the problem is transformed into an elliptic equation with concave and indefinite convex nonlinearities. We use singular spectral theory, the Leray-Schauder degree, bifurcation and monotony methods to obtain the existence results, and fixed point index in cones and a Picone identity to show the multiplicity results and the existence of a unique positive solution linearly asymptotically stable.

1. Introduction

In this work we analyze the existence, stability and multiplicity of nonnegative and non-trivial solutions of the degenerate logistic indefinite superlinear model

$$\begin{cases} \mathcal{L}w^m = \lambda w + a(x)w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $N \geq 1$; $m > 1$; $\lambda \in \mathbb{R}$ that it will be considered parameter, $a \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, changes sign and $\mathcal{L}$ is a

2000 *Mathematics Subject Classification.* 35B32; 35J25; 35K57; 92D25.

Key words and phrases. Degenerate logistic indefinite equation, singular eigenvalue problems, indefinite superlinear problems, multiplicity results.

The second author is grateful to Professors D. Arcoya and J. López Gómez for their helpful comments. The authors thank to CICYT and MCYT of Spain for research support under grants MAR98-0486 and BFM2000-0797 respectively.

second order uniformly elliptic operator of the form

$$\mathcal{L}u := -\sum_{i,j=1}^{N} D_i(a_{ij}D_j u) + \sum_{i=1}^{N} b_i(x) D_i u, \tag{2}$$

with $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$ and $b_i \in C^1(\overline{\Omega})$.

We define $a^+(x) := \max\{a, 0\}$, $a^- := \min\{a, 0\}$, and so $a = a^+ + a^-$ and

$$A_+ := \{x \in \Omega : a^+(x) > 0\}, \quad A_- := \{x \in \Omega : a^-(x) < 0\}, \quad A_0 := \Omega \setminus (\overline{A}_+ \cup \overline{A}_-)$$

and assume that $A_\pm$ are open and sufficiently smooth, and that $a^\pm$ are bounded away from zero on compact subsets of $A_\pm$.

Equation (1) can be regarded as a model of a steady-state single species inhabiting in Ω, so $w(x)$ stands for the population density. The parameter λ represents the growth rate of the species and $a(x)$ describes the limiting effects of crowding in the species in A_- and the intraspecific cooperation in A_+. Observe that in A_0 the population is free from crowding and symbiosis effects. Finally, $\mathcal{L}$ measures the diffusivity and the external transport effects of the species. The term $m > 1$ was introduced in [18] by describing the dynamics of biological population whose mobility depends upon their density. In this context, $m > 1$ means that the diffusion, the rate of movement of the species from high density regions to low density ones, is slower than in the linear case ($m = 1$), which seems give more realistic models, see [18].

The change of variable $u := w^m$ transforms (1) into

$$\begin{cases} \mathcal{L}u = \lambda u^q + a(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with $q = 1/m$ and $p = 2/m$. Along this work we suppose

$$0 < q < 1 < p \tag{H}$$

so, we are assuming that $1 < m < 2$, we call this case the slow diffusion. When $q = 1$, that is $m = 1$, (3) has been studied extensively in the last years, see for example [2], [3], [4], [6], [11], [12], [13], [17], [21], [22] and [24]. Roughly speaking, in these works it was proved that from the trivial solution $u = 0$ at $\lambda = \sigma_1[\mathcal{L}]$ bifurcates an unbounded continuum of positive solutions supercritically (resp. subcritically) if

$$D := \int_\Omega a\varphi_1^p \varphi_1^* < 0 \quad (\text{resp.} > 0,)$$

where $\sigma_1[\mathcal{L}] = \sigma_1[\mathcal{L}^*]$, φ_1 and φ_1^* stand for the principal eigenvalue and the principal eigenfunction of $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ in Ω under homogeneous Dirichlet boundary conditions. Moreover, in [6] and [11], assuming some restrictions on p and on the decay of a^+ near ∂A_+, the authors obtained a priori bounds for λ in compact interval of $\mathbb{R}$. So, if for example $D < 0$, in [6] was shown the existence of positive solution for $\lambda \in (\infty, \lambda^*]$ for some $\lambda^* > \sigma_1[\mathcal{L}]$ and the existence of, at least, two positive solutions for $\lambda \in (\sigma_1[\mathcal{L}], \lambda^*)$. Recently, the existence of a unique solution linearly asymptotically stable in $(\sigma_1[\mathcal{L}], \lambda^*)$ and multiplicity results in such interval have been showed in [17].

The results when $q < 1$ are completely different. Indeed, in the specific case $\mathcal{L} = -\Delta$ and $a(x) = 1$, Ambrosetti, Brezis and Cerami, in the pioneer work [7], proved the existence of, at least, two positive solutions of (3) in the interval $(0, \lambda^*)$ if $p < (N+2)/(N-2)$ and where λ^* is the supremum of the set

$$\Lambda := \{\lambda > 0 : (3) \text{ has a positive solution.}\}$$

To obtain this result, the authors used the sub-supersolution and variational methods. More recently, in [10] the authors have proved that from the trivial solutions $u = 0$ emanates an unbounded continuum of positive solutions at $\lambda = 0$. Unlike the case $q = 1$, this continuum emanates supercritically independent of the sign of D, in fact, the bifurcation direction only depends on value of p. This has been proved in [10] even when the operator $\mathcal{L}$ is quasilinear. Then, if $a(x) \geq a_0 > 0$ and $p < (N+2)/(N-2)$ they proved the existence of nonnegative solution in $(-\infty, \lambda^*)$ and of, at least, two positive solutions in $(0, \lambda^*)$. See also similar results obtained in [8] when the operator is the p-Laplacian and [1] when the boundary conditions are Neumann. In all these works, a does not change sign. When a changes sign, recently in [23] the authors have proved the existence of a weak nonnegative solution if $\lambda \leq 0$ making use of a direct variational approach, see [26] for the case of Neumann boundary conditions.

In this work, we improve and generalize the above results. We consider a non-selfadjoint operator $\mathcal{L}$, so it is well-known that the variational methods do not work, and a function a changing sign. As in [10], we prove that an unbounded continuum of nonnegative solutions emanates from the trivial solution $u = 0$ at $\lambda = 0$ supercritically. Moreover, we prove the existence of a minimal solution for $\lambda \in (0, \lambda^*)$ and the existence of $\overline{\lambda} \geq \lambda^*$ such that for $\lambda > \overline{\lambda}$, (3) does not admit positive solution. Using the results of Section 4 in [6], and assuming some restrictions on p and a^+ (see Theorem 5.1), we obtain a priori bounds for the positive solutions of (3) for compact intervals of λ. Finally, under these restrictions, we obtain a unique positive solution linearly asymptotically stable in $(0, \lambda^*)$, the existence of, at least, two positive solution in $(0, \lambda^*)$ and other multiplicity result in this interval (see Theorem 6.9). These results were strongly motivated by [17].

In order to obtain this result, we have used the Picone identity, so we assume that $\mathcal{L}$ is selfadjoint. Finally, we would like to point out that the stability results are obtained by linearizing at a positive solution. Observe that, since $q<1$, the linearized problem is a problem with a potential blowing up near $\partial\Omega$. So, we need some spectral theory with singular potential. For that, we have included some results obtained in [16] and [19].

An outline of this paper is as follows: In Section 2 we have collected some spectral theory with singular potential. In Section 3 we study problem (3) in the case $a^+=0$. These results come from [15] and will be used in the next sections. In Section 4 we apply the Leray-Schauder degree and bifurcation theory to show the existence of an unbounded continuum of nonnegative solution emanating supercritically at $\lambda=0$ from the trivial solution $u=0$. In Section 5 we obtain a priori bounds of the positive solutions of (3) for compact intervals of λ. Finally, in Section 6 we obtain multiplicity and stability results.

2. Singular eigenvalue problem

In this section we collect some results about the existence of principal eigenvalue for a singular linear eigenvalue problem of the form

$$\begin{cases} (\mathcal{L}+M(x))u=\sigma u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{4}$$

where $M\in C^1(\Omega)$ but it can blow-up near $\partial\Omega$ at a controlled way. The next result was proved in [19].

Theorem 2.1. *Suppose $M\in C^1(\Omega)$ and there exist two constants $K>0$ and $\varepsilon>0$ for which*

$$|M(x)|\le \frac{K}{[\operatorname{dist}(x,\partial\Omega)]^{2-\varepsilon}} \quad x\in\Omega. \tag{5}$$

Then, there exists a unique value of σ, denoted by $\sigma_1^\Omega[\mathcal{L}+M]$ and called principal eigenvalue of (4), for which (4) possesses a weak positive solution (in $H_0^1(\Omega)\cap L^\infty(\Omega)$), unique up to multiplicative constants, denoted by φ_1^Ω and called principal eigenfunction of (4).

Moreover, by elliptic regularity, $\varphi_1^\Omega\in C_0^1(\overline{\Omega})$, $\varphi_1^\Omega(x)>0$ for each $x\in\Omega$ and $\frac{\partial\varphi_1^\Omega}{\partial n}(x)<0$ for each $x\in\partial\Omega$, where n stands for the outward unit normal to Ω at x.

Furthermore, $\sigma_1^\Omega[\mathcal{L}+M]$ is increasing with respect to M and decreasing with respect to Ω, and if $\sigma_1^\Omega[\mathcal{L}+M]>0$ then $u=0$ is the unique weak solution of

$$\begin{cases} (\mathcal{L}+M(x))u=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

The following result was shown in [20] when $M\in L^\infty(\Omega)$, and in [16] when M satisfies (5).

Definition 2.2. *A function $\varphi\in C^2(\Omega)\cap C^1(\overline{\Omega})$ is said a supersolution of $\mathcal{L}+M$ if $(\mathcal{L}+M)\varphi\ge 0$ in Ω and $\varphi\ge 0$ on $\partial\Omega$. If in addition, $(\mathcal{L}+M)\varphi>0$ in Ω or $\varphi>0$ on $\partial\Omega$, then it is said that φ is a strict supersolution.*

Theorem 2.3. *Assume that M satisfies* (5)*. Then:*

(1) *$\sigma_1^\Omega[\mathcal{L}+M]>0$ if, and only if, $\mathcal{L}+M$ admits a positive strict supersolution.*
(2) *If there exists $\varphi\in C^2(\Omega)\cap C^1(\overline{\Omega})$ with $\varphi>0$ in Ω such that $\varphi=0$ on $\partial\Omega$ and $(\mathcal{L}+M)\varphi<0$ in Ω, then*

$$\sigma_1^\Omega[\mathcal{L}+M]<0.$$

We do not write the superindex Ω, when no confusion arises.

3. The sublinear case: $a^+\equiv 0$.

In this section we study the sublinear case, that is, when $a^+\equiv 0$. The following result characterizes the existence, uniqueness and linear stability in this case.

Theorem 3.1. *Assume $a^+\equiv 0$. Then, there exists a unique positive solution of* (3) *if, and only if, $\lambda>0$. Moreover, if we denote it by $\theta_{[\lambda,a^-]}$, then*

$$\lim_{\lambda\downarrow 0}\|\theta_{[\lambda,a^-]}\|_\infty=0. \tag{6}$$

Furthermore, if $\lambda>0$ then $\theta_{[\lambda,a^-]}$ is linearly asymptotically stable, that is,

$$\sigma_1[\mathcal{L}+M_\lambda(x)]>0, \tag{7}$$

where $M_\lambda:=-\lambda q\theta_{[\lambda,a^-]}^{q-1}-pa^-\theta_{[\lambda,a^-]}^{p-1}$

Proof. Except (7), the result follows by a similar argument to Theorem 4.2 in [15] where the result was proved when $\mathcal{L}=-\Delta$. We are going to show (7). Firstly, observe that for $\lambda>0$, (3) satisfies the strong maximum principle, so there exists $C>0$ such that

$$C\mathrm{dist}(x,\partial\Omega)\le\theta_{[\lambda,a^-]}(x),\qquad \text{for all } x\in\Omega,$$

and so, M_λ satisfies (5). On the other hand, by (H) and Theorem 2.1, it follows

$$0 = \sigma_1[\mathcal{L} - \lambda\theta^{q-1}_{[\lambda,a^-]} - a^-\theta^{p-1}_{[\lambda,a^-]}] < \sigma_1[\mathcal{L} + M_\lambda].$$

This completes the proof. □

The following result will be used in the next sections.

Lemma 3.2. *Assume $a^+ \equiv 0$. Then,*

$$\lim_{\lambda\downarrow 0} \sigma_1[\mathcal{L} + M_\lambda] = \sigma_1[\mathcal{L} - qz^{q-1}] > 0, \tag{8}$$

where z is the unique positive solution of

$$\begin{cases} \mathcal{L}z = z^q & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Proof. The existence of a unique positive solution for (9) follows by the sub-supersolution method, see [15] for details. Moreover, again by the strong maximum principle, z^{q-1} satisfies (5) and so it is well-defined $\sigma_1[\mathcal{L} - z^{q-1}]$. By (H) and Theorem 2.1, we have

$$0 = \sigma_1[\mathcal{L} - z^{q-1}] < \sigma_1[\mathcal{L} - qz^{q-1}]. \tag{10}$$

In order to prove (8), by (6) it is sufficient to show that

$$\xi_\lambda := \lambda^{1/(q-1)}\theta_{[\lambda,a^-]} \to z \quad \text{as } \lambda \downarrow 0. \tag{11}$$

It is not hard to prove that ξ_λ satisfies

$$\mathcal{L}\xi_\lambda = \xi_\lambda^q + a^-\lambda^{(p-1)/(1-q)}\xi_\lambda^p \quad \text{in } \Omega, \qquad \xi_\lambda = 0 \quad \text{on } \partial\Omega.$$

By (H), it follows (11), and thanks to (10) we obtain the result. □

4. Bifurcation from the trivial solution

In this section we will show that a bifurcation from the trivial solution of (3) occurs at $\lambda = 0$. For that, we consider the Banach space $X := C_0(\overline{\Omega})$, denote $B_\rho := \{u \in X : \|u\|_\infty < \rho\}$ and take $K > 0$ sufficiently large. We extend the function $f(\lambda, x, s) := \lambda s^q + a(x)s^p + Ks$ by taking $f(\lambda, x, s) := 0$ if $s < 0$. Note that f can take negative values. Finally, we define the map

$$\mathcal{K}_\lambda : X \mapsto X; \qquad \mathcal{K}_\lambda(u) := u - (\mathcal{L} + K)^{-1}(f(\lambda, x, u))$$

where $(\mathcal{L}+K)^{-1}$ is the inverse of the operator $\mathcal{L}+K$ under homogeneous Dirichlet boundary conditions, which is well-defined since $\sigma_1[\mathcal{L} + K] > 0$. Indeed, since positive constants are supersolutions of $\mathcal{L}$, then

$$\sigma_1[\mathcal{L}] > 0,$$

whence it follows that $\sigma_1[\mathcal{L}+K]>0$. Now, we can prove that u is a nonnegative solution of (3) if, and only if, u is a zero of the map $\mathcal{K}_\lambda$. It is clear that every nonnegative solution is a zero of $\mathcal{K}_\lambda$; conversely, if u is a zero of $\mathcal{K}_\lambda$ then, multiplying (3) by u^-, we obtain

$$\int_\Omega \sum_{i,j=1}^N a_{ij} D_i(u^-) D_j(u^-) + \int_\Omega (K - \frac{1}{2}\sum_{i=1}^N D_i b_i)(u^-)^2 \le 0, \tag{12}$$

and so, since $\mathcal{L}$ is a second uniformly elliptic operator, it follows that $u^- \equiv 0$. Observe that a nonnegative solution $u \in X$ of (3), it belongs to $C^{1+\nu}(\Omega) \cap C_0^1(\overline{\Omega})$ for $\nu := \min\{\alpha, q\}$.
The main result of this section is:

Theorem 4.1. *The value $\lambda = 0$ is the only bifurcation point from the trivial solutions for* (3)*. Moreover, there exists a continuum $\mathcal{C}_0$ of nonnegative solutions of* (3) *unbounded in $\mathbb{R}\times X$ emanating from* $(0,0)$*. In addition, $\mathcal{C}_0$ bifurcates to the right of $\lambda = 0$, i.e., it is supercritical.*

In order to prove this result we use the Leray-Schauder degree of $\mathcal{K}_\lambda$ on B_ρ with respect to zero, denoted by $\deg(\mathcal{K}_\lambda, B_\rho)$, and the index of the isolated zero u of $\mathcal{K}_\lambda$, denoted by $i(\mathcal{K}_\lambda, u)$. In the following results, we use homotopies which were used in [10], see also [9].

Lemma 4.2. *If $\lambda < 0$, then $i(\mathcal{K}_\lambda, 0) = 1$.*

Proof. Fix $\lambda < 0$. Define the map

$$\mathcal{H}_1 : [0,1] \times X \mapsto X; \qquad \mathcal{H}_1(t,u) := (\mathcal{L}+K)^{-1}(tf(\lambda,x,u)).$$

We claim that there exists $\delta > 0$ such that

$$u \neq \mathcal{H}_1(t,u)$$

for $u \in \overline{B}_\delta$, $u \neq 0$ and $t \in [0,1]$. Indeed, suppose that there exist sequences $u_n \in X\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$ such that

$$u_n = \mathcal{H}_1(t_n, u_n).$$

We know that $u_n \ge 0$. Since $\|u_n\|_\infty \to 0$ and $\lambda < 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, it holds

$$\mathcal{L}u_n \le 0 \qquad \text{in } \Omega,$$

which is impossible.

Taking now $\varepsilon \in (0,\delta]$, the homotopy defined by $\mathcal{H}_1$ is admissible and so,

$$\begin{aligned} i(\mathcal{K}_\lambda, 0) = \quad & \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_1(1,\cdot), B_\varepsilon) = \deg(I - \mathcal{H}_1(0,\cdot), B_\varepsilon) = \\ = \quad & \deg(I, B_\varepsilon) = 1. \end{aligned}$$

□

Lemma 4.3. *If $\lambda > 0$, then $i(\mathcal{K}_\lambda, 0) = 0$.*

Proof. Fix $\lambda > 0$ and $\phi \in X$, $\phi > 0$. We define the map

$$\mathcal{H}_2 : [0,1] \times X \mapsto X; \qquad \mathcal{H}_2(t,u) := (\mathcal{L} + K)^{-1}(f(\lambda, x, u) + t\phi).$$

We will show that there exists $\delta > 0$ such that $u \neq \mathcal{H}_2(t,u)$ for all $u \in \overline{B}_\delta$, $u \neq 0$ and $t \in [0,1]$. Indeed, suppose the contrary: there exist sequences $u_n \in X \setminus \{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$ such that

$$u_n = \mathcal{H}_2(t_n, u_n).$$

Since $t_n\phi \geq 0$, multiplying by u^-, and by a similar argument to the used in (12), we obtain that $u_n \geq 0$. Moreover since $\lambda > 0$, by the strong maximum principle $u_n > 0$. We fix $M \geq \sigma_1[\mathcal{L}]$. Since $\|u_n\|_\infty \to 0$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we get

$$\mathcal{L}u_n = \lambda u_n^q + a(x)u_n^p + t_n\phi > Mu_n + t_n\phi,$$

and so,

$$(\mathcal{L} - M)u_n > 0.$$

So, u_n is a positive strict supersolution of $\mathcal{L} - M$, and by Theorem 2.3, we get $\sigma_1[\mathcal{L} - M] > 0$, and so $M < \sigma_1[\mathcal{L}]$. This is impossible.

This proves that the homotopy defined by $\mathcal{H}_2$ is admissible. Then, if we take $\varepsilon \in (0,\delta]$ we have

$$i(\mathcal{K}_\lambda, 0) = \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_2(0,\cdot), B_\varepsilon) = \deg(I - \mathcal{H}_2(1,\cdot), B_\varepsilon) = 0.$$

This last equality is true because the problem $\mathcal{L}u = \lambda u^q + a(x)u^p + \phi$ has no solution in $\overline{B}_\varepsilon$ because we have shown that $u \neq \mathcal{H}_2(1,u)$ for all $u \in \overline{B}_\delta$, $u \neq 0$. □

Proof of Theorem 4.1: The fact that $\lambda = 0$ is a bifurcation point follows by Lemma 4.2 and Lemma 4.3. Moreover, from Lemma 4.2, (3) does not have bifurcation points in $(-\infty, 0)$. Assume that there exists a sequence of solutions (λ_n, u_n) such that $\lambda_n \to \lambda_0 > 0$ and $\|u_n\|_\infty \to 0$. We take $M \geq \sigma_1[\mathcal{L}]$, so there exists $n_0 \in \mathbb{N}$ such that

$$\lambda_n u_n^q + a(x)u_n^p > Mu_n \qquad \text{for all } n \geq n_0.$$

As in the proof of Lemma 4.3, we obtain that $\sigma_1[\mathcal{L} - M] > 0$, a contradiction.

Now, even though our map $\mathcal{K}_\lambda$ does not satisfy exactly the hypotheses of Theorem 1.3 in [25], the proof can be modified to obtain the result, see Theorem 3.1 in [1] and Theorem 4.4 in [10], and we can conclude the existence of a continuum of solutions of (3) such that meets $(0,0)$ either infinity or $(\lambda',0)$ with $\lambda' \neq 0$. We can discard the last possibility by the above reasoning, and so the existence of an unbounded continuum of solutions of (3) follows.

We are going to prove that the bifurcation is supercritical, for which plays an essential role that $p>1$. Indeed, assume that there exists a sequence (λ_n, u_n) of solutions of (3) such that $\lambda_n \leq 0$ and $u_n \geq 0$, $u_n \neq 0$ with $\lambda_n \to 0$ and $\|u_n\|_\infty \to 0$. Since $\sigma_1[\mathcal{L}]>0$, there exists a sufficiently small $\varepsilon>0$ such that

$$\sigma_1[\mathcal{L}-\varepsilon]>0. \tag{13}$$

For such $\varepsilon>0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for $n \geq n_0$, we get

$$\mathcal{L}u_n = \lambda_n u_n^q + a(x)u_n^p \leq a(x)u_n^p < \varepsilon u_n$$

whence by (13) we obtain a contradiction. □

The next result shows that for λ large, (3) has no solution.

Proposition 4.4. *There exists $\overline{\lambda}>0$ such that for $\lambda>\overline{\lambda}$, (3) has no solution.*

Proof. We fix $\delta>0$ sufficiently small and define the set

$$D^\delta := \{x \in A_+ : dist(x, \partial A_+) > \delta\} \neq \emptyset.$$

Then, there exists a positive constant $c^+(\delta)>0$ such that

$$a^+(x) \geq c^+(\delta) > 0 \qquad \text{in } \overline{D}^\delta.$$

Observe that D^δ has only finitely many connected components, say D_i^δ, $i = 1,\dots,r$.
Since $\lambda>0$, the strong maximun principle assures that any nonnegative and nontrivial solution of (3) is in fact strictly positive. So, by Theorem 2.1 is well-defined $\sigma_1[\mathcal{L}-\lambda u^{q-1} - a(x)u^{p-1}]$, and we have

$$0 = \sigma_1[\mathcal{L}-\lambda u^{q-1} - a(x)u^{p-1}] < \sigma_1^{D_1^\delta}[\mathcal{L}-\lambda u^{q-1} - a(x)u^{p-1}]. \tag{14}$$

Let $\varphi_1^{D_1^\delta}$ the principal eigenfunction associated with $\mathcal{L}$ in D_1^δ. We claim that there exists $\overline{\lambda}>0$ such that for $\lambda>\overline{\lambda}$, $\varphi_1^{D_1^\delta}$ is a strict subsolution of $\mathcal{L}_1 := \mathcal{L}-\lambda u^{q-1} - a(x)u^{p-1}$ in D_1^δ, and so by Theorem 2.3

$$\sigma_1^{D_1^\delta}[\mathcal{L}_1] < 0$$

which is a contradiction with (14). It remains to show the claim. Observe that in D_1^δ we have that

$$\mathcal{L}_1\varphi_1^{D_1^\delta} = (\sigma_1^{D_1^\delta}[\mathcal{L}] - \lambda u^{q-1} - a(x)u^{p-1})\varphi_1^{D_1^\delta} < (\sigma_1^{D_1^\delta}[\mathcal{L}] - \lambda u^{q-1} - c^+(\delta)u^{p-1})\varphi_1^{D_1^\delta} < 0$$

provided $\lambda > \overline{\lambda}$ with

$$\overline{\lambda} = \frac{(\sigma_1^{D_1^\delta}[\mathcal{L}])^{(p-q)/(p-1)}}{(c^+(\delta))^{(1-q)/(p-1)}}\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)}\left(\frac{p-1}{p-q}\right).$$

□

5. A priori bounds

In this section we obtain a priori bounds of the nonnegative solutions of (3) under some restrictions on p and the behaviour of a^+ near ∂A_+. For that, we assume that $\Omega\backslash\overline{A}_+$ is smooth and we follow the results of Section 4 of [6].

Theorem 5.1. *Suppose that there exist a function $h : \overline{A}_+ \mapsto \mathbb{R}^+$, continuous and bounded away from zero in a neighborhood of ∂A_+, and a constant $\gamma \geq 0$ such that*

$$a^+(x) = h(x)(dist(x, \partial A_+))^\gamma, \qquad in\ A_+.$$

Also assume that

$$p < \frac{N+1+\gamma}{N-1}$$

and

$$p < \frac{N+2}{N-2} \qquad if\ N \geq 3.$$

Then, for every compact interval $\Lambda \subset \mathbb{R}$ there exists a positive constant C such that

$$\|u\|_\infty \leq C,$$

for any nonnegative solution (λ, u) of (3) with $\lambda \in \Lambda$.

Proof. We divide the proof in two steps.
Step 1: A priori bounds on $\overline{A}_+$. For this step, we can repeat exactly the arguments of Lemma 4.2 and Theorem 4.3 of [6], where an adequate rescaling Gidas-Spruck argument and a new Liouville type theorem are used, see also Sections 2 and 3 in [11].
Step 2: A priori bounds on Ω. Define

$$R := \sup_{\lambda\in\Lambda}\sup_{x\in\overline{A}_+} u(x) < \infty.$$

We consider the problem

$$\begin{cases} \mathcal{L}z = \lambda z^q & \text{in } \Omega\backslash\overline{A}_+, \\ z = R & \text{on } \partial(\Omega\backslash\overline{A}_+). \end{cases} \tag{15}$$

We claim that there exists a unique nonnegative solution z_λ of (15). Then, it is clear that a solution u of (3) is a subsolution of (15) in $\Omega\backslash\overline{A}_+$, then by the uniqueness of the nonnegative solution of (15) we get

$$\|u\|_{L^\infty(\Omega\backslash\overline{A}_+)} \leq \|z_\lambda\|_{L^\infty(\Omega\backslash\overline{A}_+)},$$

whence the result follows.

It remains to prove the claim. For the existence we use the sub-supersolution method. Indeed, $\underline{z} := 0$ is a subsolution of (15). Now, let w be the unique positive solution of

$$\begin{cases} \mathcal{L}w = 1 & \text{in } \Omega\backslash\overline{A}_+, \\ w = R & \text{on } \partial(\Omega\backslash\overline{A}_+). \end{cases}$$

Then, $\overline{z} := Kw$ is a supersolution of (15) if K is sufficiently large.

For the uniqueness, firstly assume that $\lambda \geq 0$ and suppose that (15) possesses a further positive solution $v \neq u$. By the mean value theorem, we get

$$\mathcal{L}(u-v) = \lambda(u^q - v^q) = \lambda q \int_0^1 [tu + (1-t)v]^{q-1}\, dt\, (u-v) \quad \text{in } \Omega\backslash\overline{A}_+.$$

Hence,

$$\begin{cases} (\mathcal{L} - \lambda q M(x))(u-v) = 0 & \text{in } \ \Omega\backslash\overline{A}_+, \\ u - v = 0 & \text{on } \ \partial(\Omega\backslash\overline{A}_+), \end{cases}$$

where

$$M(x) := \int_0^1 [tu + (1-t)v]^{q-1}\, dt$$

Since $\lambda \geq 0$, by the strong maximum principle, u and v are strictly positive, and so M verifies (5). Moreover, it satisfies the following estimate

$$qM < u^{q-1} \qquad \text{in } \ \Omega\backslash\overline{A}_+.$$

Thus, according to Theorem 2.1

$$\sigma_1^{\Omega\backslash\overline{A}_+}[\mathcal{L} - \lambda q M] > \sigma_1^{\Omega\backslash\overline{A}_+}[\mathcal{L} - \lambda u^{q-1}] > 0,$$

this last inequality because the positive solution u of (15) is a positive strict supersolution of $\mathcal{L} - \lambda u^{q-1}$ in $\Omega\backslash\overline{A}_+$ under homogeneous Dirichlet boundary conditions. Therefore, $u - v = 0$ must be the unique solution of (15). This contradicts $u \neq v$ and shows the uniqueness in this case.

When $\lambda < 0$, the maximum principle implies that $z < R$ for any nonnegative solution z of (15). Hence, since $\overline{z} := R$ is supersolution of (15), it follows the existence of a maximal nonnegative solution z^* of (15) such that for any other nonnegative solution z of (15) it holds

$$0 \leq z \leq z^* < R.$$

Suppose that (15) possesses a further nonnegative solution $z < z^*$ and let $x_0 \in \Omega \backslash \overline{A}_+$ be such that $0 < z^*(x_0) - z(x_0) := \max_{x \in \Omega \backslash \overline{A}_+} \{z^*(x) - z(x)\}$. Then,

$$0 \leq -\sum_{i,j=1}^{N} a_{ij}(x_0) D_{ij}(z^* - z)(x_0) = \lambda((z^*(x_0))^q - z^q(x_0)) < 0,$$

because $\lambda < 0$, which gives us a contradiction. $\square$

Remark 5.2. *Other conditions can be imposed on p and a^+ to obtain a priori bounds for the positive solutions of* (3) *for compact interval of* $\mathbb{R}$*, see Introduction and Sections* 4*,* 5 *and* 6 *in* [6]*. On the other hand, if Ω_0 has Lebesgue measure zero and $\nabla a(x) \neq 0$ in Ω_0, we can obtain a priori estimates for the solutions for all $p < (N+2)/(N-2)$, $N \geq 3$, following* [14]*.*

6. Structure of the interval of existence. Multiplicity and stability result.

In order to show the stability and multiplicity results, we introduce some notations. Let e denote the unique positive solution of

$$\mathcal{L}e = 1 \qquad \text{in } \Omega, \quad e = 0 \qquad \text{on } \partial\Omega.$$

We denote

$$C_e(\overline{\Omega}) := \{u \in X : \text{for which there exists } \kappa > 0 \text{ such that } -\kappa e \leq u \leq \kappa e\},$$

endowed with the norm

$$\|u\|_e := \inf\{\kappa > 0 : -\kappa e \leq u \leq \kappa e\}$$

and ordered by its cone of positive functions P, which is normal and has nonempty interior, see [5]. Moreover, when $\lambda \geq 0$, we can define

$$\mathcal{K} : \mathbb{R}_+ \times P \mapsto P; \qquad \mathcal{K}(\lambda, u) := (\mathcal{L} + K)^{-1}(\lambda u^q + a(x)u^p + Ku)$$

for K sufficiently large. Then, $\mathcal{K}$ is compact on bounded sets, $\mathcal{K} \in C^1$ and strongly positive. Moreover, given (λ, u) a positive solution of (3) with $\lambda \geq 0$, the right derivative $\mathcal{K}'(\lambda, u) := (\partial_1 \mathcal{K}(\lambda, u), \partial_2 \mathcal{K}(\lambda, u))$ is strongly positive. Observe that the solution of (3) are the fixed point of $\mathcal{K}$. On the other hand, it is well-defined the fixed point index respect to P of an isolated fixed point u_0 of $\mathcal{K}$, denoted by

$i_P(\mathcal{K}, u_0)$, see Section 11 of [5] for the definition and the main properties of this concept. Furthermore, for any $\rho > 0$ we denote $P_\rho := B^e_\rho \cap P$, where B^e_ρ is the ball of radius ρ in $C_e(\overline{\Omega})$. Finally, for any (λ, u) positive solution of (3) with $\lambda \geq 0$, we call

$$R_\lambda := -q\lambda u^{q-1} - pa(x)u^{p-1}. \tag{16}$$

We say that (λ, u) is linearly asymptotically stable (resp. unstable) if $\sigma_1[\mathcal{L}+R_\lambda] > 0$ (resp. < 0) and linearly neutrally stable if $\sigma_1[\mathcal{L} + R_\lambda] = 0$.

In this section we want to describe the set of parameters λ for which (3) has a nonnegative solution, that is the set

$$\Lambda := \{\lambda \in \mathbb{R} : \exists u \in P,\, u \neq 0 \quad \text{solution of (3)}\}, \qquad \lambda^* := \sup \Lambda.$$

By Theorem 4.1 and Proposition 4.4

$$0 < \lambda^* \leq \overline{\lambda} < \infty.$$

The next result shows that λ^* goes infty when $\|a^+\|_\infty$ goes 0.

Proposition 6.1. *Suppose $\lambda > 0$. There exists $\varepsilon := \varepsilon(\lambda)$ such that if $\|a^+\|_\infty \leq \varepsilon$, then $\lambda^* \geq \lambda$.*

Proof. We define the map $\mathcal{F} : C^\alpha(\overline{A}_+) \times \operatorname{int} P \mapsto X$ by

$$\mathcal{F}(f, u) := \mathcal{L}u - \lambda u^q - fu^p - a^- u^p.$$

Observe that $\mathcal{F} \in C^1(C^\alpha(\overline{A}_+) \times \operatorname{int} P; X)$. Moreover, by Theorem 3.1, we get that $\mathcal{F}(0, \theta_{[\lambda,a^-]}) = 0$. In addition,

$$\partial_2 \mathcal{F}(0, \theta_{[\lambda,a^-]})\xi = \mathcal{L}\xi - \lambda q \theta^{q-1}_{[\lambda,a^-]}\xi - pa^- \theta^{p-1}_{[\lambda,a^-]}\xi,$$

which implies that $\partial_2 \mathcal{F}(0, \theta_{[\lambda,a^-]})$ is nonsingular by (7). The result follows from the implicit function theorem. $\square$

The following results have been proved in [7] (see Lemma 3.2, Lemma 3.4 and Theorem 2.2) when $\mathcal{L} = -\Delta$ and $a(x) = 1$. There, the authors used Lemma 3.3, which can not be applied in our case.

Lemma 6.2. *For all $\lambda \in (0, \lambda^*)$, (3) has a positive solution.*

Proof. Take $\lambda_0 \in (0, \lambda^*)$. Then, there exists $\mu \in (\lambda_0, \lambda^*]$ such that (3) has a positive solution w. By the strong maximum principle, w is strictly positive, and so there exists $\varepsilon > 0$ sufficiently small such that

$$\varepsilon \varphi_1 \leq w \quad \text{in } \Omega.$$

Moreover, $(\varepsilon\varphi_1, w)$ is a sub-supersolution of (3) for $\lambda < \mu$ and ε sufficiently small. This completes the proof. □

Lemma 6.3. *For all $\lambda \in (0, \lambda^*)$, (3) has a minimal positive solution, denoted by u_λ. Moreover, it holds*

$$\sigma_1[\mathcal{L} - q\lambda u_\lambda^{q-1} - pa(x)u_\lambda^{p-1}] \geq 0. \tag{17}$$

Proof. Given any positive solution u of (3), u is a supersolution of (3) with $a^+ \equiv 0$, and so $u \geq \theta_{[\lambda,a^-]}$. We take the pair $(\underline{u}, \overline{u}) := (\theta_{[\lambda,a^-]}, z)$ where z is the positive solution of (3) built in Lemma 6.2. Then, $(\underline{u}, \overline{u})$ is a sub-supersolution of (3). We can consider the monotone iteration

$$(\mathcal{L}+M)u_{n+1} = (\lambda u_n^q + a(x)u_n^p + Mu_n) \quad \text{in } \Omega, \quad u_{n+1} = 0 \quad \text{on } \partial\Omega, \quad \text{with } u_0 = \underline{u},$$

with M sufficiently large. Then, it is not hard to show that $u_n \uparrow u_\lambda$ and that u_λ is the minimal solution of (3).

The inequality (17) follows by Proposition 20.4 in [5], see also Lemma 3.5 in [7]. □

Proposition 6.4. *There exists $\beta > 0$ such that for all $\lambda \in (0, \lambda^*)$, there exists at most a solution u of (3) such that $\|u\|_\infty \leq \beta$.*

Proof. We define

$$\sigma := \min_{\lambda\in[0,\lambda^*]} \sigma_1[\mathcal{L} - q\lambda\theta_{[\lambda,a^-]}^{q-1} - pa^-\theta_{[\lambda,a^-]}^{p-1}].$$

By Theorem 3.1 and Lemma 3.2, we get $\sigma > 0$. Take

$$\beta := \left(\frac{\sigma}{p(a^+)_M}\right)^{1/(p-1)},$$

where $(a^+)_M := \max_{x\in\overline{\Omega}} a^+(x)$. Assume there exists a second positive solution w such that $0 < u_\lambda \leq w \leq \beta$. Let $\Phi = w - u_\lambda$ be, then since

$$\theta_{[\lambda,a^-]} \leq u_\lambda \leq w$$

and applying the mean value theorem, we have

$$(\mathcal{L} - q\lambda\theta_{[\lambda,a^-]}^{q-1} - pa^-\theta_{[\lambda,a^-]}^{p-1} + p(a^-\theta_{[\lambda,a^-]}^{p-1} - az^{p-1}))\Phi \leq 0$$

for some z such that $u_\lambda \leq z \leq w \leq \beta$. Taking account Theorem 2.3 and the definition of β, we obtain that

$$\sigma_1[\mathcal{L} - q\lambda\theta_{[\lambda,a^-]}^{q-1} - pa^-\theta_{[\lambda,a^-]}^{p-1} + p(a^-\theta_{[\lambda,a^-]}^{p-1} - az^{p-1})] =$$

$$= \sigma_1[\mathcal{L} - q\lambda\theta_{[\lambda,a^-]}^{q-1} - pa^-\theta_{[\lambda,a^-]}^{p-1} + p(a^-(\theta_{[\lambda,a^-]}^{p-1} - z^{p-1}) - a^+z^{p-1})] \geq$$

$$\geq \sigma_1[\mathcal{L} - q\lambda\theta^{q-1}_{[\lambda,a^-]} - pa^-\theta^{p-1}_{[\lambda,a^-]} - p(a^+)_M\beta^{p-1}] > 0,$$

and hence, $\Phi = 0$. □

The next result will be used in the proof of the main result of this section.

Lemma 6.5. *Let u a nonnegative and nontrivial solution of* (3) *with $\lambda = 0$. Then, u is linearly unstable, i.e.,*

$$\sigma_1[\mathcal{L} - pa(x)u^{p-1}] < 0.$$

Proof. Observe that any nonnegative and nontrivial solution of (3) with $\lambda = 0$ is in fact, by the strong maximum principle, strictly positive. It holds

$$(\mathcal{L} - pa(x)u^{p-1})u^p = p(1-p)u^{p-2}\sum_{i,j=1}^{N} a_{ij}D_iuD_ju < 0$$

and so, by Theorem 2.3 the result follows. □

The following result is consequence of Propositions 20.6, 20.7 and 20.8 of [5]. Recall the definition of R_λ in (16).

Lemma 6.6. *Let (λ_0, u_0) be a positive solution of* (3) *with $\lambda = \lambda_0 > 0$.*

(1) *If*

$$\sigma_1[\mathcal{L} + R_{\lambda_0}] > 0,$$

then, there exists $\varepsilon > 0$ and a differentiable mapping $u : I := (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \mapsto P$ such that $u(\lambda_0) = u_0$ and $(\lambda, u(\lambda))$ is a positive solution of (3) *for each $\lambda \in I$. Moreover, the mapping $\lambda \mapsto u(\lambda)$ is increasing and there exists a neighborhood $\mathcal{V}$ of (λ_0, u_0) in $\mathbb{R} \times P$ such that if $(\lambda, u) \in \mathcal{V}$ is a solution of* (3)*, then $(\lambda, u) = (\lambda, u(\lambda))$ for some $\lambda \in I$.*

(2) *If*

$$\sigma_1[\mathcal{L} + R_{\lambda_0}] = 0,$$

let Φ_0 be the principal eigenfunction associated with $\sigma_1[\mathcal{L} + R_{\lambda_0}]$. Then, there exists $\varepsilon > 0$ and a differentiable mapping $(\lambda, u) : J := (-\varepsilon, \varepsilon) \mapsto \mathbb{R} \times P$ such that $(\lambda(0), u(0)) = (\lambda_0, u_0)$ and for each $s \in J$, $(\lambda(s), u(s))$ is a positive solution of (3)*. Moreover,*

$$\lambda(s) = \lambda_0 + s^2\lambda_2 + O(s^3), \qquad u(s) = u_0 + s\Phi_0 + s^2\Psi_0 + O(s^3), \tag{18}$$

as $s \simeq 0$ and $\int_\Omega \Phi_0\Psi_0 = 0$. Moreover, there exists a neighborhood $\mathcal{W}$ of (λ_0, u_0) in $\mathbb{R} \times P$ such that $(\lambda, u) \in \mathcal{W}$ is a solution of (3)*, then $(\lambda, u) = (\lambda(s), u(s))$ for some $s \in I$. Moreover,*

$$sign\,\lambda'(s) = sign\,\sigma_1[\mathcal{L} + R_{\lambda(s)}]. \tag{19}$$

Proposition 6.7. *Assume $\mathcal{L}$ selfadjoint ($b_i = 0$ in* (2)*) and let (λ_0, u_0) be a positive solution of* (3) *with $\lambda = \lambda_0 > 0$, such that $\sigma_1[\mathcal{L} + R_{\lambda_0}] = 0$. Then,*

$$\lambda_2 < 0,$$

where λ_2 is defined (18).

Proof. By Lemma 6.6, for $s \in J$, we have

$$\mathcal{L}(u_0 + s\Phi_0 + s^2\Psi_0 + O(s^3)) = (\lambda_0 + s^2\lambda_2 + O(s^3))(u_0 + s\Phi_0 + s^2\Psi_0 + O(s^3))^q + \\ +a(x)(u_0 + s\Phi_0 + s^2\Psi_0 + O(s^3))^p.$$

Now, differentiating twice with respect s, taking account that

$$(\mathcal{L} + R_{\lambda_0})\Phi_0 = 0,$$

we obtain

$$(\mathcal{L} + R_{\lambda_0})\Psi_0 = \lambda_2 u_0^q + \frac{1}{2}\Phi_0^2(q(q-1)\lambda_0 u_0^{q-2} + p(p-1)a(x)u_0^{p-2}),$$

and so, by the Fredholm alternative

$$\lambda_2 = \frac{1}{2}\frac{\int_\Omega \Phi_0^3 u_0^{q-2}(q(1-q)\lambda_0 + p(1-p)a(x)u_0^{p-q})}{\int_\Omega u_0^q\Phi_0}.$$

Observe that, since u_0 and Φ_0 are strictly positive, there exist $C_i > 0$, $i = 1, 2$, such that

$$\Phi_0^3 u_0^{q-2} \le C_1 \mathrm{dist}(x, \partial\Omega)^{3-2+q} \le C_1,$$

and so λ_2 is well-defined.

To prove that $\lambda_2 < 0$, the basic tool is a Picone identity (see Section 4 in [12] and Lemma 4.1 in [21], for instance). Let $u, v \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ be such that $v/u \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $\Upsilon : [0, \infty) \mapsto \mathbb{R}$ of class C^1. Then

$$\int_\Omega \Upsilon(\frac{v}{u})(v\mathcal{L}u - u\mathcal{L}v) = -\int_\Omega \Upsilon'(\frac{v}{u})u^2 \sum_{i,j=1}^N a_{ij} D_i(\frac{v}{u}) D_j(\frac{v}{u}). \tag{20}$$

We take $\Upsilon(t) = t^2$, $v = \Phi_0$ and $u = u_0$. Observe that $v/u \in C(\overline{\Omega}) \cap C^1(\Omega)$ by the strong maximum principle. Hence, by (20) and since u cannot be a multiple of v, we obtain

$$\int_\Omega \Phi_0^3 u_0^{q-2}(\lambda_0(1-q) + a(x)(1-p)u_0^{p-q}) < 0,$$

and so, since $q < 1$

$$0 < \lambda_0(1-q)\int_\Omega \Phi_0^3 u_0^{q-2} < (p-1)\int_\Omega a(x)\Phi_0^3 u_0^{p-2}$$

now, by (H)

$$\lambda_0 q(1-q)\int_\Omega \Phi_0^3 u_0^{q-2} < p(p-1)\int_\Omega a(x)\Phi_0^3 u_0^{p-2},$$

and therefore $\lambda_2 < 0$. □

As an easy consequence of Lemma 6.6, relation (19) and Proposition 6.7, we obtain:

Corollary 6.8. *Assume $\mathcal{L}$ selfadjoint and let (λ_0, u_0) be a positive solution of (3) with $\lambda = \lambda_0 > 0$, such that $\sigma_1[\mathcal{L} + R_{\lambda_0}] = 0$. Then, there exists $\varepsilon > 0$ such that for each $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, (3) has two positive solutions, one of them linearly asymptotically stable and the other one linearly unstable. Moreover, there exist a neighborhood $\mathcal{N}$ of (λ_0, u_0) in $\mathbb{R} \times P$ such that (3) does not have a positive solution in $\mathcal{N}$ for $\lambda > \lambda_0$.*

We are ready to prove the main result of this section.

Theorem 6.9. *Assume $\mathcal{L}$ selfadjoint and that the hypotheses of Theorem 5.1 are satisfied. Then,*

(1)
$$\Lambda = (-\infty, \lambda^*],$$
(2) *There exist, at least, two positive solution in $(0, \lambda^*)$,*
(3) *There exists a unique positive solution in $(0, \lambda^*)$ linearly asymptotically stable,*
(4) *If we assume that (3) has a finite number of non-degenerate positive solutions, say $u_1, \dots, u_r$, then $r = 2k$ for some $k \geq 1$, and exactly k among them have index -1, and the other k have index 1.*

Proof. To show the first paragraph it remains to prove that there exists solution for $\lambda = \lambda^*$. Let (λ_n, u_n) a sequence of solutions with $0 < \lambda_n < \lambda^*$ and $\lambda_n \to \lambda^*$. By Theorem 5.1 and a standard compactness argument, we obtain that $u_n \to u^*$, with u^* solution of (3) for $\lambda = \lambda^*$. Moreover, $u^* \neq 0$ because of $\lambda = 0$ is the unique bifurcation value from the trivial solution, hence $u^* > 0$.

We will show that the minimal solution u_λ is the unique linearly asymptotically stable. Indeed, we take $\lambda_1 > 0$ sufficiently small such that $\sigma_1[\mathcal{L} + R_{\lambda_1}] > 0$, for $(\lambda_1, u_{\lambda_1})$. This is possible by Lemma 6.3, Proposition 6.4 and Corollary 6.8. By continuation to the left of λ_1, and thanks of Proposition 6.4 and Proposition 6.7, we obtain that u_λ is asymptotically stable for $0 < \lambda \leq \lambda_1$.

Now, we prolongate to the right of λ_1 to reach a value $\lambda_2 \leq \lambda^*$ where $\sigma_1[\mathcal{L}+R_\lambda] > 0$ for $0 < \lambda < \lambda_2$ and

$$\sigma_1[\mathcal{L} + R_{\lambda_2}] = 0.$$

If $\lambda_2 = \lambda^*$, we have just proved the existence of a linearly asymptotically stable positive solution for $\lambda \in (0, \lambda^*)$. So, assume $\lambda_2 < \lambda^*$ and take $\lambda_3 \in (\lambda_2, \lambda^*)$ and consider $(\lambda_3, u_{\lambda_3})$. In any case, $\sigma_1[\mathcal{L} + R_{\lambda_3}] = 0$ or $\sigma_1[\mathcal{L} + R_{\lambda_3}] > 0$, by Corollary 6.8 we can take $\lambda_4 \in (\lambda_2, \lambda_3]$ such that

$$\sigma_1[\mathcal{L} + R_{\lambda_4}] > 0.$$

We can prolongate to the left of λ_4 by a branch, say u^λ, of linearly asymptotically stable positive solution, see Proposition 6.7. This branch can not degenerate in the branch u_λ due to the uniqueness of positive solution around the minimal solution u_λ. Neither, it can degenerate to 0 in $\lambda = 0$, because of Proposition 6.4. Hence, there exists a positive linearly asymptotically stable for $\lambda = 0$, say u^0, which is impossible by Lemma 6.5.
A similar argument can be used to show the uniqueness of positive linearly asymptotically stable solution. Moreover, Theorem 5.1 and Theorem 4.1 show the second paragraph.

Now, we take $\Gamma := [0, b]$ with $b > \lambda^*$. By Theorem 5.1, there exists a positive constant C (independent from λ) such that $\|u\|_\infty \leq C$ for all $\lambda \in \Gamma$. We take $R := C + 1$ and then for all $\lambda \in \Gamma$

$$i_P(\mathcal{K}, P_R) = 0. \tag{21}$$

Indeed, we can consider the homotopy

$$\mathcal{H} : [0,1] \times P \mapsto P, \qquad \mathcal{H}(t,u) := (\mathcal{L} + K)^{-1}((\lambda(1-t) + tb)u^q + a(x)u^p + Ku).$$

Then,

$$i_P(\mathcal{K}, P_R) = i_P(\mathcal{H}(0,\cdot), P_R) = i_P(\mathcal{H}(1,\cdot), P_R) = i_P(\mathcal{H}(1,\cdot), 0) = 0,$$

because $u = 0$ is the only solution for $\lambda > \lambda^*$ and by Lemma 4.3. Moreover, by paragraph 3 and the Leray-Schauder formula we have

$$i_P(\mathcal{K}, u_\lambda) = 1.$$

Without lost of generality we can suppose that $u_1 = u_\lambda$. Now, for each nondegenerate solutions $u_2, \ldots, u_r$ we have

$$i_P(\mathcal{K}, u_i) = (-1)^{n_i}$$

where n_i is the sum of the algebraic multiplicities of all the eigenvalues greater than one of the linearized of $\mathcal{K}$ at u_i. Since $u = 0$ has index zero by Lemma 4.3 and (21), we obtain

$$0 = 0 + 1 + \sum_{i=2}^{r} (-1)^{n_i}$$

from where the result follows. $\square$

Remark 6.10. *Observe that we only have used that $\mathcal{L}$ is selfadjoint in Proposition 6.7 in order to apply the Picone identity. So, paragraphs (1) and (2) of Theorem 6.9 are true if $\mathcal{L}$ is a general operator as* (2).

References

[1] S. Alama, Semilinear elliptic equations with sublinear indefinite nonlinearities, *Adv. in Differential Equations,* **4** (1999), 813-842.

[2] S. Alama and G. Tarantello, On the semilinear elliptic equations with indefinite nonlinearities, *Calc. Var. Partial Differential Equations,* **1** (1993), 439-475.

[3] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite sign, *J. Funct. Anal.,* **141** (1996), 159-215.

[4] S. Alama and G. Tarantello, On the solvability of a semilinear elliptic equation via an associated eigenvalue problem, *Math. Z.*, **221** (1996), 467-493.

[5] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Review,* **18** (1976), 620-709.

[6] H. Amann and J. López-Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, *J. Differential Equations,* **146** (1998), 336-374.

[7] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.,* **122** (1994), 519-543.

[8] A. Ambrosetti, J. García-Azorero and I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal.,* **137** (1996), 219-242.

[9] A. Ambrosetti, and P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, *J. Math. Anal. Appl.,* **73** (1980), 411-422.

[10] D. Arcoya, J. Carmona and B. Pellacci, Bifurcation for some quasi-linear operators, *Proc. Royal Soc. of Edin. Section A.,* **131** (2001), 733-766.

[11] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Superlinear indefinite problems and nonlinear Liouville theorems, *Topol. Methods Nonlinear Anal.,* **4** (1994), 59-78.

[12] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, *NoDEA Nonlinear Differential Equations Appl.,* **2** (1995), 553-572.

[13] H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris, Série I,* **317** (1993), 465-472.

[14] W. Chen and C. Li, Indefinite elliptic problems in a domain, *Discrete Contin. Dynam. Systems,* **3** (1997), 333-340.

[15] M. Delgado, J. López-Gómez and A. Suárez, Non-linear versus linear diffusion. From classical solutions to metasolutions. *Adv. Differ. Eq.,* **7** (2002), 1101-1124.

[16] M. Delgado and A. Suárez, Stability and uniqueness for cooperative degenerate Lotka-Volterra model, *Nonl. Anal.,* **49** (2002), 757-778.

[17] R. Gómez-Reñasco and J. López-Gómez, The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations, *J. Differential Equations*, **167** (2000), 36-72.

[18] M. E. Gurtin and R. C. MacCamy, On the diffusion of biological populations, *Math. Biosci.,* **33** (1977), 35-49.

[19] J. Hernández, F. Mancebo and J. M. Vega de Prada, On the linearization of some singular nonlinear elliptic problems and applications, *Ann. Inst. H. Poincare Anal. Non-Lineaire,* **19** (2002), 777-813.

[20] J. López-Gómez, The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems, *J. Differential Equations*, **127** (1996), 263-294.

[21] J. López-Gómez, On the existence of positive solutions for some indefinite superlinear elliptic problems, *Comm. Partial Differential Equations*, **22** (1997), 1787-1804.

[22] J. López-Gómez, Varying bifurcation diagrams of positive solutions for a class of indefinite superlinear boundary value problems, *Trans. Amer. Math. Soc.,* **352** (2000), 1825-1858.

[23] L. Moschini, S. I. Pohozaev and A. Tesei, Existence and nonexistence of solutions of nonlinear Dirichlet problems with first order terms, *J. Funct. Anal.,* **177** (2000), 365-382.

[24] T. Ouyang, On the positive solutions of semilinear equations $\Delta u+\lambda u+hu^p=0$ on compact manifods. Part II, *Indiana Univ. Math. J.,* **40** (1991), 1083-1141.

[25] P. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.*, **7** (1971), 487-513.

[26] N. Tarfulea, Positive solution of some nonlinear elliptic equation with Neumann boundary conditions, *Proc. Japan Acad. Ser. A,* **71** (1995), 161-163.

Received April 10, 2001

Revised version received December 14, 2001

(M. Delgado) Dpto. Ecuaciones Diferenciales y Análisis Numérico, Fac. Matemáticas, C/ Tarfia s/n, C.P. 41012, Univ. Sevilla, Spain.

E-mail address: delgado@numer.us.es

(A. Suárez) Dpto. Ecuaciones Diferenciales y Análisis Numérico, Fac. Matemáticas, C/ Tarfia s/n, C.P. 41012, Univ. Sevilla, Spain.

E-mail address: suarez@numer.us.es