Bol. Soc. Esp. Mat. Apl.
n^{o}40(2007), 71–102

COMPUTATIONAL METHODS IN ALGEBRA AND ANALYSIS

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Abstract

This paper describes some applications of Computer Algebra to Algebraic Analysis also known as $\mathcal{D}$-module theory, i.e. the algebraic study of the systems of linear partial differential equations. Gröbner bases for rings of linear differential operators are the main tools in the field. We start by giving a short review of the problem of solving systems of polynomial equations by symbolic methods. These problems motivate some of the later developed subjects.

Key words: *Polynomial ring, polynomial system, ring of linear differential operators, differential system, D-module, Gröbner basis, division theorem, characteristic variety, irregularity, Bernstein-Sato polynomial, logarithmic D-module, projective module, syzygy, free resolution.*

AMS subject classifications: *68W30 13D02 13Pxx 14Qxx 16S32 16Z05 32C38 33F10.*

Introduction

The nature of this article is somehow mixed. It contains for example some elementary and very well known results on commutative polynomial rings but on the other hand it also contains non trivial results about systems of linear partial differential equations. Nevertheless, anyone with a basic knowledge of ring and module theory could, at least, have a grasp on the referred matter. The interested reader will find in the text precises references for the proofs of the announced results.

Computer Algebra, *Symbolic Computation* (CA/SC in what follows) or *Computational Algebra* is a relatively new discipline. This new field of research is

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Fecha de recepción: 07/12/2005. Aceptado (en forma revisada): 17/09/2007.

The title of this paper coincides with the one of one Special Session of the First Joint Meeting AMS-RSME (June 18-21, Seville, 2003). The Special Session was organized by E. Cattani and the author.

This work has been supported by MTM2004-01165 and FQM-333.

called *Calcul Formel* in French and *Cálculo Simbólico* or *Álgebra Computacional* in Spanish. The item *Álgebra Computacional* appears in the Spanish *Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica 2004-2007, Programa Nacional de Matemáticas*[1], as subsection 3.5.

In the Mathematics Subject Classification 2000 (MSC2000) used by Mathematical Reviews and Zentralblatt MATH, CA/SC appears as `68W30 Symbolic computation and algebraic computation [See also 11Yxx, 12Y05, 13Pxx, 14Qxx, 16Z05, 17-08, 33F10]`, and it provides methods and tools for many Mathematical areas as for example

Commutative Algebra	Algebraic Geometry
Number Theory	Algebraic Analysis or $\mathcal{D}$–modules
Differential Geometry	Associative Algebras
Group Theory	Algebraic Groups and Lie Algebras
Algebraic and Differential Topology	Combinatorics
Graph Theory	Computational Geometry
Coding Theory and Cryptography	Statistic and Probability

Recent CA/SC developments deeply interact with Numerical Analysis. Moreover, the study and analysis of the algorithms arising in CA/SC are also useful in other disciplines especially in Robotics (see e.g. [82, 37]), Computer Vision (see e.g. [27]), Computer Aided Geometric Design (see e.g. [38]), Artificial Intelligence (see e.g. [3]), Chemistry, Physics and Engineering (see e.g. [23] and [1]), Biology (see e.g. [28]), and Statistics and Economics (see e.g. [63] and [77, Chaps. 6,8]). Finally the *Journal of Symbolic Computation* and *Applicable Algebra in Engineering, Communication and Computing* are international journals mainly directed to researchers who are interested in symbolic computation and a new section of the *Journal of Algebra* is titled and devoted to *Computational Algebra*. Journals as *Mathematics of Computation, Journal of Complexity, Computational Complexity, SIAM Journal of Computing, ACM Communications in Computer Algebra* publish regularly CA/SC papers.

The algorithms described in this paper have been implemented in several Computer Algebra Systems most of them freely available. The following are widely used: `Macaulay 2` [35], `CoCoA` [22], `Singular-Plural` [36], `Risa-Asir` [62], `kan` [62], `Bergman` [6], `Gap` [32]. Mathematica® MAGMA® MuPAD® and Maple® are some commercial Computer Algebra Systems of general purpose containing implementations of some of the algorithms treated in this article.

The article is intended to provide a short introduction to the use of some Computer Algebra methods in the algebraic study of linear partial differential systems. Our main tool will be Gröbner bases for linear partial differential operators. Some of the algebraic methods developed in this article have been

[1]National Plan for Scientific Research and Technological Development 2004-2007, National Mathematics Program

treated by different authors elsewhere. A list of such works should include Ch. Riquier [66] and M. Janet [45] both inspired by the works of E. Cartan. This article does not deal with the general theory of Differential Algebra (see e.g. [67], [48]).

The article does not contain proofs but for any of them a reference is given.

The structure of the article is as follows: Section 1 is devoted to the description of some problems on systems of polynomial equations and their solutions by using Gröbner bases for polynomial rings. In Section 2 we recall the notion of Gröbner basis for rings of differential operators and its application to the algebraic study of systems of linear differential equations. We focus on the calculation of the characteristic variety of a linear partial differential system and on the computation of a free resolution of the module associated with the considered system. In Section 3 we sketch some applications of Gröbner bases to the computational study of the irregularity of differential systems and to logarithmic $\mathcal{D}$–modules.

1 Getting started

1.1 Polynomial rings and polynomial systems

> On peut dire que l'origine historique et un des buts essentiels de l'Algèbre, depuis les Babyloniens, les Hindous et Diophante jusqu'à nos jours, est l'étude des *solutions des systèmes d'équations polynomiales.*[2]

In this subsection we will describe some problems related to the study of systems of polynomial equations in several variables.

Let us denote by $\mathbb{R}[x_1, \ldots, x_n]$ the set of polynomials in the variables $x_1, \ldots, x_n$ and with coefficients in the field of real numbers $\mathbb{R}$. We also write $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ if no confusion arises. The letters $f, g, h, \ldots$ or the expressions $f(x), g(x), h(x), \ldots$ (sometimes with subindexes) stand for polynomials. A polynomial in $\mathbb{R}[x]$ can be written as a finite sum

$$\sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where the coefficients c_α are real numbers. To simplify we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The degree of f, denoted by $\deg(f)$, is the maximum of the integer numbers $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for $c_\alpha \neq 0$.

The set $\mathbb{R}[x]$ is a commutative ring with respect to the addition and the product of polynomials.

Let us consider a finite system

$$\mathcal{S} \equiv \{f_1(x) = f_2(x) = \cdots = f_m(x) = 0\} \tag{1}$$

[2]Extracted from the Introduction of the book: Grothendieck, A. and Dieudonné, J.A. Éléments de géométrie algébrique I, Springer-Verlag, Berlin, (1971), ISBN:0387051139.

of polynomial equations. We denote

$$\mathcal{V}_{\mathbb{R}}(\mathcal{S}) = \{w = (w_1, \ldots, w_n) \in \mathbb{R}^n \text{ such that } f_i(w) = 0,\, i = 1, \ldots, m\}$$

the set of *real solutions* of the system $\mathcal{S}$. We also write $\mathcal{V}_{\mathbb{R}}(\mathcal{S}) = \mathcal{V}_{\mathbb{R}}(f_1, \ldots, f_m)$. The subsets of $\mathbb{R}^n$ of type $\mathcal{V}_{\mathbb{R}}(\mathcal{S})$ for some system $\mathcal{S}$, are called *real affine algebraic sets* or simply algebraic sets if no confusion is possible. For example, the graph in $\mathbb{R}^2$ of any polynomial $f(x_1)$ in one variable x_1 is an algebraic set in $\mathbb{R}^2$ as this graph is the set $\mathcal{V}_{\mathbb{R}}(x_2 - f(x_1)) = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_2 = f(a_1)\}$.

If

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2 + a_{20}x_1^2 + a_{02}x_2^2$$

is a degree 2 polynomial in two variables x_1, x_2 then the set $\mathcal{V}_{\mathbb{R}}(f(x_1, x_2))$ is nothing but a real affine conic which could be degenerate.

If each equation in the system $\mathcal{S}$ is linear –i.e. if each polynomial $f_i(x)$ has degree 1, then the solution set $\mathcal{V}_{\mathbb{R}}(\mathcal{S})$ is simply a linear affine variety in $\mathbb{R}^n$ and Linear Algebra is devoted to the study of such objects.

The first objective of Algebraic Geometry is the study of the properties of algebraic sets $\mathcal{V}(\mathcal{S})$ when $\mathcal{S}$ is a general system of polynomial equations. Let us remark that if $w = (w_1, \ldots, w_n)$ is a real solution of System (1) then w is also a solution of any equation $f(x) = 0$ deduced from the $f_i(x) = 0$ by a linear combination with polynomial coefficients, i.e. $f(w) = 0$ for $f(x) = \sum_{i=1}^m q_i(x) f_i(x)$ for any choice of $q_i(x)$ in $\mathbb{R}[x]$. The set of such linear combinations is denoted by $\langle f_1(x), \ldots, f_m(x)\rangle$ and it's called an *ideal* of the ring $\mathbb{R}[x]$ (an ideal in a commutative ring R is an additive subgroup of R closed by products with elements in R).

Formally, we can also consider non necessarily finite systems of polynomial equations but the Hilbert's Basissatz (see, e.g. [5, Th. 7.5.]) assures that each system *is equivalent* to a finite one. More precisely, let us consider a system

$$\{f(x) = 0\}_{f(x) \in T}$$

of polynomial equations where T is an arbitrary subset of $\mathbb{R}[x]$. Its solution set is denoted by $\mathcal{V}_{\mathbb{R}}(T)$. By Hilbert's Basissatz there is a finite subset $\{g_1(x), \ldots, g_r(x)\}$ in T such that

$$\mathcal{V}_{\mathbb{R}}(T) = \mathcal{V}_{\mathbb{R}}(g_1, \ldots, g_r). \tag{2}$$

Moreover, if we denote by $\langle T\rangle$ the *ideal generated* by T, i.e. the set of all linear combinations of elements in T with polynomial coefficients, the Hilbert's Basissatz states precisely that there is a finite subset $\{g_1(x), \ldots, g_r(x)\}$ in T such that

$$\langle g_1(x), \ldots, g_r(x)\rangle = \langle T\rangle \tag{3}$$

and the equality (3) implies the equality (2). A commutative ring in which any ideal is finitely generated is called a *Noetherian ring*.

In what we have done before the field $\mathbb{R}$ can be replaced by any other field $\mathbb{K}$. Then we can consider the polynomial ring $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ on the variables

x_i and with coefficients in $\mathbb{K}$. In many practical applications $\mathbb{K}$ will be one of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ while finite fields are widely used for example in Coding Theory and Cryptography.

So, let us consider a general system of polynomial equations

$$\mathcal{S} \equiv \{f_1(x) = f_2(x) = \cdots = f_m(x) = 0\} \tag{4}$$

with coefficients if the field $\mathbb{K}$ and let us denote its solution set in the affine space $\mathbb{K}^n$ by $\mathcal{V}_{\mathbb{K}}(\mathcal{S}) = \mathcal{V}(f_1, \ldots, f_m) = \{w = (w_1, \ldots, w_n) \in \mathbb{K}^n \mid f_1(w) = \cdots = f_m(w) = 0\}$.

A subset Y of the affine space $\mathbb{K}^n$ is said to be an *affine algebraic set* or simply an *algebraic set* if there exists $T \subset \mathbb{K}[x]$ such that $Y = \mathcal{V}_{\mathbb{K}}(T)$. The empty set and $\mathbb{K}^n$ are algebraic sets. The union of two algebraic sets is an algebraic set. The intersection of an arbitrary family of algebraic sets is also an algebraic set (see e.g. [41]). These last properties of algebraic sets show that they are the closed sets of a topology on $\mathbb{K}^n$ which is called the *Zariski topology*.

Taking the system $\mathcal{S}$ as input we are concerned with the following problems or questions that can be considered as effective problems in polynomial rings.

(P1) Describe an algorithm to decide whether the set $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ is empty.

(P2) If $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ is not empty, describe an algorithm to decide whether $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ is finite and to compute its cardinal.

(P3) If $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ is finite, describe an algorithm to compute the elements of $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ i.e. to solve the system $\mathcal{S}$.

(P4) If $\mathcal{V}_{\mathbb{K}}(\mathcal{S})$ is not finite, can we describe its elements using parameters?

There are no known algorithms for solving Problems P1-P4 with the above generality. For example, if $f(x, y) \in \mathbb{Q}[x, y]$ is a cubic (i.e. if $\deg(f) = 3$) then there is no currently known algorithm to decide if $\mathcal{V}_{\mathbb{Q}}(f)$ is empty; see e.g. [74].

Nevertheless a weaker form of these problems will be solved in Subsection 1.2 by using Gröbner bases techniques. In these weaker forms, which will be denoted (P1'), (P2'), (P3') and (P4'), we will assume that even if the coefficients of $\mathcal{S}$ belong to the given field $\mathbb{K}$, the solutions of the system $\mathcal{V}_{\mathbb{L}}(\mathcal{S})$ are to be searched over an algebraically closed[3] field $\mathbb{L}$ containing $\mathbb{K}$ (for example $\mathbb{L}$ could be the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$).

One of the fundamental problems on effectiveness in polynomial rings is the so called *membership problem* as defined for example by G. Hermann [42] in 1926. It can be stated as follows:

(P0) Given a finite set of polynomials $f_1(x), \ldots, f_m(x)$ in $\mathbb{K}[x]$ describe an algorithm deciding if a given polynomial $f(x)$ belongs to the ideal $\langle f_1(x), \ldots, f_m(x)\rangle$.

[3] A field $\mathbb{K}$ is algebraically closed if the roots of every polynomial in one variable $f(t) \in \mathbb{K}[t]$ are in $\mathbb{K}$.

To solve this problem G. Hermann gave an upper bound for the degree of the polynomials $q_i(x)$ appearing in an expression $f(x) = \sum_i q_i(x) f_i(x)$ if they exist, and then she used Linear Algebra to algorithmically solve problem P0. The upper bound found by G. Hermann is of type $\deg(f) + (md)^{2^n}$ where d is the maximum of the degrees of the f_i. Moreover, as shown in [57], this upper bound is sharp. Hermann's upper bound is in fact related to the degree of a *Gröbner basis* of the ideal generated by the f_i, as we will see in the next section.

1.2 Gröbner bases for polynomial rings

If $f(x_1)$ is a nonzero polynomial in $\mathbb{K}[x_1]$ then the set $\mathcal{V}_{\mathbb{K}}(f)$ is nothing but the set of roots of $f(x_1) = 0$ in $\mathbb{K}$ and its cardinal is bounded by the degree $\deg(f)$. Moreover, $\deg(f)$ equals the dimension of the quotient $\mathbb{K}$–vector space $\frac{\mathbb{K}[x_1]}{\langle f(x_1)\rangle}$.

For a general system $\mathcal{S}$ as (4) we have

Proposition 1 (see e.g. [26, Chap. 2, Th. 2.10] or [54, Chap. 2 Prop. 1.4]) *Let $\mathcal{S}$ be the system $f_1(x) = f_2(x) = \cdots = f_m(x) = 0$ and let us denote by $\langle \mathcal{S}\rangle$ the ideal in $\mathbb{K}[x]$ generated by the f_i. If $\dim_{\mathbb{K}} \frac{\mathbb{K}[x]}{\langle \mathcal{S}\rangle}$ is finite then*

$$\#\mathcal{V}_{\mathbb{K}}(\mathcal{S}) \leq \dim_{\mathbb{K}} \frac{\mathbb{K}[x]}{\langle \mathcal{S}\rangle}.$$

Here $\#Z$ denotes the cardinal of a set Z and $\frac{\mathbb{K}[x]}{\langle \mathcal{S}\rangle}$ is considered as the quotient vector space of $\mathbb{K}[x]$ by the ideal $\langle \mathcal{S}\rangle$ and $\dim_{\mathbb{K}}(\,)$ denotes the vector space dimension. The quotient $\frac{\mathbb{K}[x]}{\langle \mathcal{S}\rangle}$ also has a natural structure of commutative ring with unit with respect to the addition and the multiplication of classes modulo $\langle \mathcal{S}\rangle$.

The reciprocal of Proposition 1 is false in general. Consider $f(x_1, x_2) = x_1^2 + x_2^2$. We have $\mathcal{V}_{\mathbb{R}}(f) = \{(0,0)\} \subset \mathbb{R}^2$ but $\dim_{\mathbb{R}}(\frac{\mathbb{R}[x_1,x_2]}{\langle f\rangle}) = +\infty$. Moreover, in Theorem 3 we will see a more precise result.

The computation of the dimension of the vector space $\mathbb{K}[x]/\langle \mathcal{S}\rangle$, even when it is finite, is more complicated than in the one variable case. It is not enough to consider only the degrees $\deg(f_i)$. To deal with we will start by introducing the notion of *monomial order*. A total ordering $\prec$ on $\mathbb{N}^n$ is a monomial order if $0 = (0, \ldots, 0) \prec \alpha$ for all $\alpha \in \mathbb{N}^n$ and $\prec$ is compatible with the sum (i.e. $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for all $\gamma \in \mathbb{N}^n$). The *lexicographical order* (denoted by $<_{\rm lex}$) on $\mathbb{N}^n$ is defined as follows: $(\alpha_1, \ldots, \alpha_n) <_{\rm lex} (\beta_1, \ldots, \beta_n)$ if and only if the first nonzero component of $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is negative. The total order $<_{\rm lex}$ is a monomial order. The only monomial order in $\mathbb{N}$ is the natural order. We can translate any order $\prec$ in $\mathbb{N}^n$ to the set of monomial $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ just by writing $x^\alpha \prec x^\beta$ if and only if $\alpha \prec \beta$.

Once a monomial order has been fixed on $\mathbb{N}^n$, we can associate to each nonzero polynomial $f = f(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{K}[x]$ its *privileged exponent* $\exp_\prec(f)$ with respect to $\prec$, defined as the maximum with respect to $\prec$ of the set $\{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}$. We write simply $\exp(f)$ is no confusion is possible. One has $\exp(fg) = \exp(f) + \exp(g)$ for all nonzero $f, g \in \mathbb{K}[x]$. If $0 \neq f = f(x_i)$ only involves one variable x_i, then $\exp(f) = \deg(f)\epsilon_i$ where ϵ_i is the i-th element in the canonical basis of $\mathbb{N}^n$.

For each non empty subset A of $\mathbb{K}[x]$ let us denote by $E_\prec(A)$ (or simply $E(A)$) the set

$$E_\prec(A) = \cup_{f\in A}(\exp_\prec(f) + \mathbb{N}^n).$$

It is easy to prove the equality $E(A) + \mathbb{N}^n = E(A)$. There exists a finite subset $A' \subset A$ such that $E(A) = E(A')$ (this is a consequence of Dickson's Lemma; see e.g. [26, p. 12]). We say that $E(A)$ is generated by the set $\{\exp(f) \mid f \in A'\}$. So, if $A = I$ is an ideal of $\mathbb{K}[x]$ there exists a finite subset $G \subset I$ such that $E(I) = E(G)$ is generated by $\{\exp(g) \mid g \in G\}$.

Definition 1 *Let $I \subset \mathbb{K}[x]$ be a nonzero ideal. A finite subset $G \subset I$ is said to be a Gröbner basis of I with respect to the fixed monomial order $\prec$ if $E_\prec(I) = E_\prec(G)$.*

Example 1 *a) If the ideal $I = \langle f\rangle$ is principal and generated by a polynomial f then $E(I)$ is the (hyper)-quadrant generated by* $\exp(f)$*, i.e. $E(I) = \exp(f) + \mathbb{N}^n$ and then $\{f\}$ is a Gröbner basis of I.*
b) Let us consider $f_1 = x_1 - x_2, f_2 = x_1 + x_2$ and $I \subset \mathbb{R}[x_1, x_2]$ the ideal generated by f_1, f_2. With respect to the lexicographical order $<_{\text{lex}}$ on $\mathbb{N}^2$ we have $\exp(f_1) = \exp(f_2) = (1,0)$ *since $x_2 <_{\text{lex}} x_1$. It is easy to prove that $E(I)$ is the union of the quadrants* $(1,0)+\mathbb{N}^2$ *and* $(0,1)+\mathbb{N}^2$*. So, $\{f_1, f_2\}$ is not a Gröbner basis of I with respect to $<_{\text{lex}}$. Nevertheless, $\{f_1, x_2\}$ is a Gröbner basis of I, with respect to $<_{\text{lex}}$.*

If $E \subset \mathbb{N}^n$ we denote by $c(E) = \mathbb{N}^n \setminus E$ its complement and by $\mathbb{K}[x]^{c(E)}$ the vector space of polynomials of type $\sum_{\beta\in c(E)} c_\beta x^\beta$. For the sake of simplicity we denote $c_\prec(I)$ instead of $c(E_\prec(I))$ (or simply $c(I)$ if no confusion is possible).

Let us consider a vector $(f_1, \ldots, f_m)$ of nonzero polynomials in $\mathbb{K}[x]$. The Division Theorem in $\mathbb{K}[x]$ is stated as follows:

Theorem 2 (e.g. [26, p. 9] or [2, Th. 1.5.9]) *For each $f \in \mathbb{K}[x]$ there exists a polynomial $r \in \mathbb{K}[x]^{c(f_1,\ldots,f_m)}$ such that $f - r = \sum_i q_i f_i$ for some polynomials q_i. Here $c(f_1, \ldots, f_m) = c(\cup_{i=1}^m(\exp(f_i) + \mathbb{N}^n))$.*

The q_i and r can be chosen to be unique if they satisfy certain combinatorial conditions about their supports. The support of a polynomial $f = \sum_\alpha c_\alpha x^\alpha$ is the set $\{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}$. There exists an algorithm computing the remainder r and the quotients q_i starting from f and the f_i (see e.g. [2, Algorithm 1.5.1]).

If $n = 1$ the Division Theorem 2 is nothing but the classical Euclidean Division Theorem. As a corollary of Theorem 2 one can prove that any Gröbner basis of an ideal $I \subset \mathbb{K}[x]$ generates I.

A famous algorithm, due to B. Buchberger [12], takes as input a monomial order $\prec$ on $\mathbb{N}^n$ and a finite set $\mathcal{F} = \{f_1(x), \ldots, f_m(x)\}$ of polynomials, and computes a Gröbner basis –with respect to $\prec$, of the ideal I of $\mathbb{K}[x]$ generated by $\mathcal{F}$. As a consequence one can also compute a finite system of generators of the set $E(I)$.

The worst-case complexity of the computation of a Gröbner basis is doubly exponential on the degrees of the f_i as proved in [57]. Nevertheless, despite this theoretical bad behavior lots of invariants can be effectively computed in Algebraic Geometry using Gröbner basis theory (see e.g.[7]).

The proof of the following Theorem uses some properties of Gröbner bases. Recall the notation $c(I) = \mathbb{N}^n \setminus E(I)$ for any ideal $I \subset \mathbb{K}[x]$ and that $\#Z$ is the cardinal of the set Z. The *radical* $\sqrt{I}$ of an ideal $I \subset \mathbb{K}[x]$ is the ideal of polynomials $f \in \mathbb{K}[x]$ such that $f^e \in I$ for some integer e (which depends on f).

Theorem 3 [26, pp. 37-42] *Let $\mathbb{K}$ be a field and $\{f_1 = f_2 = \cdots = f_m = 0\}$ a polynomial system. Let $I \subset \mathbb{K}[x]$ be the ideal generated by the f_i. Then*

$$\#c(I) = \dim_{\mathbb{K}}(\frac{\mathbb{K}[x]}{I}).$$

Moreover, if $\mathbb{K}$ is algebraically closed then

1) $\#\mathcal{V}_{\mathbb{K}}(I) < +\infty$ if and only if $\dim_{\mathbb{K}}(\frac{\mathbb{K}[x]}{I}) < +\infty$

2) $\#\mathcal{V}_{\mathbb{K}}(I) \leq \dim_{\mathbb{K}}(\frac{\mathbb{K}[x]}{I})$ and equality holds if and only if I is a radical ideal (i.e. if and only if $\sqrt{I} = I$).

If the ideal $I \subset \mathbb{K}[x]$ satisfies $\dim_{\mathbb{K}}(\mathbb{K}[x]/I) < +\infty$ we will say that I is *zero dimensional.*[4]

Let us give a solution to problems P0, P1', P2', P3' and P4' described in Subsection 1.1.

Solution to P0.- Although G. Hermann gave a solution to this problem, we can give a new one by using Gröbner basis. We first compute –using Buchberger's algorithm, a Gröbner basis $\{g_1, \ldots, g_r\}$ of the ideal I generated by the f_i and then we compute the remainder r of the division of the polynomial f by $\{g_1, \ldots, g_r\}$ (see Theorem 2). Then $f \in I$ if and only if $r = 0$.

Solution to P1' and P2'.- By Buchberger's algorithm a finite system of generators of $E(I)$ and $\dim_{\mathbb{K}} \frac{\mathbb{K}[x]}{I}$ can be computed. By Hilbert's Nullstellensatz (e.g. [50, p. 16]), the set $\mathcal{V}_{\overline{\mathbb{K}}}(I)$ is empty if and only if $I = \mathbb{K}[x]$ and so, if and only if any Gröbner basis of I contains a nonzero constant polynomial. Here $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$. This result gives the answer to P1'. Moreover, by Theorem 3 we can test the finiteness of $\mathcal{V}_{\overline{\mathbb{K}}}(I)$ since $\dim_{\overline{\mathbb{K}}}(\overline{\mathbb{K}}[x]/\overline{\mathbb{K}}[x]I) = \dim_{\mathbb{K}}(\mathbb{K}[x]/I)$. To compute the exact number of solutions $\mathcal{V}_{\overline{\mathbb{K}}}(I)$ we can apply Theorem 3 again and the fact that $\sqrt{I}$ can be computed (i.e. a finite system

[4]The *Krull dimension* (see e.g. [10, Chap. 8]) of the quotient ring $\mathbb{K}[x]/I$ is zero in this case. See **Solution to P4'** in Subsection 1.2.

of generators of the radical ideal $\sqrt{I}$ can be computed in an effective way,[5] see e.g. [26, Chap. 2, Prop. 2.7]). This solves problem P2'. As suggested by one referee this can be seen as a generalization of the one variable case. Let I be the principal ideal $I = \langle t^3 \rangle \subset \mathbb{C}[t]$ then $\mathcal{V}_{\mathbb{C}}(I) = \{0\}$ has only one element but the dimension of the quotient vector space $\mathbb{C}[t]/I$ is 3. In this case $\sqrt{I} = \langle t \rangle$ and the dimension of $\mathbb{C}[t]/\sqrt{I}$ is 1.

Solution to P3'.- For any field $\mathbb{K}$, if the solution set $V := \mathcal{V}_{\overline{\mathbb{K}}}(f_1, \ldots, f_m)$ is finite then there is an algorithm based in the *elimination principle* (see [80]) to compute V in a finite field extension of the base field $\mathbb{K}$. To have a grasp of how it works, we can show that if we calculate a Gröbner basis of the ideal $\langle f_1, \ldots, f_m \rangle$ with respect to an ordering for which $x_i > x_n$ for $i = 1, \ldots, n-1$ —this is a special case of what are called *elimination orderings*— we obtain a nonzero polynomial $g(x_n)$ in the Gröbner basis. Once the roots of this polynomial $g(x_n)$ are known (maybe in a finite extension $\mathbb{K}'$ of the field $\mathbb{K}$) we can substitute them in the original system to obtain (a finite number of) systems in $x_1, \ldots, x_{n-1}$ with coefficients in $\mathbb{K}'$. This strategy is a generalization of Gaussian elimination in the linear case. It should be said that an elimination order leads to computations which are very often untractable. To avoid this bottleneck, the so-called FGLM (from J.C. Faugère, P. Gianni, D. Lazard and T. Mora) method can be applied in this case [29].

In practical situations a *numerical approximation* of a real or complex solution of a system is useful and very often even necessary if the results are accurate enough. So, in general, symbolic and numerical methods are combined to solve real or complex polynomial systems. Nevertheless, the elimination-extension methods can produce errors accumulation which are difficult to manage (see e.g. [26, p. 28-34]). To overcome these problems several strategies are used as for example the one based on the computation of the eigenvalues of some matrices attached to our starting system (cf. [26, Chap. 2, Sec. 4]).

Solution to P4'.- If $V = \mathcal{V}_{\overline{\mathbb{K}}}(f_1, \ldots, f_m)$ is infinite the previous method using *elimination strategy* fails. The key idea in this case is to consider some of the variables as "parameters" and to solve the system giving a finite number of solutions as function of these parameters. This is a generalization of what is done when the system has only finitely many solutions. This "parametric" method can be done in a systematic way applying for example Noether's Normalization Lemma (e.g. [5]) to find an integer r, $1 \le r \le n$ and new coordinates $y_1, \ldots, y_n$ such that for each point $(a_1, \ldots, a_r) \in \mathbb{K}^r$ the system defined by $\{g_1(a_1, \ldots, a_r, y_{r+1}, \ldots, y_n), \ldots, g_m(a_1, \ldots, a_r, y_{r+1}, \ldots, y_n)\}$ has finitely many solutions and we can apply the previous case to solve it. Here the polynomials $g_i(y)$ come from the $f_j(x)$ by the variable change. This procedure is also algorithmic and the integer r is called the *(Krull) dimension* (see e.g. [10, Chap. 8]) of the algebraic set V. (See e.g. [54]).

Let us finish this subsection by quoting that the Division Theorem and

[5]In our case, since $\dim_{\mathbb{K}}(\mathbb{K}[x]/I) < +\infty$, the computation of $\sqrt{I}$ is much easier than for general ideals.

Gröbner bases techniques can be generalized for submodules of a free module $\mathbb{K}[x]^r$ (see e.g. [31] or [2, Chap. 3]).

2 Rings of differential operators and systems of linear partial differential equations

For simplicity we are going to consider either the complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$ as the base field. Nevertheless, in what follows many algebraic results also hold for any base field $\mathbb{K}$ of characteristic zero.

Let us recall that a linear differential operator (LDO) in n variables, with polynomial coefficients is a finite sum of the form

$$P(x,\partial) = \sum_{\beta\in\mathbb{N}^n} p_\beta(x)\partial^\beta$$

where each $p_\beta(x)$ is a polynomial in $\mathbb{K}[x]$, $\partial = (\partial_1,\ldots,\partial_n)$ with $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$.

The set of such LDOs is denoted by $A_n(\mathbb{K})$ (or simply A_n if no confusion is possible) and it is called the *Weyl algebra* of order n with coefficients in the field $\mathbb{K}$. The expressions $P(x,\partial), Q(x,\partial), R(x,\partial),\ldots$ and $P, Q, R,\ldots$ (sometimes with subindexes) will denote LDOs.

The elements in A_n can be added and multiplied in a natural way. Leibniz's rule holds for the multiplication in the ring $A_n:\ \partial_i a(x) = a(x)\partial_i + \frac{\partial a(x)}{\partial x_i}$ for any $a(x)\in\mathbb{K}[x]$. So A_n is an associative non commutative ring with unit (the unit being the 'constant' operator $1 = x_1^0\cdots x_n^0\partial_1^0\cdots\partial_n^0$).

The *order* of a nonzero operator $P = \sum_{\beta\in\mathbb{N}^n} p_\beta(x)\partial^\beta$, denoted by $\mathrm{ord}(P)$, is the maximum of the integer numbers $|\beta| = \beta_1+\cdots+\beta_n$ for $p_\beta(x)\neq 0$ and the *principal symbol* of P is the polynomial

$$\sigma(P) = \sum_{|\beta|=\mathrm{ord}(P)} p_\beta(x)\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}.$$

Here $\xi = (\xi_1,\ldots,\xi_n)$ are new variables and so $\sigma(P)\in\mathbb{K}[x,\xi]$ is a polynomial in $2n$ variables. Sometimes we will write $\sigma(P)(x,\xi)$ to emphasize this fact. Notice that $\sigma(P)$ is homogeneous in ξ of degree $\mathrm{ord}(P)$. One has the equality $\sigma(PQ) = \sigma(P)\sigma(Q)$ for $P, Q\in A_n$ and by definition $\sigma(0) = 0$.

Remark 1 *One can also consider LDOs with coefficients in other rings as the ring $\mathcal{O}_{\mathbb{C}^n}(U)$ (resp. $\mathcal{O}_{\mathbb{R}^n}(U)$) of holomorphic (resp. analytic) functions in some open set $U\subset\mathbb{C}^n$ (resp. $U\subset\mathbb{R}^n$) or the ring of convergent power series $\mathbb{C}\{x\} = \mathbb{C}\{x_1,\ldots,x_n\}$ (or $\mathbb{R}\{x\} = \mathbb{R}\{x_1,\ldots,x_n\}$). If R is any of these rings we will denote by* $\mathrm{Diff}(R)$ *the corresponding ring of LDOs.*

One of the goals of the theory of Differential Equations is to study the existence, uniqueness and the properties of the solutions of linear partial differential systems (LPDS)

$$\begin{array}{cccccc} P_{11}(u_1) & +\cdots+ & P_{1m}(u_m) & = & v_1 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{\ell 1}(u_1) & +\cdots+ & P_{\ell m}(u_m) & = & v_\ell \end{array} \tag{5}$$

where P_{ij} are LDOs, u_j are unknown and v_i are given data. Both u_j and v_j could be functions, distributions, hyperfunctions, ...

Assume System (5) is homogeneous (i.e. $v_1 = \cdots = v_\ell = 0$). If $u = (u_1, \dots, u_m)$ is a solution of the system then u is also a solution[6] of any equation $P_1(u_1) + \cdots + P_m(u_m) = 0$ with

$$(P_1, \dots, P_m) = \sum_{i=1}^{\ell} Q_i (P_{i1}, \dots, P_{im}) \tag{6}$$

for any Q_i in A_n (or more generally $Q_i \in \mathrm{Diff}(R)$ if we are considering any of the rings of Remark 1. For simplicity in what follows, we will assume $\mathrm{Diff}(R) = A_n = A_n(\mathbb{C})$). The set of such linear combinations (see Equation (6)) with coefficients in A_n is a (left) sub-module of the free module A_n^m denoted by $A_n(\underline{P}_1, \dots, \underline{P}_\ell)$ where $\underline{P}_i$ is the vector $(P_{i1}, \dots, P_{im})$.

Here A_n^m is nothing but the set of all vectors $(P_1, \dots, P_m)$ of differential operators $P_i \in A_n$ and plays a crucial role in the whole theory of $\mathcal{D}$–modules. This role is somehow analogous to the one played by vector spaces $\mathbb{K}^m$ in Linear Algebra. It is also important to mention here that the module $A_n(\underline{P}_1, \dots, \underline{P}_\ell)$ plays the role of a vector sub-space of $\mathbb{K}^m$. We have used here the algebraic structure *module* (see e.g. [68]) (over the non commutative ring A_n) which generalizes the vector space notion to a context where the scalars are no longer in a field but in a ring (in our case this ring is A_n).

When $m = 1$ (i.e. when the system has only one unknown $u = u_1$) then System (5) reduces to (writing $P_{11} = P_1, \dots, P_{\ell 1} = P_\ell$)

$$\begin{array}{ccc} P_1(u) & = & v_1 \\ \vdots & & \vdots \\ P_\ell(u) & = & v_\ell \end{array} \tag{7}$$

and the set of linear combinations $\sum_i Q_i P_i$ with coefficients $Q_i \in A_n$ is a (left) ideal in A_n, denoted by $A_n(P_1, \dots, P_\ell)$.

B. Malgrange [55], D. Quillen [64] and the Japanese school of M. Sato (e.g. [72]) have been probably the first to associate to each system as (5) the (left) quotient A_n–module[7]

$$\frac{A_n^m}{A_n(\underline{P}_1, \dots, \underline{P}_\ell)}$$

[6] We do not need to precise here the space of the wanted solutions. The result is true for any such space.

[7] All the modules and ideals considered here will be left modules and left ideals unless otherwise stated.

and this quotient is also called a *differential system.*

As A_n is left-Noetherian (see Subsection 2.2) any finitely generated left A_n-module is isomorphic to a quotient of type

$$\frac{A_n^m}{A_n(\underline{P}_1,\ldots,\underline{P}_\ell)}.$$

As in the polynomial case, see Subsection 1.1, the attached quotient module encodes important information about the system.

Different systems could have the same associated module, i.e. the corresponding quotient modules could be isomorphic. For example, we have the isomorphism

$$\frac{A_2}{A_2(\partial_1^2+\partial_2^2)} \simeq \frac{A_2^3}{N}$$

where $N \subset A_2^3$ is the submodule generated by the family $(\partial_1,-1,0)$, $(\partial_2,0,-1)$, $(0,\partial_1,\partial_2)$. This isomorphism encodes the fact that the systems

$$(\partial_1^2+\partial_2^2)(u)=0 \tag{8}$$

and

$$\left\{\begin{array}{lcl} \partial_1(u_1)-u_2 & = & 0\\ \partial_2(u_1)-u_3 & = & 0\\ \partial_1(u_2)+\partial_2(u_3) & = & 0 \end{array}\right. \tag{9}$$

are equivalent in the sense that the computation of their solutions (wherever they lie) are equivalent problems since they can be reduced to each other. More precisely, a suitable function $u=u(x_1,x_2)$ is a solution of Equation (8) if and only if the vector $(u_1=u, u_2=\partial_1(u_1), u_3=\partial_2(u_1))$ is a solution of System (9).

The study of such A_n–modules is the object of the so-called *Algebraic Analysis*[8] or $\mathcal{D}$–module theory.[9]

In the next three Subsections we are going to recall the classical definition of characteristic vector of a linear partial differential equation (Subsection 2.1), then we will recall the definition and basic properties of Gröbner bases for LDOs and we will show how they can be used to compute the characteristic variety of a LPDS (Subsections 2.2 and 2.3).

2.1 Classical characteristic vectors

If we have just one linear partial differential equation (LPDE)

$$P(x,\partial)(u) = \left(\sum_\beta p_\beta(x)\partial^\beta\right)(u) = v$$

[8]Introduced by M. Sato. See `http://en.wikipedia.org/wiki/Mikio_Sato`

[9]`Mathematics Subject Classification 2000 (MSC2000): 32C38 Sheaves of differential operators and their modules, D-modules [See also 14F10, 16S32, 35A27, 58J15].`

with analytic coefficients $p_\beta(x)$ in some open subset $U \subset \mathbb{R}^n$, a vector $\xi_0 \in \mathbb{R}^n$ is called *characteristic* for P at $x_0 \in U$ if $\sigma(P)(x_0, \xi_0) = 0$ and the set of all such ξ_0 is called the *characteristic variety* of the operator P (or of the equation $P(u) = v$) at $x_0 \in U$ and is denoted by $\mathrm{Char}_{x_0}(P)$. Notice that here, in contrast to some textbooks, the zero vector could be characteristic. More generally, the *characteristic variety* of the operator P is by definition the set

$$\mathrm{Char}(P) = \{(x_0, \xi_0) \in U \times \mathbb{R}^n \,|\, \sigma(P)(x_0, \xi_0) = 0\}.$$

Assume $\mathrm{ord}(P) \geq 1$, then P is said to be *elliptic* at x_0 if P has no nonzero characteristic vectors at x_0 (i.e. $\mathrm{Char}_{x_0}(P) \subset \{0\}$) and it is said to be *elliptic* on U if $\mathrm{Char}(P) \subset U \times \{0\}$.

The *Laplace operator* $\sum_{i=1}^n \partial_i^2$ is elliptic on $\mathbb{R}^n$. The characteristic variety of the *wave operator* $P = \partial_1^2 - \sum_{i=2}^n \partial_i^2$ is nothing but the hyperquadric defined in $\mathbb{R}^n \times \mathbb{R}^n$ by the equation $\xi_1^2 - \sum_{i=2}^n \xi_i^2 = 0$.

Characteristic vectors are important in the study of singularities of solutions as can be seen in any classical book on Differential Equations.

To define the principal symbol and the characteristic vectors for a system (5) of linear differential equations in many variables is more involved and in general the naive approach of simply considering the principal symbols of the equations turns out to be unsatisfactory (see Example 2). We will use *graded ideals* and Gröbner bases for LDOs (see Subsections 2.2 and 2.3) to define and to compute the *characteristic variety* of a general LPDS.

2.2 Gröbner bases for rings of differential operators

The definition and construction of Gröbner bases for polynomial rings can be adapted to the case of rings of linear differential operators [11, 15], see also [70].

Let $P = P(x, \partial) = \sum_{\beta \in \mathbb{N}^n} p_\beta(x) \partial^\beta$ be a differential operator in A_n. The operator P can be rewritten as

$$P = \sum_{\alpha\beta} p_{\alpha\beta} x^\alpha \partial^\beta$$

just by considering the polynomial p_β as $p_\beta(x) = \sum_\alpha p_{\alpha\beta} x^\alpha$, with $p_{\alpha\beta} \in \mathbb{C}$.

Let us fix a monomial order $\prec$ on $\mathbb{N}^{2n}$. We call *privileged exponent* with respect to $\prec$ of a nonzero operator P –and we denote it by $\exp_\prec(P)$– the maximum $(\alpha, \beta) \in \mathbb{N}^{2n}$ such that $p_{\alpha\beta} \neq 0$. We will write simply $\exp(P)$ if no confusion is possible. The equality $\exp(PQ) = \exp(P) + \exp(Q)$ is satisfied for all nonzero $P, Q \in A_n$.

If I is a nonzero ideal in A_n, we denote (as in the polynomial case) by $E_\prec(I)$ (or simply $E(I)$) the set of privileged exponents of the nonzero elements in I. Since $E(I) + \mathbb{N}^{2n} = E(I)$ there exists a finite subset $G \subset I$ such that $E(I)$ is generated by $\{\exp(P) \,|\, P \in G\}$ (see Subsection 1.2).

Definition 2 *Let $I \subset A_n$ be a nonzero ideal. A finite subset $G = \{P_1, \ldots, P_r\} \subset I$ such that $E(I)$ is generated by $\{\exp(P_i) \,|\, i = 1, \ldots, m\}$, is called a Gröbner basis of I with respect to the fixed monomial order $\prec$.*

A Division Theorem (analogous to Theorem 2) can be proved for elements in A_n (see [11, 15]) and as a consequence, each Gröbner basis of I is a generating system of I and in particular the ring A_n is left-Noetherian.

If the ideal I is principal and generated by an operador P then $E(I)$ is the hyper-quadrant generated by $\exp(P)$ in $\mathbb{N}^{2n}$: one has $E(I) = \exp(P) + \mathbb{N}^{2n}$.

The Buchberger's algorithm for polynomials can be adapted to the ring of differential operators [11, 15], see also [70]. Considering as input a monomial order $\prec$ in $\mathbb{N}^{2n}$ and a finite set $\mathcal{F} = \{P_1, \ldots, P_m\}$ of differential operators, one can compute a Gröbner basis, with respect to $\prec$, of the ideal $I \subset A_n$ generated by $\mathcal{F}$. So, one can also compute a finite set of generators of the subset $E(I) \subset \mathbb{N}^{2n}$ (see Subsection 1.2).

The Division Theorem and the theory of Gröbner basis can be extended for submodules of free modules A_n^m for any m or more generally for submodules of $\mathcal{D}_n^m$ [15]. Here $\mathcal{D}_n$ denotes the ring $\mathrm{Diff}(\mathbb{C}\{x\})$ of LDOs with coefficients in the convergent power series ring. Moreover, Division Theorem and Gröbner basis can be also considered, in a straightforward way, for right ideals (or more generally for right submodules of a free module A_n^m or $\mathcal{D}_n^m$). In particular, A_n (resp. $\mathcal{D}_n$) is a right-Noetherian ring and so actually a Noetherian ring.

2.3 Graded ideal, characteristic variety and dimension.

Assume $I \subset A_n$ is a ideal (e.g. the ideal generated by operators $P_1, \ldots, P_m$ in the system (7)). The graded ideal $\mathrm{gr}(I)$ associated with I is defined as the ideal in $\mathbb{C}[x, \xi]$ generated by the set of principal symbols $\{\sigma(P) \mid P \in I\}$. Notice that $\mathrm{gr}(I)$ is a homogeneous polynomial ideal with respect to the (ξ)-degree.

If $I = A_n P$ is the principal ideal generated by P then $\mathrm{gr}(I)$ is also principal in $\mathbb{C}[x, \xi]$ and it is generated by $\sigma(P)$.

Definition 3 *The* characteristic variety *of the quotient A_n–module A_n/I (or of the system defined by I) –denoted by* $\mathrm{Char}(A_n/I)$*, is by definition the affine algebraic variety defined in $\mathbb{C}^{2n}$ by the ideal* $\mathrm{gr}(I) \subset \mathbb{C}[x, \xi]$.

If $I = A_n P$ is a principal ideal then the characteristic variety of A_n/I coincides with the classical characteristic variety of P (see Subsection 2.1). We will omit here, because it is more involved, the definition of the characteristic variety $\mathrm{Char}(M)$ of any finitely generated A_n–module M (see e.g. [24]) which is an affine algebraic subvariety of $\mathbb{C}^{2n}$.

Definition 4 *The* singular locus *of a finitely generated A_n–module M is the Zariski closure of the image of* $\mathrm{Char}(M) \setminus \mathbb{C}^n \times \{0\}$ *under the projection* $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, $\pi(a, b) = a$.

In general, if I is generated by a family $P_1, \ldots, P_m$, the ideal $\mathrm{gr}(I)$ could be strictly bigger than the ideal generated by $\sigma(P_1), \ldots, \sigma(P_m)$[10]. This is analogous to a common situation in Algebraic Geometry. Let $Z = \mathcal{V}_{\mathbb{C}}(J) \subset \mathbb{C}^n$

[10] See Example 2.

be the algebraic set defined by a polynomial ideal $J \subset \mathbb{C}[x]$ and assume J is generated by polynomials $f_1, \ldots, f_m$. Assume $0 \in \mathcal{V}_{\mathbb{C}}(J)$. The *tangent cone* to Z at 0 is by definition the algebraic set $C = \mathcal{V}_{\mathbb{C}}(in(J))$ where $in(J) \subset \mathbb{C}[x]$ is the ideal generated by the family $\{in(f) \,|\, f \in J\}$. Here the *initial form* $in(f)$ of a polynomial f is the homogeneous component of lowest degree in f (i.e. the sum of the monomials of minimal degree in f). In general the ideal $\langle in(f_1), \ldots, in(f_m)\rangle$ is strictly contained in $in(J)$.

Gröbner basis theory in A_n can be used to calculate $\mathrm{gr}(I)$[11]. Namely, if $P_1, \ldots, P_\ell$ is a Gröbner basis of I[12] then $\sigma(P_1), \ldots, \sigma(P_\ell)$ generate $\mathrm{gr}(I)$ and so these principal symbols define the characteristic variety of A_n/I; see [11], [15], [70].

Example 2 *If* $I = A_2(P_1, P_2)$ *with* $P_1 = x_1\partial_1 + x_2\partial_2$ *and* $P_2 = x_1\partial_2 + x_2^2\partial_1$ *then* $\mathrm{gr}(I) = \langle \xi_1, \xi_2\rangle$ *that is strictly bigger than* $\langle \sigma(P_1), \sigma(P_2)\rangle = \langle x_1\xi_1 + x_2\xi_2, x_1\xi_2 + x_2^2\xi_1\rangle$.

The following `Macaulay 2` *script can be used to compute generators of the graded ideal* $\mathrm{gr}(I)$. *The corresponding* `Macaulay 2` *command is called* `charIdeal`. *We need* `D-modules.m2` *package to this end (see [35]). Input lines in* `Macaulay` *are denoted by* `i1, i2, ...` *while the corresponding output lines are* `o1, o2, ...`

The command `R=QQ[x,y]` *defines the ring* `R` *to be the polynomial ring in the variables* `x, y` *and with rational coefficients. The command* `W=makeWA R` *defines the ring* `W` *to be the Weyl algebra of order 2 with coefficients in* `R`.

```
Macaulay 2, version 0.9.2 --Copyright 1993-2001, D. R. Grayson and
M. E. Stillman --Singular-Factory 1.3b, copyright 1993-2001, G.-M.
Greuel, et al. --Singular-Libfac 0.3.2, copyright 1996-2001, M.
Messollen

i1 : R=QQ[x,y]

i2 : load "D-modules.m2"

i3 : W=makeWA R

i4 : P1=x*dx+y*dy,P2=x*dy+y^2*dx

o4 = (x*dx + y*dy, y^2*dx + x*dy)

i5 : I=ideal(P1,P2)

o5 = ideal (x*dx + y*dy, y^2*dx + x*dy)

i6 : charIdeal I

o6 = ideal (dy, dx)
```

[11] Analogous to the commutative case: we use a (special) Gröbner basis of the polynomial ideal J to compute $in(J)$, see e.g. [49, Cor. 6.2.25].

[12] With respect to a monomial ordering compatible with the (ξ)-degree.

```
o6 : Ideal of QQ [x, y, dx, dy]

i7 : J=ideal(dx,dy)

o7 = ideal (dx, dy)

o7 : Ideal of W

i8 : J==I

o8 = true
```

Input line `i4` *defines the operators* `P1, P2` *generating the ideal I (the corresponding definition in* `Macaulay` *is the input line* `i5`*.*

The computation of the input line `i6: charIdeal I` *gives the ideal* `o6: ideal (dy, dx)`*. Notice that as remarked by* `Macaulay` *output* `o6 : Ideal of QQ [x, y, dx, dy]` *the ideal given by* `o6: ideal (dy, dx)` *is in fact an ideal of the ring* `QQ [x, y, dx, dy]` *which is considered to be a commutative polynomial ring while* `W` *is the Weyl algebra of order 2.*

In fact, the last part of the script (from `i7` *to* `o8`*) proves that the ideal I equals the ideal $A_2(\partial_1, \partial_2)$. We are using here $x = x_1$, $y = x_2$.*

In the Weyl algebra `W` *the expressions* `dx, dy` *stand for ∂_1 and ∂_2 while in* `QQ [x, y, dx, dy]` *they stand for ξ_1 and ξ_2 respectively.*

The previous computation can be also made by hand although they are not completely obvious.

If $I = A_2(P_1, P_2)$ as in Example 2 we have proven that $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle$ and then the equality $\mathrm{Char}(A_2/I) = \mathbb{C}^2 \times \{(0,0)\}$.

Let's see another example using `Macaulay 2`.

Example 3 *The following* `Macaulay 2` *script computes $\mathrm{gr}(J)$ for $J = A_2(Q_1, Q_2)$ and $Q_1 = \partial_1^2 - \partial_2$, $Q_2 = x_1\partial_1 + 2x_2\partial_2$.*

```
i2 : R=QQ[x,y]

i3 : W2=makeWA R

i4 : Q1=dx^2-dy,Q2=x*dx+2*y*dy

o4 = (dx^2 - dy, x*dx + 2y*dy)

o4 : Sequence

i5 : J = ideal (Q1,Q2)

o5 = ideal (dx^2 - dy, x*dx + 2y*dy)

o5 : Ideal of W2

i6 : charIdeal J

o6 = ideal (dx^2 , x*dx + 2y*dy)

o6 : Ideal of QQ [x, y, dx, dy]
```

The input `J = ideal(Q1, Q2)` *defines the ideal* `J` *of the Weyl algebra* `W` *generated by the linear differential operators* `Q1, Q2`*. Then the input* `i6 : charIdeal J` *computes the graded ideal* $\mathrm{gr}(J)$*. Then* $\mathrm{gr}(J)$ *is generated by the polynomials* $\xi_1^2, x_1\xi_1 + 2x_2\xi_2$ *and the characteristic variety* $\mathrm{Char}(A_2/I)$ *is the union of the two planes* $\xi_1 = x_2 = 0$ *and* $\xi_1 = \xi_2 = 0$ *in* $\mathbb{C}^4$*.*

By definition the *dimension* of a finitely generated nonzero A_n–module M, denoted by $\dim(M)$, is the dimension[13] of its characteristic variety $\dim(vchar(M))$ viewed as an algebraic variety in $\mathbb{C}^{2n}$. The modules A_2/I and A_2/J of Examples 2 and 3 have both dimension 2 since their characteristic varieties are, in the first case, the plane $\mathbb{C}^2 \times 0$ in $\mathbb{C}^4$ and the union of the planes $\xi_1 = x_2 = 0$ and $\xi_1 = \xi_2 = 0$ (again in $\mathbb{C}^4$) in the second case.

A fundamental result due to I.N. Bernstein [8] says that if $M \neq 0$ then $\dim(M) \geq n$.

If $M = A_n/I$ (and more generally if M is a quotient of a free A_n–module) the dimension of M can be computed using Gröbner basis in A_n. To this end we first notice that $\dim(A_n/I) = \dim \mathrm{Char}(A_n/I)$ if the Krull dimension of the quotient ring $\mathbb{C}[x,\xi]/\mathrm{gr}(I)$ (see e.g. [10, Chap. 8]). We first compute a system of generators of $\mathrm{gr}(I)$ –assuming that a system of generators of I is given– and then, applying again Gröbner basis computation, this time in the polynomial ring $\mathbb{C}[x,\xi]$, we compute the Krull dimension of $\mathbb{C}[x,\xi]/\mathrm{gr}(I)$[14].

Definition 5 *A finitely generated* A_n*–module* M *is said to be* holonomic *(or a* holonomic system*) if either* $M = (0)$ *or* M *is nonzero and* $\dim(M) = n$*.*

Holonomic systems generalize the classical notion of maximally overdetermined systems (see [46]). The previous examples A_2/I and A_2/J are holonomic.

Remark 2 *If* $K = A_nP$ *is the principal ideal generated by* $P \in A_n$ *and the quotient* $M = A_n/K$ *is non zero then* M *is holonomic if and only if* $n = 1$*. In fact* $\mathrm{gr}(K)$ *is just generated by the principal symbol* $\sigma(P) \in \mathbb{C}[x,\xi]$ *and the characteristic variety* $\mathrm{Char}(M)$ *is the hypersurface defined by the polynomial* $\sigma(P)(x,\xi)$ *in* $\mathbb{C}^{2n}$*. So* $\dim(M) = 2n-1$ *and* $\dim(M) = n$ *if and only if* $n = 1$*.*

Let $I \subset A_n$ be an ideal. We define, following [70], the *holonomic rank* of the ideal I as

$$\mathrm{rank}(I) = \dim_{\mathbb{C}(x)} \frac{\mathbb{C}(x)[\xi]}{\mathbb{C}(x)[\xi]\mathrm{gr}(I)}$$

where $\mathbb{C}(x)$ is the field of rational functions and $\mathrm{gr}(I) \subset \mathbb{C}[x,\xi]$ is the graded ideal associated with I.

It is easy to see that if A_n/I is holonomic then $\mathrm{rank}(I) < +\infty$ and that the converse is not true (see e.g. [70, Prop. 1.4.9]).

---

[13] We are considering here the Krull dimension (see e.g. [10, Chap. 8]).

[14] Actually only a single Gröbner basis of I is needed if the monomial ordering if suitably chosen.

2.4 Solutions of a system. Homological algebra over A_n

We start by recalling some basics of homological algebra.

A (cochain) complex $(V_\bullet, d_\bullet)$ of $\mathbb{C}$–vector spaces is a collection of $\mathbb{C}$–vector spaces V_i, $i \in \mathbb{Z}$, and $\mathbb{C}$–linear maps $d_i : V_i \to V_{i+1}$ such that $d_{i+1} \circ d_i = 0$ for all i (or equivalently if $\mathrm{im}(d_{i-1}) \subset \ker(d_i)$ for all i). We make analogous definition for complexes of A_n–modules and morphisms of A_n–modules.

Given a complex $(V_\bullet, d_\bullet)$ its cohomology in degree i (or its i-th cohomology group) is the quotient $\ker(d_i)/\mathrm{im}(d_{i-1})$.

Definition 6 *A complex $(V_\bullet, d_\bullet)$ is* exact *in degree i if $\mathrm{im}(d_{i-1}) = \ker(d_i)$ and the complex is said to be* exact *(or an* exact sequence*) if it is exact in degree i for all i.*

Let us consider a LPDE

$$P(u) = P(x, \partial)(u) = 0$$

and suppose we want to compute its solutions in some function space $\mathcal{F}$ where A_n acts naturally. The space $\mathcal{F}$ should be then an A_n–module. Typical examples of such spaces are function spaces (continuos functions, real analytic or holomorphic functions, polynomial functions ...), spaces of multivalued functions and spaces of distributions.

A central question in the theory of Differential Equations is to compute the solution set

$$Sol(P; \mathcal{F}) = \{u \in \mathcal{F} \text{ such that } P(u) = 0\}.$$

Actually, this solution vector space is nothing but the kernel of the morphism $P(\,) : \mathcal{F} \to \mathcal{F}$ defined by the action of P on $\mathcal{F}$. So one has $Sol(P; \mathcal{F}) = \ker(P(\,))$. Notice that as A_n is non commutative $P(\,)$ is only a $\mathbb{C}$–linear map.

Let us denote $M = A_n/A_nP$. We will see that $Sol(P, \mathcal{F})$ is isomorphic, as vector space, to $\mathrm{Hom}_{A_n}(M, \mathcal{F})$ the space of A_n-morphisms from M to $\mathcal{F}$[15].

Each solution $u \in Sol(P; \mathcal{F})$ determines the morphism (of A_n–modules)

$$\phi_u : M \to \mathcal{F}$$

defined by $\phi_u(\overline{Q}) = Q(u)$ for $Q \in A_n$, where $\overline{Q}$ stands for the class of Q modulo the ideal A_nP. On the other hand, each A_n–module morphism

$$\phi : M \to \mathcal{F}$$

(i.e. each $\phi \in \mathrm{Hom}_{A_n}(M, \mathcal{F})$) determines the solution

$$u_\phi = \phi(\overline{1})$$

[15] Someway, an analogous situation happens when solving a system $\mathcal{S}$ of complex polynomial equations with only finitely many solutions (that is the set $\mathcal{V}_\mathbb{C}(\mathcal{S})$ is finite). There exists a natural bijection from $\mathcal{V}_\mathbb{C}(\mathcal{S})$ to $Hom_\mathbb{C}(\mathbb{C}[x]/\langle\mathcal{S}\rangle, \mathbb{C})$ defined by attaching to each solution $\underline{a} \in \mathcal{V}_\mathbb{C}(\mathcal{S})$ the corresponding evaluation homomorphism $(\overline{g(x)} \mapsto g(\underline{a}))$

since $P(\phi(\overline{1})) = \phi(P\cdot\overline{1}) = \phi(\overline{0}) = 0$. So the solution space $Sol(P;\mathcal{F})$ is naturally isomorphic to the vector space $\mathrm{Hom}_{A_n}(M,\mathcal{F})$.

Similarly we can prove that if $I \subset A_n$ is an ideal then the solution space

$$Sol(I;\mathcal{F}) = \{u \in \mathcal{F} \mid P(u) = 0, \forall P \in I\}$$

is naturally isomorphic, as a vector space, to $\mathrm{Hom}_{A_n}(A_n/I,\mathcal{F})$.

Let us return to the case of the complete equation $P(u) = v$ where v is in $\mathcal{F}$. The obstruction to solve this equation is given by the vector space $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,))$ that is the cokernel of the map $P(\,) : \mathcal{F} \to \mathcal{F}$. That is, for a fixed $v \in \mathcal{F}$, the equation $P(u) = v$ has a solution u in $\mathcal{F}$ if and only if $v \in P(\mathcal{F})$ or equivalently if and only if the class of v in the quotient space $\mathcal{F}/P(\mathcal{F})$ is zero.

More concretely, the complete equation has a solution u for each v if and only if $\mathcal{F} = P(\mathcal{F})$ (or equivalently if and only if $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,)) = (0)$). This situation can be interpreted by using a little bit of homological algebra. This homological algebra interpretation can be also considered for more general LPDS.

We will see that $\mathrm{coker}(P(\,))$ is naturally isomorphic, as vector space, to $\mathrm{Ext}^1_{A_n}(M,\mathcal{F})$ the first extension group (in this case it is a vector space) of M by $\mathcal{F}$ (see e.g. [68]).

First of all, let us consider the natural exact sequence of modules and morphisms

$$0 \to A_n \xrightarrow{\phi_P} A_n \xrightarrow{\pi} M = \frac{A_n}{A_nP} \to 0. \tag{10}$$

where the morphism ϕ_P is defined by $\phi_P(Q) = QP$ for $Q \in A_n$ and π is the natural projection. Then by truncating the previous one we consider the complex (of A_n-modules)

$$0 \to A_n \xrightarrow{\phi_P} A_n \to 0. \tag{11}$$

We then apply to this complex the functor $\mathrm{Hom}_{A_n}(-,\mathcal{F})$ and we get the complex of vector spaces

$$0 \to \mathrm{Hom}_{A_n}(A_n,\mathcal{F}) \xrightarrow{(\phi_P)^*} \mathrm{Hom}_{A_n}(A_n,\mathcal{F}) \to 0 \tag{12}$$

where $(\phi_P)^*(\eta) = \eta \circ \phi_P$ for $\eta \in \mathrm{Hom}_{A_n}(A_n,\mathcal{F})$.

The vector space $\mathrm{Hom}_{A_n}(A_n,\mathcal{F})$ has a natural structure of A_n–module which is in fact isomorphic to $\mathcal{F}$. This is a general fact in ring theory: to each morphism $\eta \in \mathrm{Hom}_{A_n}(A_n,\mathcal{F})$ we associate $\eta(1) \in \mathcal{F}$ and this correspondence is in fact an isomorphism. Under this isomorphism the last complex can be read as

$$0 \to \mathcal{F} \xrightarrow{P()} \mathcal{F} \to 0$$

Homological algebra tells us that we have natural isomorphisms of vector spaces $\ker(P(\,)) \simeq \mathrm{Hom}_{A_n}(M,\mathcal{F}) = \mathrm{Ext}^0_{A_n}(M,\mathcal{F})$ and $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,)) \simeq \mathrm{Ext}^1_{A_n}(M,\mathcal{F})$.

Then the vector spaces $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$ for $i = 0, 1$ are called the *solutions spaces* of the equation $P(u) = 0$ (or more precisely of the A_n–module $M = A_n/A_nP$) in $\mathcal{F}$.

In order to generalize this notion of *solutions spaces* for general systems as (5) we have to consider the corresponding A_n-module $M = A_n^m/\mathrm{im}(\mathcal{P})$ where $\mathrm{im}(\mathcal{P})$ is the submodule of A_n^m generated by the rows of the matrix $(P_{ij})_{ij}$ (appearing in System (5)). By definition the *solutions spaces* of M in $\mathcal{F}$ are the vector spaces $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$ for $i = 0, \ldots, n$ defined using the right derived functors (see e.g. [68]) of the functor $\mathrm{Hom}_{A_n}(-, \mathcal{F})$.

By definition $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$ is the i-th cohomology group of the complex $\mathrm{Hom}_{A_n}(\mathcal{L}_\bullet, \mathcal{F})$ where $\mathcal{L}_\bullet$ is a *free resolution* (see Subsection 2.5) of M.

For general systems as (5) and general function spaces $\mathcal{F}$ there is no algorithm to compute the solution spaces $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$. Nevertheless, if $M = A_n/I$ is holonomic (see Definition 5) and $\mathcal{F} = \mathbb{C}[x]$ there are algorithms computing a basis of $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$ for all i, ([61], [81]).

As a consequence of Cauchy Theorem (see e.g. [70, Th. 1.4.19]) we have

$$\dim_{\mathbb{C}} Sol(I; \mathcal{O}_{\mathbb{C}^n}(U)) = \mathrm{rank}(I)$$

where the system A_n/I is holonomic and $\mathcal{O}_{\mathbb{C}^n}(U)$ stands for the space of holomorphic functions on an open set $U \subset \mathbb{C}^n \setminus Z$ where Z is the singular locus of A_n/I (see Definition 4). This result could be compared with Theorem 3.

2.5 Free resolutions

The exact sequence (10) is an example of free resolution of the A_n–module A_n/A_nP.

A free resolution of a finitely generated A_n–module M is an exact sequence

$$0 \longrightarrow \mathcal{L}_{-r} \xrightarrow{\phi_{-r}} \mathcal{L}_{-r+1} \xrightarrow{\phi_{-r+1}} \cdots \xrightarrow{\phi_{-2}} \mathcal{L}_{-1} \xrightarrow{\phi_{-1}} \mathcal{L}_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where each $\mathcal{L}_i$ is a free A_n–module of finite rank.

As an application of Gröbner basis theory over the ring A_n one can compute (see e.g. [15], [70]), starting from a system of generators $\{P_1, \ldots, P_\ell\}$ of an ideal $I \subset A_n$, a system of generators of the *first syzygy module* of $\{P_1, \ldots, P_\ell\}$ which is by definition the module

$$\mathrm{Syz}(P_1, \ldots, P_\ell) := \{(Q_1, \ldots, Q_\ell) \in A_n^\ell \mid \sum_i Q_i P_i = 0\}.$$

In fact one can also compute $\mathrm{Syz}(\underline{P}_1, \ldots, \underline{P}_\ell)$ for any finite set of vectors $\underline{P}_i$ in A_n^m, for any m.

Given $M = A_n^m/\langle \underline{P}_1, \ldots, \underline{P}_\ell \rangle$ one can consider the exact sequence

$$A_n^\ell \xrightarrow{\phi} A_n^m \longrightarrow M \longrightarrow 0$$

where ϕ is the map defined by the matrix whose rows are the vectors $\underline{P}_i$ and we have, by definition of syzygy, $\ker(\phi) = \mathrm{Syz}(\underline{P}_1, \ldots, \underline{P}_\ell) \subset A_n^\ell$.

One can compute, using Gröbner bases, a system of generators $\{\underline{S}_1, \ldots, \underline{S}_k\}$ of $\ker(\phi)$. This leads to a new exact sequence

$$A_n^k \xrightarrow{\psi} A_n^\ell \xrightarrow{\phi} A_n^m \longrightarrow M \longrightarrow 0$$

where $\psi(e_i) = \underline{S}_i$, e_i being the i-th canonical vector in A_n^k.

Let us rename $r_0 = m, r_1 = \ell, r_2 = k$, $\phi_1 = \phi, \phi_2 = \psi$. Restarting with the matrix ϕ_2 one can compute, for each $i \geq 0$ and by applying the same process, a finite sequence of modules and morphisms

$$A_n^{r_p} \xrightarrow{\phi_p} \cdots \longrightarrow A_n^{r_2} \xrightarrow{\phi_2} A_n^{r_1} \xrightarrow{\phi_1} A_n^{r_0} \longrightarrow M \to 0$$

which is exact (see Definition 6).

One can apply the same argument as in the Syzygies Hilbert Theorem (see e.g. [26, Chap. 6], [15]) to assure that there is an integer p such that $\ker(\phi_p) = 0$. This process gives up a finite *free resolution* of the given A_n–module M.

Finite free resolutions are useful to study finitely generated A_n–modules. As we have seen before, given a system as (5) we consider the associated module $M = A_n^m / \langle \underline{P}_1, \ldots, \underline{P}_\ell \rangle$ where $\underline{P}_i = (P_{i1}, \ldots, P_{im})$. The polynomial solutions $(u_1, \ldots, u_m)$ of System (5) can be simply viewed as the vector space $\mathrm{Hom}_{A_n}(M, \mathbb{C}[x])$.

If we apply the functor $\mathrm{Hom}_{A_n}(-, \mathbb{C}[x])$ to the complex

$$0 \longrightarrow A_n^{r_p} \xrightarrow{\phi_p} \cdots \longrightarrow A_n^{r_2} \xrightarrow{\phi_2} A_n^{r_1} \xrightarrow{\phi_1} A_n^{r_0} \longrightarrow 0$$

and then we compute the cohomology of the resulting complex we get the vector spaces $\mathrm{Ext}^i_{A_n}(M, \mathbb{C}[x])$ for $i = 1, \ldots, n$ which are considered as the *higher order* polynomial solutions of System (5).

As we have said before, if M is holonomic (see Definition 5) one can effectively compute, using Gröbner bases, a generating system for the vector spaces $\mathrm{Ext}^i_{A_n}(M, \mathbb{C}[x])$ for any i (see [61], [81]). Since these algorithms uses Gröbner bases computation in the Weyl algebra A_n they have a high complexity.

Syzygies and finite free resolutions are fundamental tools in Computational Algebraic Analysis. They are intensively used in the computation of the four operations –localization, local cohomology, restriction and integration, on differential systems [60] (see also [70]) and in the computation of truncated holomorphic solutions of holonomic systems as shown in [78].

3 More applications of Gröbner bases for LDOs

Gröbner basis theory is also used in many other situations related to A_n–modules. Let's just quote its use in the computational study of *projective*[16] A_n–modules. In [30] there is an algorithm to compute a basis of a projective

[16] A module M is projective provided there is a module N such that the direct sum $M \oplus N$ is a free module.

module over A_n, if it has one (see [76] for a proof that every projective A_n-module with rank greater than 1 is free). These results are also related to [53] and [43].

In Subsections 3.1 and 3.2 we will sketch two more areas of $\mathcal{D}$-modules where the use of computational methods give significant insights into the theory.

3.1 Regular singular points. Regular $\mathcal{D}$–modules and slopes

Let us consider a single equation

$$P(t, \frac{d}{dt})(u) = \left(p_0(t) + p_1(t)\frac{d}{dt} + \cdots + p_m(t)\frac{d^m}{dt^m} \right)(u) = 0 \qquad (13)$$

of order $m \geq 1$ defined on an open disc $\Delta \subset \mathbb{C}$ centered at the origin and where each coefficient $p_i(t)$ is a holomorphic function on Δ.

A point $t_0 \in \Delta$ is said to be *singular* for $P(u) = 0$ if $p_m(t_0) = 0$[17]. Assume the origin $0 \in \Delta$ is the only singular point of $P(u) = 0$ on Δ. In particular the characteristic variety of this linear differential equation is

$$\mathrm{Char}(P) = \{0\} \times \mathbb{C} \cup \Delta \times \{0\}.$$

By a classical result of Ordinary Linear Differential Equations (cf. [34, 410-411]) each multivalued holomorphic solution φ of $P(u) = 0$ on $\Delta^* = \Delta \setminus \{0\}$, can be written as

$$\varphi = \sum_{i=1}^{\ell} c_i(t) t^{\alpha_i} \log^{\nu_i} t$$

where $\alpha_i \in \mathbb{C}$, $\nu_i \in \mathbb{N}$ and $c_i(t)$ is an uniform analytic function on Δ^*.

The equation $P(u) = 0$ has a *regular singular point* at 0 if for all solutions φ multivalued on Δ^* the corresponding analytic functions $c_i(t)$ are meromorphic for all i. If some of the $c_i(t)$ has an essential singularity at 0 then the origin is said to be an *irregular singular point* for $P(u) = 0$.

The Maple command `DEtools[formal_sol]` gives the formal series solutions –up to any order, of a given ordinary differential equation at a fixed point. This is the Maple script for the Euler equation which has an irregular singular point at 0.

```
> Eu:=t^2*Dt+1;
                                  2
                        Eu := t  Dt + 1

> sol:=formal_sol(Eu,[Dt,t],S,'terms'=12,t=0);

                                              12
               sol := [[exp(1/S) (1 + O(S   )), S = t]]
```

[17] This notion agrees with the one of singular locus given in Definition 4.

The classical Gauss hypergeometric equation

$$\left(t(1-t)\frac{d^2}{dt^2} + (\gamma - t(\alpha+\beta+1))\frac{d}{dt} - \alpha\beta\right)(u) = 0$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, has two *regular singular* points in the complex line $\mathbb{C}$, namely the points $0, 1 \in \mathbb{C}$.

The following Maple script gives the two linearly independent formal solutions –up to order 4, of the given Gauss hypergeometric equation for $\alpha = \beta = 1$ and $\gamma = i \in \mathbb{C}$

```
> with(DEtools):
> L:=t*(1-t)*Dt^2+(I-3*t)*Dt-1;
                                   2
                L := t (1 - t) Dt  + (I - 3 t) Dt - 1

> sol:=formal_sol(L,[Dt,t],T,'terms'=4,t=0);
                                2                     3     4
  sol := [[1 - I T + (-1 - I) T  + (-9/5 - 3/5 I) T  + O(T ), T = t],

  (1 - I)                                  2                     3      4
[T       (1 + (2 - I) T + (5/2 - 5/2 I) T  + (5/2 - 25/6 I) T +  O(T )), T = t]]
```

From the very definition it is difficult to see whether a given point is regular singular. The fundamental theorem of Fuchs (cf. [44, 15.3]) states that the origin is a regular singular point for Equation (13) if and only if the order of the pole at 0 of the meromorphic function $p_i(t)/p_m(t)$ is less or equal than $m-i$, $i = 0, \ldots, m$. This theorem can be restated in terms of the so-called *Newton* or *Newton-Puiseux* polygon of the equation $P(u) = 0$ (see e.g. [51]).

The notion of regular singular point for a differential system as (5) in higher dimension –dated only on the last 70's, is due to several authors, especially to Z. Mebkhout, M. Kashiwara and T. Kawai. This definition of regular singular point of a differential system in dimension n is highly abstract and uses derived categories and functors.

To this end Z. Mebkhout introduced the *irregularity sheaves* of a holonomic module along hypersurfaces [58].

In [51] the notion of *slope* of a differential system at a point in $\mathbb{C}^n$ is introduced. In [52] the authors gave an equivalent definition of regular singular point (or more precisely of *regular* system) that can be effectively computed starting with a system of differential equations and using again Gröbner basis theory, provides the system is holonomic. The key point is the computation of the *slopes* of a holonomic A_n–module. A holonomic module M is said to be *regular* at a point in $\mathbb{C}^n$ if M has no slopes at that point. The computation of the slopes can be done by the so-called ACG algorithm [4]. This algorithm uses Gröbner bases as a fundamental tool. The works [17], [39, 40], [73] are related to the computation of the slopes for hypergeometric systems[18].

[18] For the theory of hypergeometric systems see [33]. The book [70] deals with the computational treatment of this kind of differential systems.

3.2 Bernstein-Sato polynomial

Let $f \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ nonconstant and let s be a new variable.

Consider the Weyl algebra $A_n(\mathbb{C}(s))$ over the field $\mathbb{C}(s)$ of rational functions in s and the $A_n(\mathbb{C}(s))$–module $N = \mathbb{C}(s)[x_1, \ldots, x_n, f^{-1}] \cdot f^s$, where f^s is considered as a formal symbol and the $A_n(\mathbb{C}(s))$-action on N is defined by

$$\partial_j(f^s) = sf^{-1}\frac{\partial f}{\partial x_j}f^s$$

the product rule and the formal differentiation of rational functions.

Let $M = A_n(\mathbb{C}(s))f^s$ be the submodule of N generated by f^s. We can see in [8] that both modules are holonomic (see Definition 5) over $A_n(\mathbb{C}(s))$, and that M has finite length. This means that there exists $Q(s) \in A_n(\mathbb{C}(s))$ such that $f^s = Q(s)(f^{s+1})$. If $b(s)$ is the common multiple of the denominators of the coefficients of $Q(s)$ and $P(s) = b(s)Q(s)$ we have

$$P(s)(f^{s+1}) = b(s)f^s \tag{14}$$

where $P(s) \in A_n[s]$ and $b(s)$ are not uniquely defined by the functional equation (14).

The set of all $b(s)$ satisfying Equation (14) for some $P(s)$ is a nonzero ideal in $\mathbb{C}[s]$ whose (monic) generator $b_f(s)$ is called *the Bernstein-Sato polynomial of f* or *the b-function of f*. It was introduced by I.N. Bernstein [8] and by M. Sato [71].

As an example of Bernstein-Sato polynomial, taking $f = \sum_{i=1}^n x_i^2$ one has the identity

$$\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}\right)(f^{s+1}) = 4b(s)f^s,$$

where $b(s) = (s+1)(s+\frac{n}{2})$. It is not difficult to prove that in this case $b(s) = b_f(s)$.

The Bernstein-Sato polynomial $b_f(s)$ is always multiple of $(s+1)$ and equality holds if f is smooth. M. Kashiwara proved that its roots are always rational and negative [47]. B. Malgrange pointed out the connection between the singularity structure of $f^{-1}(0)$ and the roots of $b_f(s)$ [56].

Until [59] there were not an algorithm for the computation of $b_f(s)$. We can use `Risa-Asir` [62] for example, to compute the b-function of a polynomial.

Another interesting information included in $b_f(s)$ is related to the module structure of the *localization* of the ring $\mathbb{C}[x]$ by f, noted by $\mathbb{C}[x]_f$, that is, the ring of rational functions with poles along f (that is along the hypersurface $f = 0$)

$$\mathbb{C}[x]_f = \left\{\frac{g}{f^m} \mid g \in \mathbb{C}[x],\, m \in \mathbb{N}\right\}.$$

$\mathbb{C}[x]_f$ is a $\mathbb{C}[x]$-module and an A_n-module in a natural way. Of course if f is not a constant $\mathbb{C}[x]_f$ is not a finitely generated $\mathbb{C}[x]$-module. We have an analogous situation in the *analytic* setting, i.e. starting from $f \in \mathbb{C}\{x\}$, the ring of convergent power series around 0, and considering $\mathbb{C}\{x\}_f$, the ring of (germs of) meromorphic functions with poles along f, as a $\mathcal{D}_n$-module, where $\mathcal{D}_n = \mathrm{Diff}(\mathbb{C}\{x\})$ is the ring of linear differential operators over $\mathbb{C}\{x\}$ (see Remark 1).

One of the main results in $\mathcal{D}$-module theory is the following theorem:

Theorem 4 ([8]) *For any $f \in \mathbb{C}[x]$, the A_n-module $\mathbb{C}[x]_f$ is finitely generated. More precisely, there exists a positive integer number α such that $\mathbb{C}[x]_f$ is the A_n-module generated by the rational function $\frac{1}{f^\alpha}$.*

The analogous version for the analytic case was proved in [9]. The main ingredient in the proof of Theorem 4 is the existence of the b-function attached to f and the positive integer postulated is $-\alpha_0$ if α_0 is the smallest integer root of $b_f(s)$. So we have the following problem:

Problem.- Let $f \in \mathbb{C}[x]$. Give a presentation of $\mathbb{C}[x]_f$ as a finitely generated A_n-module.

A way to solve the problem is to:

Compute $b_f(s)$ and α_0.

As $\mathbb{C}[x]_f \equiv A_n f^{\alpha_0}$, compute a system of generators of the annihilating ideal

$$\mathrm{ann}_{A_n}(f^{\alpha_0}) = \{P \in A_n \mid P(f^{\alpha_0}) = 0\},$$

and then

$$\mathbb{C}[x]_f \simeq \frac{A_n}{\mathrm{ann}_{A_n}(f^{\alpha_0})}.$$

There are algorithms based in Gröbner bases for LDOs to obtain $b_f(s)$ (see [59, 60]) and the annihilating ideal of a power of f [60]. Unfortunately, many interesting examples can not be treated due to the size of the intermediate computations. Nonetheless, it is possible to obtain the so called *logarithmic* $\mathcal{D}$-modules that are natural approximations to $\mathbb{C}[x]_f$.

3.3 Logarithmic $\mathcal{D}$-modules

The starting point of this approach are the works of K. Saito [69] about *logarithmic vector fields* and *logarithmic differential forms.* After [13] the connection with $\mathcal{D}$-modules of this subject became very important. Here we will treat only the global (algebraic) case of logarithmic A_n-modules to simplify.

Let $D \subset \mathbb{C}^n$ be an hypersurface —usually called a *divisor* in algebraic geometry— defined by $f \in \mathbb{C}[x]$. A vector field with polynomial coefficients

$$\delta = \sum_{i=1}^{n} a_i(x)\partial_i$$

is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathbb{C}[x]$. The $\mathbb{C}[x]$-module of logarithmic vector fields is denoted by $Der(-\log D)$.

We will consider some ideals in A_n associated to any divisor D:

$$I^{\log D} = A_n Der(-\log D)$$

$$\widetilde{I}^{\log D} = A_n\{\delta + \tfrac{\delta(f)}{f} \mid \delta \in Der(-\log D)\}.$$

Notice that if $\delta(f) = af$ then $(\delta + a)(1/f) = 0$ and so $\delta + \frac{\delta(f)}{f} \in \mathrm{ann}_{A_n}(1/f)$ and

$$\widetilde{I}^{\log D} \subset \mathrm{ann}_{A_n}(1/f).$$

We will write $\widetilde{I}^{\log D} = \mathrm{ann}^{(1)}_{A_n}(1/f)$ to emphasize that this ideal is generated by order 1 LDOs.

It is important to notice that $Der(-\log D)$ can be computed –using commutative Gröbner bases– since

$$Der(-\log D) \simeq \mathrm{Syz}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f)$$

a natural isomorphism sending each logarithmic vector field $\delta = \sum_i a_i(x)\partial_i$ to the syzygy $(a_1(x), \ldots, a_n(x), -\frac{\delta(f)}{f})$.

In this context we have the following problem:

Open Problem.- If $D \subset \mathbb{C}^n$ is the divisor defined by a polynomial $f \in \mathbb{C}[x]$, when

$$\mathrm{ann}^{(1)}_{A_n}(1/f) = \mathrm{ann}_{A_n}(1/f)\,?$$

That is, when the annihilating ideal of $1/f$ is generated by elements of order 1?

Some advances have been obtained for the analytical case in [18], [19], [20], [21] and [79] for the family of *free* divisors i.e. for the divisors for which $Der(-\log D)$ is a free $\mathbb{C}[x]$–module ([69]).

This open problem is intimately related to the *Logarithmic Comparison Theorem (LCT)* (see e.g. [16], [19], [14]): we say that LCT holds for a divisor $D \subset \mathbb{C}^n$ if the cohomology of the complement of D in $\mathbb{C}^n$ is computed by the complex of logarithmic differential forms. It has been conjectured in [79] that LCT holds for D if and only if $\mathrm{ann}_{A_n}(1/f) = \mathrm{ann}^{(1)}_{A_n}(1/f)$ (at least locally).

Conclusions

We have described some applications of Computer Algebra methods to the algebraic study of systems of linear partial differential equations. Using Gröbner basis theory for linear differential operators we have described how to calculate the characteristic variety of such systems and how to construct a free resolution of the associated module. Free resolutions are used in particular in the computation of the *solution spaces* of the given system. Computational methods are also used to study the *irregularity* of a system, the Bernstein-Sato polynomials and the so-called logarithmic $\mathcal{D}$–modules.

The use of Gröbner basis theory in this context is motivated by analogous situations in Commutative Algebra and Algebraic Geometry.

Acknowledgements

I would like to thank my colleagues J. Gago, M.I. Hartillo and J.M. Ucha for their help in writing this paper. I have freely used some parts of their works and ideas. I would also thank one referee for his comments, suggestions and criticisms. My especial thanks to S.R. for her intangible support.

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