Journal of
Mathematical
Inequalities
Volume 5, Number 3 (2011), 341–353

A PROOF OF A TRIGONOMETRIC INEQUALITY. A GLIMPSE INSIDE THE MATHEMATICAL KITCHEN

JUAN ARIAS DE REYNA AND JAN VAN DE LUNE

(*Communicated by I. Franjić*)

Abstract. We prove the inequality

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{r^k\cos k\phi}{k+2}<\sum_{k=1}^{\infty}(-1)^{k+1}\frac{r^k}{k+2}$$

for $0<r\leqslant 1$ and $0<\phi<\pi$.

For the case $r=1$ we give two proofs. The first one is by means of a general numerical technique (Maximal Slope Principle) for proving inequalities between elementary functions. The second proof is fully analytical. Finally we prove a general rearrangement theorem and apply it to the remaining case $0<r<1$.

Some of these inequalities are needed for obtaining general sharp bounds for the errors committed when applying the Riemann-Siegel expansion of Riemann's zeta function.

1. Introduction

In this note we prove a useful trigonometric inequality by two different methods.

The first (applying the Maximal Slope Principle) may be a model for proving many intricate inequalities. The second is purely analytical and we explain the path we have followed. For example, at the start of Section 6 where we use some heuristical Eulerian methods, we only explain how we arrived at the differential equation (15), which afterwards receives a standard proof in Proposition 6.1.

2. The problem to be dealt with in this note

The main goal of this note is to prove that for $0<r\leqslant 1$ and $0<\varphi<\pi$

$$\frac{r\cos\varphi}{3}-\frac{r^2\cos 2\varphi}{4}+\frac{r^3\cos 3\varphi}{5}-+\cdots<\frac{r}{3}-\frac{r^2}{4}+\frac{r^3}{5}-+\cdots \tag{1}$$

We soon recognized that this is not a trivial problem, and still hold that view.

Mathematics subject classification (2010): Primary 26D05, 26D15, secondary 42A05.
The first author is supported by grant MTM2009-08934.

Paper JMI-05-30

3. Motivation

In one of our studies [1] of the error(s), inherent in using the Riemann-Siegel formula for the Riemann ζ function (see, for example, Edwards [2] or Gabcke [4]), we encountered the following problem: Find a sharp bound of the integral

$$\int_C (1-z)^{-\sigma} e^{-x^2 f(z)} \frac{dz}{z^{k+1}} \quad \text{where} \quad f(z) := -\frac{\log(1-z)}{z^2} - \frac{1}{z} - \frac{1}{2}. \tag{2}$$

Here k is a natural number, σ and x denote arbitrary real numbers, C is a simple circular contour around $z=0$ with radius $r \in (0,1]$, and $\log(1-z)$ is the principal logarithm: $\log(1-z) := -\sum_{k=1}^{\infty} \frac{z^k}{k}$ for $|z| \leqslant 1$, $z \neq 1$.

The usual technical paper proceeds, as directly as possible, to the final result. However, it occurred to us that an interested reader might appreciate a glimpse inside the mathematical kitchen. To this end, our note will provide the reader a detailed summary of the struggles we encountered along the way to our final solution.

4. Reduction of the problem

We soon recognized that our problem concerning the integral in (2) may be reduced to finding a suitable sharp upper bound of $-\mathrm{Re}\, f(z)$ for $|z| = r$, i. e., a suitable sharp upper bound of $-\mathrm{Re}\, f(re^{i\varphi})$ for $-\pi < \varphi < \pi$.

It is easily seen that $\mathrm{Re}\, f(re^{i\varphi})$ is an even function of φ, so that we may restrict ourselves to $0 \leqslant \varphi < \pi$. The reader may know that in such cases we have a habit of first making a Plot (using Mathematica) of the function(s) in question.

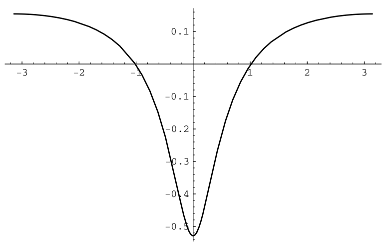

Figure 1: Plot of $-\mathrm{Re}\, f(re^{i\varphi})$ for $r = 0.7$ and $\varphi \in (-\pi, \pi)$.

After having made various such plots of $-\mathrm{Re}\, f(re^{i\varphi})$ we decided to be satisfied with showing that $-\mathrm{Re}\, f(re^{i\varphi})$ is maximal for $\varphi = \pi$ or, equivalently, that

$$-\mathrm{Re}\, f(re^{i\varphi}) < -\mathrm{Re}\, f(re^{\pi i}) = -\mathrm{Re}\, f(-r) \quad \text{for all} \quad 0 < \varphi < \pi. \tag{3}$$

(Although in [1] this inequality was actually needed only for $r = 1$, $r = 8/9$ and $r = 0.883$, we are striving for some generality here.)

Using the power series expansion of $\log(1-z)$ we may write (3) as (1). As said before, proving inequality (1) will be our main goal in this note. (There are no serious convergence problems in (1).)

5. Application of the Maximal Slope Principle

Suppose we have a differentiable real function $h(x)$ on an interval $[a,b]$ with $|h'(x)| \leqslant M$ for all $x \in [a,b]$. (Here we assume $M > 0$, because otherwise we are not dealing with a serious problem.) As a simple application of the Mean Value Theorem, the Maximal Slope Principle (MSP) now asserts the following: If, for example, $h(b) > 0$ then $h(x)$ is also positive for all $x \in (x_1, b)$ where $x_1 := \max(a, b - \frac{h(b)}{M})$. (Just draw a picture !)

Note that if $x_1 > a$ and $h(x_1) > 0$ we may repeat this procedure (until we reach an $x_1 \leqslant a$).

5.1. Some Kitchen Prep Work

Of the many useful applications of the MSP we briefly mention a few examples:

- Flett's function $F(t) := \sum_{n=1}^{\infty} \frac{\sin(t/n)}{n}$ has no zeros in the t-interval $(0,48)$. The first zero is found at $t = 48.418454\ldots$ (see[7]).

- For all $n \in [2,10]$ the function $Q(x) := \frac{1^x+2^x+3^x+\cdots+n^x+(n+1)^x}{1^x+2^x+3^x+\cdots+n^x}$ is log-convex (in x) on the entire real line $\mathbf{R}$.

  (To this we might add our conjecture that $Q(x)$ is log-convex (in x) on $\mathbf{R}$ for all $n \in \mathbf{N}$.)

- By means of the MSP one may prove (or disprove) excruciatingly complicated inequalities $L < R$ where L and R are exponential polynomials.

- The MSP may also be used to locate zeroes of real functions such as, for example, $R_{23}(t) := \sum_{n=1}^{23} \frac{\cos(t \log n)}{n}$ (see [8]).

5.2. Application of the MSP method

Following in the footsteps of Hilbert and Pólya, we apply the MSP to the function $-\mathrm{Re}\, f(re^{i\varphi})$ for the simplest case $r = 1$. In [3, pp. 126–127.] we read: 'Courant describes Hilbert's method of dealing with problems as follows: *He was a most concrete, intuitive mathematician who invented, and very consciously used, a principle: namely, if you want to solve a problem first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as it can be made, without losing any of its punch, and then you solve it. The generalization is a triviality, which you do not need to pay too much attention to. This principle of Hilbert's proved extremely useful for him and also for others who learned it from him; unfortunately it has been forgotten.*'.

In the present case ($r = 1$) we thus have to show that $-\mathrm{Re}\, f(e^{i\varphi}) < -\mathrm{Re}\, f(-1)$ for $0 < \varphi < \pi$. It is clear that in this inequality we may replace φ by $\pi - \varphi$, so that we may just as well prove that

$$-\mathrm{Re}\, f(-e^{-i\varphi}) < -\mathrm{Re}\, f(-1) \qquad \text{for } 0 < \varphi < \pi. \tag{4}$$

Writing

$$u(\varphi) := -\operatorname{Re} f(-e^{-i\varphi}) = \operatorname{Re}\left(e^{2i\varphi}\log(1+e^{-i\varphi}) - e^{i\varphi} + \tfrac{1}{2}\right) \tag{5}$$

we may also write our inequality as $u(\varphi) < u(0)$.

We have (for $-\pi < \varphi < \pi$)

$$\begin{aligned}
u(\varphi) &= \operatorname{Re}\left[e^{2i\varphi}\log\left(e^{-i\varphi/2}(e^{i\varphi/2}+e^{-i\varphi/2})\right) - e^{i\varphi} + \frac{1}{2}\right]\\
&= \operatorname{Re}\left[(\cos 2\varphi + i\sin 2\varphi)\left(\log\left(2\cos\frac{\varphi}{2}\right) - \frac{i\varphi}{2}\right) - e^{i\varphi} + \frac{1}{2}\right]\\
&= \cos(2\varphi)\log\left(2\cos\frac{\varphi}{2}\right) + \frac{\varphi}{2}\sin(2\varphi) - \cos\varphi + \frac{1}{2}.
\end{aligned}$$

Now we define

$$h(\varphi) := u(0) - u(\varphi) = \tag{6}$$

$$= \log 2 - 1 + \cos\varphi - \frac{\varphi}{2}\sin(2\varphi) - \cos(2\varphi)\log\left(2\cos\frac{\varphi}{2}\right). \tag{7}$$

We have just seen that we have to show that $h(\varphi) > 0$ for $0 < \varphi < \pi$.

Before applying the MSP to $h(\varphi)$ we first show that $h(\varphi)$ is positive on the intervals $0 < \varphi \leqslant \frac{1}{3}$ and $3 \leqslant \varphi < \pi$.

LEMMA 5.1. *$h(\varphi) > 0$ for $0 < \varphi \leqslant \frac{1}{3}$.*

Proof. We will use the elementary inequalities

$$1 - \frac{t^2}{2!} \leqslant \cos t \leqslant 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \quad \text{and} \quad \sin t \leqslant t - \frac{t^3}{3!} + \frac{t^5}{5!}, \qquad (t \geqslant 0).$$

Then, with $x = \frac{\varphi^2}{8} - \frac{\varphi^4}{384}$ we have $\cos\frac{\varphi}{2} \leqslant 1 - x$ so that

$$\frac{1}{\cos\frac{\varphi}{2}} \geqslant \frac{1}{1-x} > 1 + x \tag{8}$$

and

$$\begin{aligned}
h(\varphi) &= \log 2 - 1 + \cos\varphi - \frac{\varphi}{2}\sin(2\varphi) - \cos(2\varphi)\log\left(2\cos\frac{\varphi}{2}\right)\\
&> \left(2\varphi^2 - \frac{2}{3}\varphi^4\right)\log 2 + (1-2\varphi^2)\log(1+x) - \frac{\varphi}{2}\left(2\varphi - \frac{(2\varphi)^3}{3!} + \frac{(2\varphi)^5}{5!}\right) - \frac{1}{2}\varphi^2.
\end{aligned}$$

Now we use $\log(1+x) > \frac{x}{1+x/2}$ and simplify, yielding

$$h(\varphi) > \frac{\varphi^2 p(\varphi)}{23040 + 1440\varphi^2 - 30\varphi^4} \tag{9}$$

where $p(\varphi)$ is the polynomial

$$\begin{aligned} p(\varphi) := 4\varphi^8 &+ (20\log 2 - 212)\varphi^6 - (1020\log 2 + 1947)\varphi^4 \\ &+ (7380 - 12480\log 2)\varphi^2 + (46080\log 2 - 31680). \end{aligned} \tag{10}$$

The real roots of $p(\varphi)$ are $\pm 0.392976\ldots$ and $\pm 7.78294\ldots$, and $p(\varphi)$ is positive for $0 < \varphi < 0.392976\ldots$. The denominator of $h(\phi)$ has only two real roots at $\pm 7.78849\ldots$.

So, $h(\varphi) > 0$ for $0 < \varphi < 0.392976$, in particular for $0 < \varphi \leqslant \frac{1}{3}$.

LEMMA 5.2. *$h(\varphi) > 0$ for $3 \leqslant \varphi < \pi$.*

Proof. For $3 \leqslant \varphi < \pi$ we have

$$0 < 2\cos\frac{\varphi}{2} < 2\cos\frac{3}{2}, \quad \log\Big(2\cos\frac{\varphi}{2}\Big) < \log\Big(2\cos\frac{3}{2}\Big) < 0,$$
$$0 < \cos 6 < \cos 2\varphi < 1, \quad \sin 6 < \sin 2\varphi < 0, \quad -1 < \cos\varphi < \cos 3 < 0$$

so that

$$\begin{aligned} h(\varphi) &= \log 2 - 1 + \cos\varphi - \frac{\varphi}{2}\sin(2\varphi) - \cos(2\varphi)\log\Big(2\cos\frac{\varphi}{2}\Big) \\ &> \log 2 - 1 - 1 - \cos 6 \times \log\Big(2\cos\frac{3}{2}\Big) \approx 0.570891. \end{aligned}$$

Now we can apply the MSP to $h(\varphi)$ on the interval $\frac{1}{3} \leqslant \varphi \leqslant 3$. First, we have to determine the maximal slope of $h(\varphi)$ on this interval.

LEMMA 5.3. *For $\frac{1}{3} \leqslant \varphi \leqslant 3$ we have $|h'(\varphi)| < 20$.*

Proof. We have $h(\varphi) = u(0) - u(\varphi) = u(0) + \operatorname{Re} f(-e^{i\varphi})$. If we put $z = e^{i\varphi}$, then $\frac{d}{d\varphi} = iz\frac{d}{dz}$. Hence

$$\begin{aligned} h'(\varphi) &= \operatorname{Re}\Big\{iz\frac{d}{dz}\Big(-\frac{\log(1+z)}{z^2} + \frac{1}{z} - \frac{1}{2}\Big)\Big\} \\ &= \operatorname{Re} i\Big(\frac{2\log(1+z)}{z^2} - \frac{1}{z(1+z)} - \frac{1}{z}\Big). \end{aligned}$$

It follows that for $\frac{1}{3} \leqslant \varphi \leqslant 3$ we have

$$\begin{aligned} |h'(\varphi)| &\leqslant 2|\log(1+z)| + \frac{1}{|1+z|} + 1 \\ &\leqslant 2(\big|\log|1+e^{3i}|\big| + \pi) + \frac{1}{|1+e^{3i}|} + 1 < 11 + 8 + 1 = 20. \end{aligned} \tag{11}$$

Now applying the MSP (repeatedly) on the interval $[1/3, 3]$ we find (in 4163 steps) that indeed $h(\varphi) > 0$ on this interval. The procedure can be speeded up considerably by introducing a more flexible $M = M(t)$.

```
h[φ_] := Log[2] - 1 + Cos[φ] - φ/2 Sin[2 φ] - Cos[2 φ] Log[2 Cos[φ/2]];
t = 3;
step = 0; h0 = h[1 / 3];
While[t ≥ 1 / 3, f = N[h[t]]; If[f > h0, δ = f/20; t -= δ; step += 1,
   Print["WRONG: At t = ", t, "  we have f ≤ h0"] ; Abort[]]];
Print["We have reached the value t = ", t];
Print["# of steps = ", step];

We have reached the value t = 0.333282

# of steps = 4163
```

The other cases $r = \frac{8}{9}$ and $r = 0.883$ may be dealt with in a similar manner.

Note: The above program is only an indication, for a complete proof we must study the errors in the computations. In the computer all numbers are dyadic. So, what we need is a sequence of dyadic numbers $b = t_1 > t_2 > \cdots > t_m$ (without loss of generality we may assume that b is dyadic) such that $t_{k+1} > t_k - h(t_k)/M$, for $k = 1$, $2, \dots m-1$, with $h(t_k) > 0$ for all $1 \leqslant k \leqslant m$ and such that $t_m < a$. In our case a more careful program will reveal that in the same number of steps (4163) we get a $t_m < 1/3$, so that essentially the above computation is correct.

6. Once again the case $r = 1$: Our Eulerian approach

We will now show that $h(\varphi)$ as defined in (6) is strictly convex for $0 < \varphi < \pi$. Since $h'(0) = 0$ this will solve our problem for $r = 1$.

In view of the power series for $\log(1+z)$ our inequality may also be written in the following interesting way

$$u(\varphi) = \frac{\cos\varphi}{3} - \frac{\cos 2\varphi}{4} + \frac{\cos 3\varphi}{5} - + \cdots < \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \cdots = \log 2 - \frac{1}{2},$$
$$0 < \varphi < \pi. \tag{12}$$

Now we present a *heuristic* approach —a technique often used by Euler himself. We write the left hand side of (12) as

$$u(\varphi) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(n\varphi)}{n+2}. \tag{13}$$

Differentiating we find

$$\begin{aligned} u''(\varphi) &= \sum_{n=1}^{\infty} (-1)^n \frac{n^2 \cos n\varphi}{n+2} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 4}{n+2} \cos n\varphi + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n+2} \cos n\varphi \\ &= \sum_{n=1}^{\infty} (-1)^n (n-2) \cos n\varphi - 4 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+2} \cos n\varphi. \end{aligned}$$

Hence

$$\begin{aligned} u''(\varphi)+4u(\varphi) &= \sum_{n=1}^{\infty}(-1)^n(n-2)\cos n\varphi \\ &= \sum_{n=1}^{\infty}(-1)^n n\cos n\varphi + 2\sum_{n=1}^{\infty}(-1)^{n-1}\cos n\varphi \end{aligned}$$

(see Hardy [5, Section 1.2, p. 2])

$$= -\frac{1}{2}\frac{d}{d\varphi}\tan\frac{\varphi}{2} + 2\times\frac{1}{2} = 1-\frac{1}{2}\frac{d}{d\varphi}\tan\frac{\varphi}{2}$$

so that

$$u''(\varphi)+4u(\varphi) = 1-\frac{1}{2}\frac{d}{d\varphi}\tan\frac{\varphi}{2} \tag{14}$$

which may also be written as

$$u''(\varphi)+4u(\varphi) = 1-\frac{1}{2}\frac{1}{1+\cos\varphi}. \tag{15}$$

Fully independent of the above Eulerian deduction, one may prove (by direct verification) that this differential equation for $u(\varphi)$ is valid indeed.

PROPOSITION 6.1. *The function u defined in* (5) *satisfies the differential equation* (15).

Proof. Since $u(\varphi)$ is even we have

$$u(\varphi) = \mathrm{Re}\left(e^{-2i\varphi}\log(1+e^{i\varphi}) - e^{-i\varphi}+\tfrac{1}{2}\right) = \mathrm{Re}\left(\frac{\log(1+z)}{z^2}-\frac{1}{z}+\frac{1}{2}\right) \tag{16}$$

where $z=e^{i\varphi}$. Then we have $\frac{d}{d\varphi} = iz\frac{d}{dz}$. In this way we easily get

$$\begin{aligned} u'(\varphi) &= \mathrm{Re}\left(\frac{i}{z}+\frac{i}{z(1+z)}-\frac{2i\log(1+z)}{z^2}\right), \\ u''(\varphi) &= \mathrm{Re}\left(\frac{1}{z}+\frac{1}{(1+z)^2}+\frac{3}{z(1+z)}-\frac{4\log(1+z)}{z^2}\right) \end{aligned}$$

so that

$$u''(\varphi)+4u(\varphi) = \mathrm{Re}\left(\frac{z(1+2z)}{(1+z)^2}\right). \tag{17}$$

One may verify that

$$\begin{aligned} \frac{z(1+2z)}{(1+z)^2} &= \frac{z}{1+z}+\left(\frac{z}{1+z}\right)^2 = \frac{e^{i\varphi/2}}{e^{i\varphi/2}+e^{-i\varphi/2}}+\left(\frac{e^{i\varphi/2}}{e^{i\varphi/2}+e^{-i\varphi/2}}\right)^2 \\ &= \frac{e^{i\varphi/2}}{2\cos\varphi/2}+\frac{e^{i\varphi}}{4\cos^2\varphi/2}. \end{aligned}$$

Taking real parts we get

$$u''(\varphi)+4u(\varphi)=\frac{1}{2}+\frac{\cos\varphi}{4\cos^2\varphi/2}=\frac{1}{2}+\frac{\cos\varphi}{2(1+\cos\varphi)}=1-\frac{1}{2(1+\cos\varphi)}. \tag{18}$$

We also have (14). In fact

$$1-\frac{1}{2}\frac{d}{d\varphi}\tan\frac{\varphi}{2}=1-\frac{1}{4}\frac{1}{\cos^2\varphi/2}=1-\frac{1}{2(1+\cos\varphi)}$$

PROPOSITION 6.2. *The function h may be represented by a power series*

$$h(\varphi)=\sum_{k=1}^{\infty}\frac{d_k}{(2k)!}(2\varphi)^{2k} \tag{19}$$

where for $k\geqslant 1$

$$d_k=(-1)^k\frac{3}{4}-(-1)^k\log 2+(-1)^{k+1}\sum_{j=1}^{k}\Bigl(1-\frac{1}{2^{2j}}\Bigr)\frac{B_{2j}}{2j} \tag{20}$$

where the B_j ($B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}, \dots$) are the Bernoulli numbers.

Proof. By (7) we know that h is analytic for $|\varphi|<\pi$, so that (19) is valid for $|\varphi|<\pi$. To determine the coefficients, observe that, because $h(\varphi)=u(0)-u(\varphi)$ by (14), we have

$$h''(\varphi)+4h(\varphi)=4u(0)-1+\frac{1}{2}\frac{d}{d\varphi}\tan\frac{\varphi}{2} \tag{21}$$

so that

$$4\sum_{k=1}^{\infty}2k(2k-1)\frac{d_k}{(2k)!}(2\varphi)^{2k-2}+4\sum_{k=1}^{\infty}\frac{d_k}{(2k)!}(2\varphi)^{2k}$$
$$=4\log 2-3+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)(2^{2k}-1)B_{2k}}{(2k)!}\varphi^{2k-2}.$$

Equating coefficients of equal powers of φ we get

$$4d_1=4\log 2-3+\frac{3}{2}B_2=4\log 2-\frac{11}{4} \tag{22}$$

and for $k\geqslant 1$

$$4(2k+2)(2k+1)\frac{2^{2k}d_{k+1}}{(2k+2)!}+4\frac{2^{2k}d_k}{(2k)!}=(-1)^k\frac{(2k+1)(2^{2k+2}-1)B_{2k+2}}{(2k+2)!} \tag{23}$$

which simplifies to

$$d_{k+1}=-d_k+(-1)^k(1-2^{-2k-2})\frac{B_{2k+2}}{2k+2}, \qquad k\geqslant 2. \tag{24}$$

Now we can prove formula (20) by induction. First, for $k=1$, (20) gives the correct value of d_1. Assuming that (20) is true for k we get

$$\begin{aligned} d_{k+1} &= -d_k + (-1)^k(1-2^{-2k-2})\frac{B_{2k+2}}{2k+2} = \\ &= -(-1)^k\frac{3}{4} + (-1)^k\log 2 - (-1)^{k+1}\sum_{j=1}^{k}\Big(1-\frac{1}{2^{2j}}\Big)\frac{B_{2j}}{2j} \\ &\quad + (-1)^k(1-2^{-2k-2})\frac{B_{2k+2}}{2k+2} \\ &= (-1)^{k+1}\frac{3}{4} - (-1)^{k+1}\log 2 + (-1)^{k+2}\sum_{j=1}^{k+1}\Big(1-\frac{1}{2^{2j}}\Big)\frac{B_{2j}}{2j} \end{aligned}$$

so that (20) is also true for d_{k+1}.

PROPOSITION 6.3. *All coefficients d_k in the Taylor expansion* (19) *are strictly positive.*

Proof. Recall the well known formula [2]

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2\cdot(2n)!} \tag{25}$$

so that we may write (20) as

$$d_k = (-1)^{k+1}\sum_{j=1}^{k}(-1)^{j+1}\Big(1-\frac{1}{2^{2j}}\Big)\frac{2\cdot(2j-1)!\zeta(2j)}{(2\pi)^{2j}} + (-1)^k\frac{3}{4} - (-1)^k\log 2. \tag{26}$$

Therefore, since $\log 2 < 3/4$,

$$d_k \geqslant \\ \Big(1-\frac{1}{2^{2k}}\Big)\frac{2\cdot(2k-1)!\zeta(2k)}{(2\pi)^{2k}} - \sum_{j=1}^{k-1}\Big(1-\frac{1}{2^{2j}}\Big)\frac{2\cdot(2j-1)!\zeta(2j)}{(2\pi)^{2j}} - \Big(\frac{3}{4}-\log 2\Big).$$

So, we only need to prove that

$$\sum_{j=1}^{k-1}\Big(1-\frac{1}{2^{2j}}\Big)\frac{2\cdot(2j-1)!\zeta(2j)}{(2\pi)^{2j}} + \Big(\frac{3}{4}-\log 2\Big) < \Big(1-\frac{1}{2^{2k}}\Big)\frac{2\cdot(2k-1)!\zeta(2k)}{(2\pi)^{2k}}. \tag{27}$$

But we have

$$\sum_{j=1}^{k-1}\Big(1-\frac{1}{2^{2j}}\Big)\frac{2\cdot(2j-1)!\zeta(2j)}{(2\pi)^{2j}} \leqslant 2\zeta(2)\sum_{j=1}^{k-1}\frac{(2j-1)!}{(2\pi)^{2j}}. \tag{28}$$

For $k \geqslant 8$ the last term of the sum in the right hand side of (28) is the greatest, so that

$$2\zeta(2)\sum_{j=1}^{k-1}\frac{(2j-1)!}{(2\pi)^{2j}} \leqslant 2(k-1)\zeta(2)\frac{(2k-3)!}{(2\pi)^{2k-2}} = \zeta(2)\frac{(2k-2)!}{(2\pi)^{2k-2}}, \qquad k \geqslant 8.$$

Also, it is easy to check that

$$\zeta(2)\frac{(2k-2)!}{(2\pi)^{2k-2}} > \frac{3}{4} - \log 2, \qquad (k \geqslant 8). \tag{29}$$

It follows that for $k \geqslant 8$ inequality (27) would be a consequence of

$$2\zeta(2)\frac{(2k-2)!}{(2\pi)^{2k-2}} < \Big(1 - \frac{1}{2^{2k}}\Big)\frac{2\cdot(2k-1)!\zeta(2k)}{(2\pi)^{2k}}. \tag{30}$$

This follows from the inequality

$$2\zeta(2)\frac{(2k-2)!}{(2\pi)^{2k-2}} < \frac{2^{16}-1}{2^{16}}\cdot\frac{2\cdot(2k-1)!}{(2\pi)^{2k}}. \tag{31}$$

So, we only need to show that

$$\frac{2^{16}}{2^{16}-1}(2\pi)^2\zeta(2) < (2k-1) \tag{32}$$

which is true for $k \geqslant 33$.

It remains to prove that $d_k > 0$ for $1 \leqslant k \leqslant 32$. Each of the numbers d_k is of the form $\frac{a}{b} \pm \log 2$. Each inequality $d_k > 0$ can be written as $\log 2 > r$ or $\log 2 < s$ where r and s are certain rational numbers. It is easy to see that $\max r = \frac{177}{256}$ and $\min s = \frac{89}{128}$ and we check that in fact

$$0.691406 \approx \frac{177}{256} < \log 2 \approx 0.693147 < \frac{89}{128} \approx 0.695313$$

finishing the proof that $d_k > 0$ for all $k \geqslant 1$.

7. The general case

For $0 < r < 1$ we want to prove that $-\operatorname{Re} f(re^{i\varphi}) \leqslant -\operatorname{Re} f(-r)$ for $-\pi < \varphi \leqslant \pi$. As before we change variables putting $\pi - \varphi$ instead of φ. So, we want to prove that $-\operatorname{Re} f(-re^{-i\varphi}) \leqslant -\operatorname{Re} f(-r)$.

Because $-\operatorname{Re} f(-re^{-i\varphi}) = -\operatorname{Re} f(-re^{i\varphi})$ we will show that

$$-\operatorname{Re} f(-re^{i\varphi}) \leqslant -\operatorname{Re} f(-r), \qquad -\pi < \varphi \leqslant \pi. \tag{33}$$

For $|z| < 1$ we define

$$\begin{aligned} U(z) = U(re^{i\varphi}) &:= \operatorname{Re} f(-re^{i\varphi}) - \operatorname{Re} f(-1) \\ &= -\operatorname{Re} f(-1) - \operatorname{Re}\Big(\frac{\log(1+z)}{z^2} - \frac{1}{z} + \frac{1}{2}\Big), \qquad z = re^{i\varphi}. \end{aligned} \tag{34}$$

Then U is a harmonic function on the unit disc $\Delta := \{z : |z| < 1\}$. In fact it extends to a continuous function on $\Delta \setminus \{-1\}$. This extension will also be denoted by U. The values of $U(e^{i\varphi})$ at the boundary of Δ coincide with those of $h(\varphi)$ as defined by (6). Our problem is to show that for $0 < r < 1$ and $-\pi < \varphi < \pi$ we have $U(re^{i\varphi}) \geqslant U(r)$.

Because $h \in \mathscr{L}^1(0, 2\pi)$ we have

$$U(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} h(t)P_r(\varphi - t)\,dt, \qquad 0 < r < 1, \quad -\pi < \varphi < \pi \tag{35}$$

where $P_r(t) := \frac{1-r^2}{1+r^2-2r\cos t}$ is the Poisson kernel.

Our claim will now follow from some (slightly adapted) theorems on rearrangements as described in the book by Hardy-Littlewood-Pólya on inequalities [6, Theorems 368 and 378]. Since the theorems there do not apply directly to our situation we prove the following:

PROPOSITION 7.1. *Let F and G be measurable positive periodic functions on* $\mathbf{R}$*, with period 2π. We assume that F and G are even, and that F is non decreasing and G non increasing on $(0, \pi)$.*

If $T\colon (0, 2\pi] \to (0, 2\pi]$ is a Borel measurable function that preserves Lebesgue measure, i. e. for any Borel set $B \subset (0, 2\pi]$ we have $|T^{-1}(B)| = |B|$, then

$$\int_0^{2\pi} F(t)G(t)\,dt \leqslant \int_0^{2\pi} F(t)G(T(t))\,dt. \tag{36}$$

Proof. Consider first the case in which F and G only take the values 0 and 1. Then, the hypotheses of the Proposition imply that G is the characteristic function of an interval I with center at 0 and F the characteristic function of an interval J with center at π (considering the functions F and G as defined on the circle (group)). Then

$$|I \cap J| = \int_0^{2\pi} F(t)G(t)\,dt \quad \text{and} \quad |I \cap M| = \int_0^{2\pi} F(t)G(T(t))\,dt$$

where $M = T^{-1}(J)$ is a measurable set of measure $|M| = |J|$. If $I \cap J = \emptyset$ there is nothing to prove. In the other case we will have

$$|I| + |J| - |I \cap J| = |I \cup J| = 2\pi \quad \text{and} \quad |I| + |M| - |I \cap M| = |I \cup M| \leqslant 2\pi$$

and it follows that $|I \cup J| \leqslant |I \cap M|$.

In the general case F and G can be written as the suprema of increasing sequences of step functions of type $F = \lim F_r$, with $F_r := \sum_{k=1}^{n} a_k \chi_{J_k}$ and $G = \lim G_r$ with $G_r := \sum_{k=1}^{m} b_k \chi_{I_k}$, where $a_k \geqslant 0$, $b_k \geqslant 0$, the J_k being intervals centered at π and the I_k intervals centered at 0.

Then the result for intervals implies

$$\int_0^{2\pi} F_r(t)G_r(t)\,dt \leqslant \int_0^{2\pi} F_r(t)G_r(T(t))\,dt.$$

Applying the Monotone Convergence Theorem we get (36).

THEOREM 7.2. *For* $0<r<1$ *and* $0<\varphi<\pi$ *we have* $-\operatorname{Re} f(re^{i\varphi}) < -\operatorname{Re} f(-r)$.

Proof. The inequality is equivalent to

$$\operatorname{Re} f(-r) - \operatorname{Re} f(-1) = U(r) < U(re^{i\varphi}) = \operatorname{Re} f(-re^{i\varphi}) - \operatorname{Re} f(-1).$$

We can apply Proposition 7.1 to the representation (35). In fact our $h(t)$ is even, positive and non decreasing on $(0,\pi)$, and the Poisson kernel $P_r(t) = \frac{1-r^2}{1+r^2-2r\cos t}$ is even, positive and non increasing on $(0,\pi)$. Also the translation $t \mapsto \varphi - t$ is measure preserving on the circle. So Proposition 7.1 yields $U(r) \leqslant U(re^{i\varphi})$.

To show that the inequality is strict for $0<\varphi<\pi$, we consider a small $\delta>0$ such that $0<a:=\varphi/2-\delta<\varphi/2<b:=\varphi/2+\delta<\pi$, and also a small $\varepsilon>0$ such that $0<a-\varepsilon<a+\varepsilon<\varphi/2<b-\varepsilon<b+\varepsilon<\pi$. Consider the intervals $I_a := [a-\varepsilon, a+\varepsilon]$ and $I_b := [b-\varepsilon, b+\varepsilon]$. The transformation $t \mapsto \varphi - t$ transforms I_a into I_b and I_b into I_a. Now consider the transformation T such that $T(t) = \varphi - t$ when $t \notin I_a \cup I_b$. For $t \in I_a$ we define $T(t) = 2a - t$ and for $t \in I_b$ we put $T(t) = 2b - t$ (t and $2b-t$ are symmetrical with respect to b). It is clear that T preserves the measure of $(-\pi,\pi]$ (considered as the circle). We will prove that

$$\int_{-\pi}^{\pi} h(t)P_r(\varphi - t)\,dt \leqslant \int_{-\pi}^{\pi} h(t)P_r(T(t))\,dt < \int_{-\pi}^{\pi} h(t)P_r(\varphi - t)\,dt \tag{37}$$

thereby concluding the proof.

The first inequality is simply a new application of Proposition 7.1. We only need to confirm the second inequality in (37). By definition $T(t) = \varphi - t$ except on $I_a \cup I_b$ so that

$$\begin{aligned} D :=& \int_{-\pi}^{\pi} h(t)P_r(\varphi - t)\,dt - \int_{-\pi}^{\pi} h(t)P_r(T(t))\,dt = \\ &= \int_{I_a\cup I_b} h(t)P_r(\varphi - t)\,dt - \int_{I_a\cup I_b} h(t)P_r(T(t))\,dt = \int_{a-\varepsilon}^{a+\varepsilon} h(t)P_r(\varphi - t)\,dt \\ &+ \int_{b-\varepsilon}^{b+\varepsilon} h(t)P_r(\varphi - t)\,dt - \int_{a-\varepsilon}^{a+\varepsilon} h(t)P_r(T(t))\,dt - \int_{b-\varepsilon}^{b+\varepsilon} h(t)P_r(T(t))\,dt. \end{aligned} \tag{38}$$

Now we change variables so that all integrals are taken over the same interval $(-\varepsilon,\varepsilon)$. Observing that $a = \varphi/2 - \delta$ and $\varphi - a = b$,

$$\begin{aligned} D = \int_{-\varepsilon}^{\varepsilon} h(a+t)P_r(b-t)\,dt + \int_{-\varepsilon}^{\varepsilon} h(b+t)P_r(a-t)\,dt - \\ - \int_{-\varepsilon}^{\varepsilon} h(a+t)P_r(a-t)\,dt - \int_{-\varepsilon}^{\varepsilon} h(b+t)P_r(b-t)\,dt \end{aligned} \tag{39}$$

and we find that

$$D = \int_{-\varepsilon}^{\varepsilon} \big(h(b+t) - h(a+t)\big)\big(P_r(a-t) - P_r(b-t)\big)\,dt. \tag{40}$$

Here we always have $0 < a+t < b+t < \pi$, and $0 < a-t < b-t < \pi$ so that the integrand is strictly positive. We thus have $D > 0$, completing the proof.

Acknowledgement. The authors would like to thank Foster Dieckhoff (Kansas City, Missouri, USA) and Maarten van Swaay (formerly at Kansas State University, Manhattan, Kansas, USA) for their linguistic assistance in preparing this note, and for their interest in the subject.

REFERENCES

[1] J. ARIAS DE REYNA, *High Precision Computation of Riemann's Zeta Function by the Riemann-Siegel Formula, I*, Math. Comp., **80** (2011), 995–1009.
[2] H. M. EDWARDS, *Riemann's zeta function*, Reprint of the 1974 original [Academic Press, New York], Dover Publications Inc., Mineola, NY, 2001.
[3] M. FITZGERALD & I. JAMES, *The mind of the Mathematician*, The Johns Hopkins University Press, Baltimore, 2007.
[4] W. GABCKE, *Neue Herleitung und explizite Restabschätzung der Riemann-Siegel-Formel*, Dissertation, Göttingen, 1979.
[5] G. HARDY, *Divergent Series*, Oxford University Press, 1949.
[6] G. HARDY, J. E. LITTLEWOOD & G. PÓLYA, *Inequalities*, Second Ed., Cambridge University Press, Cambridge, 1991.
[7] J. VAN DE LUNE, *A note on the zeros of Flett's function*, Afdeling Zuivere Wiskunde, Report ZW 167. Mathematisch Centrum, Amsterdam, 1981. `http://repository.cwi.nl/search/fullrecord.php?publnr=6776`
[8] J. VAN DE LUNE & H. H. J. TE RIELE, *Explicit computation of special zeros of partial sums of Riemann's zeta function*, Afdeling Zuivere Wiskunde, Report NW 44. Mathematisch Centrum, Amsterdam, 1977. `http://repository.cwi.nl/search/fullrecord.php?publnr=9024`

(Received April 2, 2010)

Juan Arias de Reyna
Facultad de Matemáticas
Universidad de Sevilla
Apdo. 1160
41080-Sevilla
Spain
e-mail: arias@us.es

Jan van de Lune
Langebuorren 49
9074 CH Hallum
The Netherlands
e-mail: j.vandelune@hccnet.nl