Anisotropic estimates and strong solutions of the Primitive Equations.

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1 Introduction.

Large number of geophysical fluids are modeled by the so-called "primitive equations". This model is obtained formally from the Navier-Stokes equations, with anisotropic (eddy) viscosity, assuming two important simplifications: "hydrostatic pressure" (depending linearly on the depth) and "rigid lid hypothesis" (fix water surface), see [10], [14] and references therein cited.

The model: For simplicity, we take constant density and assume that the effects due to the temperature (and salinity) can be decoupled from the dynamic of the flow. Then, we have a three-dimensional flow induced by the wind tension on the surface and by the centripetal and Coriolis forces. When the Earth curvature is not considered, we can use Cartesian coordinates instead of spherical coordinates (see in Lions-Temam-Wang [12] the model with spherical coordinates), hence the Lipschitz-continuous domain Ω is given by

$$\Omega = \{(\vec{x}, z) \in \mathbb{R}^3;\ \vec{x} \in \omega, -D(\vec{x}) < z < 0\}, \tag{1}$$

where $\omega \subseteq \mathbb{R}^2$ is an open domain and $D : \overline{\omega} \to \mathbb{R}_+$ is the depth function. The different boundaries of Ω (surface, bottom and sidewalls) are respectively: $\Gamma_s = \{(\vec{x}, 0);\ \vec{x} \in \omega\}$, $\Gamma_b = \{(\vec{x}, -D(\vec{x}));\ \vec{x} \in \omega\}$ and $\Gamma_l = \{(\vec{x}, z);\ \vec{x} \in \partial\omega, -D(\vec{x}) < z <$

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$0\}$ (here, $\partial\omega$ denotes the boundary of ω). The (three-dimensional) model is:

$$(EP)\begin{cases} \partial_t \vec{u} + (\vec{u}\cdot\nabla_H)\vec{u} + u_3\partial_z\vec{u} + \alpha\vec{u}^\perp & \\ \qquad -\nu_h\Delta_H\vec{u} - \nu_v\partial^2_{zz}\vec{u} + \nabla_H p_s = \vec{F} & \text{in } (0,T)\times\Omega, \\ \nabla_H\cdot\left(\displaystyle\int_{-D(\vec{x})}^0 \vec{u}(t;\vec{x},z)dz\right) = 0 & \text{in } (0,T)\times\omega, \\ \vec{u}_{|t=0} = \vec{u}_0 & \text{in } \Omega, \\ \nu_v\partial_z\vec{u}_{|\Gamma_s} = \vec{\tau}, \quad \vec{u}_{|\Gamma_b\cup\Gamma_l} = \vec{0} & \text{in } (0,T). \end{cases}$$

Here, we denote $\vec{x} = (x,y)$, $\nabla_H = (\partial_x, \partial_y)$ and $\Delta_H = \partial^2_{xx} + \partial^2_{yy}$. The unknowns are the horizontal components of the velocity $\vec{u} = (u_1, u_2) : (0,T)\times\Omega \to \mathbb{R}^2$ and the surface pressure $p_s : (0,T)\times\omega \to \mathbb{R}$ (in fact, p_s is a potential function, since it also includes centripetal and gravity effects, see [14]), whereas the vertical component of the velocity is

$$u_3(t;\vec{x},z) = -\int_{-D(\vec{x})}^z \nabla_H\cdot\vec{u}(t;\vec{x},s)ds, \quad \text{for } t\in(0,T) \text{ and } (\vec{x},z)\in\Omega. \tag{2}$$

Moreover, ν_h and $\nu_v > 0$ are positive constants, representing respectively horizontal and vertical (eddy) viscosity coefficients, $\vec{F} : (0,T)\times\Omega \to \mathbb{R}^2$ is a given function (depending on the temperature and the salinity, for instance) and $\vec{\tau} : (0,T)\times\Gamma_s \to \mathbb{R}^2$ represents the horizontal stress on the surface produced by the wind. Finally, $\alpha\vec{u}^\perp = \alpha(-u_2, u_1)^t$ models Coriolis effects. The no-slip condition is assumed on the bottom and vertical slipping is permitted on the sidewalls.

Most of the results for these equations have been obtained by means of isotropic estimates (i.e. using norms with the same regularity in all spatial directions), see [2], [12], [10] and [9]. For 2D domains and using isotropic estimates, we obtained in [9]: existence of global strong solution (see Definition 1.6) for small data (via a Galerkin method), local strong solution in time for small depth (via Schauder's Fixed Point Theorem) and uniqueness of the weak solution assuming that there exists a strong solution. We also explained in [9] that similar results in 3D domains cannot be obtained if we use the same kind of estimates.

The main new results: In this work, we use anisotropic estimates that let us obtain the following results. The precise notions of weak solution and strong solution (jointly with the space V) will be given in the next subsection.

First, we obtain existence of global strong solution for small data, given by the following theorem:

Theorem 1.1 "Global strong solution for small data". *Let $\omega \subset \mathbb{R}^d$ ($d = 1$ or 2) be a C^2 domain and $D \in C^3(\overline{\omega})$ such that $D \geq D_{\min} > 0$ in $\overline{\omega}$. Suppose that*

$\vec{u}_0 \in V$, $\vec{F} = \vec{f}_1 + \vec{f}_2$ with $\vec{f}_1 \in L^2(0,T;L^2(\Omega)^d)$ and $\vec{f}_2 \in L^\infty(0,T;L^2(\Omega)^d)$, and $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$ with $\vec{\tau}_1 \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$ and $\vec{\tau}_2 \in L^\infty(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$ for some $\varepsilon > 0$, such that $\partial_t\vec{\tau}_1 \in L^2(0,T;H^{-1/2}(\Gamma_s)^d)$ and $\partial_t\vec{\tau}_2 \in L^\infty(0,T;H^{-1/2}(\Gamma_s)^d)$. If, moreover the data verify the **"smallness conditions"**:

$$(H)\begin{cases} \|\vec{f}_1\|_{L^2_T(L^2)} < c\,\nu^{3/2}, & \|\vec{f}_2\|_{L^\infty_T(L^2)} < c\,\nu^2, \\ \|\vec{\tau}_1\|_{L^2_T(H_0^{1/2+\varepsilon})} < c\,\nu^{3/2}, & \|\partial_t\vec{\tau}_1\|_{L^2_T(H^{-1/2})} < c\,\nu^{5/2}, \\ \|\vec{\tau}_2\|_{L^\infty_T(H_0^{1/2+\varepsilon})} < c\,\nu^2, & \|\partial_t\vec{\tau}_2\|_{L^\infty_T(H^{-1/2})} < c\,\nu^3, \\ \|\vec{u}_0\|_{H^1} < c\,\nu\sqrt{\dfrac{\nu}{\bar{\nu}}}, & \|\vec{\tau}_1(0)\|_{H^{-1/2}} < c\,\nu^2\sqrt{\dfrac{\nu}{\bar{\nu}}}, \\ \|\vec{\tau}_2(0)\|_{H^{-1/2}} < c\,\nu^2\sqrt{\dfrac{\nu}{\bar{\nu}}}, & \end{cases}$$

where $\nu = \min\{\nu_h, \nu_v\}$, $\bar{\nu} = \max\{\nu_h, \nu_v\}$ and c is a small enough constant (depending only on Ω), then there exists a (unique) strong solution $(\vec{u}, p_s)$ of (EP) in $(0,T)$ (p_s is unique up to an additive function depending only on t).

Remark 1.1 *We denote $L^q_T(L^p) = L^q(0,T;L^p(\Omega))$, where T can be equal to $+\infty$ and $q \in (1,+\infty]$; $H^{-1/2} = H^{-1/2}(\Gamma_s)$ and $H_0^{1/2+\varepsilon} = H_0^{1/2+\varepsilon}(\Gamma_S)$.*

Secondly, we obtain existence of local strong solution for any data. In the 2D case, this result improves the results obtained in [9] (by relaxing the small depth hypothesis). The 3D case is new.

Theorem 1.2 "Local strong solution for any data." *Let $\omega \subset \mathbb{R}^d$ ($d = 1$ or 2) be a C^2 domain and $D \in C^3(\overline{\omega})$ such that $D \geq D_{\min} > 0$ in $\overline{\omega}$. Suppose that $\vec{u}_0 \in V$, $\vec{F} \in L^2(0,T;L^2(\Omega)^d)$ and $\vec{\tau} \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$, for some $\varepsilon > 0$, such that $\partial_t\vec{\tau} \in L^2(0,T;H^{-1/2}(\Gamma_s)^d)$. Then, there exists $T_* \in (0,T]$ and a unique strong solution $(\vec{u}, p_s)$ of (EP) in $(0,T_*)$.*

Third, we get uniqueness of weak solution assuming existence of a strong one. The precise notion of the space $L^\infty_z L^2_{\vec{x}}$ will be given in Subsection 3.1.

Theorem 1.3 "Uniqueness of strong/weak solution." *Assume $\Omega \subseteq \mathbb{R}^3$. Let $\vec{u}$ be a weak solution of (EP) in $(0,T)$. If there exists a weak solution $\underline{\vec{u}}$ of (EP) in $(0,T)$ with the same initial conditions, such that it verifies the additional regularity:*

$$\begin{aligned} &\nabla_H \underline{\vec{u}} \in L^2(0,T;L^\infty_z L^2_{\vec{x}}) \\ &\partial_z \underline{\vec{u}} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2), \end{aligned} \tag{3}$$

then both solutions coincide on $[0,T)$.

And finally, we obtain convergence of global strong solution to a stationary strong solution (under small data hypothesis), when $t \uparrow +\infty$.

Theorem 1.4 "Convergence of the 3D-evolution solution to a 3D-stationary solution". *Let $\vec{u}$ be a strong solution of (EP) in $(0, +\infty)$ with second member $\vec{F} = \vec{f}_1 + \vec{f}_2$, where $\vec{f}_1 \in L^2(0, +\infty; L^2(\Omega)^2)$ and $\vec{f}_2 \in L^2(\Omega)^2$ is independent of t, and Newman boundary condition $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$, where $\vec{\tau}_1 \in L^2(0, +\infty; H_0^{1/2+\varepsilon}(\Gamma_s)^2)$ for some $\varepsilon > 0$, such that $\partial_t \vec{\tau}_1 \in L^2(0, +\infty; H^{-1/2}(\Gamma_s)^2)$, and $\vec{\tau}_2 \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$ for some $\varepsilon > 0$. Assuming* **"smallness conditions"** *(H) with $T = +\infty$, if $\vec{v}$ is the stationary strong solution of (EP) with second member $\vec{f}_2$ and Newman boundary condition $\vec{\tau}_2$, then $\vec{u}(t) \to \vec{v}$ in the $H^1(\Omega)$-norm as $t \uparrow +\infty$.*

1.1 Definitions and auxiliary results.

To give a variational formulation to problem (EP), we define the following function spaces:

$$C_{b,l}^\infty(\Omega) = \{\varphi \in C^\infty(\Omega);\ supp(\varphi) \text{ is a compact set } \subseteq \overline{\Omega} \backslash (\Gamma_b \cup \Gamma_l)\},$$

$$H_{b,l}^1(\Omega) = \overline{C_{b,l}^\infty(\Omega)}^{H^1} = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_b \cup \Gamma_l\}, \quad H_{b,l}^{-1}(\Omega) = \text{dual of } H_{b,l}^1(\Omega),$$

$$\mathcal{V} = \{\vec{\varphi} \in C_{b,l}^\infty(\Omega)^d;\ \nabla_H \cdot \langle \vec{\varphi} \rangle = 0 \text{ in } \omega\}, \quad \text{where } \langle \vec{\varphi} \rangle(\vec{x}) = \int_{-D(\vec{x})}^0 \vec{\varphi}(\vec{x}, z) dz,$$

$$H = \overline{\mathcal{V}}^{L^2} = \{\vec{v} \in L^2(\Omega)^d;\ \nabla_H \cdot \langle \vec{v} \rangle = 0 \text{ in } \omega,\ \langle \vec{v} \rangle \cdot \vec{n}_{\partial\omega} = 0\},$$

$$V = \overline{\mathcal{V}}^{H^1} = \{\vec{v} \in H^1(\Omega)^d;\ \nabla_H \cdot \langle \vec{v} \rangle = 0 \text{ in } \omega,\ \vec{v}_{|\Gamma_b \cup \Gamma_l} = \vec{0}\}.$$

Definition 1.5 *Let $\vec{u}_0 \in H$, $\vec{F} \in L^2(0, T; H_{b,l}^{-1}(\Omega)^d)$ and $\vec{\tau} \in L^2(0, T; H^{-1/2}(\Gamma_s)^d)$ be given. We say that $\vec{u} : (0, T) \times \Omega \to \mathbb{R}^2$ is a* **weak solution of** *(EP)* **in** *$(0, T)$ if $\vec{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$, verifies the variational formulation:*

$$\int_0^T \int_\Omega -\vec{u} \cdot \Big(\partial_t \vec{\varphi} + (\vec{u} \cdot \nabla_H) \vec{\varphi} + u_3 \partial_z \vec{\varphi} \Big) + \nu_h \nabla_H \vec{u} : \nabla_H \vec{\varphi} + \nu_v \partial_z \vec{u} \cdot \partial_z \vec{\varphi} + \alpha \vec{u}^\perp \cdot \vec{\varphi}$$

$$= \int_\Omega \vec{u}_0 \cdot \vec{\varphi}(0) + \int_0^T \Big\{ \langle \vec{F}, \vec{\varphi} \rangle_\Omega + \langle \vec{\tau}, \vec{\varphi} \rangle_{\Gamma_s} \Big\} dt, \quad \forall\, \vec{\varphi} \in C^1([0, T]; \mathcal{V}) \ s.t.\ \vec{\varphi}(T) = \vec{0},$$

and, moreover $\vec{u}$ satisfies the "energy inequality" a.e. $t \in (0, T)$,

$$\frac{1}{2} \|\vec{u}(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\vec{u}\|_V^2 ds \le \frac{1}{2} \|\vec{u}_0\|_{L^2(\Omega)}^2 + \int_0^t \Big\{ \langle \vec{F}, \vec{u} \rangle_\Omega + \langle \vec{\tau}, \vec{u} \rangle_{\Gamma_s} \Big\} ds. \qquad (4)$$

In the case $T = +\infty$, we say that $\vec{u}$ is a weak solution of (EP) in $(0, +\infty)$ if $\vec{u}$ is a weak solution of (EP) in $(0, T)$, $\forall T < +\infty$.

Here, $\langle \cdot, \cdot \rangle_\Omega$ denotes duality between $H^{-1}_{b,l}(\Omega)$ and $H^1_{b,l}(\Omega)$, whereas $\langle \cdot, \cdot \rangle_{\Gamma_s}$ denotes duality between $H^{-1/2}(\Gamma_s)$ and $H^{1/2}(\Gamma_s)$. On the other hand, the V-norm will be denoted by $\|\vec{u}\|^2_V = \nu_h \|\nabla_H \vec{u}\|^2_{L^2(\Omega)} + \nu_v \|\partial_z \vec{u}\|^2_{L^2(\Omega)}$, and the $H^1(\Omega)$-norm will be denoted by $\|\vec{u}\|^2_{H^1(\Omega)} = \|\nabla_H \vec{u}\|^2_{L^2(\Omega)} + \|\partial_z \vec{u}\|^2_{L^2(\Omega)}$.

Definition 1.6 *Let $\vec{u}_0 \in V$, $\vec{F} \in L^2(0,T;L^2(\Omega)^d)$, $\vec{\tau} \in L^2(0,T;H^{1/2}(\Gamma_s)^d)$ and $\partial_t \vec{\tau} \in L^2(0,T;H^{-1/2}(\Gamma_s)^d)$ be given. Let $\vec{u}$ be a weak solution of (EP) in $(0,T)$, we say that $\vec{u}$ is a* **strong solution** *of (EP) in $(0,T)$ if it verifies the additional regularity:*

$$\vec{u} \in C([0,T];V) \cap L^2(0,T;H^2(\Omega)^d \cap V), \quad \partial_t \vec{u} \in L^2(0,T;H).$$

Remark 1.2 *In the 2D case, besides changing vectorial notation by scalar notation, one important difference with respect to the 3D case is that $\omega \subseteq \mathbb{R}$ is an open interval which simplifies the function spaces of free divergence. Indeed, conditions $\nabla_H \cdot \langle \vec{v} \rangle = 0$ in ω and $\langle \vec{v} \rangle \cdot n_{\partial\omega} = 0$ are replaced by $\langle v \rangle = 0$ in ω.*

Auxiliary results: In this work, we will frequently consider the evolution linear problem (Stokes with hydrostatic pressure):

$$(S) \left\{ \begin{array}{rcl} \partial_t \vec{v} - \nu_h \Delta_H \vec{v} - \nu_v \partial^2_{zz} \vec{v} + \nabla_H q_s &=& \vec{f} \quad \text{in } (0,T) \times \Omega, \\ \nabla_H \cdot \langle \vec{v} \rangle &=& 0 \quad \text{in } (0,T) \times \omega, \\ \vec{v}|_{t=0} &=& \vec{v}_0 \quad \text{in } \Omega, \\ \nu_v \partial_z \vec{v} = \vec{\tau} \text{ on } (0,T) \times \Gamma_s, \quad \vec{v} &=& \vec{0} \quad \text{on } (0,T) \times (\Gamma_b \cup \Gamma_l), \end{array} \right.$$

and the stationary problem associated to (S), that it will be called (S_{st}).
We denote by C different positive constants, always independent of ν_h and ν_v.

Theorem 1.7 "Weak solution of (S_{st})**"** *Let $\omega \subseteq \mathbb{R}^d$ ($d = 1$ or 2) and let $\Omega \subseteq \mathbb{R}^{d+1}$, defined as in (1), be a Lipschitz-continuous domain. If $\vec{f} \in H^{-1}_{b,l}(\Omega)^d$ and $\vec{\tau} \in H^{-1/2}(\Gamma_s)^d$, then the problem (S_{st}) has a unique solution $\vec{v} \in H^1(\Omega)^d$. Moreover, there exists a constant $C = C(\Omega) > 0$ such that*

$$\|\vec{v}\|^2_{H^1(\Omega)} \leq \frac{C}{\nu^2} \left\{ \|\vec{\tau}\|^2_{H^{-1/2}(\Gamma_s)} + \|\vec{f}\|^2_{H^{-1}_{b,l}(\Omega)} \right\}. \tag{5}$$

In [2], [4] and [10], there are different proofs of this result.

Theorem 1.8 "Weak solution of (S)**"** *([12]) Let ω and Ω as in Theorem 1.7. If $\vec{f} \in L^2(0,T;H_{b,l}^{-1}(\Omega)^d)$ and $\vec{\tau} \in L^2(0,T;H^{-1/2}(\Gamma_s)^d)$, then there is a (unique) weak solution $\vec{v}$ of problem (S) in $(0,T)$.*

Theorem 1.9 "Strong solution of (S_{st})**"** *([16]) Let $\omega \subseteq \mathbb{R}^d$ ($d = 1$ or 2) be a C^2 domain and $D \in C^3(\overline{\omega})$ with $D \geq D_{\min} > 0$ in $\overline{\omega}$. If $\vec{f} \in L^2(\Omega)^d$ and $\vec{\tau} \in H_0^{1/2+\varepsilon}(\Gamma_s)^d$ (for some $\varepsilon > 0$), then there exists a (unique) strong solution $\vec{v}$ of (S_{st}) (i.e. $\vec{v} \in H^2(\Omega)^d \cap V$). Moreover, there exists $C = C(\Omega) > 0$ such that:*

$$\|\vec{v}\|_{H^2(\Omega)}^2 \leq \frac{C}{\nu^2}\left\{\|\vec{f}\|_{L^2(\Omega)}^2 + \|\vec{\tau}\|_{H_0^{1/2+\varepsilon}(\Gamma_s)}^2\right\}. \tag{6}$$

Finally, the next result gives a linear version of Theorem 1.1 and will be used in the case $\vec{\tau}_2 = \vec{0}$ and $\vec{f}_2 = \vec{0}$ to lift the data $\vec{v}_0$, $\vec{\tau}_1$ and $\vec{f}_1$.

Theorem 1.10 "Strong solution of (S)**"** *Let $\omega \subseteq \mathbb{R}^d$ ($d = 1$ or 2) be a C^2 domain and $D \in C^3(\overline{\omega})$ with $D \geq D_{\min} > 0$ in $\overline{\omega}$. If $\vec{v}_0 \in V$, $\vec{f} = \vec{f}_1 + \vec{f}_2$ with $\vec{f}_1 \in L^2((0,T) \times \Omega)^d$ and $\vec{f}_2 \in L^\infty(0,T;L^2(\Omega)^d)$, $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$ with $\vec{\tau}_1 \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$ and $\vec{\tau}_2 \in L^\infty(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$ (for some $\varepsilon > 0$), such that $\partial_t\vec{\tau}_1 \in L^2(0,T;H^{-1/2}(\Gamma_s)^d)$ and $\partial_t\vec{\tau}_2 \in L^\infty(0,T;H^{-1/2}(\Gamma_s)^d)$, then there exists a (unique) strong solution $\vec{v}$ of (S) in $(0,T)$. Moreover, there exists $C = C(\Omega) > 0$ such that*

$$\begin{aligned}
\|\vec{v}\|_{L_T^\infty(H^1)}^2 &\leq \frac{\bar{\nu}}{\nu}\left\{\|\vec{v}_0\|_{H^1}^2 + \frac{C}{\nu^2}\left(\|\vec{\tau}_1(0)\|_{H^{-1/2}}^2 + \|\vec{\tau}_2(0)\|_{H^{-1/2}}^2\right)\right\} \\
&+ \frac{C}{\nu}\left\{\|\vec{f}_1\|_{L_T^2(L^2)}^2 + \|\vec{\tau}_1\|_{L_T^2(H_0^{1/2+\varepsilon})}^2\right\} + \frac{C}{\nu^3}\|\partial_t\vec{\tau}_1\|_{L_T^2(H^{-1/2})}^2 \\
&+ \frac{C}{\nu^2}\left\{\|\vec{f}_2\|_{L_T^\infty(L^2)}^2 + \|\vec{\tau}_2\|_{L_T^\infty(H_0^{1/2+\varepsilon})}^2\right\} + \frac{C}{\nu^4}\|\partial_t\vec{\tau}_2\|_{L_T^\infty(H^{-1/2})}^2,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\|\vec{v}\|_{L_T^2(H^2)}^2 &\leq \frac{\bar{\nu}}{\nu}\left\{\frac{1}{\nu}\|\vec{v}_0\|_{H^1}^2 + \frac{C}{\nu^3}\left(\|\vec{\tau}_1(0)\|_{H^{-1/2}}^2 + \|\vec{\tau}_2(0)\|_{H^{-1/2}}^2\right)\right\} \\
&+ \frac{C}{\nu^2}\left\{\|\vec{f}_1\|_{L_T^2(L^2)}^2 + \|\vec{\tau}_1\|_{L_T^2(H_0^{1/2+\varepsilon})}^2\right\} + \frac{C}{\nu^4}\|\partial_t\vec{\tau}_1\|_{L_T^2(H^{-1/2})}^2 \\
&+ \frac{CT}{\nu^2}\|\vec{f}_2\|_{L_T^\infty(L^2)}^2 + \frac{CT}{\nu^2}\|\vec{\tau}_2\|_{L_T^\infty(H_0^{1/2+\varepsilon})}^2 + \frac{CT}{\nu^4}\|\partial_t\vec{\tau}_2\|_{L_T^\infty(H^{-1/2})}^2,
\end{aligned} \tag{8}$$

$$
\begin{aligned}
\|\partial_t \vec{v}\|^2_{L^2_T(L^2)} \;&\leq\; \frac{\bar{\nu}}{\nu}\left\{\nu\|\vec{v}_0\|^2_{H^1} + \frac{C}{\nu}\left(\|\vec{\tau}_1(0)\|^2_{H^{-1/2}} + \|\vec{\tau}_2(0)\|^2_{H^{-1/2}}\right)\right\} \\
&+\; C\|\vec{f}_1\|^2_{L^2_T(L^2)} + \frac{C}{\nu^2}\|\partial_t\vec{\tau}_1\|^2_{L^2_T(H^{-1/2})} \\
&+\; CT\|\vec{f}_2\|^2_{L^\infty_T(L^2)} + \frac{CT}{\nu^2}\|\partial_t\vec{\tau}_2\|^2_{L^\infty_T(H^{-1/2})}.
\end{aligned}
\tag{9}
$$

Proof of Theorem 1.10: A proof of the existence can be seen in [9]. Here, we only sketch the proof of the continuous dependence estimates, specifying the dependence of these estimates on the viscosities.
We set $\vec{v} = \vec{y}_1 + \vec{y}_2 + \vec{e}_1 + \vec{e}_2$, where $\vec{e}_i(t)$ $(i = 1, 2)$ are the solutions of the stationary problems:

$$
(S_i)\begin{cases}
-\nu_h\Delta_H\vec{e}_i - \nu_v\partial^2_{zz}\vec{e}_i + \nabla_H q_i \;=\; \vec{0} \;\text{ in } \Omega, \quad \nabla_H\cdot\langle\vec{e}_i\rangle \;=\; 0 \;\text{ in } \omega, \\
\nu_v\partial_z\vec{e}_i \;=\; \vec{\tau}_i(t) \;\text{ on } \Gamma_s, \quad \vec{e}_i \;=\; \vec{0} \;\text{ on } \Gamma_b\cup\Gamma_l,
\end{cases}
$$

and $\vec{y}_1$, $\vec{y}_2$ are the solutions of the evolution problems:

$$
(E_1)\begin{cases}
\partial_t\vec{y}_1 - \nu_h\Delta_H\vec{y}_1 - \nu_v\partial^2_{zz}\vec{y}_1 + \nabla_H p_1 \;=\; \vec{f}_1 - \partial_t\vec{e}_1 \;\text{ in } (0,T)\times\Omega, \\
\nabla_H\cdot\langle\vec{y}_1\rangle \;=\; 0 \;\text{ in } (0,T)\times\omega, \qquad \vec{y}_1|_{t=0} \;=\; \vec{v}_0 - \vec{e}_1(0) - \vec{e}_2(0) \;\text{ in } \Omega, \\
\nu_v\partial_z\vec{y}_1 \;=\; \vec{0} \;\text{ on } (0,T)\times\Gamma_s, \qquad \vec{y}_1 \;=\; \vec{0} \;\text{ on } (0,T)\times(\Gamma_b\cup\Gamma_l),
\end{cases}
$$

$$
(E_2)\begin{cases}
\partial_t\vec{y}_2 - \nu_h\Delta_H\vec{y}_2 - \nu_v\partial^2_{zz}\vec{y}_2 + \nabla_H p_2 \;=\; \vec{f}_2 - \partial_t\vec{e}_2 \;\text{ in } (0,T)\times\Omega, \\
\nabla_H\cdot\langle\vec{y}_2\rangle \;=\; 0 \;\text{ in } (0,T)\times\omega, \qquad \vec{y}_2|_{t=0} \;=\; \vec{0} \;\text{ in } \Omega, \\
\nu_v\partial_z\vec{y}_2 \;=\; \vec{0} \;\text{ on } (0,T)\times\Gamma_s, \qquad \vec{y}_2 \;=\; \vec{0} \;\text{ on } (0,T)\times(\Gamma_b\cup\Gamma_l),
\end{cases}
$$

Then, we have the following estimates for $\vec{e}_1$ and $\vec{e}_2$:

$$
\begin{aligned}
\|\vec{e}_1(t)\|^2_{H^1} &\leq \frac{C}{\nu^2}\|\vec{\tau}_1(t)\|^2_{H^{-1/2}} \leq \frac{C}{\nu^2}\left\{\|\vec{\tau}_1(0)\|^2_{H^{-1/2}} + 2\int_0^t \|\vec{\tau}_1\|_{H^{-1/2}}\|\partial_t\vec{\tau}_1\|_{H^{-1/2}}\right\} \\
&\leq \frac{C}{\nu^2}\|\vec{\tau}_1(0)\|^2_{H^{-1/2}} + \frac{C}{\nu^3}\|\partial_t\vec{\tau}_1\|^2_{L^2_t(H^{-1/2})} + \frac{C}{\nu}\|\vec{\tau}_1\|^2_{L^2_t(H_0^{1/2+\varepsilon})}, \quad \forall t\in[0,T],
\end{aligned}
$$

$$
\|\vec{e}_1\|_{L^2_T(H^2)} \;\leq\; \frac{C}{\nu}\|\vec{\tau}_1\|_{L^2_T(H_0^{1/2+\varepsilon})}, \quad \|\partial_t\vec{e}_1\|_{L^2_T(H^1)} \leq \frac{C}{\nu}\|\partial_t\vec{\tau}_1\|_{L^2_T(H^{-1/2})},
$$

$$\|\vec{e}_2\|^2_{L^\infty_T(H^1)} \leq \frac{C}{\nu^2}\|\vec{\tau}_2\|^2_{L^\infty_T(H^{-1/2})}$$
$$\|\vec{e}_2\|^2_{L^2_T(H^2)} \leq \frac{C\,T}{\nu^2}\|\vec{\tau}_2\|^2_{L^\infty_T(H^{1/2+\varepsilon})}, \quad \|\partial_t\vec{e}_2\|^2_{L^2_T(H^1)} \leq \frac{C\,T}{\nu^2}\|\partial_t\vec{\tau}_2\|^2_{L^\infty_T(H^{-1/2})},$$

Now, we define hydrostatic Stokes operator A, i.e. $A\vec{v} = \vec{u}$ means that (see [9]):

$$\begin{cases} -\nu_h\Delta_H\vec{v} - \nu_v\partial^2_{zz}\vec{v} + \nabla_H p_s = \vec{u} \text{ in } \Omega, \\ \nabla_H\cdot\langle\vec{v}\rangle = \vec{0} \text{ in } \omega, \\ \nu_v\partial_z\vec{v} = \vec{0} \text{ on } \Gamma_s, \quad \vec{v} = \vec{0} \text{ on } \Gamma_l\cup\Gamma_b. \end{cases}$$

Taking $A\vec{y}_1$ as a test function in the variational formulation of (E_1), we obtain:

$$\frac{d}{dt}\|\vec{y}_1(t)\|^2_V + \|A\vec{y}_1(t)\|^2_{L^2(\Omega)} \leq \|\vec{f}_1(t)\|^2_{L^2(\Omega)} + \|\partial_t\vec{e}_1(t)\|^2_{L^2(\Omega)}.$$

Thus, integrating in time, using that

$$\nu\|\vec{y}_1(t)\|^2_{H^1(\Omega)} \leq \|\vec{y}_1(t)\|^2_V \leq \bar{\nu}\|\vec{y}_1(t)\|^2_{H^1(\Omega)},$$

$$\|\vec{y}_1(t)\|^2_{H^2(\Omega)} \leq \frac{C}{\nu^2}\|A\vec{y}_1(t)\|^2_{L^2(\Omega)}, \tag{10}$$

and the previous estimates for $\partial_t\vec{e}_1$, we get:

$$\|\vec{y}_1\|^2_{L^\infty_T(H^1)} \leq \frac{\bar{\nu}}{\nu}\|\vec{y}_1(0)\|^2_{H^1} + \frac{C}{\nu}\left(\|\vec{f}_1\|^2_{L^2_T(L^2)} + \frac{1}{\nu^2}\|\partial_t\vec{\tau}_1\|^2_{L^2_T(H^{-1/2})}\right),$$

$$\|\vec{y}_1\|^2_{L^2_T(H^2)} \leq \frac{\bar{\nu}}{\nu^2}\|\vec{y}_1(0)\|^2_{H^1(\Omega)} + \frac{C}{\nu^2}\left(\|\vec{f}_1\|^2_{L^2_T(L^2)} + \frac{1}{\nu^2}\|\partial_t\vec{\tau}_1\|^2_{L^2_T(H^{-1/2})}\right).$$

In the same way, we have for $\vec{y}_2$:

$$\frac{d}{dt}\|\vec{y}_2(t)\|^2_V + \|A\vec{y}_2(t)\|^2_{L^2(\Omega)} \leq C\left(\|\vec{f}_2\|^2_{L^\infty_T(L^2)} + \|\partial_t\vec{e}_2\|^2_{L^\infty_T(L^2)}\right). \tag{11}$$

Therefore, using that

$$\nu\|\vec{y}_2(t)\|^2_V \leq C_1\|A\vec{y}_2(t)\|^2_{L^2(\Omega)}, \tag{12}$$

multiplying by $exp\left(\dfrac{\nu}{C_1}t\right)$ and integrating in time, one has:

$$\|\vec{y}_2\|^2_{L^\infty_T(H^1)} \leq \frac{C}{\nu^2}\left(\|\vec{f}_2\|^2_{L^\infty_T(L^2)} + \frac{1}{\nu^2}\|\partial_t\vec{\tau}_2\|^2_{L^\infty_T(H^{-1/2})}\right). \tag{13}$$

Now, using (10) for $\vec{y}_2(t)$, from $(11)-(13)$, we deduce:

$$\|\vec{y}_2\|^2_{L^2_T(H^2)} \leq \frac{CT}{\nu^2}\left(\|\vec{f}_2\|^2_{L^\infty_T(L^2)} + \frac{1}{\nu^2}\|\partial_t\vec{\tau}_2\|^2_{L^\infty_T(H^{-1/2})}\right).$$

All the above bounds let us obtain (7) and (8). Expression (9) is easily obtained taking $\partial_t\vec{y}_i$ $(i=1,2)$ as test functions in (E_i) and arguing as before. ■

Remark 1.3 *If $T=+\infty$, we only can handle expression* (7). *Therefore, only estimates in $L^\infty(0,+\infty;H^1(\Omega)^d)$ can be obtained.*

2 The 2D case.

We start with the proof of the existence results (Theorems 1.1 and 1.2) in the 2D case and postpone the 3D case to the next section, where we need to prove more precise anisotropic Sobolev inequalities.

2.1 Some 2D anisotropic spaces and related estimates.

We introduce the following 2D anisotropic function spaces.

Definition 2.1 *Given $p,q\in[1,+\infty]$, a function u belongs to $L^p_xL^q_z(\Omega)$ if:*

$$u(x,\cdot)\in L^q(-D(x),0) \quad \text{and} \quad \|u(x,\cdot)\|_{L^q(-D(x),0)}\in L^p(\omega).$$

Moreover, its norm is given by the expression:

$$\|u\|_{L^p_xL^q_z(\Omega)} = \Big\|\|u(x,\cdot)\|_{L^q(-D(x),0)}\Big\|_{L^p(\omega)}$$

Remark 2.1 *The most useful norms that we will use in the 2D case will be:*

$$\begin{aligned}\|u\|_{L^\infty_xL^2_z(\Omega)} &= \sup_{x\in\omega}\left(\|u(x,\cdot)\|_{L^2(-D(x),0)}\right)\\ \|u\|_{L^2_xL^\infty_z(\Omega)} &= \left\|\sup_{z\in(-D(x),0)}|u(x,z)|\right\|_{L^2(\omega)}.\end{aligned}$$

Remark 2.2 *For sake of simplicity, we sometimes denote $L^p_xL^q_z$ instead of $L^p_xL^q_z(\Omega)$, and L^p instead of $L^p(\Omega)$, when there is no risk of confusion.*

Now, we enunciate several lemmas that will be frequently used in this work. We denote $D_{\max}=\max_{\overline{\omega}} D$.

Lemma 2.2 "Anisotropic regularity for the vertical velocity". *Let $v : \Omega \to \mathbb{R}$ be a function such that $\partial_x v \in L^2(\Omega)$ and define $v_3(x,z) = -\int_{-D(x)}^{z} \partial_x v(x,s)ds$, then one verifies:*

$$\|v_3\|_{L^2_x L^\infty_z} \le D_{\max}^{1/2} \|\partial_x v\|_{L^2(\Omega)}.$$

Proof: From the definition of v_3,

$$\|v_3(x,\cdot)\|_{L^\infty(-D(x),0)} \le \int_{-D(x)}^{0} |\partial_x v(x,s)|ds.$$

Then,

$$\begin{aligned}
\|v_3\|^2_{L^2_x L^\infty_z} &\le \int_\omega \left(\int_{-D(x)}^{0} |\partial_x v(x,s)|ds \right)^2 dx \\
&\le \int_\omega \left(\int_{-D(x)}^{0} |\partial_x v(x,s)|^2 ds \right) D(x)dx \le D_{\max} \|\partial_x v\|^2_{L^2(\Omega)} \qquad \blacksquare.
\end{aligned}$$

Lemma 2.3 "Vertical Poincaré's inequalities". *Let $u \in L^2(\Omega)$ be a function such that $\partial_z u \in L^2(\Omega)$. Then, $u \in L^2_x L^\infty_z$ and verifies the following estimates:*

(**a**) $\|u\|^2_{L^2_x L^\infty_z} \le 2\, \|u\|_{L^2(\Omega)} \|\partial_z u\|_{L^2(\Omega)}$ *if $(u n_z)|_{\Gamma_b} = 0$,*
(**b**) $\|u\|^2_{L^2_x L^\infty_z} \le C \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$ *for some constant $C = C(\Omega) > 0$.*

Remark 2.3 *We denote $\vec{n} = (n_x, n_z)$ the outward normal vector to $\partial\Omega$. Notice that since $u \in L^2(\Omega)$ and $\partial_z u \in L^2(\Omega)$, then $u n_z \in H^{-1/2}(\partial\Omega)$.*

Proof: In case (**a**), by hypothesis, $u(x, -D(x)) = 0$. Then,

$$\begin{aligned}
u(x,z)^2 &= \int_{-D(x)}^{z} \partial_z (u(x,s)^2) ds \;=\; 2 \int_{-D(x)}^{z} u(x,s) \partial_z u(z,s) ds \\
&\le 2\, \|u(x,\cdot)\|_{L^2(-D(x),0)} \|\partial_z u(x,\cdot)\|_{L^2(-D(x),0)}.
\end{aligned}$$

Taking essential supremum in $z \in (-D(x), 0)$,

$$\|u(x,\cdot)\|^2_{L^\infty(-D(x),0)} \le 2\, \|u(x,\cdot)\|_{L^2(-D(x),0)} \|\partial_z u(x,\cdot)\|_{L^2(-D(x),0)},$$

and integrating on $x \in \omega$,

$$\|u\|^2_{L^2_x L^\infty_z} \le 2\, \|u\|_{L^2(\Omega)} \|\partial_z u\|_{L^2(\Omega)}.$$

As for case (**b**), we consider the following "Extension Theorem" (see [6] for instance):

"Let Ω be a $C^{0,1}$-domain in $\mathbb{R}^2$. Given a domain $\Omega' \supset\supset \Omega$, there exists a (linear) extension operator E from $H^1(\Omega)$ into $H^1_0(\Omega')$ such that $Eu|_\Omega = u$ and

$$\|Eu\|_{H^1(\Omega')} \leq C\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{14}$$

where $C = C(\Omega, \Omega') > 0$. Moreover, one verifies:

$$\|Eu\|_{L^2(\Omega')} \leq C\|u\|_{L^2(\Omega)} \quad \forall u \in H^1(\Omega)." \tag{15}$$

It is easy to demonstrate that:

$$\|u\|^2_{L^2_x L^\infty_z(\Omega)} \leq \|Eu\|^2_{L^2_x L^\infty_z(\Omega')}.$$

On the other hand, applying case (**a**) to Eu we have:

$$\|Eu\|^2_{L^2_x L^\infty_z(\Omega')} \leq 2\,\|Eu\|_{L^2(\Omega')}\|\partial_z(Eu)\|_{L^2(\Omega')}.$$

Taking into account the two previous estimates and using (14) and (15), we arrive at (**b**).

■

Lemma 2.4 "Horizontal Poincaré's inequalities". *Let $u \in L^2(\Omega)$ be a function such that $\partial_x u \in L^2(\Omega)$. Then, $u \in L^\infty_x L^2_z(\Omega)$ and verifies the following estimate:*

(**a**) $\|u\|^2_{L^\infty_x L^2_z} \leq 2\,\|u\|_{L^2(\Omega)}\|\partial_x u\|_{L^2(\Omega)}$ *if* $(un_x)|_{\Gamma_b \cup \Gamma_l} = 0$,

(**b**) $\|u\|^2_{L^\infty_x L^2_z} \leq C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$, *for some constant* $C = C(\Omega) > 0$.

Remark 2.4 *Notice that, since $u \in L^2(\Omega)$ and $\partial_x u \in L^2(\Omega)$, then $un_x \in H^{-1/2}(\partial\Omega)$.*

Proof: In the case (**a**), given $(x, z) \in \Omega$, we consider $x_0 \in \partial\omega^{i_0}_z$ where $\omega_z = \{x \in \omega/\,(x, z) \in \Omega\} = \bigcup_i \omega^i_z$, being $(\omega^i_z)_{i\in I_z}$ the connexe components of ω_z and $x \in \omega^{i_0}_z$. By hypothesis, $u(x_0, z) = 0$. Then,

$$\begin{aligned} u(x,z)^2 &= \int_{x_0}^x \partial_x(u(s,z)^2)ds \;=\; 2\int_{x_0}^x u(s,z)\partial_x u(s,z)ds \\ &\leq 2\,\|u(\cdot,z)\|_{L^2(\omega_z)}\|\partial_x u(\cdot,z)\|_{L^2(\omega_z)}. \end{aligned}$$

Integrating in $z \in (-D(x), 0)$ and taking essential supremum in $x \in \omega$, the inequality (**a**) of this Lemma is obtained.

Case (**b**) follows the same argument as Lemma 2.3 (**b**), changing $\partial_z u$ by $\partial_x u$ and applying Lemma 2.4 (**a**) instead of Lemma 2.3 (**a**). ■

2.2 Existence of strong solution of the 2D hydrostatic problem.

Proof of Theorem 1.1 in the 2D case (d = 1):
We are going to focus on the study of the strong regularity for problem (EP), considered in a domain $\Omega \subset \mathbb{R}^2$ and data $F = f_1 + f_2$, with $f_1 \in L^2(0,T;L^2(\Omega))$ and $f_2 \in L^\infty(0,T;L^2(\Omega))$, $\tau = \tau_1 + \tau_1$ with $\tau_1 \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s))$ and $\tau_2 \in L^\infty(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)$ for some $\varepsilon > 0$, such that $\partial_t \tau_1 \in L^2(0,T;H^{-1/2}(\Gamma_s))$ and $\partial_t \tau_2 \in L^\infty(0,T;H^{-1/2}(\Gamma_s)$. To do this, first we lift up the non homogeneous Neumann boundary condition τ_2 by taking $(e^\infty(t), q_s^\infty(t))$ the strong solution of the steady hydrostatic Stokes problem:

$$\left\{ \begin{array}{l} -\nu_h \partial^2_{xx} e^\infty - \nu_v \partial^2_{zz} e^\infty + \partial_x q_s^\infty \;=\; 0 \;\text{ in } \Omega, \quad \langle e^\infty \rangle \;=\; 0 \;\text{ in } \omega, \\ \nu_v \partial_z e^\infty \;=\; \tau_2(t) \;\text{ on } \Gamma_s, \quad e^\infty \;=\; 0 \;\text{ on } \Gamma_b \cup \Gamma_l. \end{array} \right.$$

Then, we lift up τ_1, the horizontal force f_1 and the initial condition by taking (e, q_s) the strong solution of hydrostatic Stokes evolution problem,

$$(E) \left\{ \begin{array}{l} \partial_t e - \nu_h \partial^2_{xx} e - \nu_v \partial^2_{zz} e + \partial_x q_s \;=\; f_1 \;\text{ in } (0,T) \times \Omega, \\ \langle e \rangle \;=\; 0 \;\text{ in } (0,T) \times \omega, \quad e|_{t=0} \;=\; u_0 - e^\infty(0) \;\text{ in } \Omega, \\ \nu_v \partial_z e \;=\; \tau_1 \;\text{ on } (0,T) \times \Gamma_s, \quad e \;=\; 0 \;\text{ on } (0,T) \times (\Gamma_b \cup \Gamma_l). \end{array} \right.$$

Therefore, we have to study the resulting problem (R) that verifies $(w, \pi_s) = (u - e - e^\infty, p_s - q_s - q_s^\infty)$, where (u, p_s) is a (possible) strong solution of (EP):

$$(R) \left\{ \begin{array}{l} \partial_t w - \nu_h \partial^2_{xx} w - \nu_v \partial^2_{zz} w + (w + e + e^\infty) \partial_x (w + e + e^\infty) \\ \;+ (w_3 + e_3 + e_3^\infty) \partial_z (w + e + e^\infty) + \partial_x \pi_s \;=\; f_2 - \partial_t e^\infty \;\text{ in } (0,T) \times \Omega, \\ \langle w \rangle \;=\; 0 \;\text{ in } (0,T) \times \omega, \quad w|_{t=0} \;=\; 0 \;\text{ in } \Omega, \\ \nu_v \partial_z w \;=\; 0 \;\text{ on } (0,T) \times \Gamma_s, \quad w \;=\; 0 \;\text{ on } (0,T) \times (\Gamma_b \cup \Gamma_l), \end{array} \right.$$

where $w_3 = -\int_{-D(x)}^z \partial_x w \, ds$ and $e_3 = -\int_{-D(x)}^z \partial_x e \, ds$ (similarly for e_3^∞).

Existence and weak estimates of approximate solutions of (R): We approximate w by w_m, the Galerkin approximations in the m-dimensional spaces $V_m = \{z_1, .., z_m\}$, where $\{z_1, .., z_m, ...\}$ is a basis of orthonormal (in H^1) eigenfunctions of the 2D hydrostatic operator A. Then, we consider the variational formulation for w_m with test functions in V_m:

$$(R)_m \begin{cases} \displaystyle\int_\Omega \partial_t w_m v_m d\Omega + \int_\Omega A w_m v_m d\Omega + \int_\Omega (w_m + e + e^\infty)\partial_x (w_m + e + e^\infty) v_m d\Omega \\ \qquad\qquad \displaystyle + \int_\Omega (w_{m3} + e_3 + e_3^\infty)\partial_z (w_m + e + e^\infty) v_m d\Omega \\ \qquad\qquad \displaystyle = \int_\Omega f_2 v_m d\Omega - \int_\Omega \partial_t e^\infty v_m d\Omega, \quad \forall v_m \in V_m, \\ w_m(0) = 0. \end{cases}$$

Obviously, one has $w_m = u_m - e - e^\infty$, where u_m is the corresponding Galerkin approximation of problem (EP). Standard weak estimates of (u_m) can be obtained in a standard way. Then, weak estimates of (w_m) and (w_{m3}) follow from weak estimates of (u_m) and weak regularity of e and e^∞. Since (w_m) is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$, $(\partial_t w_m)$ in $L^2(0,T;(V \cap H^2(\Omega))')$ and (w_{m3}) in $L^2(0,T;L^2(\Omega))$, we can extract a subsequence that converges weakly to a limit function w (and w_3), which is a weak solution of (R). Therefore, it suffices to obtain strong estimates for (w_m) (i.e. (w_m) is bounded in $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ and $(\partial_t w_m)$ in $L^2(0,T;L^2(\Omega))$) to ensure that w is also a strong solution of (R), and consequently, u is a strong solution of (EP).

Strong estimates for the approximate solutions of (R): Taking $v_m = Aw_m(t) \in V_m$ as test functions in $(R)_m$, we arrive at:

$$\begin{cases} \displaystyle \frac{1}{2}\frac{d}{dt}\|w_m\|_V^2 + \|Aw_m\|_{L^2(\Omega)}^2 = -\int_\Omega (w_m + e + e^\infty)\partial_x w_m A w_m \, d\Omega \\ \displaystyle -\int_\Omega (w_m + e + e^\infty)\partial_x e A w_m \, d\Omega - \int_\Omega (w_m + e + e^\infty)\partial_x e^\infty A w_m d\Omega \\ \displaystyle -\int_\Omega ((w_{m3} + e_3 + e_3^\infty)\partial_z w_m A w_m \, d\Omega - \int_\Omega ((w_{m3} + e_3 + e_3^\infty)\partial_z e A w_m \, d\Omega \\ \displaystyle -\int_\Omega ((w_{m3} + e_3 + e_3^\infty)\partial_z e^\infty A w_m d\Omega + \int_\Omega f_2 A w_m d\Omega - \int_\Omega \partial_t e^\infty A w_m d\Omega \equiv \sum_{i=1}^{8} I_i \end{cases} \tag{16}$$

Using Lemma 2.4 (**a**) for w_m, e and e^∞, Lemma 2.3 (**b**) for $\partial_x w_m$, $\partial_x e$ and $\partial_x e^\infty$ (because $(\partial_x w_m)|_{\Gamma_b}$, $(\partial_x e)|_{\Gamma_b}$, $(\partial_x e^\infty)|_{\Gamma_b} \neq 0$ and $(\partial_x w_m)|_{\Gamma_s}$, $(\partial_x e)|_{\Gamma_s}$, $(\partial_x e^\infty)|_{\Gamma_s} \neq 0$ in general), taking into account (10) for $w_m(t)$ and estimate:

$$\|w_m(t)\|_{H^1(\Omega)} \leq \frac{1}{\nu^{1/2}}\|w_m(t)\|_V, \tag{17}$$

we bound I_1, I_2 and I_3 in the form:

$$\begin{aligned}
I_1 &\le \left(\|w_m\|_{L^\infty_x L^2_z} + \|e\|_{L^\infty_x L^2_z} + \|e^\infty\|_{L^\infty_x L^2_z}\right) \|\partial_x w_m\|_{L^2_x L^\infty_z} \|Aw_m\|_{L^2(\Omega)} \\
&\le \frac{C}{\nu^{1/2}} \left(\|w_m\|^{1/2}_{L^2(\Omega)} \|w_m\|^{1/2}_{H^1(\Omega)} + \|e\|^{1/2}_{L^2(\Omega)} \|e\|^{1/2}_{H^1(\Omega)} + \|e^\infty\|^{1/2}_{L^2(\Omega)} \|e^\infty\|^{1/2}_{H^1(\Omega)}\right) \\
&\times \|w_m\|^{1/2}_{H^1(\Omega)} \|Aw_m\|^{3/2}_{L^2(\Omega)} \\
&\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{C}{\nu^2} \left(\|w_m\|^4_{H^1(\Omega)} \|w_m\|^2_{L^2(\Omega)}\right. \\
&+ \left.\left(\|e\|^2_{L^2(\Omega)} \|e\|^2_{H^1(\Omega)} + \|e^\infty\|^2_{L^2(\Omega)} \|e^\infty\|^2_{H^1(\Omega)}\right) \|w_m\|^2_{H^1(\Omega)}\right) \\
&\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{C}{\nu^5} \|w_m\|^6_V + \frac{C}{\nu^3} \left(d_1^4 + d_2^4\right) \|w_m\|^2_V,
\end{aligned}$$

where $d_1 = \|e\|_{L^\infty_T(H^1)}$ and $d_2 = \|e^\infty\|_{L^\infty_T(H^1)}$ whose expressions are bounded by using (5) and (7).

$$\begin{aligned}
I_2 &\le \left(\|w_m\|_{L^\infty_x L^2_z} + \|e\|_{L^\infty_x L^2_z} + \|e^\infty\|_{L^\infty_x L^2_z}\right) \|\partial_x e\|_{L^2_x L^\infty_z} \|Aw_m\|_{L^2(\Omega)} \\
&\le C \left(\|w_m\|^{1/2}_{L^2(\Omega)} \|w_m\|^{1/2}_{H^1(\Omega)} + \|e\|^{1/2}_{L^2(\Omega)} \|e\|^{1/2}_{H^1(\Omega)} + \|e^\infty\|^{1/2}_{L^2(\Omega)} \|e^\infty\|^{1/2}_{H^1(\Omega)}\right) \\
&\times \|e\|^{1/2}_{H^1(\Omega)} \|e\|^{1/2}_{H^2(\Omega)} \|Aw_m\|_{L^2(\Omega)} \\
&\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{C}{\nu} \|e\|_{H^1(\Omega)} \|e\|_{H^2(\Omega)} \|w_m\|^2_V \\
&+ C \left(\|e\|_{L^2(\Omega)} \|e\|^2_{H^1(\Omega)} + \|e^\infty\|_{L^2(\Omega)} \|e^\infty\|_{H^1(\Omega)} \|e\|_{H^1(\Omega)}\right) \|e\|_{H^2(\Omega)} \\
&\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{Cd_1}{\nu} \|e\|_{H^2(\Omega)} \|w_m\|^2_V + C \left(d_1^3 + d_1 d_2^2\right) \|e\|_{H^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
I_3 &\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{C}{\nu} \|e^\infty\|_{H^1(\Omega)} \|e^\infty\|_{H^2(\Omega)} \|w_m\|^2_V \\
&+ C \left(\|e\|_{L^2(\Omega)} \|e\|_{H^1(\Omega)} \|e^\infty\|_{H^1(\Omega)} + \|e^\infty\|_{L^2(\Omega)} \|e^\infty\|^2_{H^1(\Omega)}\right) \|e^\infty\|_{H^2(\Omega)} \\
&\le \frac{1}{16} \|Aw_m\|^2_{L^2(\Omega)} + \frac{Cd_2}{\nu} \|e^\infty\|_{H^2(\Omega)} \|w_m\|^2_V + C \left(d_1^2 d_2 + d_2^3\right) \|e^\infty\|_{H^2(\Omega)}
\end{aligned}$$

Using Lemma 2.2 for w_{m_3}, e_3 and e_3^∞, and Lemma 2.4 (**b**) for $\partial_z w_m$, $\partial_z e$ and ∂e^∞, we obtain the following estimates for I_4, I_5 and I_6:

$$
\begin{aligned}
I_4 &\leq \|w_{m3} + e_3 + e_3^\infty\|_{L_x^2 L_z^\infty} \|\partial_z w_m\|_{L_x^\infty L_z^2} \|Aw_m\|_{L^2(\Omega)} \\
&\leq \frac{C}{\nu^{1/2}} \left(\|w_m\|_{H^1(\Omega)} + \|e\|_{H^1(\Omega)} + \|e^\infty\|_{H^1(\Omega)}\right) \|w_m\|_{H^1(\Omega)}^{1/2} \|Aw_m\|_{L^2(\Omega)}^{3/2} \\
&\leq \frac{1}{14}\|Aw_m\|_{L^2(\Omega)}^2 + \frac{C}{\nu^2}\left(\|w_m\|_{H^1(\Omega)}^6 + \left(\|e\|_{H^1(\Omega)}^4 + \|e^\infty\|_{H^1(\Omega)}^4\right)\|w_m\|_{H^1(\Omega)}^2\right) \\
&\leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + \frac{C}{\nu^5}\|w_m\|_V^6 + \frac{C}{\nu^3}\left(d_1^4 + d_2^4\right)\|w_m\|_V^2
\end{aligned}
$$

$$
\begin{aligned}
I_5 &\leq \left(\|w_{m3}\|_{L_x^2 L_z^\infty} + \|e_3\|_{L_x^2 L_z^\infty} + \|e_3^\infty\|_{L_x^2 L_z^\infty}\right) \|\partial_z e\|_{L_x^\infty L_z^2} \|Aw_m\|_{L^2(\Omega)} \\
&\leq C\left(\|w_m\|_{H^1(\Omega)} + \|e\|_{H^1(\Omega)} + \|e^\infty\|_{H^1(\Omega)}\right) \|e\|_{H^1(\Omega)}^{1/2} \|e\|_{H^2(\Omega)}^{1/2} \|Aw_m\|_{L^2(\Omega)} \\
&\leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + C\left(\|e\|_{H^1(\Omega)}\|e\|_{H^2(\Omega)}\|w_m\|_{H^1(\Omega)}^2\right. \\
&+ \left.\|e\|_{H^1(\Omega)}^3\|e\|_{H^2(\Omega)} + \|e^\infty\|_{H^1(\Omega)}^2\|e\|_{H^1(\Omega)}\|e\|_{H^2(\Omega)}\right) \\
&\leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + \frac{C}{\nu}d_1\|e\|_{H^2(\Omega)}\|w_m\|_V^2 + C\left(d_1^3 + d_1 d_2^2\right)\|e\|_{H^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
I_6 &\leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + C\left\{\|e^\infty\|_{H^1(\Omega)}\|e^\infty\|_{H^2(\Omega)}\|w_m\|_{H^1(\Omega)}^2\right. \\
&+ \left.\|e^\infty\|_{H^1(\Omega)}\|e^\infty\|_{H^2(\Omega)}\|e\|_{H^1(\Omega)}^2 + \|e^\infty\|_{H^1(\Omega)}^3\|e^\infty\|_{H^2(\Omega)}\right\} \\
&\leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + \frac{C}{\nu}d_2\|e^\infty\|_{H^2(\Omega)}\|w_m\|_V^2 + C\left(d_1^2 d_2 + d_2^3\right)\|e^\infty\|_{H^2(\Omega)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_7 &\leq \|f_2\|_{L^2(\Omega)}\|Aw_m\|_{L^2(\Omega)} \leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + C\|f_2\|_{L^2(\Omega)}^2 \\
I_8 &\leq \|\partial_t e^\infty\|_{L^2(\Omega)}\|Aw_m\|_{L^2(\Omega)} \leq \frac{1}{16}\|Aw_m\|_{L^2(\Omega)}^2 + C\|\partial_t e^\infty\|_{L^2(\Omega)}^2
\end{aligned}
$$

Combining all the above bounds, we arrive at:

$$
\begin{aligned}
&\frac{d}{dt}\|w_m\|_V^2 + \|Aw_m\|^2_{L^2(\Omega)} \leq \frac{C}{\nu^5}\|w_m\|_V^6 + \frac{C}{\nu^3}\left(d_1^4+d_2^4\right)\|w_m\|_V^2 \\
&\quad + \frac{C}{\nu}\left(d_1\|e\|_{H^2(\Omega)} + d_2\|e^\infty\|_{H^2(\Omega)}\right)\|w_m\|_V^2 + C\left(\|f_2\|^2_{L^2(\Omega)} + \|\partial_t e^\infty\|^2_{L^2(\Omega)}\right) \\
&\quad + C\left(d_1^3 + d_1 d_2^2\right)\|e\|_{H^2(\Omega)} + C\left(d_1^2 d_2 + d_2^3\right)\|e^\infty\|_{H^2(\Omega)} \\
&\quad \leq \frac{C}{\nu^5}\|w_m\|_V^6 + \frac{C}{\nu^3}\left(d_1^4+d_2^4\right)\|w_m\|_V^2 + C\left(\|f_2\|^2_{L^2(\Omega)} + \|\partial_t e^\infty\|^2_{L^2(\Omega)}\right) \\
&\quad + C\left\{\left(d_1^3 + d_1 d_2^2\right)\|e\|_{H^2(\Omega)} + \left(d_1^2 d_2 + d_2^3\right)\|e^\infty\|_{H^2(\Omega)}\right\} \\
&\quad + C\nu^2\left(\|e\|^2_{H^2(\Omega)} + \|e^\infty\|^2_{H^2(\Omega)}\right)
\end{aligned} \tag{18}
$$

Using the inequality (12) for $w_m(t)$, we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\|w_m\|_V^2 + \frac{1}{C_1}\left(\nu - \frac{C}{\nu^5}\|w_m\|_V^4 - \frac{C(d_1^4+d_2^4)}{\nu^3}\right)\|w_m\|_V^2 \\
&\quad \leq C\nu^2\left(\|e\|^2_{H^2(\Omega)} + \|e^\infty\|^2_{H^2(\Omega)}\right) + C\left(\|f_2\|^2_{L^2(\Omega)} + \|\partial_t e^\infty\|^2_{L^2(\Omega)}\right) \\
&\quad + C\left\{\left(d_1^3 + d_1 d_2^2\right)\|e\|_{H^2(\Omega)} + \left(d_1^2 d_2 + d_2^3\right)\|e^\infty\|_{H^2(\Omega)}\right\}
\end{aligned} \tag{19}
$$

As d_1^2 and d_2^2 have the following bounds:

$$
\left\{
\begin{array}{rcl}
d_1^2 & \leq & \dfrac{C}{\nu}\left\{\|f_1\|^2_{L^2_T(L^2)} + \|\tau_1\|^2_{L^2_T(H_0^{1/2+\varepsilon})}\right\} + \dfrac{C}{\nu^3}\|\partial_t\tau_1\|^2_{L^2_T(H^{-1/2})} \\
& + & \dfrac{\bar\nu}{\nu}\left\{\|u_0\|^2_{H^1} + \dfrac{C}{\nu^2}\left(\|\tau_1(0)\|^2_{H^{-1/2}} + \|\tau_2(0)\|^2_{H^{-1/2}}\right)\right\}, \\
d_2^2 & \leq & \dfrac{C}{\nu^2}\|\tau_2\|^2_{L^\infty_T(H^{-1/2})} \leq \dfrac{C}{\nu^2}\|\tau_2\|^2_{L^\infty_T(H_0^{1/2+\varepsilon})}
\end{array}
\right.
$$

(see (7) and the inequality for e_2 in the proof of Theorem 1.10), hypothesis (H) allows us to bound:

$$
\begin{array}{rcl}
d_1,\, d_2 & < & Cc\,\nu, \\
\|e^\infty\|_{L^\infty_T(H^2)} & \leq & \dfrac{C}{\nu}\|\tau_2\|_{L^\infty_T(H_0^{1/2+\varepsilon})} < Cc\,\nu, \\
\|f_2\|^2_{L^\infty_T(H^2)} & < & c\,\nu^2,
\end{array}
$$

$$\|\partial_t e^\infty\|^2_{L^\infty_T(L^2)} \;\leq\; \frac{C}{\nu}\|\partial_t \tau_2\|^2_{L^\infty_T(H^{-1/2})} \;<\; Cc\,\nu^2,$$

where c is the small constant that appears in (H) (depending only on the size of the data). Therefore, we can rewrite (19) as

$$\begin{aligned} \frac{d}{dt}\|w_m\|^2_V \;&+\; \frac{\nu}{C_1}\left(1 - \frac{C}{\nu^6}\|w_m\|^4_V - Cc^4\right)\|w_m\|^2_V \\ &\leq\; C\left(\phi(c)\nu^4 + \nu^2\|e(t)\|^2_{H^2(\Omega)}\right), \end{aligned} \tag{20}$$

where $\phi(c) = c^2(1 + c^2 + c^4)$.

Then we choose a small constant $\gamma > 0$ satisfying the following two conditions:

$$\begin{cases} C\,(\gamma^4 + c^4) \;<\; \dfrac{1}{2}, \\ \quad\;\; C\,\phi(c) \;<\; \gamma^2 \end{cases} \tag{21}$$

(this choice is possible since c is small enough).

Then we can conclude that $\|w_m(t)\|_V \leq \gamma\nu^{3/2}$, $\forall t \in [0,T]$ (here, we assume $T < +\infty$ for simplicity). To prove this statement, we argue by contradiction. Suppose there exists some instant in $(0,T)$ where the bound $\gamma\nu^{3/2}$ is reached. Let t^* be the smallest of these instants, i.e. $\|w_m(t)\|_V < \gamma\nu^{3/2}$, $\forall t \in [0, t^*)$ and $\|w_m(t^*)\|_V = \gamma\nu^{3/2}$. Then, $\forall t \in [0, t^*]$, from (20) and $(21)_1$, we have:

$$\frac{d}{dt}\|w_m(t)\|^2_V + \frac{\nu}{2C_1}\|w_m(t)\|^2_V \leq C\left(\phi(c)\nu^4 + \nu^2\|e(t)\|^2_{H^2(\Omega)}\right).$$

If we multiply by $exp\left(\dfrac{\nu}{2C_1}t\right)$ and integrate in time (recall that $w_m(0) = 0$), one has:

$$\|w_m(t)\|^2_V \leq C\left(\phi(c)\nu^3 + \nu^2\int_0^t \|e(s)\|^2_{H^2(\Omega)}ds\right) \quad \forall t \in [0, t^*]$$

Now, using (H) and estimate (8), we can bound $\|e\|^2_{L^2_T(H^2)}$ (recall that there is no force in $L^\infty_T(L^2)$ in this case) as follows:

$$\|e\|^2_{L^2_T(H^2)} < Cc^2\nu,$$

and we get:

$$\|w_m(t)\|^2_V < C\phi(c)\nu^3, \quad \forall t \in [0, t^*],$$

hence we get a contradiction, taking into account $(21)_2$. Then, one has that (w_m) is bounded in $L^\infty(0,T;V)$. The estimates of (w_m) in $L^2(0,T;H^2(\Omega))$ and of $(\partial_t w_m)$ in $L^2(0,T;L^2(\Omega))$ can be deduced from this last estimate (see [9]).

Therefore, we have demonstrated the existence of a strong solution v of (EP) in $(0,T)$, where T can be equal to $+\infty$, under the smallness hypothesis (H). ■

Proof of Theorem 1.2 **in the** $2D$ **case:**
If we call $y(t) = \|w_m(t)\|_V^2$, dividing (18) (recall that there is no force in $L_T^\infty(L^2)$) by $(1+y(t))^3$, we obtain:

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}[(1+y(t))^{-2}] + (1+y(t))^{-3}\|Aw_m(t)\|^2_{L^2(\Omega)} \\ \leq C\left(\frac{1}{\nu^5} + \frac{d_1^4}{\nu^3} + d_1^3\|e(t)\|_{H^2(\Omega)} + \nu^2\|e(t)\|^2_{H^2(\Omega)}\right). \end{aligned} \tag{22}$$

Integrating (22) on $(0,t)$ and taking into account that $\|e\|^2_{H^2(\Omega)} \in L^1(0,T)$, we get the following inequality:

$$\begin{aligned} \frac{1}{2(1+y(0))^2} &+ \int_0^t \frac{\|Aw_m(s)\|^2_{L^2(\Omega)}}{(1+y(s))^3}ds \leq \\ &\leq \frac{1}{2(1+y(t))^2} + \frac{C}{\nu^3}\left(\frac{1}{\nu^2} + d_1^4\right)t + Cd_1^3\|e\|_{L_t^2(H^2)}\sqrt{t} + C\nu^2\|e\|^2_{L_t^2(H^2)}. \end{aligned} \tag{23}$$

Therefore, a sufficient condition for having $y(t)$ bounded independently of m is:

$$\frac{C}{\nu^3}\left(\frac{1}{\nu^2} + d_1^4\right)t + Cd_1^3\|e\|_{L_t^2(H^2)}\sqrt{t} + C\nu^2\|e\|^2_{L_t^2(H^2)} < \frac{1}{2(1+y(0))^2}. \tag{24}$$

Indeed, if we have (24), then:

$$0 < \frac{1}{2(1+y(0))^2} - \left(\frac{C}{\nu^3}\left(\frac{1}{\nu^2} + d_1^4\right)t + Cd_1^3\|e\|_{L_t^2(H^2)}\sqrt{t} + C\nu^2\|e\|^2_{L_t^2(H^2)}\right) \leq \frac{1}{2(1+y(t))^2}.$$

Now, we pull out the $(1+y(t))$ factor to obtain:

$$\begin{aligned} 1+y(t) \leq (1+y(0)) \quad &\Big(1 - 2(1+y(0))^2 \\ &\times \left(\frac{C}{\nu^3}\left(\frac{1}{\nu^2} + d_1^4\right)t + Cd_1^3\|e\|_{L_t^2(H^2)}\sqrt{t} + C\nu^2\|e\|^2_{L_t^2(H^2)}\right)\Big)^{-1/2}. \end{aligned}$$

Coming back to (24), since $\|e\|_{L^2(0,t;H^2(\Omega))}$ is a continuous function with respect to t (that vanishes for $t=0$), we can always find a T_* small enough to verify condition (24) for all $t \in [0, T_*]$. ■

Remark 2.5 *In the context of Theorem* 1.2, *one can obtain Hausdorff estimates for the singular times (times of blow up in* $H^1(\Omega)$*-norm), imposing more regularity in* *time for F, τ and $\partial_t\tau$. For instance, if $F \in L^\infty(0,T;L^2(\Omega))$, $\tau \in L^\infty(0,T;H_0^{1/2+\varepsilon}(\Gamma_s))$ and $\partial_t\tau \in L^\infty(0,T;H^{-1/2}(\Gamma_s))$, then we can estimate the Hausdorff dimension by 1/2 (see [5]). Finally, for intermediate regularities L^q in time one obtain a dimension $\leq d(q) = \dfrac{q}{2(q-1)}$ (see [8]).*

3 The 3D case.

In this section, we give the necessary changes to handle the nonlinear terms in the 3D case. We start by some 3D anisotropic estimates.

3.1 Some 3D anisotropic spaces and related estimates.

Definition 3.1 *Given $p, q \in [1, +\infty]$, it will be said that a function $\vec{u}$ belongs to $L^q_z L^p_{\vec{x}}(\Omega)$ if:*

$$\vec{u}(\cdot, z) \in L^q(\omega_z) \quad \textit{and} \quad \|\vec{u}(\cdot, z)\|_{L^q(\omega_z)} \in L^p(-D_{\max}, 0),$$

and its norm is given by the expression:

$$\Big\| \|\vec{u}(\cdot, z)\|_{L^q(\omega_z)} \Big\|_{L^p(-D_{\max},0)}$$

Remark 3.1 *The most useful norms that we will use in the 3D case will be:*

$$\begin{aligned} \|\vec{u}\|_{L^2_z L^4_{\vec{x}}(\Omega)} &= \left(\int_{-D_{\max}}^0 \|\vec{u}(\cdot, z)\|^2_{L^4(\omega_z)} dz \right)^{1/2} \\ \|\vec{u}\|_{L^\infty_z L^4_{\vec{x}}(\Omega)} &= \sup_{z \in (-D_{\max}, 0)} \|\vec{u}(\cdot, z)\|_{L^4(\omega_z)}, \end{aligned}$$

Lemma 3.2 *Let $\vec{v} : \Omega \to \mathbb{R}^2$ be a function such that $\nabla_H \cdot \vec{v} \in L^2(\Omega)$ and define $v_3(\vec{x}, z) = -\int_{-D(\vec{x})}^z \nabla_H \cdot \vec{v}(\vec{x}, s) ds$, then one verifies:*

$$\|v_3\|_{L^\infty_z L^2_{\vec{x}}} \leq D^{1/2}_{\max} \|\nabla_H \cdot \vec{v}\|_{L^2(\Omega)}.$$

Proof: Using definition of v_3, we have:

$$|v_3(\vec{x}, z)|^2 = \left| \int_{-D(\vec{x})}^z \nabla_H \cdot \vec{v}(\vec{x}, s) ds \right|^2 \leq \left(\int_{-D(\vec{x})}^z |\nabla_H \cdot \vec{v}(\vec{x}, z)|^2 ds \right) D_{\max}$$

Therefore integrating in x, we get

$$\int_{\omega_z} |v_3(\vec{x}, z)|^2 \, d\vec{x} \leq D_{\max} \int_{\omega_z} \int_{-D(\vec{x})}^z |\nabla_H \cdot \vec{v}(\vec{x}, s)|^2 ds d\vec{x} \leq D_{\max} \|\nabla_H \cdot \vec{v}\|^2_{L^2(\Omega)}.$$

Taking the essential supremum in $z \in (-D_{\max}, 0)$, we conclude the proof. ■

Lemma 3.3 "Interpolation inequalities"

(**a**) *Let $v \in L^2(\Omega)$ be a function such that $\partial_z v \in L^2(\Omega)$ and $(vn_z)|_{\Gamma_b} = 0$. Then, $v \in L^\infty_z L^2_{\vec{x}}(\Omega)$ and verifies the estimate:*

$$\|v\|^2_{L^\infty_z L^2_{\vec{x}}} \leq 2\, \|v\|_{L^2(\Omega)} \|\partial_z v\|_{L^2(\Omega)}. \tag{25}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^\infty_z L^2_{\vec{x}}(\Omega)$, and there exists $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^\infty_z L^2_{\vec{x}}} \leq C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega). \tag{26}$$

(**b**) *Let $v \in L^2(\Omega)$ be a function such that $\nabla_H v \in L^2(\Omega)^2$ and $(vn_{\vec{x}})|_{\Gamma_b \cup \Gamma_l} = 0$. Then, $v \in L^2_z L^4_{\vec{x}}(\Omega)$ and verifies the estimate:*

$$\|v\|^2_{L^2_z L^4_{\vec{x}}} \leq 4\, \|v\|_{L^2(\Omega)} \|\nabla_H v\|_{L^2(\Omega)}. \tag{27}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^2_z L^4_{\vec{x}}$, and there exists $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^2_z L^4_{\vec{x}}} \leq C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \tag{28}$$

Proof:
(**a**) If $(vn_z)|_{\Gamma_b} = 0$, then $v(\vec{x}, -D(\vec{x})) = 0$. So that,

$$v(\vec{x}, z)^2 = 2 \int_{-D(\vec{x})}^{z} v(\vec{x}, s) \partial_z v(\vec{x}, s) ds.$$

Therefore, integrating in ω_z, we arrive at:

$$\begin{aligned} \int_{\omega_z} |v(\vec{x}, z)|^2 dx &\leq 2 \int_{\omega_z} \int_{-D(x)}^{z} |v(\vec{x}, z)||\partial_z v(\vec{x}, z)| ds d\vec{x} \\ &\leq 2 \int_\Omega |v||\partial_z v| d\Omega \leq 2\, \|v\|_{L^2(\Omega)} \|\partial_z v\|_{L^2(\Omega)}. \end{aligned}$$

Hence, taking essential supremum in $z \in (-D_{\max}, 0)$, we arrive at (25).

Remark 3.2 *In the previous argument, hypothesis $(vn_z)|_{\Gamma_b} = 0$ can be changed by $v|_{\Gamma_s} = 0$.*

Now, if we consider the case $v \in H^1(\Omega)$, without hypothesis on its trace, using the Extension Theorem that appears in Lemma 2.3 (**b**) and the previous argument for the extended function, we can prove (26).

(**b**) Let v be a function such that $vn_{\vec{x}}|_{\Gamma_b\cup\Gamma_l} = 0$. Then, $v(\cdot,z) \in H^1_0(\omega_z)$ for a.e. z in $(-D_{\max}, 0)$ and we can easily obtain:

$$\|v(\cdot,z)\|^2_{L^4(\omega_z)} \leq 4\,\|v(\cdot,z)\|_{L^2(\omega_z)}\|\nabla_H v(\cdot,z)\|_{L^2(\omega_z)}. \tag{29}$$

Indeed, to prove (29) it suffices to prove the following inequality (this is an adaptation of a result in [13]):

$$\|\varphi\|^2_{L^2(\omega)} \leq \|\partial_x\varphi\|_{L^1(\omega)}\|\partial_y\varphi\|_{L^1(\omega)}, \quad \forall\,\varphi \in W^{1,1}_0(\omega), \tag{30}$$

and apply this inequality to $\varphi = |v|^2$. To prove (30), we express φ as:

$$\varphi(x,y) = \varphi(x,y_0) + \int_{y_0}^{y} \partial_y\varphi(x,t)dt$$

and

$$\varphi(x,y) = \varphi(x_0,y) + \int_{x_0}^{x} \partial_x\varphi(s,y)ds$$

where x_0 (resp. y_0) is one point of the intersection of $\omega_y = \{s/(s,y) \in \overline{\omega}\}$ (resp. $\omega_x = \{t/(x,t) \in \overline{\omega}\}$) and $\partial\omega$. Since $\varphi = 0$ on $\partial\omega$, multiplying the two above equalities, we have:

$$|\varphi(x,y)|^2 \leq \left(\int_{\omega_x} |\partial_y\varphi(x,t)|dt\right)\left(\int_{\omega_y} |\partial_x\varphi(s,y)|ds\right).$$

Integrating on $(x,y) \in \omega$, we get (30).
Integrating (29) on $z \in (-D_{\max}, 0)$, we get

$$\begin{aligned}
\|v\|^2_{L^2_z L^4_{\vec{x}}} &\leq 4\int_{-D_{\max}}^{0} \|v(\cdot,z)\|_{L^2(\omega_z)}\|\nabla_H v(\cdot,z)\|_{L^2(\omega_z)}dz \\
&\leq 4\left(\int_{-D_{\max}}^{0} \|v(\cdot,z)\|^2_{L^2(\omega_z)}dz\right)^{1/2}\left(\int_{-D_{\max}}^{0} \|\nabla_H v(\cdot,z)\|^2_{L^2(\omega_z)}dz\right)^{1/2} \\
&= 4\,\|v\|_{L^2(\Omega)}\|\nabla_H v\|_{L^2(\Omega)}.
\end{aligned}$$

For a more general function $v \in H^1(\Omega)$, we use Extension Operator as in Lemma 2.3 (**b**), and extends $v \in H^1(\Omega)$ to a function $Ev \in H^1(\tilde{\Omega})$ with $Ev|_{\Gamma_b\cup\Gamma_l} = 0$. Then, from (27) we obtain:

$$\|Ev\|^2_{L^2_z L^4_{\vec{x}}} \leq C(\Omega)\|Ev\|_{L^2(\tilde{\Omega})}\|Ev\|_{H^1(\tilde{\Omega})}.$$

Now, applying properties of Extension Operator (see proof of Lemma 2.3), (28) holds.

■

Lemma 3.4 "New estimate for v_3". *Let $\vec{v} \in L^2(\Omega)^2$ be a function such that $\nabla_H \cdot \vec{v} \in H^1(\Omega)$. Then, if we consider v_3 defined in function of $\nabla_H \cdot \vec{v}$ as in Lemma 3.2, we have that $v_3 \in L_z^\infty L_{\vec{x}}^4(\Omega)$ and verifies the estimate:*

$$\|v_3\|_{L_z^\infty L_{\vec{x}}^4} \leq C(\Omega) \|\nabla_H \cdot \vec{v}\|_{L^2(\Omega)}^{1/2} \|\nabla_H \cdot \vec{v}\|_{H^1(\Omega)}^{1/2}$$

Proof: We will use that if $p > q$,

$$\|u\|_{L_{\vec{x}}^p L_z^q} \leq \|u\|_{L_z^q L_{\vec{x}}^p}.$$

Then, from Lemma 3.3 (**b**),

$$\|\nabla_H \cdot \vec{v}\|_{L_{\vec{x}}^4 L_z^2}^2 \leq \|\nabla_H \cdot \vec{v}\|_{L_z^2 L_{\vec{x}}^4}^2 \leq C(\Omega) \|\nabla_H \cdot \vec{v}\|_{L^2(\Omega)}^2 \|\nabla_H \cdot \vec{v}\|_{H^1(\Omega)}^2,$$

and as $v_3(\vec{x}, z) = -\int_{-D(\vec{x})}^z \nabla_H \cdot \vec{v}(\vec{x}, s) ds$,

$$\|v_3(\vec{x}, \cdot)\|_{L^\infty(-D(\vec{x}),0)} \equiv \sup_{z \in (-D_{\max}, 0)} |v_3(\vec{x}, z)| \leq \int_{-D(\vec{x})}^0 |\nabla_H \cdot \vec{v}(\vec{x}, z)| dz$$

$$\leq D_{\max}^{1/2} \left(\int_{-D(\vec{x})}^0 |\nabla_H \cdot \vec{v}(\vec{x}, z)|^2 dz \right)^{1/2} \equiv D_{\max}^{1/2} \|\nabla_H \cdot \vec{v}(\vec{x}, \cdot)\|_{L^2(-D(\vec{x}),0)}.$$

So we can easily finish the proof, using that $\|v_3\|_{L_z^\infty L_{\vec{x}}^4} \leq \|v_3\|_{L_{\vec{x}}^4 L_z^\infty}$. ∎

3.2 Global strong solution for small data.

Proof of Theorem 1.1 in the 3D case: For simplicity, we take $\vec{\tau}_2 = \vec{0}$, and then $\vec{e}^\infty = \vec{0}$.
As in the 2D case, it suffices to obtain the strong estimate of (w_m) in $L^\infty(0, T; H^1(\Omega)^2)$, where w_m is the approximate solution of (R) in the 3D case. In all this Section, we drop the subindex m. Starting from the variational formulation $(R)_m$ and taking $\vec{v} = A\vec{w}$ as a test function, we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\vec{w}\|_V^2 + \|A\vec{w}\|_{L^2(\Omega)}^2 = -\int_\Omega ((\vec{w} + \vec{e}) \cdot \nabla_H)\vec{w} \cdot A\vec{w} \, d\Omega$$

$$- \int_\Omega ((\vec{w} + \vec{e}) \cdot \nabla_H)\vec{e} \cdot A\vec{w} \, d\Omega - \int_\Omega (w_3 + e_3) \partial_z \vec{w} \cdot A\vec{w} \, d\Omega$$

$$- \int_\Omega (w_3 + e_3) \partial_z \vec{e} \cdot A\vec{w} \, d\Omega + \int_\Omega \vec{f}_2 \cdot A\vec{w} d\Omega \equiv \sum_{i=1}^5 I_i.$$

Using interpolation inequality $\|\vec{v}\|_{L^4(\Omega)} \le C\|\vec{v}\|^{1/4}_{L^2(\Omega)}\|\vec{v}\|^{3/4}_{H^1(\Omega)}$, (10) and (17), we estimate I_1 and I_2 in the form:

$$\begin{aligned}
I_1 &\le \|A\vec{w}\|_{L^2(\Omega)}\left(\|\vec{w}\|_{L^4(\Omega)} + \|\vec{e}\|_{L^4(\Omega)}\right)\|\nabla\vec{w}\|_{L^4(\Omega)} \\
&\le C\|A\vec{w}\|_{L^2(\Omega)}\left(\|\nabla\vec{w}\|^{3/4}_{L^2(\Omega)}\|\vec{w}\|^{1/4}_{L^2(\Omega)}\right. \\
&+ \left.\|\vec{e}\|^{3/4}_{H^1(\Omega)}\|\vec{e}\|^{1/4}_{L^2(\Omega)}\right)\|\nabla_H\vec{w}\|^{3/4}_{H^1(\Omega)}\|\nabla_H\vec{w}\|^{1/4}_{L^2(\Omega)} \\
&\le \frac{C}{\nu^{3/4}}\|A\vec{w}\|^{7/4}_{L^2(\Omega)}\left(\|\vec{w}\|_{H^1(\Omega)} + \|\vec{e}\|_{H^1(\Omega)}\right)\|\vec{w}\|^{1/4}_{H^1(\Omega)} \\
&\le \frac{C}{\nu^{7/8}}\|A\vec{w}\|^{7/4}_{L^2(\Omega)}\left(\frac{1}{\nu^{1/2}}\|\vec{w}\|_V + \|\vec{e}\|_{H^1(\Omega)}\right)\|\vec{w}\|^{1/4}_V \\
&\le \frac{1}{10}\|A\vec{w}\|^2_{L^2(\Omega)} + C\left(\frac{1}{\nu^{11}}\|\vec{w}\|^8_V + \frac{1}{\nu^7}\|\vec{e}\|^8_{H^1(\Omega)}\right)\|\vec{w}\|^2_V
\end{aligned}$$

In an analogous way, we can obtain estimates for I_2:

$$\begin{aligned}
I_2 &\le \|A\vec{w}\|_{L^2(\Omega)}\left(\|\vec{w}\|_{L^4(\Omega)} + \|\vec{e}\|_{L^4(\Omega)}\right)\|\nabla_H\vec{e}\|_{L^4(\Omega)} \\
&\le C\|A\vec{w}\|_{L^2(\Omega)}\left(\|\nabla\vec{w}\|^{3/4}_{L^2(\Omega)}\|\vec{w}\|^{1/4}_{L^2(\Omega)} + \|\vec{e}\|^{3/4}_{H^1(\Omega)}\|\vec{e}\|^{1/4}_{L^2(\Omega)}\right)\|\nabla_H\vec{e}\|^{1/4}_{L^2(\Omega)}\|\nabla_H\vec{e}\|^{3/4}_{H^1(\Omega)} \\
&\le \frac{1}{10}\|A\vec{w}\|^2_{L^2(\Omega)} + \frac{C}{\nu}\|\vec{e}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)}\|\vec{w}\|^2_V + C\|\vec{e}\|^{5/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)}
\end{aligned}$$

Using Lemma 3.4 for w_3 and e_3 and Lemma 3.3 (estimate (28)) for $\partial_z\vec{w}$ and $\partial_z\vec{e}$, we estimate I_3 and I_4. To be more transparent, we separate $I_3 = J_1 + J_2$, where

$$J_1 = -\int_\Omega (w_3\cdot\partial_z)\vec{w}\cdot A\vec{w}d\Omega,$$

and

$$J_2 = -\int_\Omega (e_3\cdot\partial_z)\vec{w}\cdot A\vec{w}d\Omega.$$

$$\begin{aligned}
|J_1| &\le \|A\vec{w}\|_{L^2(\Omega)}\|w_3\|_{L^\infty_z L^4_{\vec{x}}}\|\partial_z\vec{w}\|_{L^2_z L^4_{\vec{x}}} \\
&\le \frac{C}{\nu}\|A\vec{w}\|^2_{L^2(\Omega)}\|\vec{w}\|_{H^1(\Omega)} \le \frac{C}{\nu^{3/2}}\|A\vec{w}\|^2_{L^2(\Omega)}\|\vec{w}\|_V.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
|J_2| &\le C\|A\vec{w}\|_{L^2(\Omega)}\|\vec{w}\|^{1/2}_{H^2(\Omega)}\|\vec{w}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{1/2}_{H^2(\Omega)}\|\vec{e}\|^{1/2}_{H^1(\Omega)} \\
&\le \frac{1}{10}\|A\vec{w}\|^2_{L^2(\Omega)} + \frac{C}{\nu^3}\|\vec{w}\|^2_V\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)}.
\end{aligned}$$

And similar estimates for I_4:

$$\begin{aligned} I_4 &\leq \left(\|w_3\|_{L^\infty_z L^4_{\vec{x}}} + \|e_3\|_{L^\infty_z L^4_{\vec{x}}}\right) \|\partial_z \vec{e}\|_{L^2_z L^4_{\vec{x}}} \|A\vec{w}\|_{L^2(\Omega)} \\ &\leq C\left(\|\vec{w}\|^{1/2}_{H^1(\Omega)}\|\vec{w}\|^{1/2}_{H^2(\Omega)} + \|\vec{e}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{1/2}_{H^2(\Omega)}\right)\|\vec{e}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{1/2}_{H^2(\Omega)}\|A\vec{w}\|_{L^2(\Omega)} \\ &\leq \frac{1}{10}\|A\vec{w}\|^2_{L^2(\Omega)} + \frac{C}{\nu^3}\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)}\|\vec{w}\|^2_V + C\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)}. \end{aligned}$$

Finally for I_5 we get:

$$I_5 \leq \frac{1}{10}\|A\vec{w}\|^2_{L^2(\Omega)} + C\|\vec{f}_2\|^2_{L^2(\Omega)}.$$

Putting together all these bounds, we arrive at:

$$\begin{aligned} \frac{d}{dt}\|\vec{w}\|^2_V &+ \|A\vec{w}\|^2_{L^2(\Omega)} \leq \frac{C}{\nu^{3/2}}\|A\vec{w}\|^2_{L^2(\Omega)}\|\vec{w}\|_V + \frac{C}{\nu^{11}}\|\vec{w}\|^{10}_V \\ &+ \frac{C}{\nu}\left(\frac{1}{\nu^6}\|\vec{e}\|^8_{H^1(\Omega)} + \|\vec{e}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)} + \frac{1}{\nu^2}\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)}\right)\|\vec{w}\|^2_V \\ &+ C\left(\|\vec{e}\|^{5/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)} + \|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)} + \|\vec{f}_2\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{31}$$

Now, imposing $\|\vec{w}(t)\|_V < \gamma\nu^{3/2}$, we can control the term $C\|A\vec{w}\|^2_{L^2(\Omega)}\|\vec{w}\|_V$. Afterwards, using hypothesis (H) on the data, we follow the same kind of reasoning as in the $2D$ case, hence we can deduce that $\|\vec{w}(t)\|_V < \gamma\nu^{3/2}$, $\forall t \in [0, T]$ (for some small constant $\gamma > 0$) and finish the proof. ■

3.3 Local strong solution for any data.

Proof of Theorem 1.2 in the 3D case:
Arguing as for the proof of existence of global strong solution for small data, we arrive at (31). As $\vec{w}_m(0) = \vec{0}$ and $\vec{w}_m$ is continuous in time with values in H^1, we can choose a time T^1_m such that:

$$\|\vec{w}_m(t)\|_V \leq \frac{\nu^{3/2}}{2C}, \quad \forall t \in [0, T^1_m].$$

Then, we want to show that T^1_m can be chosen such that T^1_m is bounded from below independently from m. Integrating the expression (31) between 0 and t, and using that $t \in [0, T_m^1]$, we obtain

$$\begin{aligned} \|\vec{w}_m(t)\|_V^2 &+ \int_0^t \|A\vec{w}_m(s)\|_{L^2(\Omega)}^2 ds \\ &\le C\left\{\nu^4 + \frac{d_1^8}{\nu^4} + d_1^4 + d_1^2\nu^2 + \nu^2\|\vec{e}\|_{H^2(\Omega)}^2\right\}. \end{aligned}$$

Next, we can take a time T^2 such that:

$$C\left\{\left(\nu^4 + \frac{d_1^8}{\nu^4} + d_1^4 + d_1^2\nu^2\right)T^2 + \nu^2\|\vec{e}\|_{L^2_{T^2}(H^2(\Omega))}^2\right\} < \frac{\nu^3}{4C^2}.$$

Hence, we see that for all m, T_m^1 can be chosen to be equal to T^2. The existence proof can then be carried out very easily. ∎

Remark 3.3 *The argument of Hausdorff estimates for singular times are not possible in the 3D case, due to the term* $\frac{C}{\nu^{3/2}}\|A\vec{w}\|_{L^2(\Omega)}^2\|\vec{w}\|_V$ *on the right hand side of* (31) *(which did not appear in* (18)*).*

4 Uniqueness of weak/strong solution of hydrostatic problem.

In this Section, we are going to prove that any weak solution $\vec{u}$ coincides with a more regular solution $\underline{\vec{u}}$, whenever this regular solution exists. We will do our study in the 3D case and then we will only state the results for the 2D case.

Proof of Theorem 1.3**:** Using definition 1.5 for almost every $t \in (0, T)$ the **energy inequality** (4) is verified. Observe that, starting from the variational formulation of $\vec{u}$ (Definition 1.5), one has that $\partial_t\vec{u} \in L^{4/3}(0, T; W')$, where $W = \{\vec{\psi} \in V;\ \partial_z\vec{\psi} \in H^1(\Omega)^2\}$. Indeed, the more difficult term to handle is $\int_\Omega u_3\partial_z\vec{\psi}\cdot\vec{u}\,d\Omega$, that can be controlled using that $u_3 \in L^2(0, T; L_z^\infty L_{\vec{x}}^2)$ (Lemma 3.2) and $\vec{u} \in L^4(0, T; L_z^2 L_{\vec{x}}^4)$ (Lemma 3.3). Therefore, $\vec{\psi} = \underline{\vec{u}}$ can be taken as test function in the variational formulation of $\vec{u}$.

Following the same arguments that in [9], one has:

$$\left\{\begin{array}{l} \frac{1}{2}\|(\vec{u} - \underline{\vec{u}})(t)\|_{L^2(\Omega)}^2 + \nu\int_0^t \|\vec{u} - \underline{\vec{u}}\|_{H^1(\Omega)}^2 ds \\ \\ \le -\int_0^t \int_\Omega ((\vec{u} - \underline{\vec{u}})\cdot\nabla_H\underline{\vec{u}} + (u_3 - \underline{u}_3)\partial_z\underline{\vec{u}})\cdot\vec{u}\,d\Omega\,ds \end{array}\right. \tag{32}$$

Then, we have to bound the second member (32). Using the free divergence condition for $\underline{\vec{u}}$,

$$-\int_0^t \int_\Omega ((\vec{u}-\underline{\vec{u}})\cdot\nabla_H\underline{\vec{u}} + (u_3-\underline{u}_3)\partial_z\underline{\vec{u}})\cdot\vec{u}\, d\Omega\, ds$$

$$= -\int_0^t \int_\Omega ((\vec{u}-\underline{\vec{u}})\cdot\nabla_H\underline{\vec{u}} + (u_3-\underline{u}_3)\partial_z\underline{\vec{u}})\cdot(\vec{u}-\underline{\vec{u}})\, d\Omega\, ds \equiv I_1 + I_2.$$

Using bounds from Lemmas of Section 3.1, we get the following estimates:

$$\begin{aligned} I_1 &\le \int_0^t \int_\Omega |\vec{u}-\underline{\vec{u}}|^2 |\nabla_H\underline{\vec{u}}|\, d\Omega\, ds \le \int_0^t \|\vec{u}-\underline{\vec{u}}\|^2_{L^2_zL^4_{\vec{x}}} \|\nabla_H\underline{\vec{u}}\|_{L^\infty_z L^2_{\vec{x}}}\, ds \\ &\le C\int_0^t \|\vec{u}-\underline{\vec{u}}\|_{L^2(\Omega)} \|\vec{u}-\underline{\vec{u}}\|_{H^1(\Omega)} \|\nabla_H\underline{\vec{u}}\|_{L^\infty_z L^2_{\vec{x}}}\, ds \\ &\le \frac{\nu}{4}\int_0^t \|\vec{u}-\underline{\vec{u}}\|^2_{H^1(\Omega)}\, ds + \frac{C}{\nu}\int_0^t \|\nabla_H\underline{\vec{u}}\|^2_{L^\infty_z L^2_{\vec{x}}} \|\vec{u}-\underline{\vec{u}}\|^2_{L^2(\Omega)}\, ds. \end{aligned}$$

$$\begin{aligned} I_2 &\le \int_0^t \int_\Omega |u_3-\underline{u}_3||\partial_z\underline{\vec{u}}||\vec{u}-\underline{\vec{u}}|\, d\Omega\, ds \\ &\le C(\Omega)\int_0^t \|\vec{u}-\underline{\vec{u}}\|^{3/2}_{H^1(\Omega)} \|\vec{u}-\underline{\vec{u}}\|^{1/2}_{L^2(\Omega)} \|\partial_z\underline{\vec{u}}\|^{1/2}_{L^2(\Omega)} \|\partial_z\underline{\vec{u}}\|^{1/2}_{H^1(\Omega)} ds \\ &\le \frac{\nu}{4}\int_0^t \|\vec{u}-\underline{\vec{u}}\|^2_{H^1(\Omega)}\, ds + \frac{C}{\nu}\int_0^t \|\partial_z\underline{\vec{u}}\|^2_{L^2(\Omega)} \|\partial_z\underline{\vec{u}}\|^2_{H^1(\Omega)} \|\vec{u}-\underline{\vec{u}}\|^2_{L^2(\Omega)}\, ds. \end{aligned}$$

Arranging the constants from all the previous bounds, inequality (32) becomes:

$$\left\{ \begin{aligned} &\frac{1}{2}\|(\vec{u}-\underline{\vec{u}})(t)\|^2_{L^2(\Omega)} + \nu\int_0^t \|(\vec{u}-\underline{\vec{u}})(s)\|^2_{H^1(\Omega)} ds \\ &\le \frac{C}{\nu}\int_0^t \left\{ \|\nabla_H\underline{\vec{u}}(s)\|^2_{L^\infty_z L^2_{\vec{x}}} + \|\partial_z\underline{\vec{u}}(s)\|^2_{L^2(\Omega)} \|\partial_z\underline{\vec{u}}(s)\|^2_{H^1(\Omega)} \right\} \|(\vec{u}-\underline{\vec{u}})(s)\|^2_{L^2(\Omega)} ds \end{aligned} \right. \tag{33}$$

Therefore, if $\underline{\vec{u}}$ verifies, besides the weak regularity, the additional regularity (3), then the inequality (33) yields the uniqueness of $\underline{\vec{u}}$ by using a Gronwall Lemma ∎

Remark 4.1 *Using Lemma* 3.3 (**a**) *for* $\nabla_H\underline{\vec{u}} \in H^1(\Omega)$, *we obtain the following estimate:*

$$\|\nabla_H\underline{\vec{u}}\|^2_{L^\infty_z L^2_{\vec{x}}} \le C\|\nabla_H\underline{\vec{u}}\|_{L^2(\Omega)} \|\nabla_H\underline{\vec{u}}\|_{H^1(\Omega)}.$$

Then, we can conclude that if there exists a strong solution as constructed on its time of existence, every weak solution of (EP) *coincides with this strong solution. In [9], using isotropic estimates, we obtained uniqueness imposing* $\partial_z \underline{\vec{u}} \in L^8(0,T;L^4(\Omega)^2)$, *that is a regularity that, in general, a strong solution does not verify.*

Remark 4.2 (Constant depth) *If the depth is constant,* $\nabla_H \underline{\vec{u}}|_{\Gamma_b} = \vec{0}$, *we can use the first inequality in Lemma* 3.3 (**a**) *for* $\nabla_H \underline{\vec{u}}$, *obtaining:*

$$\|\nabla_H \underline{\vec{u}}\|^2_{L^\infty_z L^2_{\vec{x}}} \leq C \|\nabla_H \underline{\vec{u}}\|_{L^2(\Omega)} \|\partial_z (\nabla_H \underline{\vec{u}})\|_{L^2(\Omega)}.$$

Therefore, it is only necessary to make the additional regularity assumption for $\partial_z \underline{\vec{u}}$ *given in* (3).

Remark 4.3 (2D case) *In the 2D case, the authors have obtained (see [9]) uniqueness of the weak solution, if there exists a weak solution* $\bar{u}$ *of* (EP) *in* $(0,T)$ *with the additional regularity:*

$$\partial_z \bar{u} \in L^4(0,T;L^4(\Omega)).$$

Now, using 2D anisotropic estimates, we can deduce that it is sufficient to impose:

$$\partial_z \bar{u} \in L^4(0,T;L^2(\Omega)).$$

5 Asymptotic behaviour of solutions.

As in the previous section, we will do the study in the 3D case. Results in the 2D case are similar and easier.

Proof of Theorem 1.4**:** Let $\vec{v}$ be the solution of the stationary problem:

$$(EP)_{st} \left\{ \begin{array}{rcll} -\nu_h \Delta_H \vec{v} - \nu_v \partial^2_{zz} \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + v_3 \partial_z \vec{v} + \nabla_H p_s & = & \vec{f}_2 & \text{in } \Omega, \\ \nabla_H \cdot \langle \vec{v} \rangle & = & 0 & \text{in } \omega, \\ \nu_v \partial_z \vec{v}_{|\Gamma_s} = \vec{\tau}_2, \quad \vec{v}_{|\Gamma_b \cup \Gamma_l} & = & \vec{0}. & \end{array} \right.$$

If data $\vec{f}_2$ and $\vec{\tau}_2$ are small enough in $L^2(\Omega)^2$ and $H_0^{1/2+\varepsilon}(\Gamma_s)^2$-norms respectively, then there exists a unique solution for $(EP)_{st}$ and the following estimates for weak and strong regularity solutions are verified:

$$\|\vec{v}\|^2_{H^1(\Omega)} \leq \frac{C}{\nu^2} \left\{ \|\vec{f}_2\|^2_{H^{-1}(\Omega)} + \|\vec{\tau}_2\|^2_{H^{-1/2}(\Gamma_s)} \right\},$$

$$\|\vec{v}\|^2_{H^2(\Omega)} \leq \frac{C}{\nu^2}\left\{\|\vec{f}_2\|^2_{L^2(\Omega)} + \|\vec{\tau}_2\|^2_{H_0^{1/2+\varepsilon}(\Gamma_s)}\right\}.$$

Let $\vec{e}$ be the solution of (S) with $\vec{f} = \vec{f}_1$, $\vec{e}(0) = \vec{u}_0 - \vec{v}$ and $\vec{\tau} = \vec{\tau}_1$. The problem verified by $\vec{w} = \vec{u}(t) - \vec{e}(t) - \vec{v}$ is the following:

$$\begin{cases} \partial_t \vec{w} - \nu_h \Delta_H \vec{w} - \nu_v \partial^2_{zz}\vec{w} + (\vec{w}\cdot\nabla_H)\vec{w} + w_3\partial_z \vec{w} \\ \quad +((\vec{v}+\vec{e})\cdot\nabla_H)\vec{w} + (v_3+e_3)\partial_z\vec{w} \\ +(\vec{w}\cdot\nabla_H)(\vec{v}+\vec{e}) + w_3\partial_z(\vec{v}+\vec{e}) + (\vec{e}\cdot\nabla_H)\vec{v} + e_3\partial_z\vec{v} \\ \quad +((\vec{v}+\vec{e})\cdot\nabla_H)\vec{e} + (v_3+e_3)\partial_z\vec{e} = 0 \text{ in } (0,T)\times\Omega, \\ \nabla_H\cdot\langle\vec{w}\rangle = 0 \text{ in } (0,T)\times\omega, \quad \vec{w}(t=0) = 0 \text{ in } \Omega, \\ \nu_v\partial_z\vec{w} = 0 \text{ on } (0,T)\times\Gamma_s, \quad \vec{w} = 0 \text{ on } (0,T)\times(\Gamma_b\cup\Gamma_l). \end{cases}$$

Taking $A\vec{w}$ as a test function, we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\vec{w}\|^2_V + \|A\vec{w}\|^2_{L^2(\Omega)} &= -\int_\Omega((\vec{w}\cdot\nabla_H)\vec{w})\cdot A\vec{w}d\Omega - \int_\Omega w_3\partial_z\vec{w}\cdot A\vec{w}d\Omega \\ &- \int_\Omega((\vec{v}+\vec{e})\cdot\nabla_H)\cdot\vec{w}\cdot A\vec{w}d\Omega - \int_\Omega(v_3+e_3)\partial_z\vec{w}\cdot A\vec{w}d\Omega \\ &- \int_\Omega((\vec{w}\cdot\nabla_H)(\vec{v}+\vec{e})\cdot A\vec{w}d\Omega - \int_\Omega w_3\partial_z(\vec{v}+\vec{e})\cdot A\vec{w}d\Omega \\ &- \int_\Omega((\vec{e}\cdot\nabla_H)\vec{v}\cdot A\vec{w}d\Omega - \int_\Omega e_3\partial_z\vec{v}\cdot A\vec{w}d\Omega \\ &- \int_\Omega((\vec{v}+\vec{e})\cdot\nabla_H\vec{e}\cdot A\vec{w}d\Omega - \int_\Omega(v_3+e_3)\partial_z\vec{e}\cdot A\vec{w}d\Omega \end{aligned}$$

Bounding each term in an analogous way to the previous ones, we obtain:

$$\begin{aligned} \frac{d}{dt}\|\vec{w}\|^2_V + \|A\vec{w}\|^2_{L^2(\Omega)} &\leq \frac{C}{\nu^{3/2}}\|A\vec{w}\|^2_{L^2(\Omega)}\|\vec{w}\|_V + \frac{C}{\nu^{11}}\|\vec{w}\|^{10}_V \\ &+ (a_1(t)+a_2(t))\|\vec{w}\|^2_V + b(t), \end{aligned}$$

where

$$\begin{cases} a_1(t) &=& C\left\{\dfrac{1}{\nu^7}\left(\|\vec{v}\|^8_{H^1(\Omega)}+\|\vec{e}\|^8_{L^\infty(H^1)}\right)+\dfrac{1}{\nu^3}\|\vec{v}\|^2_{H^1(\Omega)}\|\vec{v}\|^2_{H^2(\Omega)}+\dfrac{1}{\nu}\|\vec{v}\|^{1/2}_{H^1(\Omega)}\|\vec{v}\|^{3/2}_{H^2(\Omega)}\right\} \\ a_2(t) &=& C\left\{\dfrac{1}{\nu^3}\|\vec{e}\|^2_{L^\infty(H^1)}\|\vec{e}\|^2_{H^2(\Omega)}+\dfrac{1}{\nu}\|\vec{e}\|^{1/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)}\right\} \\ &\leq& \dfrac{C}{\nu}\left(\dfrac{d_1^2}{\nu^2}+1\right)\|\vec{e}\|^2_{H^2(\Omega)}\leq\dfrac{C}{\nu}\|\vec{e}\|^2_{H^2} \\ b(t) &=& C\left\{\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{v}\|^{1/2}_{H^1(\Omega)}\|\vec{v}\|^{3/2}_{H^2(\Omega)}+\|\vec{e}\|_{H^1}\|\vec{v}\|_{H^1(\Omega)}\|\vec{v}\|_{H^2(\Omega)}\|\vec{e}\|_{H^2(\Omega)}\right. \\ &+& \left.\|\vec{v}\|^2_{H^1}\|\vec{e}\|^{1/2}_{H^1}\|\vec{e}\|^{3/2}_{H^2}+\|\vec{e}\|^{5/2}_{H^1(\Omega)}\|\vec{e}\|^{3/2}_{H^2(\Omega)}+\|\vec{e}\|^2_{H^1(\Omega)}\|\vec{e}\|^2_{H^2(\Omega)}\right\} \\ &\leq& C\left\{\|\vec{v}\|^{1/2}_{H^1(\Omega)}\|\vec{v}\|^{3/2}_{H^2(\Omega)}+\|\vec{v}\|_{H^1(\Omega)}\|\vec{v}\|_{H^2(\Omega)}+\|\vec{v}\|^2_{H^1}+\|\vec{e}\|^2_{H^1(\Omega)}\right\}\|\vec{e}\|^2_{H^2(\Omega)} \\ &\leq& C\nu^2\|\vec{e}\|^2_{H^2(\Omega)}. \end{cases}$$

We notice that $a_1 \in L^\infty(0,\infty)$ and that $a_2 \in L^1(0,\infty)$. Then, from hypothesis (H) we can deduce that: $\forall t \in [0,+\infty)$

$$\|\vec{w}(t)\|_V < \gamma\nu^{3/2}, \tag{34}$$

for some small constant $\gamma > 0$.
Now, taking into account (34) and smallness condition for the data (H), we obtain the following inequality:

$$y'(t)+\frac{\nu}{2C}y(t)\leq C\nu^2\|\vec{e}(t)\|^2_{H^2(\Omega)}$$

where we call $y(t)=\|\vec{w}(t)\|^2_V$ and we have used (12) for $\vec{w}$. To get this inequality, we use that $a_2(t)\|\vec{w}\|^2_V+b(t)\leq C\nu^2\|\vec{e}\|^2_{H^2(\Omega)}$ and absorb the other terms with the $\|A\vec{w}\|^2_{L^2(\Omega)}$. Then, we get

$$y(t)\leq\int_0^t e^{\frac{\nu}{2C}(s-t)}\|\vec{e}(s)\|^2_{H^2(\Omega)}ds.$$

As $\|\vec{e}\|^2_{H^2(\Omega)}\in L^1(0,+\infty)$, for all $\delta>0$ there exists $T_*\in[0,+\infty)$ such that

$$\int_{T_*}^{+\infty}\|\vec{e}(t)\|^2_{H^2(\Omega)}dt<\delta.$$

Then,

$$e^{-\frac{\nu}{2C}t}\int_0^t e^{\frac{\nu}{2C}s}\|\vec{e}(s)\|^2_{H^2(\Omega)}ds\leq e^{-\frac{\nu}{2C}t}e^{\frac{\nu}{2C}T_*}\int_0^{T_*}\|\vec{e}(s)\|^2_{H^2(\Omega)}ds+\int_{T_*}^t\|\vec{e}(s)\|^2_{H^2(\Omega)}ds.$$

Therefore, as $t \uparrow +\infty$, we can conclude that $y(t) = \|\vec{u}(t) - \vec{e}(t) - \vec{v}\|_V^2 \to 0$.
Then, it suffices to prove that $\|\vec{e}(t)\|_V \to 0$ as $t \uparrow +\infty$. Considering the completely homogeneous problem verified by $\vec{e}-\vec{e}_1$ (being $\vec{e}_1$ the solution of stationnary problem (S_1) given in the proof of Theorem 1.10) and taking as test function $A(\vec{e}-\vec{e}_1)$, one has:

$$\frac{d}{dt}\|(\vec{e}-\vec{e}_1)(t)\|_V^2 + \|A(\vec{e}-\vec{e}_1)(t)\|_{L^2(\Omega)}^2 \le C\left(\|\vec{f}_1(t)\|_{L^2(\Omega)}^2 + \|\partial_t \vec{e}_1(t)\|_{L^2(\Omega)}^2\right). \quad (35)$$

Adding in both parts of (35), $\frac{d}{dt}\|\vec{e}_1(t)\|_V^2 + \|A\vec{e}_1(t)\|_{L^2(\Omega)}^2$, taking into account (12) for $(\vec{e}-\vec{e}_1)(t)$ and $\vec{e}_1(t)$ and that:

$$\frac{d}{dt}\|\vec{e}_1(t)\|_V^2 \le \|\vec{e}_1(t)\|_V \|\partial_t \vec{e}_1(t)\|_V \le C\left(\nu\|\vec{e}_1(t)\|_V^2 + \frac{1}{\nu}\|\partial_t \vec{e}_1(t)\|_V^2\right),$$

we obtain for $z(t) = \|(\vec{e}-\vec{e}_1)(t)\|_V^2 + \|\vec{e}_1(t)\|_V^2$ the inequality:

$$z'(t) + \frac{\nu}{C}z(t) \le C\left(\|\vec{f}_1(t)\|_{L^2(\Omega)}^2 + \|A\vec{e}_1(t)\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\partial_t \vec{e}_1(t)\|_V^2\right).$$

Multiplying by $exp\left(\frac{\nu}{C}t\right)$, integrating in $(0,t)$, using that $\|\vec{e}\|_V^2 \le 2z(t)$ and taking into account that

$$\|\partial_t \vec{e}_1(s)\|_V^2 \le \frac{C}{\nu}\|\partial_t \vec{\tau}_1(s)\|_{H^{-1/2}(\Gamma_s)}^2 \quad \text{and} \quad \|A\vec{e}_1(s)\|_{L^2(\Omega)}^2 \le C\|\vec{\tau}_1(s)\|_{H_0^{1/2+\varepsilon}(\Gamma_s)}^2,$$

we obtain:

$$\begin{aligned}\|\vec{e}(t)\|_V^2 \;\le\; & \frac{C\bar{\nu}}{\nu}e^{-\frac{\nu}{C}t}\left(\|\vec{u}_0 - \vec{v}\|_{H^1(\Omega)}^2 + \frac{1}{\nu^2}\|\vec{\tau}_1(0)\|_{H^{-1/2}(\Gamma_s)}^2\right) \\ & + \; C\int_0^t e^{-\frac{\nu}{C}(t-s)}\left\{\|\vec{f}_1(s)\|_{L^2(\Omega)}^2 + \|\vec{\tau}_1(s)\|_{H_0^{1/2+\varepsilon}(\Gamma_s)}^2 + \frac{1}{\nu^2}\|\partial_t \vec{\tau}_1(s)\|_{H^{-1/2}(\Gamma_s)}^2\right\}\end{aligned}$$

Then, since the term between brackets is in $L^1(0,+\infty)$, we do as before and we prove the convergence to 0. ■

Remark 5.1 *The smallness condition (H) is only necessary to insure the existence of a global strong solution. The result is also true if we assume the existence of a global strong solution in $(0,+\infty)$ and only impose smallness condition for $\vec{\tau}_2$ and $\vec{f}_2$.*

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References

[1] P. Azérad & F. Guillén, *Équations de Navier-Stokes en bassin peu profond: l'approximation hydrostatique,* C. R. Acad. Sci. Paris, t. 329, Série I, p. 961-966, 1999.

[2] O. Besson & M. R. Laydi, *Some Estimates for the Anisotropic Navier-Stokes Equations and for the Hydrostatic Approximation*, Mathematical Modelling and Numerical Analysis, Vol. n. 7, p. 855-865, 1992.

[3] H. Brézis, *Analyse Fonctionnelle. Théorie et applications,* Masson, 1993.

[4] T. Chacón & F. Guillén, *An intrinsic analysis of existence of solutions for the hydrostatic approximation of the Navier-Stokes equations,* C. R. Acad. Sci. Paris, t. 330, Série I, p. 841-846, 2000.

[5] P. Constantin & C. Foias, *Navier-Stokes Equations,* The University of Chicago Press, Chicago, 1988.

[6] D. Gilbarg & N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[7] F. Guillén & M. V. Redondo, *Convergencia de algunos esquemas numéricos hacia el modelo evolutivo de Ecuaciones Primitivas,* Actas XVI CEDYA, VI CMA, Universidad de Las Palmas de Gran Canaria, p. 1165-1172, 1999.

[8] F. Guillén-González & M. A. Rodríguez-Bellido, *On the Singular Times with Nonlinear Viscosity,* submitted.

[9] F. Guillén-González & M. A. Rodríguez-Bellido, *On the strong solutions of the 2D Primitive Equations problem,* submitted.

[10] R. Lewandowski, *Analyse Mathématique et Océanographie*, Masson, 1997.

[11] P. L. Lions, *Mathematical Topics in Fluids Mechanics, Vol. 1: Incompressible Models,* Clarendon Press, Oxford, 1996.

[12] J. L. Lions, R. Teman, S. Wang, *On the equations of the large scale Ocean,* Nonlinearity, 5, p. 1007-1053, 1992.

[13] L. Nirenberg, *On Elliptic Partial Differential Equations,* Ann. Scuo. Norm. Sup. Pisa, **13** (3), pp. 115-162, 1959.

[14] J. Pedlosky, *Geophysical fluid dynamics,* Springer-Verlag, 1987.

[15] R. Teman, *Navier-Stokes Equations: Theory and Numerical Analysis,* North Holland, Amsterdam, New York, 1977.

[16] M. Ziane, *Regularity Results for Stokes Type Systems,* Applicable Analysis, Vol. 58, p. 263-292, 1995.