CONVERGENCE AND ERROR ESTIMATES OF TWO ITERATIVE METHODS FOR THE STRONG SOLUTION OF THE INCOMPRESSIBLE KORTEWEG MODEL

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Abstract

We show the existence of strong solutions for a fluid model with Korteweg tensor, which is obtained as limit of two iterative linear schemes. The different unknowns are sequentially decoupled in the first scheme and in parallel form in the second one. In both cases, the whole sequences are bounded in strong norms and convergent towards the strong solution of the system, by using a generalization of the Banach's Fixed Point Theorem. Moreover, we explicit a priori and a posteriori error estimates (respect to the weak norms), which let us to compare both schemes.

Keywords: Korteweg model; iterative method; strong solution, Banach's Fixed Point Theorem; Cauchy's sequence; convergence; a priori and a posteriori error estimates.

AMS Subject Classification: 35B45, 35B65, 35Q35, 65M12, 65M15, 76D03

1 Introduction

We will study a Navier-Stokes type model with a specific stress tensor introduced for the first time by D.J. Korteweg, see Ref. [4], in 1901. This model describes the stress induced by a nonuniform density distribution (or concentration) in a viscous fluid, which is related to the capillarity effects due to the interfaces where this concentration varies rapidly.

Although these Korteweg stresses were introduced more than a hundred years ago, models of this kind have raised a growing interest over the past decades, where diffuse interface methods

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have been widely studied. This interest is motivated by a large range of physical applications, from binary fluids to near-critical single-component fluids, new experimental evidence of the effect of Korteweg stress, e.g. in micro-fluids or in low gravity experiments (e.g., see the survey Ref. [1] and the references therein). Most contributions to the mathematical study of Korteweg models are based on compressible fluids.[2] In this paper, we are interested in a Korteweg model for incompressible fluids.

Suppose that we study a complex fluid confined into a three-dimensional smooth enough domain Ω, and we denote $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$. Then, we consider the following model:

$$
(KM) \left\{
\begin{array}{rcll}
\partial_t \rho + \mathbf{u} \cdot \nabla \rho &=& 0 & \text{in } Q, \\[2ex]
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi &=& -k\, \nabla \rho\, \Delta \rho & \text{in } Q, \\[2ex]
\nabla \cdot \mathbf{u} &=& 0 & \text{in } Q, \\[2ex]
\mathbf{u} &=& \mathbf{0} & \text{on } \Sigma, \\[2ex]
\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \rho|_{t=0} &=& \rho_0 & \text{in } \Omega,
\end{array}
\right.
$$

The unknowns in the (KM)-system are the solenoidal velocity $\mathbf{u} : Q \to \mathbb{R}^3$, the pressure of the fluid $\pi : Q \to \mathbb{R}$, and the density $\rho : Q \to \mathbb{R}$. With regard to the coefficients, $\nu > 0$ is the fluid viscosity and $k > 0$ is the capillarity coefficient. Finally, $\mathbf{u}_0 : \Omega \to \mathbb{R}^3$ and $\rho_0 : \Omega \to \mathbb{R}$ are initial data.

The way the model has been written is a consequence of a manipulation made over the right-hand side of the system for $\mathbf{u}$, that in divergence form was $-k\, \nabla \cdot (\nabla \rho \otimes \nabla \rho)$, where $\nabla \rho \otimes \nabla \rho$ is called the Korteweg stress tensor ($\otimes$ denotes the tensor product). Indeed, it suffices to use the equality

$$
-k\, \nabla \cdot (\nabla \rho \otimes \nabla \rho) = -k\, \nabla (\frac{1}{2} |\nabla \rho|^2) - k\, \nabla \rho\, \Delta \rho,
$$

and to include $-k\, \nabla(\frac{1}{2}|\nabla \rho|^2)$ into the pressure term.

The model (KM) is a very primitive one, although it does include some of the key ideas of the original problem of free interface motion in the mixture of fluids. The density is nonuniform and the velocity is assumed to be solenoidal. For instance, this assumption can be valid if two incompressible fluids are mixed; indeed, no diffusion is considered, so that the density (or concentration) of a particle can be regarded as constant along its trajectory. This fact is also assumed in the density-dependent Navier-Stokes model.[6]

An important feature of the model (KM) is its dissipative character, because the system admits (at least formally) the following energy equality (since multiplying the ρ-equation by

$-k\Delta\rho$ and the $\mathbf{u}$-system by $\mathbf{u}$ all the nonlinear convective terms vanish):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|\mathbf{u}|^2 + k|\nabla\rho|^2\Big) + \nu\int_\Omega |\nabla\mathbf{u}|^2 = 0. \tag{1.1}$$

In particular, this equality implies that the total energy (that is, the kinetic energy $\int_\Omega |\mathbf{u}|^2$ plus the elastic energy $\int_\Omega k|\nabla\rho|^2$) decreases with respect to the time.

In this paper, we describe and compare two iterative methods of a Fixed Point type for the model (KM). Firstly, for each scheme, we obtain some estimates in regular norms (based on Ref. [7] and Ref. [8]) under two restrictive conditions (either small time or small data). Secondly, we obtain the convergence of the sequence of approximate solutions towards the strong solution of (KM) by means of Cauchy's argument in weak norms. As a consequence, the existence of local in time regular solutions of (KM) is also proved. By the way, a priori and a posteriori error estimates for both schemes are obtained. This procedure has been already applied in Ref. [3] to a more regular model with mass diffusion.

Several studies about the regularity of this type of models have been made. For instance, in order to model two miscible liquids brought in contact, the same Korteweg tensor is considered in Ref. [5] together with a convection-diffusion equation for the density. In fact, the tensor $\nabla\rho\Delta\rho$ acts like an effective interfacial tension that relaxes with the time due to the mass diffusion. In this case, the existence of local in time regular solutions for the resulting parabolic system is obtained. However, the mathematical study of (KM) becomes more difficult than that of the parabolic model considered in Ref. [5], because in (KM) there is no diffusion for the density (in our case, a parabolic and hyperbolic coupled system is considered). The existence of local in time regular solutions for (KM) is obtained in Ref. [8] using the Schauder's Fixed Point Theorem.

Now, we will prove the same type of result as in Ref. [8], but the construction of solutions is based on two iterative methods. The convergence of both schemes and error estimates are proved using an easy but tricky generalization of the Banach fixed point theorem (see Theorem 3.1 below), under smallness assumptions on time or on initial data (in a Besov space for the velocity).

This work is organized as follows. To finish this Section 1, we introduce the functional spaces, equivalent norms and the notation for some weighted norms that we will use in the sequel. Section 2 is devoted to the design of the two iterative schemes. In Section 3, the main result is stated as an application of a generalization of the Banach's Fixed Point Theorem, announced in the paper as Theorem 3.1 and proved in Appendix A. The proof of this main result for the two iterative schemes is done in Sections 4 and 5, respectively. In the previous two schemes an explicit treatment of the convective terms for the velocity system and for density equation is considered. The case of a semi-explicit treatment of the convective term is studied in Section 6. Finally, in the Appendices B and C, the proofs of two technical results are done.

These two results are applied to obtain estimates for the second scheme, which are a bit more technical than the estimates of the first scheme.

1.1 Functional spaces and equivalent norms

In general, the notation will be abridged. We shall set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we shall denote by $L_T^p(X)$ the Banach space $L^p(0,T;X)$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^3$.

We shall denote by $\mathbf{H}$ and $\mathbf{V}$ the spaces of type L^2 and H^1 respectively, associated to the incompressibility and Dirichlet homogeneous boundary conditions for the velocity, that is to say:

$$\begin{aligned} \mathbf{H} &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \; : \; \nabla \cdot \mathbf{u} = 0, \; \mathbf{u} \cdot \mathbf{n}_{|\partial\Omega} = 0\} \\ \mathbf{V} &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) \; : \; \nabla \cdot \mathbf{u} = 0, \; \mathbf{u}_{|\partial\Omega} = 0\} \end{aligned}$$

Moreover, $\mathbf{W}^{1,p}(\Omega) \cap \mathbf{V}$ and $\mathbf{W}^{2,p}(\Omega) \cap \mathbf{V}$ (with $p > 3$) will be the spaces where the estimates will be obtained, and we shall take $\|\nabla \mathbf{v}\|_{L^p(\Omega)}$ and $\|\Delta \mathbf{v}\|_{L^p(\Omega)}$ as the norms equivalent to the usual norms in these spaces.

For the study of the concentration of the fluid ρ, we shall consider the space:

$$W_N^{k,p}(\Omega) = \left\{ \rho \in W^{k,p}(\Omega) \; : \; \int_\Omega \rho(t;\mathbf{x})\, d\Omega = \int_\Omega \rho_0 \, d\Omega, \, \forall t \right\} \equiv \overline{\rho}_0 + W_{N,0}^{k,p}(\Omega),$$

where $W_{N,0}^{k,p}(\Omega) = \left\{ \rho \in W^{k,p}(\Omega) : \int_\Omega \rho(t;\mathbf{x})\, d\Omega = 0 \right\}$. For $p = 2$, we denote $W_{N,0}^{k,2}(\Omega) = H_{N,0}^k(\Omega)$.

Without loss of generality (and by simplicity in the mathematical analysis), one can work in the space of densities with $\overline{\rho} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \rho \, d\Omega \equiv 0$ (considering, if it is necessary, $\rho - \overline{\rho}_0$ as the new unknown for the density). Thus, $\|\nabla \rho\|_{L^p(\Omega)}$ and $\|\Delta \rho\|_{L^p(\Omega)}$ define norms in $W^{1,p}(\Omega)$ and $W^{2,p}(\Omega)$, respectively, equivalent to the usual norms.

Definition 1.1 *We shall say that $(\rho, \mathbf{u}, \pi)$ is a strong solution for the Korteweg's model if it satifies the system (KM) and has the following regularity, for some $p > 3$:*

$$\begin{cases} \rho \in L^\infty(0,T; W^{2,p}(\Omega)), \quad \pi \in L^p(0,T; W^{1,p}(\Omega)), \\ \\ \mathbf{u} \in L^\infty(0,T; \mathbf{W}^{1,p}(\Omega)) \cap L^p(0,T; \mathbf{W}^{2,p}(\Omega)), \quad \partial_t \mathbf{u} \in L^p(0,T; \mathbf{L}^p(\Omega)). \end{cases}$$

In order to consider the regularity $L^p(0,T;\mathbf{W}^{2,p}(\Omega)) \cap L^\infty(0,T;\mathbf{W}^{1,p}(\Omega))$ for the non-stationary Stokes problem, the initial data for the velocity $\mathbf{u}_0$ must be considered in the space of Besov type $\mathbf{B}_p^{2-2/p}(\Omega)$ (see Ref. [7] for more details) which is a halfway space between $\mathbf{W}^{1,p}(\Omega)$ and $\mathbf{W}^{2,p}(\Omega)$. More concretely, a $L^p(Q)$-estimate for the time-dependent Stokes problem, see inequality (4.18) below, will be used in an essential way.

1.2 Notation of weighted norms

For convenience, we shall consider the following notation for the spaces related to the density and velocity respectively:

$$W(T) = L^\infty(0,T;H^1_{N,0}(\Omega)),$$
$$\mathbf{Y}(T) = \{\mathbf{u}/\ \mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})\}.$$

They will be endowed with the following weighted norms in $W(T)$ and $\mathbf{Y}(T)$ (equivalent to the usual norms):

$$\begin{aligned}
\|\rho\|_{W(T)} &:= \|\rho\|_{L^\infty_{T,w}(H^1)} = \sup_{s\in(0,T)} \left\{ e^{-\frac{D}{2\,\delta}\,s}\,|\nabla\rho(s)| \right\} \\
\|\mathbf{u}\|_{\mathbf{Y}(T)} &:= \|\mathbf{u}\|_{L^\infty_{T,w}(\mathbf{H})\times L^2_{T,w}(\mathbf{V})} \\
&= \left(\sup_{s\in(0,T)} \left\{ e^{-\frac{D}{\delta}\,s}\,|\mathbf{u}(s)|^2 \right\} + \int_0^T e^{-\frac{D}{\delta}\,s}\,|\nabla\mathbf{u}(s)|^2\,ds \right)^{1/2}
\end{aligned}$$

being $\delta > 0$ a small enough constant, $D > 0$ a constant which will be stated in the sequel (and will be different for each scheme) and $|\cdot|$ denotes the $L^2(\Omega)$-norm. Finally, the norm in $W(T) \times \mathbf{Y}(T)$ is defined as:

$$\|(\rho,\mathbf{u})\|^2_{W(T)\times\mathbf{Y}(T)} = \sup_{s\in(0,T)} \left\{ e^{-\frac{D}{\delta}\,s}\,\left(|\nabla\rho(s)|^2 + |\mathbf{u}(s)|^2\right) \right\} + \int_0^T e^{-\frac{D}{\delta}\,s}\,|\nabla\mathbf{u}(s)|^2\,ds$$

2 Description of the Schemes

In this paper, we are going to approximate solutions of problem (KM) by means of two iterative schemes. In both schemes, the density will be computed by applying a simple linear transport scheme. The velocity and pressure will be obtained by solving a Stokes equation, where the source term arising from the nonlinear term in the Navier-Stokes equation, will be expressed in terms of the velocity computed at the previous step. In the first scheme density and velocity will be computed successively, but in the second one both computations can be made independently and computed in parallel.

2.1 Scheme 1 (sequential)

The first iterative process in order to approach the (KM) problem is the following one:

Initialization: We consider an adequate initial function $\mathbf{u}^0(t)$, for each $t \in [0,T]$.

Step $n \geq 1$: First, given $\mathbf{u}^{n-1}$, to find ρ^n such that:

$$\left\{ \begin{array}{rcll} \partial_t\rho^n + (\mathbf{u}^{n-1}\cdot\nabla)\rho^n &=& 0 & \text{in } Q, \\ \rho^n|_{t=0} &=& \rho_0 - \bar{\rho}_0 & \text{in } \Omega. \end{array} \right. \tag{2.2}$$

Afterwards, given $(\rho^n, \mathbf{u}^{n-1})$, to find $(\mathbf{u}^n, \pi^n)$ such that:

$$\begin{cases} \partial_t \mathbf{u}^n - \nu\Delta\mathbf{u}^n + \nabla\pi^n &= -k\,\nabla\rho^n\,\Delta\rho^n - (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1} \quad \text{in } Q, \\ \nabla\cdot\mathbf{u}^n &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } Q, \\ \mathbf{u}^n|_{\partial\Omega} &= \mathbf{0} \quad \text{in } (0,T), \quad \mathbf{u}^n|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega. \end{cases} \tag{2.3}$$

2.2 Scheme 2 (parallel)

The second iterative method is a parallel scheme, designed as follows:

Initialization: Let $\rho^0(t) \equiv \rho_0$ and consider $\mathbf{u}^0(t)$ for all $t \in [0,T]$, an adequate initial function.

Step $n \geq 1$: Given $(\rho^{n-1}, \mathbf{u}^{n-1})$, to find ρ^n and $(\mathbf{u}^n, \pi^n)$ satisfying the following problems, respectively:

$$\begin{cases} \partial_t\rho^n + (\mathbf{u}^{n-1}\cdot\nabla)\rho^n &= 0 \qquad\qquad \text{in } (0,T)\times\Omega, \\ \rho^n|_{t=0} &= \rho_0 - \bar{\rho}_0 \quad \text{in } \Omega, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \partial_t \mathbf{u}^n - \nu\Delta\mathbf{u}^n + \nabla\pi^n &= -k\,\nabla\rho^{n-1}\,\Delta\rho^{n-1} - (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1} \quad \text{in } Q, \\ \nabla\cdot\mathbf{u}^n &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } Q, \\ \mathbf{u}^n|_{\partial\Omega} &= \mathbf{0} \quad \text{in } (0,T), \qquad \mathbf{u}^n|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega. \end{cases} \tag{2.5}$$

Notice that problems (2.4) and (2.5) are independent, therefore they can be computed at the same time. Problems (2.2) or (2.4) are linear transport problems and (2.3) and (2.5) are time-dependent Stokes problems.

From now on, we will consider $\nu = k = 1$, for simplicity.

2.3 Initial data

Throughout this work, we shall consider the initial data $\mathbf{u}_0 \in \mathbf{B}_p^{2-2/p}(\Omega)$ and $\rho_0 \in W^{2,p}(\Omega)$. Moreover, we shall need the initialization for the velocity $\mathbf{u}^0 \in L^\infty(0,T;\mathbf{W}^{1,p}(\Omega)) \cap L^p(0,T;\mathbf{W}^{2,p}(\Omega)\cap \mathbf{V})$ with $p > 3$. Then, we can take $\rho^0 \equiv \rho_0$, but we cannot take $\mathbf{u}^0 \equiv \mathbf{u}_0$ because $\mathbf{u}_0 \notin \mathbf{W}^{2,p}(\Omega)$ in general (see Remark 4.4). In order to obtain estimates for the sequence $(\rho^n, \mathbf{u}^n, \pi^n)$, we shall need to impose the following constraints relating the final time T and the initial data for both schemes, either $\mathbf{u}^0$ for Scheme 1 or $(\rho^0, \mathbf{u}^0)$ for Scheme 2:

- **Scheme 1:** There exists a constant $K > 0$ such that:

$$\|\mathbf{u}^0\|^p_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} \leq K^p \tag{2.6}$$

and

$$C_1 \, \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}} + C_2 \, T \, \Big[K^{2p} + \|\rho_0 - \overline{\rho}_0\|^{2p}_{W^{2,p}} \exp\Big(2 \, C_3 \, T^{1/p'} \, K \Big) \Big] \le K^p, \tag{2.7}$$

where C_1, C_2, C_3 are the constants furnished by Lemma 4.1 below, concretely in (4.17), and p' is the conjugate exponent of p.

- **Scheme 2:** There exist two constants K_1, $K_2 > 0$ such that:

$$\|\rho_0 - \overline{\rho}_0\|^p_{W^{2,p}} \le K_1^p, \tag{2.8}$$

$$\|\mathbf{u}^0\|^p_{L^\infty_T(\mathbf{W}^{1,p}) \cap L^p_T(\mathbf{W}^{2,p})} \le K_2^p, \tag{2.9}$$

and:

$$\left\{ \begin{array}{rcl} \|\rho_0 - \overline{\rho}_0\|_{W^{2,p}} \exp\Big(C_3 \, T^{1/p'} \, K_2 \Big) & \le & K_1 \\ C_1 \, \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}} + C_2 \, T \, (K_1^{2p} + K_2^{2p}) & \le & K_2^p, \end{array} \right. \tag{2.10}$$

where C_1, C_2, C_3 are the constants furnished by Lemma 5.1 below, concretely in (5.37) and (5.38).

Remark 2.1 (Smallness over T or over initial data for Scheme 1) *Hypothesis* (2.6)−(2.7) *are verified choosing either final time T or initial data small enough. Concretely,*

- **small time***: Let $(\rho_0, \mathbf{u}_0)$ and $\mathbf{u}^0$ be fixed, then we take $K > 0$ such that both $C_1 \, \|\mathbf{u}_0\|_{\mathbf{B}_p^{2-2/p}} \le K^p/2$ and (2.6) holds. Therefore, we choose $T_* > 0$ small enough such that:*

$$C_2 \, T_* \, \Big[K^{2p} + \|\rho_0 - \bar{\rho}_0\|^{2p}_{W^{2,p}} \exp\Big(2 \, C_3 \, T_*^{1/p'} K \Big) \Big] \quad \le \quad \frac{K^p}{2},$$

where p' is the conjugate exponent of p.

- **small data:** *Now, the time T is fixed. Therefore, we choose K, $\|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}}$ and $\|\rho_0 - \bar{\rho}_0\|_{W^{2,p}}$ small enough such that both (2.6) and* (2.7) *holds.*

Remark 2.2 (Smallness over T or over initial data for Scheme 2) *Hypotheses* (2.8)−(2.10) *are verified choosing T or initial data small enough. Concretely,*

- **small time***: Let K_1, K_2 be taken as $K_1 \ge C_3 \, \|\rho_0 - \bar{\rho}_0\|_{W^{2,p}}$ and $K_2^p \ge 2 \, C_1 \, \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}}$, then we choose $T_* > 0$ small enough such that:*

$$C_2 \, T_* \, \Big[K_1^{2p} + K_2^{2p} \Big] \le \frac{K_2^p}{2}, \qquad \|\rho_0 - \bar{\rho}_0\|_{W^{2,p}} \exp\Big(C_3 \, T_*^{1/p} \, K_2 \Big) \le K_1.$$

- **small data:** *Now, the time T is fixed. Therefore, we choose K_1, K_2, $\|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}}$, $\|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p}) \cap L^p_T(\mathbf{W}^{2,p})}$ and $\|\rho_0 - \bar{\rho}_0\|_{W^{2,p}}$ small enough such that both* (2.8) − (2.10) *holds.*

3 The Main Result

In this Section, we shall present the main result of this paper, which will be valid for both iterative schemes of approximation to the solution of (KM), obtaining:

- Existence of strong solution for (KM) via the convergence of both iterative methods.
- A priori and a posteriori error estimates for the iterative sequences obtained in both schemes.

The key is to apply the following abstract theorem, which is a generalization of the Banach's Fixed Point Theorem (see A for a proof).

Theorem 3.1 *Let X be a Banach space, A be a nonempty closed subset of X and $R : A \subseteq X \to X$ be a continuous operator such that $R(A) \subseteq A$ and:*

$$\|R^{n+s}x - R^n x\|_X \leq G(n)\, \|R^s x - x\|_X \quad \forall\, x \in A,\, \forall s, n \in \mathbb{N}, \tag{3.11}$$

where $G(n) > 0$ is such that $\sum_{n\geq 0} G(n) = G < +\infty$. Then, the successive approximation method:

$$\begin{cases} x^0 \in A \\ x^{n+1} = Rx^n \quad \forall n \geq 0, \end{cases}$$

is globally convergent to a fixed point $\widehat{x} \in A$, that is:

$$\forall\, x^0 \in A, \quad R^n\, x^0 \to \widehat{x} \ \text{ in } X \text{ with } \widehat{x} \in A \text{ and } \widehat{x} = R\,\widehat{x}.$$

Moreover, the following a priori and a posteriori error estimates hold:

$$\|\widehat{x} - R^n x^0\|_X \leq G(n)\, G\, \|Rx^0 - x^0\|_X \tag{3.12}$$

$$\|\widehat{x} - R^n x^0\|_X \leq G\, \|R^n x^0 - R^{n-1} x^0\|_X \tag{3.13}$$

Remark 3.2 *In general, uniqueness of fixed point of R in A cannot be assured. For instance, this happens if we consider the trivial case $R \equiv Id$.*

Remark 3.3 *Notice that the Banach's Fixed Point Theorem for contractive mappings is a particular case of Theorem 3.1, for $G(n) = \alpha^n$, being $\alpha : 0 < \alpha < 1$ the constant of contractiveness of R and $G = (1 - \alpha)^{-1}$.*

The precise main result of this paper can be written as follows:

Theorem 3.4 *Let $\mathbf{u}_0 \in \mathbf{B}_p^{2-2/p}(\Omega)$, $\rho_0 \in W^{2,p}(\Omega)$ and $(\rho^n, \mathbf{u}^n)_{n\geq 0}$ be the sequence of functions obtained from Scheme 1 or Scheme 2 (see (2.2) – (2.3) or (2.4) – (2.5), respectively). Then, under hypotheses of smallness either for the final time $T > 0$ or for the initial data $(\mathbf{u}_0, \rho_0 - \bar{\rho}_0)$ given in §2.3, the whole sequence $(\rho^n, \mathbf{u}^n, \pi^n)$ converges towards the (unique) strong solution $(\rho, \mathbf{u}, \pi)$ of (KM). Moreover, the following a priori and a posteriori error estimates (in weak norms) hold for each $t \in [0, T]$ and for each n:*

$$\|(\rho - \rho^n, \mathbf{u} - \mathbf{u}^n)\|_{W(t)\times \mathbf{Y}(t)} \leq G(n)\, G\, \Psi(\rho^1 - \rho^0, \mathbf{u}^1 - \mathbf{u}^0), \tag{3.14}$$

$$\|(\rho - \rho^n, \mathbf{u} - \mathbf{u}^n)\|_{W(t)\times \mathbf{Y}(t)} \leq G\, \Psi(\rho^n - \rho^{n-1}, \mathbf{u}^n - \mathbf{u}^{n-1}), \tag{3.15}$$

with

$$G(n) = \begin{cases} (\delta\, E(t))^{n/2} & \text{if Scheme 1,} \\ e^{\frac{K}{2\,\delta^2}\, t}\, (\delta\, E(t))^{n/2} & \text{if Scheme 2,} \end{cases}$$

$$G = \sum_{n\geq 0} G(n) = \begin{cases} (1 - (\delta\, E(t))^{1/2})^{-1} & \text{if Scheme 1,} \\ e^{\frac{K\,t}{2\,\delta^2}} (1 - (\delta\, E(t))^{1/2})^{-1} & \text{if Scheme 2,} \end{cases}$$

being $\delta > 0$ the small enough constant that appears in the weighted norm of $W(t)$ and $\mathbf{Y}(t)$, $E(t) = e^{C\, t^{1/p'}}$, with $C > 0$, a constant independent of δ and t, and

$$\Psi(\rho, \mathbf{u}) = \begin{cases} \|\mathbf{u}\|_{L^2_{t,w}(\mathbf{H}^1)} & \text{if Scheme1,} \\ \|(\rho, \mathbf{u})\|_{W(t)\times \mathbf{Y}(t)} & \text{if Scheme 2.} \end{cases}$$

Remark 3.5 (Comparison between both schemes) *As usual, a more explicit iterative method has an easier implementation but worse estimates. Observe that the main difference between both schemes arises from the constant that appears in the a priori and a posteriori error estimates, which is amplified by the factor $e^{\frac{K\,t}{2\,\delta^2}}$ in Scheme 2 respect to Scheme 1. Notice that this factor is large for small δ.*

Remark 3.6 *The method of proof that we present in this paper is an alternative to that obtained in Ref. [8], where the Schauder's fixed point Theorem was used. Here, besides the existence result, we reach the global convergence for two successive approximation methods and some (a priori and a posteriori) error estimates, which allows us to do a quantitative comparison.*

4 Study for Scheme 1

Now, we focus on Scheme 1 defined by (2.2) – (2.3). Before studying the error estimates in the weak norms, we will see some estimates for the sequence $(\rho^n, \mathbf{u}^n, p^n)$ in regular norms.

4.1 Strong estimates for $(\rho^n, \mathbf{u}^n, \pi^n)$ under smallness assumptions

We will start this Subsection with the proof of a recursive inequality.

Lemma 4.1 *The sequence $(\rho^n, \mathbf{u}^n, \pi^n)$ obtained by Scheme 1 satisfies the following recursive inequalities (in strong norms), for each $t \in (0,T)$:*

$$\|\rho^n\|^p_{L^\infty_t(W^{2,p})} \le \|\rho_0 - \bar\rho_0\|^p_{W^{2,p}} \exp\left(C_3 \|\mathbf{u}^{n-1}\|_{L^1_t(W^{2,p})}\right), \tag{4.16}$$

$$\|\mathbf{u}^n\|^p_{L^\infty_t(\mathbf{W}^{1,p})} + \|\partial_t \mathbf{u}^n\|^p_{L^p_t(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L^p_t(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L^p_t(W^{1,p})}$$

$$\le C_1 \|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\, t \left\{ \|\mathbf{u}^{n-1}\|^{2p}_{L^\infty_t(\mathbf{W}^{1,p})} + \|\rho^n\|^{2p}_{L^\infty_t(W^{2,p})} \right\}. \tag{4.17}$$

Remark 4.2 *Similar estimates were made in Ref. [8] in order to prove the strong regularity for the Korteweg model (KM), via Schauder's fixed point argument.*

Proof. We take the laplacian in the equation $(2.2)_1$ and multiply by $|\Delta\rho^n|^{p-2}\Delta\rho^n$, obtaining:

$$\frac{1}{p}\frac{d}{dt}\|\Delta\rho^n\|^p_{L^p(\Omega)} \le \int_\Omega |\Delta\mathbf{u}^{n-1}||\nabla\rho^n||\Delta\rho^n|^{p-1} d\Omega + 2\int_\Omega |\nabla\mathbf{u}^{n-1}||D^2\rho^n||\Delta\rho^n|^{p-1} d\Omega$$

$$- \int_\Omega (\mathbf{u}^{n-1}\cdot\nabla)\,\Delta\rho^n\,|\Delta\rho^n|^{p-2}\,\Delta\rho^n\, d\Omega := I_1 + I_2 + I_3.$$

We bound the I_i-terms as follows (recall that $p > 3$):

$$I_1 \le \|\Delta\mathbf{u}^{n-1}\|_{\mathbf{L}^p(\Omega)}\|\nabla\rho^n\|_{L^\infty(\Omega)}\|\Delta\rho^n\|^{p-1}_{L^p(\Omega)} \le C\,\|\Delta\mathbf{u}^{n-1}\|_{\mathbf{L}^p(\Omega)}\|\Delta\rho^n\|^p_{L^p(\Omega)}$$

$$I_2 \le C\,\|\nabla\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}\|D^2\rho^n\|^p_{L^p(\Omega)} \le C\,\|\mathbf{u}^{n-1}\|_{\mathbf{W}^{2,p}(\Omega)}\|\Delta\rho^n\|^p_{L^p(\Omega)}$$

$$I_3 = -\frac{1}{p}\int_\Omega (\mathbf{u}^{n-1}\cdot\nabla)\,|\Delta\rho^n|^p\, d\Omega = 0 \quad (\text{since } \nabla\cdot\mathbf{u}^{n-1} = 0 \text{ and } \mathbf{u}^{n-1}|_{\partial\Omega} = 0)$$

which led us to the expression $\frac{d}{dt}\|\Delta\rho^n(t)\|^p_{L^p(\Omega)} \le C\,\|\mathbf{u}^{n-1}\|_{\mathbf{W}^{2,p}(\Omega)}\,\|\Delta\rho^n(t)\|^p_{L^p(\Omega)}$. Thus, integrating directly we obtain (4.16). Notice that the fact that $I_3 = 0$ is fundamental to obtain (4.16). In fact, although the scheme for a more explicit scheme respect to the ρ^n-equation, replacing $\mathbf{u}^{n-1}\cdot\nabla\rho^n$ by $\mathbf{u}^{n-1}\cdot\nabla\rho^{n-1}$ in (2.2), can be easily computed, the corresponding $I_3 \neq 0$, and in this case it is not clear how to obtain an inequality like (4.16).

Using the non-hilbertian estimates for the Stokes problem (2.3), see Ref. [7][Th. 15, p. 102], we found that for any $p > 1$:

$$\|\partial_t\mathbf{u}^n\|^p_{L^p_t(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L^p_t(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L^p_t(W^{1,p})}$$

$$\le C_1\left(\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + \|(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}\|^p_{L^p_t(\mathbf{L}^p)} + \|\Delta\rho^n\,\nabla\rho^n\|^p_{L^p_t(L^p)}\right) \tag{4.18}$$

In this previous inequality is where the boundedness of $\|\mathbf{u}_0\|_{\mathbf{B}_p^{2-2/p}}$ is used.

Using the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (since $p > 3$), we can bound the last two terms of (4.18) as follows: For the last one,

$$\|\Delta\rho^n \nabla\rho^n\|^p_{L^p(\Omega)} \le \|\nabla\rho^n(s)\|^p_{L^\infty(\Omega)}\|\Delta\rho^n(s)\|^p_{L^p(\Omega)} \le C\,\|\Delta\rho^n\|^{2p}_{L^p(\Omega)},$$

hence, integrating in time,

$$\begin{aligned}\int_0^t \|\Delta\rho^n(s)\,\nabla\rho^n(s)\|^p_{L^p(\Omega)}\,ds &\le \int_0^t \|\nabla\rho^n\|^p_{L^\infty(\Omega)}\|\Delta\rho^n\|^p_{L^p(\Omega)} \\ &\le C\,t\,\|\rho^n\|^{2p}_{L^\infty_t(W^{2,p})} \qquad (4.19)\end{aligned}$$

And for the last but one term, we have:

$$\begin{aligned}\int_0^t \|(\mathbf{u}^{n-1}(s)\cdot\nabla)\mathbf{u}^{n-1}(s)\|^p_{\mathbf{L}^p(\Omega)}\,ds &\le \int_0^t \|\mathbf{u}^{n-1}(s)\|^p_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{u}^{n-1}(s)\|^p_{\mathbf{L}^p(\Omega)}\,ds \\ \le C\int_0^t \|\mathbf{u}^{n-1}(s)\|^p_{\mathbf{W}^{1,p}(\Omega)}\,\|\mathbf{u}^{n-1}(s)\|^p_{\mathbf{W}^{1,p}(\Omega)}\,ds &\le C\,t\,\|\mathbf{u}^{n-1}\|^{2p}_{L^\infty_t(\mathbf{W}^{1,p})} \qquad (4.20)\end{aligned}$$

Applying (4.19) and (4.20) in (4.18), we arrive at:

$$\begin{aligned}\|\partial_t\mathbf{u}^n\|^p_{L^p_t(\mathbf{L}^p)} &+ \|\mathbf{u}^n\|^p_{L^p_t(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L^p_t(W^{1,p})} \\ &\le C\,\|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}} + C\,t\,\|\mathbf{u}^{n-1}\|^{2p}_{L^\infty_t(\mathbf{W}^{1,p})} + C\,t\,\|\rho^n\|^{2p}_{L^\infty_t(\mathbf{W}^{2,p})} \qquad (4.21)\end{aligned}$$

Observe that it is possible to show that (see for instance Ref. [8]):

$$\|\mathbf{u}^n\|^p_{L^\infty_t(\mathbf{W}^{1,p})} \le \|\mathbf{u}^n_0\|^p_{\mathbf{W}^{1,p}} + C\left(\|\partial_t\mathbf{u}^n\|^p_{L^p_t(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L^p_t(\mathbf{W}^{2,p})}\right). \qquad (4.22)$$

Therefore, from inequalities (4.21) and (4.22), we can deduce (4.17). ∎

In order to obtain strong estimates for the scheme, an induction argument can be applied. We can deduce that choosing $(\rho^{n-1}, \mathbf{u}^{n-1})$ in a convenient bounded set, we can get that $(\rho^n, \mathbf{u}^n)$ remains in that set. Consequently, a good choice of $(\rho^0, \mathbf{u}^0)$ guarantees that all the sequence $(\rho^n, \mathbf{u}^n)$ remains in the same bounded set. The concrete result is:

Lemma 4.3 *Let $\rho_0 \in W^{2,p}(\Omega)$, $\mathbf{u}_0 \in \mathbf{B}_p^{2-2/p}(\Omega) \cap \mathbf{V}$ and $\mathbf{u}^0 \in L^\infty(0,T;\mathbf{W}^{1,p}) \cap L^p(0,T;\mathbf{W}^{2,p})$ be given functions such that $\mathbf{u}^0(0) = \mathbf{u}_0$. If there exists a constant $K > 0$ and a time $T > 0$ such that $\mathbf{u}^0$ and $(\rho_0, \mathbf{u}_0)$ verify constraints (2.6) and (2.7) respectively, then for each $n \ge 1$ the solution $(\rho^n, \mathbf{u}^n, \pi^n)$ of (2.2) – (2.3) satisfies:*

$$\rho^n \text{ is bounded in } L^\infty(0,T;W^{2,p}),$$

$(\mathbf{u}^n, \pi^n)$ *is bounded in* $[L^\infty(0,T;\mathbf{W}^{1,p}) \cap L^p(0,T;\mathbf{W}^{2,p})] \times L^p(0,T;W^{1,p}(\Omega))$.

More concretely, for any $n \geq 0$*, there hold:*

$$\|\rho^n\|^p_{L^\infty_T(W^{2,p})} \leq \|\rho_0 - \bar{\rho}_0\|^p_{W^{2,p}} \exp\Big(C_3\, T^{1/p'}\, K\Big), \tag{4.23}$$

$$\|\mathbf{u}^n\|^p_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} + \|\partial_t \mathbf{u}^n\|^p_{L^p_T(\mathbf{L}^p)} + \|\pi^n\|^p_{L^p_T(W^{1,p})} \leq K^p. \tag{4.24}$$

Remark 4.4 *Observe that we have imposed that* $\mathbf{u}^0 \in L^p(0,T;\mathbf{W}^{2,p})$ *in the hypotheses of Lemma* 4.3*. This implies that, in principle, we cannot use* $\mathbf{u}^0(t) = \mathbf{u}_0$ *for all* $t \in [0,T]$*. In the case of having* $\mathbf{u}_0 \in \mathbf{W}^{1,p}(\Omega)\backslash\mathbf{W}^{2,p}(\Omega)$*, this assumption needs the regularization of the initial data* $\mathbf{u}_0$*. For instance, we can take* $\mathbf{u}^0$ *the solution of the Stokes problem:* $\partial_t \mathbf{u}^0 - \Delta\mathbf{u}^0 + \nabla\pi^0 = 0$*,* $\nabla\cdot\mathbf{u}^0 = 0$*,* $\mathbf{u}^0(0) = \mathbf{u}_0$*. Moreover, in this case, constraints* (2.6)*-*(2.7) *reduces to* (2.7)*.*

Proof.[Proof of Lemma 4.3] We use the induction process. Starting from (4.17) for $n = 1$ and using (4.16) for $n = 1$, we have:

$$\begin{aligned}
&\|\mathbf{u}^1\|^p_{L^\infty_T(\mathbf{W}^{1,p})} + \|\partial_t\mathbf{u}^1\|^p_{L^p_T(\mathbf{L}^p)} + \|\mathbf{u}^1\|^p_{L^p_T(\mathbf{W}^{2,p})} + \|\pi^1\|^p_{L^p_T(L^p)} \\
&\quad \leq C_1\,\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\,T\,\Big\{\|\mathbf{u}^0\|^{2p}_{L^\infty_T(\mathbf{W}^{1,p})} + \|\rho^1\|^{2p}_{L^\infty_T(\mathbf{W}^{2,p})}\Big\} \\
&\quad \leq C_1\,\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\,T\,\Big\{K^{2p} + \|\rho_0 - \bar{\rho}_0\|^{2p}_{W^{2,p}(\Omega)} \exp\Big(2\,C_3\,T^{1/p'}\,K\Big)\Big\}
\end{aligned}$$

Thanks to (2.7), we arrive at (4.24) for $n = 1$. From (4.16) and (4.24) for $n = 1$, we deduce (4.23) for $n = 1$.

Now, we do the induction pass from $n-1$ to n. Starting from (4.17) for n, using (4.16) for n and applying the induction hypothesis (4.24) for $n-1$ (which implies, in particular, that $\|\mathbf{u}^{n-1}\|_{L^\infty_t(\mathbf{W}^{1,p})} \leq K$), we have:

$$\begin{aligned}
&\|\mathbf{u}^n\|^p_{L^\infty_T(\mathbf{W}^{1,p})} + \|\partial_t\mathbf{u}^n\|^p_{L^p_T(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L^p_T(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L^p_T(W^{1,p})} \\
&\leq\; C_1\,\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\,T\,\Big\{K^{2p} + \|\rho_0 - \bar{\rho}_0\|^{2p}_{W^{2,p}} \exp\Big(2\,C_3\,T^{1/p'}\,K\Big)\Big\}.
\end{aligned}$$

From hypothesis (2.7), we conclude (4.24) for n. From (4.16) and (4.24) for n, we deduce (4.23) for n. ∎

4.2 Error estimates for Scheme 1

We are going to estimate the Cauchy's sequence $(\rho^{(n,s)}, \mathbf{u}^{(n,s)}, \pi^{(n,s)})$, defined as

$$\rho^{(n,s)} = \rho^{n+s} - \rho^n, \quad \mathbf{u}^{(n,s)} = \mathbf{u}^{n+s} - \mathbf{u}^n \quad \text{and} \quad \pi^{(n,s)} = \pi^{n+s} - \pi^n.$$

First, we describe the problems verified by $\rho^{(n,s)}$, $\mathbf{u}^{(n,s)}$ and $\pi^{(n,s)}$, respectively:

$$\begin{cases} \partial_t \rho^{(n,s)} + (\mathbf{u}^{(n-1,s)} \cdot \nabla)\rho^{n+s} + (\mathbf{u}^{n-1} \cdot \nabla)\rho^{(n,s)} = 0 \\ \rho^{(n,s)}|_{t=0} = 0 \end{cases} \tag{4.25}$$

$$\begin{cases} \partial_t \mathbf{u}^{(n,s)} - \Delta \mathbf{u}^{(n,s)} + \nabla \pi^{(n,s)} + (\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{n+s-1} \\ \quad +(\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{(n-1,s)} = -\nabla \rho^{n+s}\, \Delta \rho^{(n,s)} - \nabla \rho^{(n,s)}\, \Delta \rho^{n} \\ \nabla \cdot \mathbf{u}^{(n,s)} = 0, \quad \mathbf{u}^{(n,s)}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{u}^{(n,s)}|_{t=0} = \mathbf{0} \end{cases} \tag{4.26}$$

Lemma 4.5 *Assuming hypothesis from Lemma 4.3, the following inequality holds for each $\delta > 0$:*

$$\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})\|^2_{W(t)\times \mathbf{Y}(t)} \le \delta\, E(t)\, \|\mathbf{u}^{(n-1,s)}\|^2_{L^2_{t,w}(\mathbf{V})}, \quad t \in [0,T]. \tag{4.27}$$

Recall that the weighted norms $W(t) \times \mathbf{Y}(t)$ and $L^2_{t,w}(\mathbf{V})$ depend on δ, and $E(t) = e^{C\, t^{1/p'}}$ with $C > 0$ (a constant independent of δ and t).

From Lemma 4.5, we arrive easily at the following

Corollary 4.6 *Under the hypotheses of Lemma 4.5, the sequence $(\rho^{(n,s)}, \mathbf{u}^{(n,s)})$ satisfies:*

$$\|\mathbf{u}^{(n,s)}\|_{L^2_{t,w}(\mathbf{V})} \le (\delta\, E(t))^{n/2}\, \|\mathbf{u}^{(0,s)}\|_{L^2_{t,w}(\mathbf{V})}, \quad t \in [0,T] \tag{4.28}$$

$$\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})\|_{W(t)\times \mathbf{Y}(t)} \le (\delta\, E(t))^{n/2}\, \|\mathbf{u}^{(0,s)}\|_{L^2_{t,w}(\mathbf{V})}, \quad t \in [0,T] \tag{4.29}$$

Proof.[Proof of Lemma 4.5] Multiplying (4.25) by $-\Delta\rho^{(n,s)}$ and (4.26) by $\mathbf{u}^{(n,s)}$, and adding both expressions, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|\nabla \rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\right) + |\nabla \mathbf{u}^{(n,s)}|^2 \\ &= -\int_\Omega (\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{n+s-1} \cdot \mathbf{u}^{(n,s)}\, d\Omega - \int_\Omega (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{(n-1,s)} \cdot \mathbf{u}^{(n,s)}\, d\Omega \\ &+ \int_\Omega (\mathbf{u}^{n-1} \cdot \nabla)\rho^{(n,s)}\, \Delta\rho^{(n,s)}\, d\Omega - \int_\Omega (\mathbf{u}^{(n,s)} \cdot \nabla)\rho^{(n,s)}\, \Delta\rho^{n}\, d\Omega \\ &+ \int_\Omega \left[(\mathbf{u}^{(n-1,s)} - \mathbf{u}^{(n,s)}) \cdot \nabla\right] \rho^{n+s}\, \Delta\rho^{(n,s)}\, d\Omega := \sum_{i=1}^{5} I_i \end{aligned} \tag{4.30}$$

Now, we are going to bound the right hand side terms, taking into account scheme estimates of Lemma 4.3. For the nonlinear terms in $\mathbf{u}$ we do:

$$\begin{aligned} |I_1| &\le \|\mathbf{u}^{(n-1,s)}\|_{\mathbf{L}^6(\Omega)} \|\nabla \mathbf{u}^{n+s-1}\|_{\mathbf{L}^3(\Omega)} |\mathbf{u}^{(n,s)}| \\ &\le \delta |\nabla \mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta} \|\nabla \mathbf{u}^{n+s-1}\|^2_{\mathbf{L}^3(\Omega)} |\mathbf{u}^{(n,s)}|^2 \\ &\le \delta |\nabla \mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}\, |\mathbf{u}^{(n,s)}|^2 \end{aligned}$$

$$
\begin{aligned}
|I_2| &\leq \|\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}|\nabla\mathbf{u}^{(n-1,s)}|\,|\mathbf{u}^{(n,s)}| \\
&\leq \delta|\nabla\mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}\|\mathbf{u}^{n-1}\|^2_{\mathbf{L}^\infty(\Omega)}|\mathbf{u}^{(n,s)}|^2 \\
&\leq \delta|\nabla\mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}\,|\mathbf{u}^{(n,s)}|^2
\end{aligned}
$$

Observe that, integrating by parts in I_3 and using that $\int_\Omega(\mathbf{u}^{n-1}\cdot\nabla)\rho^{(n,s)}\cdot\nabla\rho^{(n,s)}\,d\Omega = 0$, we obtain:

$$
\begin{aligned}
|I_3| &= \left|-\int_\Omega(\nabla\mathbf{u}^{n-1}\cdot\nabla)\rho^{(n,s)}\cdot\nabla\rho^{(n,s)}\,d\Omega\right| \\
&\leq \int_\Omega|\nabla\mathbf{u}^{n-1}|\,|\nabla\rho^{(n,s)}|^2\,d\Omega \leq \|\nabla\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}|\nabla\rho^{(n,s)}|^2.
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &\leq \|\mathbf{u}^{(n,s)}\|_{\mathbf{L}^6(\Omega)}|\nabla\rho^{(n,s)}|\|\Delta\rho^n\|_{L^3(\Omega)} \leq \delta|\nabla\mathbf{u}^{(n,s)}|^2 + \frac{C}{\delta}\|\Delta\rho^n\|^2_{L^3(\Omega)}|\nabla\rho^{(n,s)}|^2 \\
&\leq \delta|\nabla\mathbf{u}^{(n,s)}|^2 + \frac{C}{\delta}\,|\nabla\rho^{(n,s)}|^2
\end{aligned}
$$

Integrating I_5 by parts, we get:

$$
\begin{aligned}
I_5 &= \int_\Omega(\nabla(\mathbf{u}^{(n,s)} - \mathbf{u}^{(n-1,s)})\cdot\nabla)\rho^{n+s}\,\nabla\rho^{(n,s)}\,d\Omega \\
&+ \int_\Omega\Big((\mathbf{u}^{(n,s)} - \mathbf{u}^{(n-1,s)})\cdot\nabla\Big)(\nabla\rho^{n+s})\,\nabla\rho^{(n,s)}\,d\Omega,
\end{aligned}
$$

hence:

$$
\begin{aligned}
&|I_5| \leq \Big\{|\nabla(\mathbf{u}^{(n,s)} - \mathbf{u}^{(n-1,s)})|\|\nabla\rho^{n+s}\|_{L^\infty(\Omega)} \\
&+\|\mathbf{u}^{(n,s)} - \mathbf{u}^{(n-1,s)}\|_{\mathbf{L}^6(\Omega)}\|D^2\rho^{n+s}\|_{L^3(\Omega)}\Big\}\,|\nabla\rho^{(n,s)}| \\
&\leq \delta\left(|\nabla\mathbf{u}^{(n,s)}|^2 + |\nabla\mathbf{u}^{(n-1,s)}|^2\right) + \frac{C}{\delta}\left(\|\nabla\rho^{n+s}\|^2_{L^\infty(\Omega)} + \|D^2\rho^{n+s}\|^2_{L^3(\Omega)}\right)|\nabla\rho^{(n,s)}|^2 \\
&\leq \delta\left(|\nabla\mathbf{u}^{(n,s)}|^2 + |\nabla\mathbf{u}^{(n-1,s)}|^2\right) + \frac{C}{\delta}\,|\nabla\rho^{(n,s)}|^2
\end{aligned}
$$

Thus, again using scheme estimates given in Lemma 4.3, we arrive at the inequality:

$$
\begin{aligned}
&\frac{d}{dt}\Big(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\Big) + |\nabla\mathbf{u}^{(n,s)}|^2 \\
&\leq \frac{C_3}{\delta}\,\left(\delta\,\|\nabla\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)} + 1\right)\,\Big(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\Big) + \delta|\nabla\mathbf{u}^{(n-1,s)}|^2
\end{aligned}
$$

which can be written as:

$$
a'_{n,s}(t) + b_{n,s}(t) \leq \frac{C_3}{\delta}\,a_{n,s}(t) + c_{n-1}(t)\,a_{n,s}(t) + \delta\,b_{n-1,s}(t), \tag{4.31}
$$

being:

$$\left\{\begin{array}{rcl} a_{n,s}(t) &=& |\nabla \rho^{(n,s)}(t)|^2 + |\mathbf{u}^{(n,s)}(t)|^2 \\ \\ b_{n,s}(t) &=& |\nabla \mathbf{u}^{(n,s)}(t)|^2 \\ \\ c_{n-1}(t) &=& C_3 \, \|\nabla \mathbf{u}^{n-1}(t)\|_{\mathbf{L}^\infty(\Omega)} \end{array}\right.$$

Then, multiplying by $e^{-\frac{C_3}{\delta}\,t}$, we rewrite (4.31) in the form:

$$A'_{n,s}(t) + B_{n,s}(t) \le c_{n-1}(t)\, A_{n,s}(t) + \delta\, B_{n-1,s}(t),$$

where:

$$\left\{\begin{array}{rcl} A_{n,s}(t) &=& a_{n,s}(t)\, e^{-\frac{C_3}{\delta}\,t} = e^{-\frac{C_3}{\delta}t}\, \|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})(t)\|^2_{H^1\times\mathbf{H}} \\ \\ B_{n,s}(t) &=& b_{n,s}(t)\, e^{-\frac{C_3}{\delta}\,t} = e^{-\frac{C_3}{\delta}t}\, \|\mathbf{u}^{(n,s)}(t)\|^2_{\mathbf{V}} \end{array}\right. \tag{4.32}$$

Using the Gronwall's Lemma and that $A_{n,s}(0) = 0$, we can deduce:

$$A_{n,s}(t) + \int_0^t B_{n,s}(\sigma)\, e^{\int_\sigma^t c_{n-1}(\alpha)\,d\alpha}\, d\sigma \le \delta \int_0^t B_{n-1,s}(\sigma)\, e^{\int_\sigma^t c_{n-1}(\alpha)\,d\alpha}\, d\sigma. \tag{4.33}$$

Now well, taking into account that $c_{n-1} \in L^p(0,T)$ $(p > 3)$,

$$\int_\sigma^t c_{n-1}(\alpha)\, d\alpha \le C\,(t-\sigma)^{1/p'} \le C\, t^{1/p'}, \qquad \forall t \in [0,T]\ (T > 0).$$

Therefore, bounding from below in the left hand side of (4.33), and from above on the right hand side of (4.33), and calling $E(t) = e^{C\,t^{1/p'}}$, we get:

$$A_{n,s}(t) + \int_0^t B_{n,s}(\sigma)\, d\sigma \le \delta\, E(t) \int_0^t B_{n-1,s}(\sigma)\, d\sigma, \qquad t \in [0,T].$$

Observe that the right hand side of previous expression is a increasing function on t. Therefore, we can deduce:

$$\sup_{\sigma\in[0,t]} A_{n,s}(\sigma) + \int_0^t B_{n,s}(\sigma)\, d\sigma \le \delta\, E(t) \int_0^t B_{n-1,s}(\sigma)\, d\sigma, \qquad t \in [0,T].$$

Taking into account (4.32),

$$\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})\|^2_{L^\infty_{t,w}(H^1\times\mathbf{H})} = \sup_{\sigma\in(0,T)} A_{n,s}(\sigma)$$

and

$$\|\mathbf{u}^{(n,s)}\|^2_{L^2_{t,w}(\mathbf{V})} = \int_0^t B_{n,s}(\sigma)\, d\sigma,$$

hence we conclude (4.27). ∎

Proof of Theorem 3.4 **for Scheme 1:** In order to apply Theorem 3.1, we identify the hypotheses appearing there:

- We define the space $\mathbf{X} = L^2(0,T;\mathbf{V})$ and its non-empty closed bounded subspace $\mathbf{A}_T \subset \mathbf{X}$ as:
$$\mathbf{A}_T = B_{\mathbf{X}}(\mathbf{0};K) = \Big\{\mathbf{u} : \, \|\mathbf{u}\|^p_{L^\infty_T(\mathbf{W}^{1,p})} + \|\partial_t \mathbf{u}\|^p_{L^p_T(\mathbf{L}^p)} + \|\mathbf{u}\|^p_{L^p_T(\mathbf{W}^{2,p})} \le K^p \Big\},$$
being $K > 0$ the constant appearing in (4.24).

- We define the operator R as:
$$\begin{array}{rcl} R \ : \ \mathbf{A}_T \subset \mathbf{X} & \to & \mathbf{X} \\ \bar{\mathbf{u}} \in \mathbf{A}_T & \mapsto & R(\bar{\mathbf{u}}) = \mathbf{u} \end{array}$$
being $\mathbf{u}$ the solution of the following linear problem (of Stokes type):
$$\left\{ \begin{array}{l} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = -(\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} - k\, \nabla\rho \,\Delta\rho \quad \text{in } Q, \\ \nabla \cdot \mathbf{u} = \mathbf{0} \quad \text{in } Q, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Sigma, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega. \end{array} \right. \tag{4.34}$$
Note that the concentration ρ appearing on the right-hand side of $(4.34)_1$ is the solution of the transport problem:
$$\left\{ \begin{array}{rcll} \partial_t \rho + (\bar{\mathbf{u}} \cdot \nabla)\rho & = & 0 & \text{in } Q, \\ \rho|_{t=0} & = & \rho_0 - \bar{\rho}_0 & \text{in } \Omega. \end{array} \right. \tag{4.35}$$

- By Lemma 4.3 we can deduce that $R(\mathbf{A}_T) \subset \mathbf{A}_T$.

- The continuity of operator R in the $\mathbf{X}$-norm can be seen in Ref. [8].

- In this case, estimate (3.11) of Theorem 3.1 is obtained in Corollary 4.6, which can be deduced from (4.28). Concretely, one has (3.11) with $G(n) = (\delta\, E(T))^{n/2}$ and $E(T) = e^{C\,T^{1/p'}}$. Condition $\sum_{n\ge 0} G(n) = G < +\infty$ is verified whenever $\delta\, E(T) < 1$, *i.e.*, if $\delta < e^{-C\,T^{1/p'}}$, being $G = \dfrac{1}{1-(\delta\, E(T))^{1/2}}$.

Thus, owing to Theorem 3.1 we can conclude the convergence of sequence $\mathbf{u}^n$ towards some limit function $\mathbf{u} \in \mathbf{A}_T$ strongly in $\mathbf{X}$. On the other hand, considering the associated density to $\mathbf{u}$ (i.e. ρ the solution of (4.35) with $\bar{\mathbf{u}} = \mathbf{u}$) due to the regularity obtained for $(\rho, \mathbf{u})$, one can prove that $(\rho, \mathbf{u})$ is the unique strong solution of (KM). The following a priori and a posteriori error estimates can be deduced as in the proof of Theorem 3.1:

- **A priori error estimates:** Given $T > 0$, (4.28) for $n = j$ and $s = 1$ is rewritten as:
$$\|\mathbf{u}^{j+1} - \mathbf{u}^j\|_{L^2_{T,w}(\mathbf{V})} \le (\delta\, E(t))^{j/2}\, \|\mathbf{u}^1 - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})}. \tag{4.36}$$

Thus,

$$\begin{aligned}
\|\mathbf{u}^s - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})} &\le \sum_{j=0}^{s-1} \|\mathbf{u}^{j+1} - \mathbf{u}^j\|_{L^2_{T,w}(\mathbf{V})} \\
&\le \sum_{j=0}^{s-1} (\delta\, E(t))^{j/2}\, \|\mathbf{u}^1 - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})} \\
&\le \frac{1}{1-(\delta\, E(t))^{1/2}}\, \|\mathbf{u}^1 - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})}
\end{aligned}$$

Therefore, from (4.29) and the previous estimate, we deduce that the sequence $(\rho^{(n,s)}, \mathbf{u}^{(n,s)})$ also satisfies:

$$\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})\|_{W(T)\times \mathbf{Y}(T)} \le \frac{(\delta E(T))^{n/2}}{1-(\delta\, E(T))^{1/2}}\, \|\mathbf{u}^1 - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})}.$$

Taking $s \uparrow +\infty$, it becomes:

$$\|(\rho - \rho^n, \mathbf{u} - \mathbf{u}^n)\|_{W(T)\times \mathbf{Y}(T)} \le \frac{(\delta E(T))^{n/2}}{1-(\delta\, E(T))^{1/2}}\, \|\mathbf{u}^1 - \mathbf{u}^0\|_{L^2_{T,w}(\mathbf{V})}.$$

- **A posteriori error estimates:** Starting from estimate (4.27) and using an estimate similar to (4.36), we obtain:

$$\begin{aligned}
\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)}\|_{W(T)\times \mathbf{Y}(T)} &\le (\delta\, E(t))^{1/2}\, \|\mathbf{u}^{n-1+s} - \mathbf{u}^{n-1}\|_{L^2_{T,w}(\mathbf{V})} \\
&\le (\delta\, E(t))^{1/2} \sum_{j=0}^{s-1} \|\mathbf{u}^{n+j} - \mathbf{u}^{n-1+j}\|_{L^2_{T,w}(\mathbf{V})} \\
&\le (\delta\, E(t))^{1/2} \sum_{j=0}^{s-1} (\delta\, E(t))^{j/2}\, \|\mathbf{u}^{n} - \mathbf{u}^{n-1}\|_{L^2_{T,w}(\mathbf{V})} \\
&\le \frac{1}{1-(\delta\, E(t))^{1/2}}\, \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^2_{T,w}(\mathbf{V})}.
\end{aligned}$$

Taking $s \uparrow +\infty$, it becomes:

$$\|(\rho - \rho^n, \mathbf{u} - \mathbf{u}^n)\|_{W(T)\times \mathbf{Y}(T)} \le \frac{1}{1-(\delta\, E(T))^{1/2}}\, \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^2_{T,w}(\mathbf{V})}.$$

5 Study for Scheme 2

5.1 Strong estimates for $(\rho^n, \mathbf{u}^n, \pi^n)$ under smallness hypothesis.

As in Subsection 4.1, we want to obtain strong estimates for $(\rho^n, \mathbf{u}^n, \pi^n)$. In this way, we will adapt the proof for Scheme 1 given there, pointing out the main differences.

Lemma 5.1 *The sequence* $(\rho^n, \mathbf{u}^n, \pi^n)$ *obtained by Scheme 2, satisfies the following recursive inequalities (in strong norms), for each* $t \in (0,T)$*:*

$$\|\rho^n\|_{L_t^\infty(W^{2,p})} \le \|\rho_0 - \bar\rho_0\|_{W^{2,p}} \exp\left(C_3 \|\mathbf{u}^{n-1}\|_{L_t^1(\mathbf{W}^{2,p})}\right) \tag{5.37}$$

$$\|\mathbf{u}^n\|^p_{L_t^\infty(\mathbf{W}^{1,p})} + \|\partial_t \mathbf{u}^n\|^p_{L_t^p(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L_t^p(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L_t^p(W^{1,p})}$$

$$\le C_1 \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}} + C_2\, t \left\{ \|\mathbf{u}^{n-1}\|^{2p}_{L_t^\infty(\mathbf{W}^{1,p})} + \|\rho^{n-1}\|^{2p}_{L_t^\infty(W^{2,p})} \right\} \tag{5.38}$$

Proof.[Sketch of the proof] The inequality for ρ^n given by (5.37) becomes from (4.16). Therefore, for $\mathbf{u}^n$ instead of (4.18), we obtain for any $p > 1$:

$$\|\partial_t \mathbf{u}^n\|^p_{L_t^p(\mathbf{L}^p)} + \|\mathbf{u}^n\|^p_{L_t^p(\mathbf{W}^{2,p})} + \|\pi^n\|^p_{L_t^p(W^{1,p})}$$

$$\le C_1 \left(\|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}} + \|(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}\|^p_{L_t^p(\mathbf{L}^p)} + \|\Delta\rho^{n-1}\,\nabla\rho^{n-1}\|^p_{L_t^p(L^p)} \right) \tag{5.39}$$

where the bound for the last term of (5.39) is now given by:

$$\|\Delta\rho^{n-1}\,\nabla\rho^{n-1}\|^p_{L^p(\Omega)} \le \|\nabla\rho^{n-1}\|^p_{L^\infty(\Omega)} \|\Delta\rho^{n-1}\|^p_{L^p(\Omega)} \le C\,\|\rho^{n-1}\|^{2p}_{W^{2,p}(\Omega)},$$

for any $p > 3$. ∎

We want to deduce that choosing $(\rho^{n-1}, \mathbf{u}^{n-1})$ into a convenient bounded set, we can guarantee that $(\rho^n, \mathbf{u}^n, \pi^n)$ remains in that set.

Lemma 5.2 *Let* $\rho_0 \in W^{2,p}(\Omega)$ *verifying* (2.8), $\mathbf{u}_0 \in \mathbf{B}_p^{2-2/p}(\Omega) \cap \mathbf{V}$ *and* $\mathbf{u}^0 \in L^\infty(0,T;\mathbf{W}^{1,p}) \cap L^p(0,T;\mathbf{W}^{2,p})$ *be given functions such that* $\mathbf{u}^0(0) = \mathbf{u}_0$ *verifying constraint* (2.9)*. If there exist two constants* K_1, $K_2 > 0$ *and a time* T *such that* $(\rho_0, \mathbf{u}_0)$ *verifies constraint* (2.10)*, then, the solution* $(\rho^n, \mathbf{u}^n, \pi^n)$ *of* (2.4)-(2.5) *satisfies:*

$$\rho^n \textit{ is bounded in}\, L^\infty(0,T;W^{2,p}),$$

$$(\mathbf{u}^n, \pi^n) \textit{ is bounded in}\, [L^\infty(0,T;\mathbf{W}^{1,p}) \cap L^2(0,T;\mathbf{W}^{2,p})] \times L^p(0,T;W^{1,p}(\Omega)).$$

More concretely, for any $n \ge 1$*, there hold:*

$$\begin{cases} \|\rho^n\|_{L_T^\infty(W^{2,p})} \le K_1 \\ \\ \|\mathbf{u}^n\|^p_{L_T^\infty(\mathbf{W}^{1,p})\cap L_T^p(\mathbf{W}^{2,p})} + \|\partial_t \mathbf{u}^n\|^p_{L_T^p(\mathbf{L}^p)} + \|\pi^n\|^p_{L_T^p(W^{1,p})} \le K_2^p \end{cases} \tag{5.40}$$

Remark 5.3 *Again* $\mathbf{u}^0(t)$ *cannot be taken as* $\mathbf{u}_0$*, due to the regularity imposed in* (2.9)*.*

Proof. We prove (5.40) by induction from (5.39). In the case $n=1$, (5.39) gives

$$\left\{\begin{array}{l} \|\rho^1\|_{L^\infty_T(W^{2,p})} \le \|\rho_0-\bar\rho_0\|_{W^{2,p}} \exp\Big(C_3\,\|\mathbf{u}^0\|_{L^1_T(\mathbf{W}^{2,p})}\Big), \\[2ex] \|\mathbf{u}^1\|^p_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} + \|\partial_t\mathbf{u}^1\|^p_{L^p_T(\mathbf{L}^p)} + \|\pi^1\|^p_{L^p_T(W^{1,p})} \\[2ex] \qquad \le C_1\,\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\,T\Big\{\|\mathbf{u}^0\|^{2p}_{L^\infty_T(\mathbf{W}^{1,p})} + \|\rho_0\|^{2p}_{L^\infty_T(W^{2,p})}\Big\}. \end{array}\right.$$

From (2.10), we obtain (5.40) for $n=1$.

Now, we do the induction pass from $n-1$ to n. First, we use the induction hypothesis (5.40) for $n-1$ to (5.39), obtaining:

$$\left\{\begin{array}{l} \|\rho^n\|_{L^\infty_T(W^{2,p})} \le \|\rho_0-\bar\rho_0\|_{W^{2,p}} \exp\Big(C_3\,T^{1/p'}\,K_2\Big), \\[2ex] \|\mathbf{u}^n\|^p_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} + \|\partial_t\mathbf{u}^n\|^p_{L^p_T(\mathbf{L}^p)} + \|\pi^n\|^p_{L^p_T(W^{1,p})} \\[2ex] \qquad \le C_1\,\|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\,T\Big\{K_2^{2p} + K_1^{2p}\Big\}, \end{array}\right.$$

hence (2.10) implies (5.40) for n. ∎

5.2 Error estimates for Scheme 2

In this case, the Cauchy's sequence $(\rho^{(n,s)},\mathbf{u}^{(n,s)},\pi^{(n,s)})$ (recall that $\rho^{(n,s)}=\rho^{n+s}-\rho^n$, $\mathbf{u}^{(n,s)}=\mathbf{u}^{n+s}-\mathbf{u}^n$ and $\pi^{(n,s)}=\pi^{n+s}-\pi^n$), satisfy the following problems:

$$\left\{\begin{array}{l} \partial_t\rho^{(n,s)} + (\mathbf{u}^{(n-1,s)}\cdot\nabla)\rho^{n+s} + (\mathbf{u}^{n-1}\cdot\nabla)\rho^{(n,s)} = 0, \\[2ex] \rho^{(n,s)}|_{t=0} = 0. \end{array}\right.$$

$$\left\{\begin{array}{l} \partial_t\mathbf{u}^{(n,s)} - \Delta\mathbf{u}^{(n,s)} + \nabla\pi^{(n,s)} + (\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s-1} + (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n-1,s)} \\[2ex] \qquad = -\nabla\rho^{n-1}\,\Delta\rho^{(n-1,s)} - \nabla\rho^{(n-1,s)}\,\Delta\rho^{n-1+s} \\[2ex] \nabla\cdot\mathbf{u}^{(n,s)} = 0, \qquad \mathbf{u}^{(n,s)}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}^{(n,s)}|_{t=0} = \mathbf{0}. \end{array}\right.$$

Lemma 5.4 *Assuming hypotheses from Lemma* 5.2*, the following inequality holds: for* $t\in[0,T]$

$$\|\rho^{(n,s)},\mathbf{u}^{(n,s)}\|^2_{W(t)\times\mathbf{Y}(t)}$$

$$\le E(t)\left\{\frac{C_6}{\delta}\,\|(\rho^{(n,s)},\mathbf{u}^{(n,s)})\|^2_{L^2_{t,w}(H^1\times\mathbf{H})} + \delta\,\|\mathbf{u}^{(n-1,s)}\|^2_{L^2_{t,w}(\mathbf{V})}\right\} \tag{5.41}$$

where the constant D appearing in the weighted norms $\|\cdot\|_{L^2_{t,w}(H^1\times\mathbf{H})}$ *and* $\|\cdot\|_{L^2_{t,w}(\mathbf{V})}$ *is now* $D=C_7$*,* C_7 *being the constant defined in* (5.42).

Proof. In an attempt of doing an estimate similar to Lemma 4.5, we modify slightly the bounds. Concretely, we replace (4.30) by:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\right) + |\nabla\mathbf{u}^{(n,s)}|^2 \\
&= -\int_\Omega (\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s-1}\cdot\mathbf{u}^{(n,s)}\,d\Omega - \int_\Omega (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n-1,s)}\cdot\mathbf{u}^{(n,s)}\,d\Omega \\
&- \int_\Omega \left[(\nabla\mathbf{u}^{(n-1,s)}\cdot\nabla)\rho^{n+s} + (\mathbf{u}^{(n-1,s)}\cdot\nabla)\nabla\rho^{n+s}\right]\nabla\rho^{(n,s)}\,d\Omega \\
&- \int_\Omega (\nabla\mathbf{u}^{n-1}\cdot\nabla)\rho^{(n,s)}\,\nabla\rho^{(n,s)}\;d\Omega \\
&+ \int_\Omega \left[(\nabla\mathbf{u}^{(n,s)}\cdot\nabla)\rho^{n-1} + (\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho^{n-1}\right]\nabla\rho^{(n-1,s)}\,d\Omega \\
&- \int_\Omega (\mathbf{u}^{(n,s)}\cdot\nabla)\rho^{(n-1,s)}\,\Delta\rho^{n-1+s}\,d\Omega = \sum_{i=1}^{6} J_i
\end{aligned}
$$

In order to bound J_i, we follow similar procedures to those of the previous Section, and use estimates (5.40):

$$
J_1 \le \|\mathbf{u}^{(n-1,s)}\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{u}^{n+s-1}\|_{\mathbf{L}^3(\Omega)}|\mathbf{u}^{(n,s)}| \le \delta|\nabla\mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}|\mathbf{u}^{(n,s)}|^2
$$

$$
J_2 \le \|\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}|\nabla\mathbf{u}^{(n-1,s)}||\mathbf{u}^{(n,s)}| \le \delta|\nabla\mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}|\mathbf{u}^{(n,s)}|^2
$$

$$
\begin{aligned}
J_3 &\le \left(|\nabla\mathbf{u}^{(n-1,s)}|\,\|\nabla\rho^{n+s}\|_{L^\infty(\Omega)} + \|\mathbf{u}^{(n-1,s)}\|_{\mathbf{L}^6(\Omega)}\,\|D^2\rho^{n+s}\|_{L^3(\Omega)}\right)|\nabla\rho^{(n,s)}| \\
&\le \delta|\nabla\mathbf{u}^{(n-1,s)}|^2 + \frac{C}{\delta}\left[\|\nabla\rho^{n+s}\|^2_{L^\infty(\Omega)} + \|D^2\rho^{n+s}\|^2_{L^3(\Omega)}\right]|\nabla\rho^{(n,s)}|^2
\end{aligned}
$$

$$
J_4 \le \|\nabla\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}|\nabla\rho^{(n,s)}|^2
$$

The estimate for J_5 is similar to estimate for J_1, and can be written as:

$$
\begin{aligned}
|J_5| &\le \delta|\nabla\mathbf{u}^{(n,s)}|^2 + \frac{C}{\delta}\left[\|\nabla\rho^{n-1}\|^2_{L^\infty(\Omega)} + \|D^2\rho^{n-1}\|^2_{L^3(\Omega)}\right]|\nabla\rho^{(n-1,s)}|^2 \\
|J_6| &\le \|\mathbf{u}^{(n,s)}\|_{\mathbf{L}^6(\Omega)}\,|\nabla\rho^{(n-1,s)}|\,\|\Delta\rho^{n-1+s}\|_{L^3(\Omega)} \le \delta|\nabla\mathbf{u}^{(n,s)}|^2 + \frac{C}{\delta}\,|\nabla\rho^{(n-1,s)}|^2
\end{aligned}
$$

Thus, we obtain the following expression:

$$
\begin{aligned}
&\frac{d}{dt}\left(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\right) + |\nabla\mathbf{u}^{(n,s)}|^2 \le \frac{C_1}{\delta}|\mathbf{u}^{(n,s)}|^2 \\
&+\frac{C_5}{\delta}\left(1+\delta\,\|\nabla\mathbf{u}^{n-1}\|_{\mathbf{L}^\infty(\Omega)}\right)|\nabla\rho^{(n,s)}|^2 + \frac{C_6}{\delta}\,|\nabla\rho^{(n-1,s)}|^2 + \delta\,|\nabla\mathbf{u}^{(n-1,s)}|^2
\end{aligned}
$$

which denoting

$$
C_7 = \max\{C_1,\, C_5\} \tag{5.42}
$$

can be rewritten as :

$$a'_{n,s}(t) + b_{n,s}(t) \le \frac{C_7}{\delta}\, a_{n,s}(t) + c_{n-1}(t)\, a_{n,s}(t) + \frac{C_6}{\delta}\, a_{n-1,s}(t) + \delta\, b_{n-1,s}(t), \tag{5.43}$$

being:

$$\begin{cases} a_{n,s}(t) &=& |\nabla \rho^{(n,s)}(t)|^2 + |\mathbf{u}^{(n,s)}(t)|^2, \\ \\ b_{n,s}(t) &=& |\nabla \mathbf{u}^{(n,s)}(t)|^2, \\ \\ c_{n-1}(t) &=& C_5\, \|\nabla \mathbf{u}^{n-1}(t)\|_{\mathbf{L}^\infty(\Omega)}. \end{cases}$$

Multiplying (5.43) by $e^{C_7 t/\delta}$, one gets

$$A'_{n,s}(t) + B_{n,s}(t) \le c_{n-1}(t)\, A_{n,s}(t) + \frac{C_6}{\delta}\, A_{n-1,s}(t) + \delta\, B_{n-1,s}(t) \tag{5.44}$$

where, similarly to (4.32),

$$\begin{cases} A_{n,s}(t) &=& a_{n,s}(t)\, e^{-\frac{C_7}{\delta}\,t} = e^{-\frac{C_7}{\delta}\,t}\, \|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})(t)\|^2_{H^1\times \mathbf{H}} \\ \\ B_{n,s}(t) &=& b_{n,s}(t)\, e^{-\frac{C_7}{\delta}\,t} = e^{-\frac{C_7}{\delta}\,t}\, \|\mathbf{u}^{(n,s)}(t)\|^2_{\mathbf{V}}. \end{cases}$$

Using the Gronwall's Lemma and that $A_{n,s}(0) = 0$ for (5.44), we obtain:

$$\begin{aligned} A_{n,s}(t) \quad &+ \quad \int_0^t B_{n,s}(\sigma)\, e^{\int_\sigma^t c_{n-1}(\alpha)\,d\alpha}\, d\sigma \\ &\le \quad \frac{C_6}{\delta} \int_0^t A_{n-1,s}(\sigma)\, e^{\int_\sigma^t c_{n-1}(\alpha)\,d\alpha}\, d\sigma + \delta \int_0^t B_{n-1,s}(\sigma)\, e^{\int_\sigma^t c_{n-1}(\alpha)\,d\alpha}\, d\sigma \end{aligned}$$

Since c_{n-1} is bounded in $L^p(0,T)$, then:

$$\int_s^t c_{n-1}(\sigma)\, d\sigma \le \|c_{n-1}\|_{L^p_t}\, (t-s)^{1/p'} \le \frac{C_8}{\delta}\, (t-s)^{1/p'}.$$

Using the same notation as before $E(t) = e^{\frac{C_8}{\delta}\,t^{1/p'}}$, we arrive at:

$$\begin{aligned} A_{n,s}(t) \quad &+ \quad \int_0^t B_{n,s}(\sigma)\, d\sigma \\ &\le \quad E(t) \left\{ \frac{C_6}{\delta} \int_0^t A_{n-1,s}(\sigma)\, d\sigma + \delta \int_0^t B_{n-1,s}(\sigma)\, d\sigma \right\}, \end{aligned} \tag{5.45}$$

for each $\delta > 0$. Observe that the right hand side of (5.45) is an increasing function on t, so we can deduce for all $t \in [0,T]$:

$$\sup_{\tau\in[0,t]} A_{n,s}(\tau) + \int_0^t B_{n,s}(\sigma)\, d\sigma \le E(t) \left\{ \frac{C_6}{\delta} \int_0^t A_{n-1,s}(\sigma)\, d\sigma + \delta \int_0^t B_{n-1,s}(\sigma)\, d\sigma \right\},$$

that is (5.41). ∎

In the search of a priori error estimates, we will use the following Lemma (see B for a proof):

Lemma 5.5 *Let $(f_n(t))_{n\in\mathbb{N}}$, $(g_n(t))_{n\in\mathbb{N}}$ be two given sequences of positive functions defined in $(0,T)$ verifying the inequality:*

$$f_n(t) + \int_0^t g_n(\sigma)\,d\sigma \le E(t)\Big\{\frac{K}{\delta}\int_0^t f_{n-1}(\sigma)\,d\sigma + \delta\int_0^t g_{n-1}(\sigma)\,d\sigma\Big\}, \tag{5.46}$$

for each $\delta > 0$, where $K > 0$ is a constant independent of δ and $E(t)$ is a positive function independent of δ. Then, for any $t \in [0,T]$:

$$f_n(t) + \int_0^t g_n(\sigma)\,d\sigma \le (E(t)\,\delta)^n\, e^{\frac{K}{\delta^2}\,t}\Big\{\|f_0\|_{L^\infty(0,t)} + \|g_0\|_{L^1(0,t)}\Big\} \tag{5.47}$$

Corollary 5.6 *Under the hypotheses of Lemma 5.4, the sequence $(\rho^{(n,s)}, \mathbf{u}^{(n,s)})$ satisfies:*

$$\|(\rho^{(n,s)}, \mathbf{u}^{(n,s)})\|^2_{W(T)\times\mathbf{Y}(T)} \le (E(T)\,\delta)^n\, e^{\frac{C_6\,T}{\delta^2}}\,\|(\rho^{(0,s)}, \mathbf{u}^{(0,s)})\|^2_{W(T)\times\mathbf{Y}(T)}. \tag{5.48}$$

Proof. Applying Lemma 5.5 to (5.45), inequality (5.47) can be written as:

$$A_{n,s}(t) + \int_0^t B_{n,s}(\sigma)\,d\sigma \quad\le\quad (E(t)\,\delta)^n\, e^{\frac{K}{\delta^2}\,t}\Big\{\|A_{0,s}\|_{L^\infty(0,t)} + \|B_{0,s}\|_{L^1(0,t)}\Big\}, \tag{5.49}$$

where $K = C_6$. In order to prove estimate (5.48), we observe the fact that the term appearing on the right hand side of (5.49) is an increasing function on t. ∎

Remark 5.7 *We can prove the following result, more general than Lemma 5.5 (see C for a proof):*

Lemma 5.8 *Let $(f_n), (g_n)$ and (h_n) be three sequences of positive functions defined in $(0,T)$, with (h_n) bounded in $L^r(0,T)$ with $r \in (1,+\infty)$, verifying the following inequality:*

$$f_n(t) + \int_0^t g_n(s)\,ds \le E(t)\,\left\{\frac{1}{\delta}\int_0^t h_{n-1}(s)\,f_{n-1}(s)\,ds + \delta\int_0^t g_{n-1}(s)\,ds\right\} \tag{5.50}$$

Then, for any $t \in [0,T]$:

$$\begin{aligned} & f_n(t) + \int_0^t g_n(s)\,ds \\ & \le \Big(K(r')^{1/r'}\,E(t)\,\delta\Big)^n e^{\left(\frac{C}{\delta^2}\right)^{r'}\frac{t}{r'}}\,P(r')\,K(1/r')\,\left\{\|f_0\|_{L^\infty(0,t)} + \|g_0\|_{L^1(0,t)}\right\} \end{aligned} \tag{5.51}$$

with $C > 0$ a bound of $\|h_{n-1}\|_{L^r(0,T)}$, and $K(r')$, $P(r')$, $K(1/r')$ positive constants such that: $\forall\, a,\, b \ge 0$

$$\begin{aligned} (a+b)^{r'} &\le K(r')\,\Big(a^{r'} + b^{r'}\Big), \\ a^{1/r'} + b^{1/r'} &\le P(r')\,(a+b)^{1/r'}, \\ (a+b)^{1/r'} &\le K(1/r')\,\Big(a^{1/r'} + b^{1/r'}\Big). \end{aligned}$$

Notice that Lemma 5.8 is a generalization of Lemma 5.5. In fact, when $r = +\infty$, $r' = 1$ and $\|h_n\|_{L^\infty(0,t)} \le K$ we have $K(1) = P(1) = 1$ and estimate (5.51) coincides with (5.47). When $r < +\infty$, $r' > 1$, then $K(r') > 1$. Therefore, the estimate of geometric type $\Big(K(r')^{1/r'}\,E(t)\,\delta\Big)^n$ improves if r increases.

Remark 5.9 *For the mass diffusion model studied in Ref. [3], an inequality similar to (5.50) can be obtained, but without the term $\delta \displaystyle\int_0^t g_{n-1}(\sigma)\,d\sigma$ on the right hand side. In this case, the following estimate holds:*

$$f_n(t) + \int_0^t g_n(\sigma)\,d\sigma \le C\left[\frac{(E\,t)^n}{n!}\right]^{1/r'} \|f_0\|_{L^\infty(0,t)},$$

where C, $E > 0$ are constants. Therefore, a convergence to zero faster than in (5.51) is deduced in Ref. [3], but for a more regular model with mass diffusion.

Proof.[Proof of Theorem 3.4 for Scheme 2] This time, we consider the following definitions in order to apply Theorem 3.1:

- The space $\mathbf{X} = L^\infty(0,T;H^1(\Omega)) \times L^2(0,T;\mathbf{V})$, and its non-empty closed bounded subspace $\mathbf{A}_T \subset \mathbf{X}$, for $p > 3$, is:

$$\begin{aligned} \mathbf{A}_T \;=\; \{(\rho,\mathbf{u}) :\; & \|\rho\|_{L^\infty_T(W^{2,p})} \le K_1, \\ & \|\mathbf{u}\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} + \|\partial_t\mathbf{u}\|_{L^p_T(\mathbf{L}^p)} \le K_2\}, \end{aligned}$$

  with K_1, $K_2 > 0$ the constants appearing in (5.40).

- The operator R is defined as:

$$\begin{aligned} R \;:\; \mathbf{A}_T \subset \mathbf{X} \quad & \to \quad \mathbf{X} \\ (\bar\rho,\bar{\mathbf{u}}) \in \mathbf{A}_T \quad & \mapsto \quad R(\bar\rho,\bar{\mathbf{u}}) = (\rho,\mathbf{u}) \end{aligned}$$

  with $\mathbf{u}$ the solution of the following linear problem (of Stokes type):

$$\left\{\begin{aligned} \partial_t\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi &= -(\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - k\,\nabla\bar\rho\,\Delta\bar\rho && \text{in } Q, \\ \nabla\cdot\mathbf{u} &= \mathbf{0} && \text{in } Q, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Sigma, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned}\right. \tag{5.52}$$

  and ρ the solution of the transport problem:

$$\left\{\begin{aligned} \partial_t\rho + (\bar{\mathbf{u}}\cdot\nabla)\rho &= 0 && \text{in } Q, \\ \rho|_{t=0} &= \rho_0 - \bar\rho_0 && \text{in } \Omega. \end{aligned}\right. \tag{5.53}$$

- By Lemma 5.2 we can deduce that $R(\mathbf{A}_T) \subset \mathbf{A}_T$.

- R is a continuous operator. It suffices to prove sequential continuity, i. e., if $(\bar{\rho}_n, \bar{\mathbf{u}}_n) \to (\bar{\rho}, \bar{\mathbf{u}})$ in $\mathbf{X}$, then $R(\bar{\rho}_n, \bar{\mathbf{u}}_n) \to (\bar{\rho}, \bar{\mathbf{u}})$ in $\mathbf{X}$. We denote $\mathbf{E} = L^\infty(0,T; W^{2,p}(\Omega)) \times [L^\infty(0,T; \mathbf{W}^{1,p}(\Omega)) \cap L^p(0,T; \mathbf{W}^{2,p}(\Omega))]$. Since $\mathbf{A}_T$ is a closed bounded set of $\mathbf{E}$, then $(\bar{\rho}, \bar{\mathbf{u}}) \in \mathbf{A}_T$ and $(\bar{\rho}_n, \bar{\mathbf{u}}_n) \rightharpoonup (\bar{\rho}, \bar{\mathbf{u}})$ in $\mathbf{E}$.

  We consider $(\rho_n, \mathbf{u}_n) = R(\bar{\rho}_n, \bar{\mathbf{u}}_n)$ and $(\rho, \mathbf{u}) = R(\bar{\rho}, \bar{\mathbf{u}})$. Observe that $\partial_t \rho_n$ is bounded in $L^p(0,T; W^{1,p}(\Omega))$, hence (ρ_n) is relatively compact in $C([0,T]; W^{1,p}(\Omega))$. Then, we can deduced the existence of a subsequence $(\rho_{n_k}, \mathbf{u}_{n_k}) \subset (\rho_n, \mathbf{u}_n)$ and a limit function $(\theta, \Phi) \in \mathbf{A}_T$ such that:

$$(\rho_{n_k}, \mathbf{u}_{n_k}) \to (\theta, \Phi) \quad \text{weakly in } \mathbf{E} \text{ and strongly in } \mathbf{X}. \tag{5.54}$$

  It suffices to prove that $(\theta, \Phi) = (\rho, \mathbf{u})$ (recall that $(\rho, \mathbf{u}) = R(\bar{\rho}, \bar{\mathbf{u}})$), because in this case the whole sequence $(\rho_n, \mathbf{u}_n)$ converges to $(\rho, \mathbf{u})$. We take the limit fist in the equation $\partial_t \rho_{n_k} + (\bar{\mathbf{u}}_{n_k} \cdot \nabla) \rho_{n_k} = 0$, obtaining that $\theta = \rho$ the unique solution of (5.53), and second in the system:

$$\partial_t \mathbf{u}_{n_k} - \nu \Delta \mathbf{u}_{n_k} + \nabla \pi_{n_k} = -(\bar{\mathbf{u}}_{n_k} \cdot \nabla) \bar{\mathbf{u}}_{n_k} - k \, \nabla \bar{\rho}_{n_k} \, \Delta \bar{\rho}_{n_k}, \quad \nabla \cdot \mathbf{u}_{n_k} = \mathbf{0}.$$

  Since the solution of (5.52) is unique, we conclude that $\Phi = \mathbf{u}$ and the convergence (5.54) holds for the whole sequence.

- Estimate (3.11) of Theorem 3.1 in our case is obtained from estimate (5.48) of Corollary 5.6, and $G(n) = (\delta \, E(T))^{n/2} \, e^{\frac{C_6 \, T}{2 \, \delta^2}}$, with $E(T) = e^{C \, T^{1/p'}}$. Condition $\sum_{n \geq 0} G(n) = G < +\infty$ is verified whenever that $\delta \, E(T) < 1$, *i.e.*, if $\delta < e^{-C \, T^{1/p'}}$, being $G = e^{\frac{C_6 \, t}{2 \, \delta^2}} (1 - (\delta \, E(T))^{1/2})^{-1}$.

Therefore, owing to Theorem 3.1 we can conclude the convergence of sequence $(\rho^n, \mathbf{u}^n)$ towards some limit function $(\rho, \mathbf{u}) \in \mathbf{A}_T$ strongly in $\mathbf{X}$. On the other hand, due to the regularity obtained for $(\rho, \mathbf{u})$, one can prove that $(\rho, \mathbf{u})$ is the unique strong solution of (KM). The a priori and a posteriori error estimates can be deduced directly from Theorem 3.1. ∎

6 The Treatment of the Convection Terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and $\mathbf{u} \cdot \nabla \rho$

In the analysis of both schemes, the nonlinear term for the velocity $(\mathbf{u} \cdot \nabla)\mathbf{u}$ has been considered in an explicit form, that is:

$$(\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n-1}$$

in order to economize the implementation of the schemes. If we consider a semi-implicit approximation as:

$$(\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n}$$

then the analysis for the iterative methods is even easier, but one can see that the convergence rate is the same.

When this new treatment of the nonlinear term for the velocity is made, we need to change lightly the constraints for the initialization $(\rho^0, \mathbf{u}^0)$ for both schemes as follows:

- **Scheme 1:** There exists a constant $K > 0$ such that:

$$\|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} \le K, \quad C_2\, T\, K^p \le \frac{1}{2} \tag{6.55}$$

  and

$$C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\, T\, \|\rho_0 - \bar\rho_0\|^{2p}_{W^{2,p}} \exp\Big(2\, C_3\, T^{1/p'}\, K\Big) \le \frac{K^p}{2}, \tag{6.56}$$

  where C_1, C_2, C_3 are the constants given by (4.17).

- **Scheme 2:** There exists $K_1, K_2 > 0$ such that:

$$\begin{cases} \|\rho_0 - \bar\rho_0\|_{W^{2,p}} \le K_1, \\[2ex] \|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} \le K_2, \quad C_2\, T\, K_2^p \le \dfrac{1}{2} \end{cases} \tag{6.57}$$

  and

$$\begin{cases} \|\rho_0 - \bar\rho_0\|_{W^{2,p}} \exp\Big(C_3\, T^{1/p'}\, K_2\Big) \le K_1 \\[2ex] C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} + C_2\, T\, K_1^{2p} \le K_2^p, \end{cases} \tag{6.58}$$

  where C_1, C_2, C_3 are the constants given by (5.38).

Remark 6.1 (Smallness for T or for initial data for Scheme 1) *Hypothesis* (6.55)−(6.56) *are verified choosing either T or initial data small enough. Concretely,*

- **small time***: Taking $K > 0$ such that $\|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})} \le K$ and $C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p} \le K^p/4$, we choose $T_* > 0$ small enough in order to obtain the following inequalities:*

$$C_2\, T_*\, K^p \le \frac{1}{2}, \quad C_2\, T_*\, \|\rho_0 - \bar\rho_0\|^{2p}_{W^{2,p}} \exp\Big(2\, C_3\, T_*^{1/p'} K\Big) \le \frac{K^p}{4}.$$

- **small data:** *Now, the time T is fixed and we choose K, $C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}^{2-2/p}_p}$, $\|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})}$ and $\|\rho_0 - \bar\rho_0\|_{W^{2,p}}$ small enough such that* (6.55)-(6.56) *hold.*

Remark 6.2 (Smallness for T or for initial data for Scheme 2) *Hypotheses* (6.57)-(6.58) *are verified choosing either T or initial data conveniently. Concretely,*

- **small time**: *Taking* $K_1 > 0$ *such that* $K_1 > \|\rho_0 - \bar\rho_0\|_{W^{2,p}}$ *and* $K_2 > 0$ *such that* $K_2^p \geq \max\{4\, C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}},\, \|\mathbf{u}^0\|^p_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})}\}$, *we choose* $T_* > 0$ *small enough in order to obtain the following inequalities:*

$$\left\{ \begin{array}{l} \|\rho_0 - \bar\rho_0\|_{W^{2,p}} \exp\left(C_3\, T_*^{1/p}\, K_2\right) \leq K_1, \\ \\ C_2\, T_*\, K_2^p \leq \dfrac{1}{2}, \qquad C_2\, T_*\, K_1^{2p} \leq \dfrac{K_2^p}{4}. \end{array} \right.$$

- **small data:** *Now, the time* T *is fixed. Therefore, we choose* K_1, K_2, $C_1\, \|\mathbf{u}_0\|^p_{\mathbf{B}_p^{2-2/p}}$, $\|\mathbf{u}^0\|_{L^\infty_T(\mathbf{W}^{1,p})\cap L^p_T(\mathbf{W}^{2,p})}$ *and* $\|\rho_0 - \bar\rho_0\|_{W^{2,p}}$ *small enough such that* (6.57)-(6.58) *hold.*

On the other hand, the choice of $\mathbf{u}^{n-1}\cdot\nabla\rho^n$ for the convection term in the density equation may be essential in order to conserve the well properties of the scheme. An implicit treatment, replacing the term $\mathbf{u}^{n-1}\cdot\nabla\rho^n$ by $\mathbf{u}^n\cdot\nabla\rho^n$, couples the computation of ρ^n and $\mathbf{u}^n$ in a nonlinear manner, and makes that the the inductive process to obtain scheme estimates of the Lemma 4.3 cannot be clearly formulated. But, it is important to remark that this nonlinear scheme coupled with the semi-implicit treatment of the convective term for the velocity $(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n$ conserves the energy equality (1.1), although the estimates obtained from (1.1) are not sufficient to guarantee the well-posedness of the nonlinear coupled problem in order to compute each step n of the schemes.

Finally, if an explicit treatment replacing the term $\mathbf{u}^{n-1}\cdot\nabla\rho^n$ by $\mathbf{u}^{n-1}\cdot\nabla\rho^{n-1}$ is considered, the resulting scheme can be more easily computed but it is not clear how to obtain the inequality (4.16), which is the key to arrive at the scheme estimates of the Lemma 4.3.

A Proof of Theorem 3.1

Let $x \in A$ and its corresponding sequence $(R^n x)$, which is well-defined using that $R(A) \subset A$. We start proving that $(R^n x)$ is a Cauchy sequence, for each $x \in A$. First, we write (3.11) for $n = m$ and $s = 1$, that reads:

$$\|R^{m+1}x - R^m x\|_X \leq G(m)\, \|Rx - x\|_X \qquad \forall x \in A. \tag{A.59}$$

Observe that applying $(A.59)$, we obtain:

$$\begin{aligned} \|R^s x - x\|_X &\leq \|R^s x - R^{s-1}x\|_X + \|R^{s-1}x - R^{s-2}x\|_X + \cdots + \|Rx - x\|_X \\ &\leq G(s-1)\, \|Rx - x\|_X + G(s-2)\, \|Rx - x\|_X + \cdots + G(0)\, \|Rx - x\|_X \\ &\leq \|Rx - x\|_X \sum_{j=0}^{s-1} G(j) \;\leq\; G\, \|Rx - x\|_X \end{aligned} \tag{A.60}$$

Then, from (3.11) and (A.60):

$$\|R^{n+s}x - R^n x\|_X \le G(n)\, G\, \|Rx - x\|_X, \tag{A.61}$$

hence, $(R^n x)$ is a Cauchy sequence in X, i.e., $\lim_{n\to+\infty} R^n x = \widehat{x}$ in X. Obviously, $\widehat{x} \in A$ because A is a closed set. Moreover, $\widehat{x}$ is a fixed point of R using the continuity of R. On the other hand, taking $s \uparrow +\infty$ in (A.61), inequality (3.12) is obtained. Finally, making a similar argument as in (A.60), one has:

$$\|R^{n+s}x - R^n x\|_X \le G\, \|R^n x - R^{n-1}x\|_X,$$

hence, taking $s \uparrow +\infty$, inequality (3.13) holds.

B Proof of Lemma 5.5

Applying (5.46) two times, we obtain (for each $t_1 \in [0,T]$):

$$\begin{aligned} f_n(t_1) \;&+\; \int_0^{t_1} g_n(t_2)\, d\,t_2 \le C(t_1) \left\{ \frac{K}{\delta} \int_0^{t_1} f_{n-1}(t_2)\, d\,t_2 + \delta \int_0^{t_1} g_{n-1}(t_2)\, d\,t_2\, d\,t_2 \right\} \\ &\le\; C(t_1)^2 \left\{ \frac{K}{\delta} \int_0^{t_1} \left[\frac{K}{\delta} \int_0^{t_2} f_{n-2}(t_3)\, d\,t_3 + \delta \int_0^{t_2} g_{n-2}(t_3)\, d\,t_3 \right] d\,t_2 \right. \\ &+\; \left. \delta \left[\frac{K}{\delta} \int_0^{t_1} f_{n-2}(t_2)\, d\,t_2 + \delta \int_0^{t_1} g_{n-2}(t_2)\, d\,t_2 \right] \right\}. \end{aligned}$$

Then,

$$\begin{aligned} f_n(t_1) \;&+\; \int_0^{t_1} g_n(t_2)\, d\,t_2 \le C(t_1)^2 \left\{ \left(\frac{K}{\delta}\right)^2 \int_0^{t_1} \left(\int_0^{t_2} f_{n-2}(t_3)\, d\,t_3 \right) d\,t_2 \right. \\ &+\; \left. \left(\frac{K}{\delta}\right) \delta \int_0^{t_1} \left(f_{n-2}(t_2) + \int_0^{t_2} g_{n-2}(t_3)\, d\,t_3 \right) d\,t_2 + \delta^2 \int_0^{t_1} g_{n-2}(t_2)\, d\,t_2 \right\}. \end{aligned}$$

Following this process, it is easy to arrive at the expression:

$$\begin{aligned} & f_n(t_1) + \int_0^{t_1} g_n(t_2)\, d\,t_2 \\ & \le C(t_1)^n \left\{ \left(\frac{K}{\delta}\right)^n \int_0^{t_1} \int_0^{t_2} \cdots (n \text{ times}) \cdots \int_0^{t_n} f_0(t_{n+1})\, d\,t_{n+1}\, d\,t_n \cdots d\,t_2 \right. \\ & + \left(\frac{K}{\delta}\right)^{n-1} \delta \int_0^{t_1} \int_0^{t_2} ((n-1) \text{ times}) \int_0^{t_{n-1}} \left(f_0(t_n) + \int_0^{t_n} g_0(t_{n+1}) dt_{n+1} \right) dt_n \cdots dt_2 \\ & \left. + \cdots + \left(\frac{K}{\delta}\right) \delta^{n-1} \int_0^{t_1} \left(f_0(t_2) + \int_0^{t_2} g_0(t_3)\, d\,t_3 \right) d\,t_2 + \delta^n \int_0^{t_1} g_0(t_2)\, d\,t_2 \right\} \end{aligned}$$

Then, rearranging and applying Fubini's result, we get:

$$f_n(t) + \int_0^t g_n(\sigma)\, d\sigma \le E(t)^n \left\{ \int_0^t f_0(\sigma) \left[\left(\frac{K}{\delta}\right)^n \frac{1}{(n-1)!}(t-\sigma)^{n-1} \right.\right.$$

$$\left. + \left(\frac{K}{\delta}\right)^{n-1} \delta \frac{1}{(n-2)!}(t-\sigma)^{n-2} + \cdots \left(\frac{K}{\delta}\right)^2 \delta^{n-2}\,(t-\sigma) + \left(\frac{K}{\delta}\right) \delta^{n-1} \right] d\sigma$$

$$+ \int_0^t g_0(\sigma) \left[\left(\frac{K}{\delta}\right)^{n-1} \delta \frac{1}{(n-1)!}(t-\sigma)^{n-1} + \left(\frac{K}{\delta}\right)^{n-2} \delta \frac{1}{(n-2)!}(t-\sigma)^{n-2} \right.$$

$$\left.\left. + \cdots + \left(\frac{K}{\delta}\right)^2 \delta^{n-2}\,\frac{1}{2}\,(t-\sigma)^2 + \left(\frac{K}{\delta}\right) \delta^{n-1}\,(t-\sigma) + \delta^n \right] d\sigma \right\}$$

As conclusion, we have that:

$$\begin{aligned}
f_n(t) + \int_0^t g_n(\sigma)\, d\sigma \;&\le\; E(t)^n \left\{ \int_0^t f_0(\sigma) \left(\frac{K}{\delta}\right) \left[\sum_{l=0}^{n-1} \frac{(t-\sigma)^l}{l!} \left(\frac{K}{\delta}\right)^l \delta^{n-1-l} \right] d\sigma \right. \\
&+ \left. \int_0^t g_0(\sigma) \left[\sum_{l=0}^{n-1} \frac{(t-\sigma)^l}{l!} \left(\frac{K}{\delta}\right)^l \delta^{n-l} \right] d\sigma \right\} \\
&\le\; E(t)^n \left\{ \delta^n \, \|f_0\|_{L^\infty(0,t)} \sum_{l=1}^{n} \left(\frac{K}{\delta^2}\right)^l \frac{t^l}{l!} \right. \\
&+ \left. \delta^n \, \|g_0\|_{L^1(0,t)} \sum_{l=0}^{n-1} \left(\frac{K}{\delta^2}\right)^l \frac{t^l}{l!} \right\} \\
&\le\; (E(t)\,\delta)^n \left\{ \|f_0\|_{L^\infty(0,t)} \left(e^{\frac{K}{\delta^2}\,t} - 1 \right) + \|g_0\|_{L^1(0,t)}\, e^{\frac{K}{\delta^2}\,t} \right\} \\
&\le\; (E(t)\,\delta)^n \; e^{\frac{K}{\delta^2}\,t} \left\{ \|f_0\|_{L^\infty(0,t)} + \|g_0\|_{L^1(0,t)} \right\}.
\end{aligned}$$

C The proof of Lemma 5.8

Raising (5.50) to the power r', we obtain:

$$(f_n(t))^{r'} + \left(\int_0^t g_n(s)\, ds \right)^{r'} \le E(t)^{r'} \left\{ \frac{1}{\delta} \int_0^t h_{n-1}(s)\, f_{n-1}(s)\, ds + \delta \int_0^t g_n(s)\, ds \right\}^{r'}.$$

Using Holdr's inequality, we have:

$$\begin{aligned}
\int_0^t h_{n-1}(s)\, f_{n-1}(s)\, ds \;&\le\; \left(\int_0^t (h_{n-1}(s))^r\, ds \right)^{1/r} \left(\int_0^t (f_{n-1}(s))^{r'}\, ds \right)^{1/r'} \\
&\le\; C \left(\int_0^t (f_{n-1}(s))^{r'}\, ds \right)^{1/r'},
\end{aligned}$$

being r and r' conjugate exponents. Thus,

$$\begin{aligned}
(f_n(t))^{r'} &+ \left(\int_0^t g_n(s)\,ds\right)^{r'} \\
&\le E(t)^{r'} \left\{\frac{C}{\delta}\left(\int_0^t (f_{n-1}(s))^{r'}\,ds\right)^{1/r'} + \delta \int_0^t g_{n-1}(s)\,ds\right\}^{r'} \\
&\le K(r')\,E(t)^{r'}\left\{\left(\frac{C}{\delta}\right)^{r'} \int_0^t (f_{n-1}(s))^{r'}\,ds + \delta^{r'}\left(\int_0^t g_{n-1}(s)\,ds\right)^{r'}\right\}.
\end{aligned}$$

Denoting

$$F_n(t) = (f_n(t))^{r'}, \quad G_n(t) = \left(\int_0^t g_n(s)\,ds\right)^{r'} \quad \text{and} \quad K = K(r')\,E(t)^{r'},$$

the previous expression can be written in a most simplified form as:

$$F_n(t) + G_n(t) \le K\left\{\left(\frac{C}{\delta}\right)^{r'} \int_0^t F_{n-1}(s)\,ds + \delta^{r'}\,G_{n-1}(t)\right\} \tag{C.62}$$

We will take $(C.62)$ as the starting inequality in order to obtain an estimate of $F_n(t)$ and $G_n(t)$ only depending on the initial data. In fact, applying $(C.62)$

$$\begin{aligned}
F_n(t_1) + G_n(t_1) \;&\le\; K\left\{\left(\frac{C}{\delta}\right)^{r'} \int_0^{t_1} K\left[\left(\frac{C}{\delta}\right)^{r'} \int_0^{t_2} F_{n-2}(t_3)\,dt_3 + \delta^{r'}\,G_{n-2}(t_2)\right] dt_2\right. \\
&+ \left.\delta^{r'}\,K\left[\left(\frac{C}{\delta}\right)^{r'} \int_0^{t_1} F_{n-2}(t_2)\,dt_2 + \delta^{r'}\,G_{n-2}(t_1)\right]\right\}.
\end{aligned}$$

Then, applying again $(C.62)$,

$$\begin{aligned}
F_n(t_1) &+ G_n(t_1) \le K^2\left(\frac{C}{\delta}\right)^{2r'} \int_0^{t_1}\left(\int_0^{t_2} F_{n-2}(t_3)\,dt_3\right) dt_2 \\
&+ K^2\left(\frac{C}{\delta}\right)^{r'} \delta^{r'} \int_0^{t_1} \left(F_{n-2}(t_2) + G_{n-2}(t_2)\right) dt_2 + K^2\,\delta^{2r'}\,G_{n-2}(t_1) \\
&\le K^3\left(\frac{C}{\delta}\right)^{3r'} \int_0^{t_1}\left(\int_0^{t_2}\left(\int_0^{t_3} F_{n-3}(t_4)\,dt_4\right) dt_3\right) dt_2 \\
&+ K^3\left(\frac{C}{\delta}\right)^{2r'} \delta^{r'} \int_0^{t_1}\left(\int_0^{t_2} \left[F_{n-3}(t_3) + G_{n-3}(t_3)\right] dt_3\right) dt_2 \\
&+ K^3\left(\frac{C}{\delta}\right)^{r'} \delta^{2r'} \int_0^{t_1} \left[F_{n-3}(t_2) + G_{n-3}(t_2)\right] dt_2 + K^3\,\delta^{3r'}\,G_{n-3}(t_1)
\end{aligned}$$

And so on,

$$\begin{aligned}
F_n(t_1) + G_n(t_1) \;&\le\; K^n \int_0^{t_1} F_0(s) \sum_{l=0}^{n-1} \left[\left(\frac{C}{\delta} \right)^{r'} \right]^{l+1} \frac{(t_1 - s)^l}{l!} \left(\delta^{r'} \right)^{n-1-l} ds \\
&+\; K^n \int_0^{t_1} G_0(s) \sum_{l=0}^{n-2} \left[\left(\frac{C}{\delta} \right)^{r'} \right]^{l+1} \frac{(t_1 - s)^l}{l!} \left(\delta^{r'} \right)^{n-1-l} ds \\
&+\; K^n \left(\delta^{r'} \right)^n G_0(t_1) := I_1 + I_2 + I_3 \qquad \text{(C.63)}
\end{aligned}$$

Now, we estimate the first term on the right hand side of $(C.63)$ as follows:

$$\begin{aligned}
I_1 \;&\le\; K^n \left(\delta^{r'} \right)^n \|F_0\|_{L^\infty(0,t_1)} \sum_{l=0}^{n-1} \left[\left(\frac{C}{\delta^2} \right)^{r'} \right]^{l+1} \frac{t_1^{l+1}}{(l+1)!} \\
&\le\; K^n \left(\delta^{r'} \right)^n \|F_0\|_{L^\infty(0,t_1)} \left(e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1} - 1 \right) \\
&\le\; K^n \left(\delta^{r'} \right)^n \|F_0\|_{L^\infty(0,t_1)} \, e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1}.
\end{aligned}$$

We estimate the second term on the right hand side of $(C.63)$:

$$\begin{aligned}
I_2 \;&\le\; K^n \left(\delta^{r'} \right)^n \|G_0\|_{L^\infty(0,t_1)} \sum_{l=1}^{n-1} \left[\left(\frac{C}{\delta^2} \right)^{r'} \right]^{l} \frac{t_1^{l}}{l!} \\
&\le\; K^n \left(\delta^{r'} \right)^n \|G_0\|_{L^\infty(0,t_1)} \left(e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1} - 1 \right) \\
&\le\; K^n \left(\delta^{r'} \right)^n G_0(t_1) \left(e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1} - 1 \right),
\end{aligned}$$

where we have used the fact that G_0 is an increasing in time function (because it is defined as the integral of a positive function). Therefore,

$$I_2 + I_3 \le K^n \left(\delta^{r'} \right)^n G_0(t_1) \, e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1}.$$

In summary, we arrive at:

$$F_n(t_1) + G_n(t_1) \le \left(K \, \delta^{r'} \right)^n \left\{ \|F_0\|_{L^\infty(0,t_1)} + G_0(t_1) \right\} e^{\left(\frac{C}{\delta^2}\right)^{r'} t_1}.$$

If we raise to the power $1/r'$, bounded from below the terms on the left hand side, and from above the terms on the right hand side, we obtain:

$$\begin{aligned}
f_n(t_1) + \int_0^{t_1} g_n(s) \, ds \;&\le\; P(r') \left(K^{1/r'} \delta \right)^n \left\{ \|F_0\|_{L^\infty(0,t_1)} + G_0(t_1) \right\}^{1/r'} e^{\left(\frac{C}{\delta^2}\right)^{r'} \frac{t_1}{r'}} \\
&\le\; P(r') \left(K^{1/r'} \delta \right)^n e^{\left(\frac{C}{\delta^2}\right)^{r'} \frac{t_1}{r'}} K(1/r') \left\{ \|f_0\|_{L^\infty(0,t_1)} + \|g_0\|_{L^1(0,t_1)} \right\},
\end{aligned}$$

hence we arrive at (5.51), taking into account that $K = K(r')\,E(t)^{r'}$.

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