On the very weak solution for the Oseen and Navier-Stokes equations*

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Abstract

We study the existence of very weak solutions regularity for the Stokes, Oseen and Navier-Stokes system when non-smooth Dirichlet boundary data for the velocity are considered in domains of class $C^{1,1}$. In the Navier-Stokes case, the results will be valid for external forces non necessarily small. Regularity results for more regular data will be also discussed.

Keywords: Stokes equations, Oseen equations, Navier-Stokes equations, Very weak solutions, Stationary Solutions.

AMS Subject Classification: Primary: 35Q30; Secondary: 76D03, 76D05, 76D07

1 Introduction and notations

In this work, we are interested in some questions concerning the Navier-Stokes equations, defined in Ω a bounded open set of $\mathbb{R}^3$ with boundary Γ:

$$(NS)\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla q &= \boldsymbol{f} \quad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} &= h \quad \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma,\end{cases}$$

*The second author has been partially supported by M.E.C. (Spain), Project MTM2006-07932, and by Junta de Andalucía, Project P06-FQM-02373.

where $\boldsymbol{u}$ denotes the velocity and q the pressure and both are unknown. The external force $\boldsymbol{f}$, the compressibility condition h and the boundary condition for the velocity $\boldsymbol{g}$ are given functions. The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^3$ are respectively denoted by boldface Roman and special Roman.

In the homogeneous case, $h = 0$, it has been well-known since Leray [18] (see also [19]) that if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $p \geq 2$ and for any $i = 0, \ldots, I$,

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma = 0, \tag{1.1}$$

where Γ_i denote the connected components of the boundary Γ of the open set Ω, then there exists a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (NS). In [25], Serre proved the existence of weak solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ for any $\frac{3}{2} < p < 2$ when $h = 0$ and $\boldsymbol{g}$ satisfies the above conditions. More recently, Kim [17] improves Serre's existence and regularity results on weak solutions of (NS) for any $\frac{3}{2} \leq p < 2$ (including the case $p = \frac{3}{2}$), when the boundary of Ω is connected ($I = 0$) provided h is small in an appropriate norm (due to the compatibility condition between h and $\boldsymbol{g}$, then $\boldsymbol{g}$ is also small in the corresponding appropriate norm).

On the other hand, the notion of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for Stokes or Navier-Stokes equations, corresponding to very irregular data, has been developed in the last years by Giga [15] (in a domain Ω of class $\mathcal{C}^\infty$), Amrouche & Girault [3] (in a domain Ω of class $\mathcal{C}^{1,1}$) and more recently by Galdi et al. [14], Farwig et al. [11] (in a domain Ω of class $\mathcal{C}^{2,1}$, see also Schumacher [24]). In this context, the boundary condition is chosen in $\mathbf{L}^p(\Gamma)$ (see Brown & Shen [7], Conca [9], Fabes et al. [10], Moussaoui & Zine [21], Shen [26], Savaré [23], Marusič-Paloka [20]) or more generally in $\mathbf{W}^{-1/p,p}(\Gamma)$.

The purpose of our work is to develop a unified theory of very weak solutions of the Dirichlet problem for Stokes, Oseen and Navier-Stokes equations (and also for the Laplace equation), see Theorem 4.10 and Theorem 5.4. One important question is to define rigorously the traces of the vector functions which are living in subspaces of $\mathbf{L}^p(\Omega)$ (see Lemma 2.10 and Lemma 2.11). We prove existence and regularity of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ of Stokes and Oseen equations for any $1 < p < \infty$ with arbitrary large data belonging to some Sobolev spaces of negative order. In the case of Navier-Stokes equations the existence of very weak solution is proved for arbitrary large external forces, but with a smallness condition for both h and $\boldsymbol{g}$. Uniqueness of very weak solutions is also proved for small enough data.

Existence of very weak solution $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$, for arbitrary large external forces $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $h = 0$ and arbitrary large boundary condition $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ and without assuming condition (1.1), was proved first by Marusič-Paloka in [20] (see Theorem 5) with Ω a bounded simply-connected open set of class $\mathcal{C}^{1,1}$. But the proof of Theorem 5 becomes correct only if either condition (1.1) or condition (5.56) holds. The same result was proved by Kim [17] for arbitrary large external forces $\boldsymbol{f} \in [\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)]'$, for small $h \in [W^{1,3/2}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ and where the boundary of Ω is supposed connected ($I = 0$). Remark that the space chosen for the divergence condition h is not correct, because $\mathcal{D}(\Omega)$ is not dense in $\mathbf{W}^{-1/3,3}(\Gamma)$ and his dual is not a subspace of distributions. Similar argument can be done for the space chosen for the external forces $\boldsymbol{f}$. The origin of this mistake (also present everywhere in the same paper [17]) is due to the fact that when we want to solve a boundary value problem, it is necessary to have an adequate Green formula and corresponding density lemmas.

In a close context, we also consider the case where the data, and then the solutions, belong to fractionary Sobolev spaces $W^{s,p}(\Omega)$ with s a real number possibly not integer (see Theorem 4.12)

The work is organized as follows: In the remains of this section, we recall the definitions of some spaces and their respective norms.

In §2, some preliminary results are stated, including density lemmas, general trace's results, characterization of dual spaces and trace's result for very weak solutions. In §3, we present Stokes' results related to the very weak, weak and strong solution. Some of them generalized those appearing in [3] in order to be extended to the Oseen and Navier-Stokes systems. In §4, we extend the results of §3 for the Oseen system. The first two main results in this paper are presented here: one about existence and uniqueness of very weak solution for the Oseen equations in $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ with $1 < p < \infty$ (see Theorem 4.10), and another one related to the regularity of solutions for the Oseen equations (see Theorem 4.12). We consider in particular the case where the external forces $\boldsymbol{f}$ and the divergence condition h are not regular, more precisely $\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega)$ and $h \in W^{\sigma-1,p}(\Omega)$ with $\frac{1}{p} < \sigma \leq 2$. In §5, existence of very weak solution for the Navier-Stokes system is obtained, using a fixed point technique over the Oseen system, first for the case of small data and then for arbitrary large external forces $\boldsymbol{f}$ but sufficiently small h and $\boldsymbol{g}$ in a domain possibly multiply-connected. The results is stated in Theorem 5.4. Regularity results for this system are obtained in Theorem 5.5. The complete proofs of results can be seen in [5].

In all this work, if we do not say anything else, Ω will be considered as a Lipschitz open bounded set of $\mathbb{R}^3$. When Ω is connected, we will say Ω is a domain. We will only specify the regularity of Ω when it to be different from the regularity presented above.

1.1 Functional framework

In what follows, for any $s \in \mathbb{R}$, p denotes a real number such that $1 < p < \infty$ and p' stands for its conjugate: $1/p + 1/p' = 1$. We shall denote by m the integer part of s and by σ its fractional part: $s = m + \sigma$ with $0 \leq \sigma < 1$. We denote by $W^{s,p}(\mathbb{R}^3)$ the space of all distributions v defined in $\mathbb{R}^3$ such that:

- $D^\alpha v \in L^p(\mathbb{R}^3)$, for all $|\alpha| \leq m$, when $s = m$ is a nonnegative integer

- $v \in W^{m,p}(\mathbb{R}^3)$ and
$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{3+\sigma p}}\, \mathrm{d}x\, \mathrm{d}y < \infty,$$
for all $|\alpha| = m$, when $s = m + \sigma$ is nonnegative and is not an integer.

The space $W^{s,p}(\mathbb{R}^3)$ is a reflexive Banach space equipped by the norm:

$$\|v\|_{W^{m,\,p}(\mathbb{R}^3)} = \Big(\sum_{|\alpha|\le m} \int_{\mathbb{R}^3} |D^\alpha v(x)|^p \; d\boldsymbol{x} \Big)^{1/p}$$

in the first case, and by the norm

$$\|v\|_{W^{s,\,p}(\mathbb{R}^3)} = \Big(\|v\|^p_{W^{m,p}(\mathbb{R}^3)} + \sum_{|\alpha|=m} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{3+\sigma p}}\, \mathrm{d}x\, \mathrm{d}y \Big)^{1/p},$$

in the second case. For $s < 0$, we denote by $W^{s,\,p}(\mathbb{R}^3)$ the dual space of $W^{-s,\,p'}(\mathbb{R}^3)$. In the special case of $p = 2$, we shall use the notation $H^s(\mathbb{R}^3)$ instead of $W^{s,\,2}(\mathbb{R}^3)$.

Now, we introduce the Sobolev space

$$H^{s,p}(\mathbb{R}^3) = \{v \in L^p(\mathbb{R}^3);\; (I - \Delta)^{s/2} v \in \; L^p(\mathbb{R}^3)\}.$$

It is known that $H^{s,p}(\mathbb{R}^3) = W^{s,p}(\mathbb{R}^3)$ if s is an integer or if $p = 2$. Furthermore, for any real number s, we have the following embeddings:

$$W^{s,p}(\mathbb{R}^3) \hookrightarrow H^{s,p}(\mathbb{R}^3) \quad \text{if } p \le 2 \quad \text{and} \quad H^{s,p}(\mathbb{R}^3) \hookrightarrow W^{s,p}(\mathbb{R}^3) \quad \text{if } p \ge 2.$$

The definition of the space $W^{s,p}(\Omega)$ is exactly the same as in the case of the whole space. Because $\mathcal{D}(\Omega)$ is not dense in $W^{s,p}(\Omega)$, the dual space of $W^{s,p}(\Omega)$ cannot be identified to a space of distributions in Ω. For this reason, we define $W^{s,p}_0(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$ and we denote by $W^{-s,\,p'}(\Omega)$ its dual space.

For every $s > 0$, we denote by $W^{s,p}(\overline{\Omega})$ the space of all distributions in Ω which are restrictions of elements of $W^{s,p}(\mathbb{R}^3)$ and by $\widetilde{W}^{s,p}(\Omega)$ the space of functions $u \in W^{s,p}(\overline{\Omega})$ such that the extension $\widetilde{u}$ by zero outside of Ω belongs to $W^{s,p}(\mathbb{R}^3)$.

2 Preliminary results

We present here some trace results, density results, De Rham's theorems and characterizations of some spaces, either known or designed specially for the Stokes, Oseen and Navier-Stokes problems, that will be used in the following sections.

Recall now some density results ([1, 16]):

i) The space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any real s.

ii) The space $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^{s,p}(\mathbb{R}^3)$ and in $H^{s,p}(\mathbb{R}^3)$ for any real s.

iii) The space $\mathcal{D}(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$.

iv) The space $\mathcal{D}(\Omega)$ is dense in $W^{s,p}(\Omega)$ for all $0 < s \leq 1/p$, that means that $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$.

Next result gives some properties of traces of functions living in $W^{s,\,p}(\Omega)$ ([1, 16]).

Theorem 2.1 *Let Ω be a bounded open set of class $\mathcal{C}^{k,1}$, for some integer $k \geq 0$. Let s be real number such that $s \leq k+1, s - 1/p = m + \sigma$, where $m \geq 0$ is an integer and $0 < \sigma < 1$.*
i) The following mapping

$$\begin{aligned} \gamma_0 : u \quad &\mapsto \quad u_{|\Gamma} \\ W^{s,\,p}(\Omega) \quad &\rightarrow \quad W^{s-1/p,\,p}(\Gamma) \end{aligned}$$

is continuous and surjective. When $1/p < s < 1 + 1/p$, we have Ker $\gamma_0 = W_0^{s,\,p}(\Omega)$.
ii) For $m \geq 1$, the following mapping

$$\begin{aligned} (\gamma_0, \gamma_1) : u \quad &\mapsto \quad (u_{|\Gamma}, \tfrac{\partial u}{\partial \boldsymbol{n}}_{|\Gamma}) \\ W^{s,\,p}(\Omega) \quad &\rightarrow \quad (W^{s-1/p,\,p}(\Gamma) \times W^{s-1-1/p,\,p}(\Gamma)) \end{aligned}$$

is continuous and surjective. When $1 + 1/p < s < 2 + 1/p$, we have Ker $(\gamma_0, \gamma_1) = W_0^{s,\,p}(\Omega)$.

We recall also the following embeddings:

$$W^{s,\,p}(\Omega) \hookrightarrow W^{t,\,q}(\Omega) \quad \text{for } t \leq s,\ p \leq q \quad \text{such that } s - 3/p = t - 3/q$$

and

$$W^{s,\,p}(\Omega) \hookrightarrow \mathcal{C}^{k,\,\alpha}(\overline{\Omega}) \quad \text{for } k < s - 3/p < k+1, \quad \alpha = s - k - 3/p,$$

where k is a non negative integer.

Then, we introduce the following spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}, \quad \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) = \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})^3;\ \nabla \cdot \boldsymbol{\psi} = 0\}.$$

Recall now two versions of De Rham's Theorem, the first one proved by G. de Rham [22] and the second by C. Amrouche & V. Girault [3]:

Lemma 2.2 (De Rham's Theorem for distributions) *Let Ω be any open subset of $\mathbb{R}^3$ and let $\boldsymbol{f}$ be a distribution of $\boldsymbol{\mathcal{D}}'(\Omega)$ that satisfies:*

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{v} \rangle = 0.$$

Then, there exists a distribution π in $\mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$.

Lemma 2.3 (De Rham's Theorem in $\mathbf{W}^{-m,p}(\Omega)$) *Let m be any integer, p any real number with $1 < p < \infty$. Let $\boldsymbol{f} \in \mathbf{W}^{-m,p}(\Omega)$ satisfy:*

$$\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0.$$

Then, there exists $\pi \in W^{-m+1,p}(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$. If in addition the set Ω is connected, then π is defined uniquely, up to an additive constant, and there exists a positive constant C, independent of $\boldsymbol{f}$, such that:

$$\inf_{K \in \mathbb{R}} \|\pi + K\|_{W^{-m+1,p}(\Omega)/\mathbb{R}} \leq C \, \|\boldsymbol{f}\|_{\mathbf{W}^{-m,p}(\Omega)}.$$

The two next lemmas are density results:

Lemma 2.4 *The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{H}_p(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}$.*

Proof.[Sketch of the proof] Let $\boldsymbol{\ell}$ be a linear and continuous mapping in $\mathbf{H}_p(\Omega)$ such that $\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$. Since $\mathbf{H}_p$ is a subspace of $\mathbf{L}^p(\Omega)$, we can extend $\boldsymbol{\ell}$ to $\boldsymbol{L} \in \mathbf{L}^{p'}(\Omega)$. We will suppose that Ω is bounded, connected but eventually multiply-connected (when Ω is not connected, we can repeat the procedure above in each connected component of Ω), being $\bigcup_{1 \leq i \leq I} \omega_i$ its wholes, and its boundary Γ is Lipschitz-continuous. We denote by ω_0 the exterior of Ω, by Γ_0 the exterior boundary of Ω and by Γ_i, $1 \leq i \leq I$, the other components of Γ. The duality between $\mathbf{W}^{-1/p,p'}(\Gamma_i)$ and $\mathbf{W}^{1/p,p}(\Gamma_i)$, and $\mathbf{W}^{-1/p,p'}(\Gamma_0)$ and $\mathbf{W}^{1/p,p}(\Gamma_0)$, will be denoted by $\langle \cdot, \cdot \rangle_{\Gamma_i}$ and $\langle \cdot, \cdot \rangle_{\Gamma_0}$, respectively. By De Rham's Lemma 2.3, there exists a unique $q \in W^{1,p'}(\Omega) \cap L^{p'}_0(\Omega)$ such that $\boldsymbol{L} = \nabla q$ and

$$L^{p'}_0(\Omega) = \left\{ \varphi \in L^{p'}(\Omega); \int_\Omega \varphi(\boldsymbol{x}) \, d\boldsymbol{x} = 0 \right\}.$$

Moreover,

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}), \qquad \langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = \langle q, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma = 0.$$

We extend $\boldsymbol{L}$ by zero out of Ω and denote the extension by $\widetilde{\boldsymbol{L}}$. Then, for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ such that $\nabla \cdot \boldsymbol{\varphi} = 0$ in $\mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{L}} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} = \int_\Omega \boldsymbol{L} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} = 0.$$

From that, we deduce that, thanks to De Rham's Lemma 2.2, there exists $h \in \mathcal{D}'(\mathbb{R}^3)$ verifying $\nabla h \in \mathbf{L}^{p'}(\mathbb{R}^3)$ such that $\widetilde{\boldsymbol{L}} = \nabla h$ (see Lemma 2.1 in [4]). It is clear that $h \in W^{1,p'}_{loc}(\mathbb{R}^3)$. As h is unique up to an additive constant and $\nabla h = 0$ in ω_0, we can choose this constant in such a way that $h = 0$ in ω_0. Therefore, we deduce that: $h = 0 \quad \text{in } \omega_0, \qquad h = c_i \quad \text{in each } \omega_i, \qquad h = q + c_0 \quad \text{in } \Omega,$

and thus: $q = -c_0 \quad \text{on } \Gamma_0, \qquad q = c_i - c_0 \quad \text{on } \Gamma_i,\ 1 \leq i \leq I.$

Let $j \in \{1, \ldots, I\}$ be a fixed index, choosing $\boldsymbol{v}_j \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that $\langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k} = \delta_{jk}$ for $1 \leq k \leq I$ and $\langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} = -1$, we can deduce that $c_j = 0$. In consequence, for every $\boldsymbol{v} \in \mathbf{H}_p(\Omega)$, we have:

$$\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \int_\Omega \nabla h \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0.$$

Thus, we deduce that $\boldsymbol{\ell} = \mathbf{0}$ in $\mathbf{H}'_p(\Omega)$. ∎

In the sequel, we will use the following space

$$\mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,r}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\}, \quad 1 < r,\, p < \infty, \tag{2.2}$$

and we set $\mathbf{X}_{p,p}(\Omega) = \mathbf{X}_p(\Omega)$. Observe that the space $\mathbf{X}_{p,p}(\Omega)$ were used in [3] in order to define very weak solution for the Stokes problem. In the case of Navier-Stokes problem, the generalization to the space $\mathbf{X}_{r,p}(\Omega)$ is necessary. In this sense, the proof of the next result follows from an argument appearing in [2].

Lemma 2.5 *The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$ and for all $q \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, we have*

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)}. \tag{2.3}$$

Next lemmas characterize the space $(\mathbf{X}_{r,p}(\Omega))'$ and give a density result.

Lemma 2.6 *Let $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$. Then, there exist $\mathbb{F}_0 = (f_{ij})_{1 \leq i,j \leq 3}$ such that $\mathbb{F}_0 \in \mathbb{L}^{r'}(\Omega)$ and $f_1 \in W^{-1,p'}(\Omega)$ such that*

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1. \tag{2.4}$$

Moreover,

$$\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1 \leq i,j \leq 3,\ \|f_1\|_{W^{-1,p'}(\Omega)}\}.$$

Conversely, if $\boldsymbol{f}$ satisfies (2.4), then $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$.

As a consequence of Lemma 2.5, we have the following embeddings:

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{2.5}$$

where the second embedding holds if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$.

Lemma 2.7 *Let Ω be a Lipschitz bounded open set. Then, the space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $(\mathbf{X}_{r,p}(\Omega))'$.*

One of the main difficulties for the definition of a very weak solution for Stokes, Oseen and Navier-Stokes problems is to give a meaning to the trace, because we are not in the classical variational framework. We shall use the spaces[1]:

$$\mathbf{T}_{p,r}(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta \boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}, \quad \mathbf{T}_{p,r,\sigma}(\Omega) = \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\},$$

[1]When $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively. Observe that these spaces were introduced in [2, 3]

endowed with the topology given by the norm:

$$\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'},$$

and

$$\mathbf{H}_{p,r}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} \in L^r(\Omega)\},$$

which is equipped with the graph norm. Next density lemmas will be necessary:

Lemma 2.8 *i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$.*

ii) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$.

Lemma 2.9 *The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.*

For the following two lemmas, we will need to introduce the space:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$$

that can also be described (see [3]) as:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\}. \tag{2.6}$$

Observe that the range space of the normal derivative $\gamma_1 : \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is

$$\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z} \cdot \boldsymbol{n} = 0\}.$$

In these lemmas, we prove that the tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is:

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}. \tag{2.7}$$

Recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_\tau$, and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\,\boldsymbol{n}$.

Lemma 2.10 *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$. The Green formula reads: for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi}\, d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}.$$

Lemma 2.11 *i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$.*

ii) Let $1 < p < \infty$ and $r > 1$ such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_{\boldsymbol{n}} : \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by $\gamma_{\boldsymbol{n}}$, from $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$ into $W^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ and $\varphi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{v} \cdot \nabla\varphi \, d\boldsymbol{x} + \int_\Omega \varphi \, \mathrm{div} \, \boldsymbol{v} \, d\boldsymbol{x} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} .$$

Lemma 2.12 *Let Ω be a Lipschitz bounded open set. Let $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ be given such that the condition (3.11) holds. For every $\varepsilon > 0$, there exist sequences $(h_\varepsilon) \subset \mathcal{D}(\overline{\Omega})$ and $(\boldsymbol{g}_\varepsilon) \subset \boldsymbol{\mathcal{C}}^\infty(\Gamma)$ such that*

$$\int_\Omega h_\varepsilon(\boldsymbol{x}) \, d\boldsymbol{x} = \int_\Gamma \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n} \, d\sigma \tag{2.8}$$

and verifying

$$\|h - h_\varepsilon\|_{L^r(\Omega)} \leq \varepsilon \quad \text{and} \quad \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leq \varepsilon \tag{2.9}$$

$$\|h_\varepsilon\|_{L^r(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leq 2\left(\|h\|_{L^r(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\right). \tag{2.10}$$

In all the rest of this work , if we do not say anything else, we assume that Ω is a **bounded connected open set of class $\mathcal{C}^{1,1}$**.

3 The Stokes problem

Before starting the study of the Oseen and Navier-Stokes problems, we focus on the study of the Stokes problem in order to make an appointment about all the knowing results about this system. Recall that the Stokes problem is:

$$(S) \qquad -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

with the compatibility condition:

$$\int_\Omega h(\boldsymbol{x}) \, d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}. \tag{3.11}$$

Basic results on weak and strong solutions of problem (S) in $L^p(\Omega)$ Sobolev spaces may be summarized in the following theorem (see [3], [8], [12]).

Theorem 3.1 *i) For every $\boldsymbol{f}$, h, $\boldsymbol{g}$ with $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$, and satisfying the compatibility condition (3.11), the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and $q \in L^p(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C \, (\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \tag{3.12}$$

ii) Moreover, if $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$, *then* $\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega)$, $q \in W^{1,p}(\Omega)$ *and there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\,(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}). \tag{3.13}$$

In the case of a bounded domain Ω which is only Lipschitz, the result of point i) is only valid for a more restricted p. In fact, if $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$, then there exists $\varepsilon > 0$ such that if $2 \leq p \leq 3 + \varepsilon$, and if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h = 0$ and $\boldsymbol{g} = \mathbf{0}$, then the result is valif for a ε such that $(3+\varepsilon)/(2+\varepsilon) < p < 3 + \varepsilon$ (see [7]).

We are interested in the case of singular data satisfying the following assumptions:

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))',\ h \in L^r(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),\ \text{with } \frac{1}{r} \leq \frac{1}{p} + \frac{1}{3} \text{ and } r \leq p. \tag{3.14}$$

Recall that the space $(\mathbf{X}_{r',p'}(\Omega))'$ is an intermediate space between $W^{-1,r}(\Omega)$ and $W^{-2,p}(\Omega)$ (see embeddings (2.5)).

We recall the definition and the existence result of very weak solution for the Stokes problem.

Definition 3.2 (Very weak solution for the Stokes problem) *We say that* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a* **very weak solution** *of* (S) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ *and* $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned} -\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} &= \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla\pi\, d\boldsymbol{x} = -\int_\Omega h\,\pi\, d\boldsymbol{x} &+ \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi\rangle_\Gamma, \end{aligned} \tag{3.15}$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle \cdot,\cdot\rangle_\Omega = \langle \cdot,\cdot\rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)}, \quad \langle \cdot,\cdot\rangle_\Gamma = \langle \cdot,\cdot\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}. \tag{3.16}$$

Note that $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ and $\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, that means that all the brackets and integrals have a sense.

Proposition 3.3 *Suppose that* $\boldsymbol{f}$, h, $\boldsymbol{g}$ *satisfy (3.14). Then the following two statements are equivalent:*

i) $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a very weak solution of (S),*

ii) $(\boldsymbol{u}, q)$ *satisfies the system (S) in the sense of distributions.*

Proof.[Sketch of the proof] i) Let $(\boldsymbol{u}, q)$ very weak solution to problem (S). It is clear that $-\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u} = h$ in Ω and consequently $\boldsymbol{u}$ belongs to $\mathbf{T}_{p,r}(\Omega)$. Using Lemma 2.11 point *ii)*, Lemma 2.10 and (2.3), we obtain

$$
\begin{aligned}
&- \int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} \, d\boldsymbol{x} + \langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\
&= \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega .
\end{aligned}
$$

Since for any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$,

$$
\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)},
$$

we deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\mathbf{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla \cdot \boldsymbol{u} = h$, we deduce that for any $\pi \in W^{1,p'}(\Omega)$, we have

$$
\langle \boldsymbol{u} \cdot \boldsymbol{n}, \pi \rangle_\Gamma = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma .
$$

Consequently $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u} = \boldsymbol{g}$ on Γ.
ii) The converse is a simple consequence of Lemma 2.11 point *ii)*, Lemma 2.10 and (2.3).

∎

The following result is a variation from Proposition 4.11 in [3], which was made for $\boldsymbol{f} = \mathbf{0}$ and $h = 0$. In the case $r = p$, we have

Proposition 3.4 *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ be given with*

$$
\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))', \quad h \in L^p(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),
$$

and satisfying the compatibility condition (3.11). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$
\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}. \tag{3.17}
$$

Moreover $\boldsymbol{u} \in \mathbf{T}_p(\Omega)$ and

$$
\|\boldsymbol{u}\|_{\mathbf{T}_p(\Omega)} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}.
$$

More generally, taking into account that now we use $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$ instead of $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$ and $h \in L^r(\Omega)$ instead of $h \in L^p(\Omega)$, we can adapt Proposition 3.4 obtaining:

Theorem 3.5 *Let $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfying (3.14) and (3.11). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$
\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\} \tag{3.18}
$$

Moreover $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ *and*

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}.$$

In particular, if $\boldsymbol{f} \in \mathbf{W}^{-1,r_0}(\Omega)$ *and* $h \in L^{r_0}(\Omega)$ *with* $r_0 = 3p/(3+p)$, *then* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *with the corresponding estimates.*

Although that in [14] Theorem 3 the authors obtain a similar result, observe that the domain is considered of class $\mathcal{C}^{2,1}$ instead of class $\mathcal{C}^{1,1}$, and the divergence term $h \in L^p(\Omega)$ instead of $h \in L^r(\Omega)$. Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$ which has been clearly characterized contrary to the space $\widehat{\mathbf{W}}^{1,p}(\Omega)$ appearing in [14] which is not charaterized, is completely abstract and is obtained as closure of $\mathbf{W}^{1,p}(\Omega)$ for the norm

$$\|\boldsymbol{u}\|_{\widehat{\mathbf{W}}^{1,p}(\Omega)} = \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|A_r^{-1/2} \mathcal{P}_r \Delta \boldsymbol{u}\|_{\mathbf{L}^r(\Omega)},$$

where A_r is the Stokes operator with domain equal to $\mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{L}^p_\sigma(\Omega)$ and $\mathcal{P}_r$ is the Helmholtz projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}^r_\sigma(\Omega)$.

Corollary 3.6 *Let* $\boldsymbol{f}, h, \boldsymbol{g}$ *be given satisfying (3.11) and*

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1 \quad \textit{with } \mathbb{F}_0 \in \mathbb{L}^r(\Omega), \quad f_1 \in W^{-1,p}(\Omega),\ h \in L^r(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma).$$

Then the solution $\boldsymbol{u}$ *given by Theorem 3.5 belongs to* $\mathbf{W}^{1,r}(\Omega)$. *Moreover, if* f_1 *belongs to* $L^r(\Omega)$, *then the solution* q *given by Theorem 3.5 belongs to* $L^r(\Omega)$. *In the both cases, we have the corresponding estimates.*

Remark 3.7 *i) It is clear that* $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{T}_{p,r}(\Omega)$ *when* $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, *and therefore* $\mathbf{T}_{p,r}(\Omega)$ *is an intermediate space between* $\mathbf{W}^{1,r}(\Omega)$ *and* $\mathbf{L}^p(\Omega)$.

ii) As a consequence of Proposition 3.4, we have the following Helmholtz decomposition: for any $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$, *there exist* $\boldsymbol{\psi} \in \mathbf{W}^{-1,p}(\Omega)$ *and* $q \in W^{-1,p}(\Omega)$ *such that*

$$\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\psi} + \nabla q, \ \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in } \Omega.$$

iii) In the same way, suppose that $\boldsymbol{f} = \nabla \cdot \mathbb{F}$ *with* $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $h \in L^p(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ *verifying the compatibility condition (3.11). Then, the solution* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *given by Theorem 3.5 satisfies* $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ *with the appropriate estimate.*

Corollary 3.8 *Let us consider* h *and* $\boldsymbol{g}$ *satisfying:*

$$h \in L^r(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \quad \int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_\Gamma,$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ *and* $r \leq p$. *Then, there exists at least one solution* $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ *verifying*

$$\nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma.$$

Moreover, there exists a constant $C = C(\Omega, p, r)$ such that:

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C \left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right).$$

The following corollary gives the existence of a unique Stokes solution $(\boldsymbol{u}, q)$ in fractionary Sobolev spaces of type $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$, with $0 < \sigma < 2$ by using an interpolation argument.

Corollary 3.9 *Let s be a real number such that $0 \leq s \leq 1$.*

i) Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfy the compatibility condition (3.11) with

$$\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega), \quad f_1 \in W^{s-1,p}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma), \;\; h \in W^{s,r}(\Omega),$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u},\; q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \\ \leq &C\left(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right) \end{aligned}$$

ii) Assume that

$$\boldsymbol{f} \in \mathbf{W}^{s-1,p}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{s+1-1/p,p}(\Gamma), \;\; h \in W^{s,p}(\Omega),$$

with the compatibility condition (3.11). Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u},\; q) \in \mathbf{W}^{s+1,p}(\Omega) \times W^{s,p}(\Omega)/\mathbb{R}$ with

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s+1,p}(\Omega)} + \|q\|_{W^{s,p}(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-1,p}(\Omega)} + \|h\|_{W^{s,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s+1-1/p,p}(\Gamma)}\right)$$

Remark 3.10 We can reformulate the point ii) as follows. For any

$$\boldsymbol{f} \in \mathbf{W}^{-s,p'}(\Omega), \;\; h \in W^{-s+1,p'}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{2-s-1/p',p'}(\Gamma),$$

with $0 \leq s \leq 1$, then problem (S) has a unique solution $(\boldsymbol{u},\; q) \in \mathbf{W}^{2-s,p'}(\Omega) \times W^{1-s,p'}(\Omega)/\mathbb{R}$.

The following theorem gives solutions for external forces $\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega)$ and divergence condition $h \in W^{s-1,p}(\Omega)$ with $1/p < s < 2$. If $p = 2$, we can obtain solutions in $\mathbf{H}^{1/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$, $0 < \varepsilon \leq 3/2$.

Theorem 3.11 *Let s be a real number such that $\frac{1}{p} < s \leq 2$. Let $\boldsymbol{f}, h$ and $\boldsymbol{g}$ satisfy the compatibility condition (3.11) with*

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega), \;\; h \in W^{s-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma).$$

Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\; q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}} + \|q\|_{W^{s-1,p}/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right) \tag{3.19}$$

Proof.[Sketch of the proof] Theorem 3.11 is proved by Corollary 3.9 point ii) if $1 \le s \le 2$. Using Theorem 2.1, we can suppose $\boldsymbol{g} = \mathbf{0}$. Let s be a real number such that $\frac{1}{p} < s < 1$. It remains to consider the following equivalent problem:
Find $(\boldsymbol{u},\ q) \in \mathbf{W}_0^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{W}_0^{-s+2,p'}(\Omega)$, $\forall \pi \in W^{-s+1,p'}(\Omega)$

$$\begin{aligned}\langle \boldsymbol{u},\ -\Delta \boldsymbol{w} + \nabla \pi\rangle_{\mathbf{W}_0^{s,p}(\Omega)\times \mathbf{W}^{-s,p'}(\Omega)} - \langle q, \nabla\cdot \boldsymbol{w}\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)} \\ = \langle \boldsymbol{f},\ \boldsymbol{w}\rangle_{\mathbf{W}^{s-2,p}(\Omega)\times \mathbf{W}_0^{-s+2,p'}(\Omega)} - \langle h,\ \pi\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)}.\end{aligned}$$

Note that $W_0^{-s+1,p'}(\Omega) = W^{-s+1,p'}(\Omega)$ because $-s+1 < 1/p'$. Using Riesz' representation theorem we deduce that there exists a unique $(\boldsymbol{u},\ q) \in \mathbf{W}_0^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ solution of (S) and satisfying the bound (3.19). ∎

Remark 3.12 *i) If $n = 2$, Ω convex polygon, with $\Gamma = \cup\Gamma_i, \Gamma_i$ linear segments, $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in H^s(\Gamma_i)$, for $i = 1, \ldots, I_0$ and $-1/2 < s < 1/2$, then $\boldsymbol{u} \in \mathbf{H}^r(\Omega)$ for any $r < s+1/2$ and $q \in H^{s-1/2}(\Omega)$ (see [21]).*

ii) When Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, with $n \ge 3$, $\boldsymbol{f} = \mathbf{0}$, $h = 0$, $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ (respectively $\boldsymbol{g} \in \mathbf{W}^{1,2}(\Gamma)$) , with $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n} = 0$, then $\boldsymbol{u} \in \mathbf{H}^{1/2}(\Omega)$ (respectively $\boldsymbol{u} \in \mathbf{H}^{3/2}(\Omega)$) and $q \in H^{-1/2}(\Omega)$ (respectively $q \in H^{1/2}(\Omega)$) (see Fabes et al. [10]). If $\boldsymbol{g} \in \mathbf{L}^p(\Gamma)$, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $2-\varepsilon \le p \le 2+\varepsilon$, then $\boldsymbol{u} \in \mathbf{W}^{1-1/p}(\Omega)$ and $q \in W^{-1/p}(\Omega)$. For a similar result in the case where $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ and Ω is a simply connected domain of $\mathbb{R}^2$, we can see [6].

iii) When Ω is only a bounded Lipschitz domain, with connected boundary, the same result has be proved by [26] with $\boldsymbol{f} = \mathbf{0}$ and $h = 0$ for any $p \ge 2$.

4 The Oseen problem

We want to study the existence of a generalized, strong and very weak solutions for the problem (O), given by:

$$(O) \quad -\Delta \boldsymbol{u} + \boldsymbol{v}\cdot\nabla \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma$$

where $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ $(s \ge 3)$ is given.

First, we present several results related to the existence of weak and strong solution for (O). Then, the definition of a very weak solution for (O) will be done and a proof of their existence. Finally, regularity results in fractional Sobolev intermediate spaces will appear.

Theorem 4.1 (Existence of solution for (O)**)** *Let Ω be a Lipschitz bounded domain. Let us consider*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in L^2(\Omega) \quad \text{and } \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$$

verifying the compatibility condition (3.11) for $p = 2$. Then, the problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Moreover, there exist some constants $C_1 > 0$ and $C_2 > 0$ such that:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C_1\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\Big), \tag{4.20}$$

$$\|q\|_{L^2(\Omega)/\mathbb{R}} \leq C_2\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\Big) \tag{4.21}$$

where $C_1 = C(\Omega)$ and $C_2 = C_1\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$.

Proof. In order to prove the existence of solution, first (using Lemma 3.3 in [3], for instance) we lift the boundary and the divergence data. Then, there exists $\boldsymbol{u_0} \in \mathbf{H^1}(\Omega)$ such that $\nabla\cdot\boldsymbol{u}_0 = h$ in Ω, $\boldsymbol{u}_0 = \boldsymbol{g}$ on Γ and:

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C\,\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big). \tag{4.22}$$

Therefore, it remains to find $(\boldsymbol{z}, q) = (\boldsymbol{u} - \boldsymbol{u}_0, q)$ in $\mathbf{H}^1_0(\Omega) \times L^2(\Omega)$ such that:

$$-\Delta\boldsymbol{z} - \boldsymbol{v}\cdot\nabla\boldsymbol{z} + \nabla q = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \nabla\cdot\boldsymbol{z} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{z} = \boldsymbol{0} \quad \text{on } \Gamma.$$

being $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta\boldsymbol{u}_0 + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_0$. Observe that $\widetilde{\boldsymbol{f}} \in \mathbf{H}^{-1}(\Omega)$. Since the space $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla\cdot\boldsymbol{\varphi} = 0\}$ is dense in the space $\mathbf{V} = \{\boldsymbol{z} \in \mathbf{H}^1_0(\Omega);\ \nabla\cdot\boldsymbol{z} = 0\}$, then the previous problem is equivalent to: Find $\boldsymbol{z} \in \mathbf{V}$ such that:

$$\forall\boldsymbol{\varphi} \in \mathbf{V}, \qquad \int_\Omega \nabla\boldsymbol{z}\cdot\nabla\boldsymbol{\varphi}\,d\boldsymbol{x} - b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \langle\widetilde{\boldsymbol{f}}, \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1_0(\Omega)},$$

where b is a trilinear antisymmetric form with respect to the last two variables, well-defined for $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$, $\boldsymbol{z}$, $\boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega)$. (We can recover the pressure π thanks to the De Rham's Lemma 2.3). By Lax-Milgram's Theorem we can deduce the existence of a unique $\boldsymbol{z} \in \mathbf{H}^1_0(\Omega)$ verifying:

$$\begin{aligned}\|\boldsymbol{z}\|_{\mathbf{H}^1(\Omega)} &\leq C(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}} + \|\Delta\boldsymbol{u}_0\|_{\mathbf{H}^{-1}(\Omega)} + \|\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)\|_{\mathbf{H}^{-1}(\Omega)})\\ &\leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\Big),\end{aligned}$$

which added to estimate (4.22) makes (4.20).

Now, $-\Delta\boldsymbol{z} - \boldsymbol{v}\cdot\nabla\boldsymbol{z} - \widetilde{\boldsymbol{f}} \in \mathbf{H}^{-1}(\Omega)$ and:

$$\forall\boldsymbol{\varphi} \in \mathbf{V}, \qquad \langle -\Delta\boldsymbol{z} - \boldsymbol{v}\cdot\nabla\boldsymbol{z} - \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1_0(\Omega)} = 0.$$

Thanks to De Rham's Lemma 2.3, there exists a unique $q \in L^2(\Omega)/\mathbb{R}$ such that:

$$-\Delta\boldsymbol{z} - \boldsymbol{v}\cdot\nabla\boldsymbol{z} + \nabla q = \widetilde{\boldsymbol{f}}$$

with $\|q\|_{L^2(\Omega)/\mathbb{R}} \leq C\,\|\nabla q\|_{\mathbf{H}^{-1}(\Omega)}$. Finally, estimate (4.21) follows from the previous equation and estimate for $\boldsymbol{z}$. ∎

As a consequence of Theorem 4.1, Theorem 3.1 and the inequality

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{\mathbf{L}^{6/5}(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)},$$

we can deduce the following result:

Corollary 4.2 *Let us assume*

$$\boldsymbol{f}\in\mathbf{L}^{6/5}(\Omega),\quad \boldsymbol{v}\in\mathbf{H}_3(\Omega),\quad h\in W^{1,6/5}(\Omega)\quad\text{and}\quad \boldsymbol{g}\in\mathbf{W}^{7/6,6/5}(\Gamma)$$

be given verifying the compatibility condition (3.11). Then, the solution $(\boldsymbol{u},q)$ given by Theorem 4.1 belongs to $\mathbf{W}^{2,6/5}(\Omega)\times W^{1,6/5}(\Omega)$ and verifies the following estimate:

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{2,6/5}(\Omega)} + \|q\|_{W^{1,6/5}(\Omega)/\mathbb{R}}\\ &\leq C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|h\|_{W^{1,6/5}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{7/6,6/5}(\Gamma)}\Big)\Big)\end{aligned}$$

Theorem 4.3 (Strong regularity for $p\geq 6/5$) *Let $p\geq\frac{6}{5}$,*

$$\boldsymbol{f}\in\mathbf{L}^{p}(\Omega),\quad h\in W^{1,p}(\Omega),\quad \boldsymbol{v}\in\mathbf{H}_s(\Omega)\quad\text{and}\quad \boldsymbol{g}\in\mathbf{W}^{2-1/p,p}(\Gamma),$$

be given with

$$s=3\quad\text{if } p<3,\qquad s=p\quad\text{if } p>3,\qquad s=3+\varepsilon\quad\text{if } p=3, \tag{4.23}$$

for some arbitrary $\varepsilon>0$, and satisfying the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x} = \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,d\sigma.$$

Then, the unique solution of (O) given by Theorem 4.1 verifies $(\boldsymbol{u},q)\in\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$. Moreover, there exists a constant $C>0$ such that:

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leq C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\times\\ &\times\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\Big(\|h\|_{W^{1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\Big)\Big)\end{aligned} \tag{4.24}$$

Proof. First, by Corollary 4.2, we can suppose $p\geq 6/5$ and then we have the following embeddings:

$$\mathbf{L}^p(\Omega)\hookrightarrow\mathbf{H}^{-1}(\Omega),\quad W^{1,p}(\Omega)\hookrightarrow L^2(\Omega),\quad\text{and}\quad \mathbf{W}^{2-1/p,p}(\Gamma)\hookrightarrow\mathbf{H}^{1/2}(\Gamma).$$

Thanks to the regularity of $\boldsymbol{f}$ and Theorem 4.1 there exists a unique solution $(\boldsymbol{u},q)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$ verifying the following estimates:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\Big) \tag{4.25}$$

and

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} + \|q\|_{L^2(\Omega)/\mathbb{R}} &\le C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \times \\ &\times \Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\Big).\end{aligned} \tag{4.26}$$

Observe that, *a priori*, the regularity for the Oseen problem cannot be deduced from the Stokes one. This follows from the fact that $\boldsymbol{v}\cdot\nabla\boldsymbol{u} = \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}) \in \mathbf{H}^{-1}(\Omega)$.

In order to obtain the strong solution in $\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$, first we apply Lemma 2.4 to function $\boldsymbol{v}$, and we take for any $\lambda > 0$, $\boldsymbol{v}_\lambda$ as the velocity of the convection term, where $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that $\nabla\cdot\boldsymbol{v}_\lambda = 0$ and $\|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \le \lambda$. Therefore, we search for $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ solution of the problem:

$$(O_\lambda)\begin{cases} -\Delta\boldsymbol{u}_\lambda - \boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda + \nabla q_\lambda = \boldsymbol{f} & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u}_\lambda = h & \text{in } \Omega,\\ \boldsymbol{u}_\lambda = \boldsymbol{g} & \text{on } \Gamma.\end{cases}$$

From above we can obtain a unique solution $(\boldsymbol{u}_\lambda, q_\lambda)$ bounded in $\mathbf{H}^1_0(\Omega)\times L^2(\Omega)/\mathbb{R}$ independently from λ. Then, we obtain again estimates (4.25) and (4.26). As $\boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda \in \mathbf{L}^2(\Omega)$, if $\boldsymbol{f}$ and h are regular enough, then using the Stokes regularity we deduce that $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{H}^2(\Omega)\times H^1(\Omega)$ if $2 \le p$ and $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ if $6/5 < p \le 2$. A bootstrap argument moreover shows that $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ if $2 < p$.

Thus, we focus on the getting of a strong estimate for $(\boldsymbol{u}_\lambda, q_\lambda)$. Let $\varepsilon > 0$ with $0 < \lambda < \varepsilon/2$. We consider

$$\boldsymbol{v}_\lambda = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \widetilde{\boldsymbol{v}}\star\rho_{\varepsilon/2}, \quad \text{and} \quad \boldsymbol{v}_{\lambda,2}^\varepsilon = \boldsymbol{v}_\lambda - \widetilde{\boldsymbol{v}}\star\rho_{\varepsilon/2}. \tag{4.27}$$

where $\widetilde{\boldsymbol{v}}$ is the extension of $\boldsymbol{v}$ by zero to $\mathbb{R}^3$ and ρ_ε is the classical mollifier. By regularity estimates for the Stokes problem, we have

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} &\le C\,(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} \\ &+\ \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}).\end{aligned} \tag{4.28}$$

Now, we use the decomposition (4.27) in order to bound the term $\|\boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}$. We observe first that

$$\|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{\mathbf{L}^s(\Omega)} \le \|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{v} - \widetilde{\boldsymbol{v}}\star\rho_{\varepsilon/2}\|_{\mathbf{L}^s(\Omega)} \le \lambda + \varepsilon/2 < \varepsilon.$$

Recall that

$$W^{2,p}(\Omega) \hookrightarrow W^{1,k}(\Omega) \tag{4.29}$$

for any $k \in [1, p*]$, with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$, if $p < 3$, for any $k \ge 1$ if $p = 3$ and for any $k \in [1, \infty]$ if $p > 3$. Moreover the embedding

$$W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \tag{4.30}$$

is compact for any $q \in [1, p*[$ if $p < 3$, for any $q \in [1, \infty[$ if $p = 3$ and for $q \in [1, \infty]$ if $p > 3$. Then, using the Hölder inequality and the Sobolev embedding, we obtain

$$\|\boldsymbol{v}^{\varepsilon}_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}^{\varepsilon}_{\lambda,2}\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^k(\Omega)} \leq C\,\varepsilon \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} \tag{4.31}$$

where $\frac{1}{k} = \frac{1}{p} - \frac{1}{s}$, which is well defined (see the defintion of the real number s). For the second estimate, we consider two cases.

i) Case $p \leq 2$. Let $r \in]3, \infty]$ be such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$ and $t \geq 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ satisfy

$$\begin{aligned} \|\boldsymbol{v}^{\varepsilon}_{1} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leq \|\boldsymbol{v}^{\varepsilon}_{1}\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)} \\ &\leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Using the estimate (4.25), we have

$$\begin{aligned} \|\boldsymbol{v}^{\varepsilon}_{1} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \Big(&\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} \; + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \\ &\times (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}) \Big). \end{aligned} \tag{4.32}$$

From (4.32) and (4.31), we deduce that

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leq C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \times \\ &\times \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\, (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}) \Big) \end{aligned} \tag{4.33}$$

ii) Case $p > 2$. First, we choose the exponent q given in (4.30) such that $q > 2$. For any ε', we known that there exists $C_{\varepsilon'} > 0$ such that

$$\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} \; + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}.$$

Let first consider $p < 3$ and choose $q < p*$ and close of $p*$. Then, there exist $r > 3$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $t > 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ satisfying

$$\begin{aligned} \|\boldsymbol{v}^{\varepsilon}_{1} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leq \|\boldsymbol{v}^{\varepsilon}_{1}\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \\ &\leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)}. \end{aligned}$$

If $p \geq 3$,

$$\begin{aligned} \|\boldsymbol{v}^{\varepsilon}_{1} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leq \|\boldsymbol{v}^{\varepsilon}_{1}\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \\ &\leq \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \|\rho_{\varepsilon/2}\|_{L^1(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)}, \end{aligned}$$

where we choose $q = \infty$ if $p > 3$ and q large enough if $p = 3$. In the both cases, in order to control the first term on the right hand side of (4.28) with the term on the left hand side, we fix ε and ε' small enough to obtain

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \Big\{ &\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} \\ &+ \; C_{\varepsilon'} \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\Omega)} \\ &\times \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)\, (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}) \Big\}. \end{aligned} \tag{4.34}$$

Thus, we deduce that $(\boldsymbol{u}_\lambda, q_\lambda)$ satisfies (4.33), where we replace $\|\boldsymbol{v}\|_{\mathbf{L}^3}$ by $\|\boldsymbol{v}\|_{\mathbf{L}^s}$.

The estimate (4.33) is uniform with respect to λ, and therefore we can extract subsequences, that we still call $\{\boldsymbol{u}_\lambda\}_\lambda$ and $\{q_\lambda\}_\lambda$, such that if $\lambda \to 0$,

$$\boldsymbol{u}_\lambda \longrightarrow \boldsymbol{u} \quad \text{weakly in } \mathbf{W}^{2,p}(\Omega),$$

and for the pressure, there exists a sequence of real numbers k_λ such that

$$q_\lambda + k_\lambda \to q \quad \text{weakly in } W^{1,p}(\Omega).$$

It is easy to verify that $(\boldsymbol{u}, q)$ is solution of (O) satisfying estimate (4.24) and this solution is unique. ∎

Thanks to the strong regularity, we can deduce the following regularity:

Theorem 4.4 (Regularity in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$, $p > 1$) *Let us consider*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

be given verifying the compatibility condition (3.11). Then, the problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, there exists some constant $C > 0$ such that:

i) if $p \geq 2$, then

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq \\ &\qquad \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right) \end{aligned} \tag{4.35}$$

holds,

ii) if $p < 2$, then

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \times \\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right) \end{aligned} \tag{4.36}$$

holds.

Proof.[Sketch of the proof] *i)* First case: $p \geq 2$. Let $(\boldsymbol{u}_0, q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ be the solution of:

$$-\Delta \boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_0 = h \quad \text{in } \Omega, \qquad \boldsymbol{u}_0 = \boldsymbol{g} \quad \text{on } \Gamma.$$

verifying the estimate:

$$\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|q_0\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right) \tag{4.37}$$

and $(\boldsymbol{z}, \theta) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$ verifying:

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \quad \text{and} \quad \nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

with $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$ and satisfying the estimate

$$\begin{aligned} \|\boldsymbol{z}\|_{\mathbf{W}^{2,t}(\Omega)} + \|\theta\|_{W^{1,t}(\Omega)/\mathbb{R}} &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0\|_{\mathbf{L}^t(\Omega)} \\ &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \end{aligned} \tag{4.38}$$

Here, we have applied Theorem 4.3 because of $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \in \mathbf{L}^t(\Omega)$. Observe that $\frac{6}{5} \leq t < 3$, if and only if $p \geq 2$.

Thanks to the embedding $\mathbf{W}^{2,t}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$, the pair $(\boldsymbol{u}, q) = (\boldsymbol{z}+\boldsymbol{u}_0, \theta+q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies the problem (O). Estimate (4.35) follows from (4.37) and (4.38).
ii) Second case: $p < 2$. We use duality argument. ∎

Using quickly the reasoning given in Theorem 4.3, we can improve estimates (4.35) and (4.36) for some values of p:

Proposition 4.5 *Under the assumptions of Theorem 4.4 and supposing that* $\frac{6}{5} \leq p \leq 6$, *the solution* $(\boldsymbol{u}, q)$ *satisfies the estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \times \\ &\times \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big)\Big) \end{aligned} \tag{4.39}$$

Moreover assuming $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ *on* Γ, *then the estimate* (4.39) *holds for any* $1 < p < \infty$.

Remark 4.6 *If we suppose that* $\boldsymbol{v} \in \mathbf{H}_p(\Omega)$, *then estimate* (4.39), *where we replace the norm* $\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}$ *by* $\|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}$, *holds when* $p > 6$ *(and then also, by duality argument, when* $p < 6/5$ *and* $\boldsymbol{v} \in \mathbf{H}_{p'}(\Omega)$*).*

Corollary 4.7 (Strong regularity for $1 < p < 6/5$) *Let* $1 < p < 6/5$ *and let us*

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and } \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

be satisfied the compatibility condition (3.11). Then, the solution given by Theorem 4.4 satisfies $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *and the estimate*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\Big)\Big) \end{aligned} \tag{4.40}$$

holds.

Proof.[Sketch of the proof] Let us consider $1 < p < 6/5$ and $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ be the solution given by Theorem 4.4. Then

$$\mathbf{L}^p(\Omega) \hookrightarrow \mathbf{W}^{-1,r}(\Omega), \quad W^{1,p}(\Omega) \hookrightarrow \mathbf{L}^r(\Omega), \quad \mathbf{W}^{2-1/p,p}(\Gamma) \hookrightarrow \mathbf{W}^{1-1/r,}(\Gamma)$$

where $r \in]\frac{3}{2}, 2[$ satisfies $\frac{1}{r} = \frac{1}{p} - \frac{1}{3}$. From Theorem 4.4, we deduce that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$ and then $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^p(\Omega)$. By Stokes regularity let us to conclude that $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. To obtain estimate (4.40), we proceed similarly to the proof of Theorem 4.3. ∎

We can summarize Theorem 4.3 and Corollary 4.7 by the following theorem:

Theorem 4.8 (Strong regularity) *Let $\boldsymbol{f}, h, \boldsymbol{g}$ be such that*

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

verifying the compatibility condition (3.11) and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ be with s defined by (4.23). Then, the solution given by Theorem 4.4 satisfies $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies estimate (4.24).

The concepts of weak and strong solutions are known for the Oseen equations. Now, we define and prove the existence of a very weak solution for the Oseen equations.

Definition 4.9 (Very weak solution for the Oseen problem) *Let $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfying (3.14) and (3.11) and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ for s as (4.42). We say that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a* **very weak solution** *of (O) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,*

$$\begin{aligned}\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}) \, d\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla \pi \, d\boldsymbol{x} = - \int_\Omega h \, \pi \, d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma,\end{aligned} \tag{4.41}$$

where the dualities on Ω and Γ are defined by (3.16).

As for the Stokes problem, the previous duality have sense. Moreover, note that $\mathbf{W}^{1,p'}(\Omega) \hookrightarrow \mathbf{L}^{p'*}(\Omega)$ and then the integral $\int_\Omega \boldsymbol{u} \cdot (\boldsymbol{v} \cdot \nabla) \boldsymbol{\varphi} \, d\boldsymbol{x}$ is well defined.

Theorem 4.10 (Very weak solution for Oseen equations) *Let us $\boldsymbol{f}, h, \boldsymbol{g}$ satisfy (3.11),*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \ \ h \in L^r(\Omega), \ \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \ \text{with} \ \frac{1}{r} = \frac{1}{p} + \frac{1}{s}$$

and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with*

$$s = 3 \quad \text{if } p > 3/2, \quad s = p' \quad \text{if } p < 3/2, \quad s = 3 + \varepsilon \quad \text{if } p = 3/2. \tag{4.42}$$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ *verifying the following estimates:*

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big) \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big), \tag{4.43}$$

$$\begin{aligned} \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\, &\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)^2 \\ &\times \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big). \end{aligned} \tag{4.44}$$

Proof. First, we shall prove that if the pair $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfies the two first equations of (O), then $\boldsymbol{u}$ belongs to $\mathbf{T}_{p,r}(\Omega)$ and thus the boundary condition $\boldsymbol{u} = \boldsymbol{g}$ on Γ makes sense. Hence, if a pair $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ satisfies the two first equations of (O), because of $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ with $\nabla \cdot \boldsymbol{v} = 0$ and thanks (again) to Lemma 2.6, then $\Delta \boldsymbol{u} = \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla q - \boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$. Therefore, $\boldsymbol{u} \in \mathbf{T}_{p,r,\sigma}(\Omega)$ and its tangential trace belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. Moreover, as $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $\nabla \cdot \boldsymbol{u} \in L^r(\Omega)$, then $\boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma \in W^{-1/p,p}(\Gamma)$, and the whole trace $\boldsymbol{u}|_\Gamma \in \mathbf{W}^{-1/p,p}(\Gamma)$ can be identified with $\boldsymbol{u}|_\Gamma = \boldsymbol{g}$.

It suffices to consider the case where $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = 0$ and $\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = 0$, the general case is similar to the proof given in the end of Proposition 3.4. The result can be deduced (see [5]) applying the Riesz's Lemma. ∎

Similarly to Corollary 3.9, we can prove:

Corollary 4.11 *i) Let* σ *be a real number such that* $0 < \sigma < 1$*. Let* $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$*,* h *and* $\boldsymbol{g}$ *satisfy the compatibility condition (3.11) with*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma), \;\; h \in W^{\sigma,r}(\Omega),$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$ *and* $r \leq p$*. Let us consider* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with*

$$s = 3 \quad \text{if } p > 3/2, \quad s = p' \quad \text{if } p < 3/2, \quad s = 3 + \varepsilon \quad \text{if } p = 3/2.$$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u},\ q)$ *belonging to* $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ *and satisfying the estimate*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1/p,p}(\Omega)/\mathbb{R}} \leq C\,(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}) \\ &\times \Big(\|\mathbb{F}_0\|_{\mathbf{W}^{\sigma,r}(\Omega)} + \|f_1\|_{W^{\sigma-1,p}(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)})(\|h\|_{W^{\sigma,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Gamma)})\Big). \end{aligned}$$

ii) If moreover $\mathbb{F}_0, f_1, \boldsymbol{g}, h$ satisfy that

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma+1,r}(\Omega), \quad f_1 \in W^{\sigma,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{\sigma+1-1/p,p}(\Gamma), \ h \in W^{\sigma+1,r}(\Omega),$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{s}$ and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$, where

$$s = 3 \quad \text{if } p < 3, \quad s = p \quad \text{if } p > 3, \quad s = 3 + \varepsilon \quad \text{if } p = 3,$$

then $(\boldsymbol{u},\ q) \in \mathbf{W}_0^{\sigma+1,p}(\Omega) \times W^{\sigma,p}(\Omega)$ and satisfies

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma+1,p}(\Omega)} + \|q\|_{W^{\sigma,p}(\Omega)/\mathbb{R}} \\ \leq &C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \times \Big(\|\mathbb{F}_0\|_{\mathbf{W}^{\sigma+1,r}(\Omega)} + \|f_1\|_{W^{\sigma,p}(\Omega)} \\ &\qquad + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)})(\|h\|_{W^{\sigma+1,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma+1-1/p,p}(\Gamma)})\Big). \end{aligned}$$

Theorem 4.12 (Regularity for Oseen equations) *Let σ be a real number such that $\frac{1}{p} < \sigma \leq 2$. Let $\boldsymbol{f},\ h$ and $\boldsymbol{g}$ satisfy the compatibility condition (3.11) with*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \ h \in W^{\sigma-1,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Let $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ satisfy (4.42). Then, the Oseen problem (O) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)}\right) + \|h\|_{W^{\sigma-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Omega)}).$$

Proof. The proof is similar to proof of Theorem 3.11. It suffices to study the new term containing the function $\boldsymbol{v}$. ∎

Remark 4.13 *i) When $\boldsymbol{f} \in \mathbf{W}^{1/p-2,p}(\Omega)$, we can conjecture that $\boldsymbol{u} \notin \mathbf{W}^{1/p,p}(\Omega)$.*

ii) If $1/p < \sigma < 1, \boldsymbol{f} \in W^{\sigma-2,p}(\Omega), \boldsymbol{g} \in W^{\sigma-1/p,p}(\Gamma)$, then the solution $(\boldsymbol{u}, q)$ of (O) belongs to $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$. This assumptions are weaker than those of Corollary 4.11 point i). Moreover, they are optimal for this case.

iii) If $0 \leq \sigma \leq 1/p$, Theorem 4.12 cannot be applied. Indeed, the trace mapping is not continuous (and not surjective) from $\mathbf{W}^{\sigma,p}(\Omega)$ into $\mathbf{W}^{\sigma-1/p,p}(\Gamma)$. If we like to solve Problem (O) with boundary condition $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$, it is necessary to suppose that $\boldsymbol{f}$ and h are more regular, precisely we must assume $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$ with $\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega)$, and $h \in W^{\sigma,r}(\Omega)$, where $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. The solution is then obtained by Corollary 4.11 point i).

5 The Navier-Stokes problem

First of all, we give the definition of a very weak solution for the Navier-Stokes equations.

Definition 5.1 (Very weak solution for the Navier-Stokes problem) *Let* $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$, $h \in L^r(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ *satisfy the compatibility condition* (3.11). *We say that* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a* **very weak solution** *of* (NS) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ *and* $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned}\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi})\, d\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla \pi \, d\boldsymbol{x} = -\int_\Omega h\, \pi \, d\boldsymbol{x} + \langle (\boldsymbol{g} \cdot \boldsymbol{n}), \pi \rangle_\Gamma,\end{aligned} \tag{5.45}$$

where the dualities on Ω *and* Γ *are defined in* (3.16).

In the stationary Navier-Stokes equations, the data h and $\boldsymbol{g}$ play an special role, making possible or not the existence of a very weak solution. If h and $\boldsymbol{g}$ are small enough, then the result is true. Until we now, we think that it is not possible to eliminate this latest condition.

Therefore, we present first three results related to the existence of very weak solution: the two first for the small external forces case (following the scheme used by Marusič-Paloka [20]) and the third one for the general Navier-Stokes case, always supposing that h and $\boldsymbol{g}$ are small enough in their respective norms. Last result involves the regularity for the Navier-Stokes equations.

Theorem 5.2 (Very weak solution for Navier-Stokes, small data case) *Let us consider* $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ *be given verifying* (3.11).

i) There exists a constant $\alpha_1 > 0$ *such that, if*

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_1, \tag{5.46}$$

then, there exists a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *to the problem* (NS) *verifying the following estimates:*

$$\| \boldsymbol{u} \|_{\mathbf{L}^3(\Omega)} \leq C \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right) \tag{5.47}$$

$$\begin{aligned}\| q \|_{W^{-1,3}/\mathbb{R}} \leq C_1 \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2})]'} \\ + 2(1 + C_2) C \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}]'} + \| h \|_{L^{3/2}} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}} \right)\end{aligned} \tag{5.48}$$

where $C > 0$ *is the constant given by* (4.43), $\alpha_1 = \min \left\{ (2C)^{-1}, (2C^2)^{-1} \right\}$, C_1 *and* C_2 *are constants of Sobolev embeddings.*

ii) Moreover there exists a constant $\alpha_2 \in]0, \alpha_1]$ such that this solution is unique, up to an additive constant for q, if

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_2. \tag{5.49}$$

Proof. i) **Existence**. The existence of a very weak solution is made through the application of the Banach's fixed point theorem. We do this fixed point over the Oseen equations, written in an adequate manner. We are searching for a fixed point for the application T,

$$\begin{cases} T: \quad \mathbf{H}_3(\Omega) \to \quad \mathbf{H}_3(\Omega) \\ \qquad\qquad \boldsymbol{v} \mapsto \quad T\boldsymbol{v} = \boldsymbol{u} \end{cases} \tag{5.50}$$

where given $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $T\boldsymbol{v} = \boldsymbol{u}$ is the unique solution of (O) given by Theorem 4.10. We also need to define a neighborhood $\mathbf{B}_r$, in the form:

$$\mathbf{B}_r = \{\boldsymbol{v} \in \mathbf{H}_3(\Omega);\ \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \leq r\}. \tag{5.51}$$

In order to prove the contraction of the operator, we must prove that: there exists $\theta \in]0, 1[$ such that

$$\|T\boldsymbol{v}_1 - T\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \leq \theta\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}. \tag{5.52}$$

Searching for an estimate of $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}$, we observe that for each $i = 1, 2$, we have

$$\begin{aligned} -\Delta \boldsymbol{u}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{u}_i + \nabla q_i = \boldsymbol{f} \quad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_i = h \quad & \text{in } \Omega, \\ \boldsymbol{u}_i = \boldsymbol{g} \quad & \text{on } \Gamma, \end{aligned}$$

with the estimates

$$\begin{aligned} \|\boldsymbol{u}_i\|_{\mathbf{L}^3(\Omega)} \leq C \left(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)}\right) & \\ \times \Big(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}\Big), \end{aligned} \tag{5.53}$$

where $C > 0$ is the constant given by (4.43). Moreover, for estimating the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we look for the problem verified by $(\boldsymbol{u}, q) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, q_1 - q_2)$, which is:

$$-\Delta \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_2 \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}_1 = T\boldsymbol{v}_1$, $\boldsymbol{u}_2 = T\boldsymbol{v}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Using the very weak estimates (4.43) made for the Oseen problem successively for $\boldsymbol{u}$ and for $\boldsymbol{u}_2$, we obtain that:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} &\leq C \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \|(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_2\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\ &\leq C^2\beta \ \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \left(1 + \|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}, \end{aligned}$$

where $\beta = \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}$. Thus, we obtain estimate (5.52) considering $C^2 \beta \, (1+r)^2 < 1$ which is verified, for example, taking:

$$r = \left(2\, C^2 \, \beta\right)^{-1/2} - 1 \qquad \text{with} \qquad \beta < (2\, C^2)^{-1}. \tag{5.54}$$

Therefore, if (5.54) is verified, using again estimate (4.43) we conclude that the fixed point $\bar{\boldsymbol{u}} \in \mathbf{L}^3(\Omega)$ verifies:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \le C\beta \, \left(1 + \|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\right).$$

If we also choose β such that $\beta < (2\, C)^{-1}$, then:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \le C\beta(1 - C\beta)^{-1} \le 2\, C\, \beta < 1.$$

Setting $\alpha_1 = \min\left\{(2C)^{-1}, (2C^2)^{-1}\right\}$, then estimate (5.47) is satisfied. For the estimate of the associated pressure, we deduce from the equations $\nabla \bar{q} = \Delta \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} + \boldsymbol{f}$ and (5.47) that:

$$\begin{aligned} \|\dot{\bar{q}}\|_{W^{-1,3}(\Omega)/\mathbb{R}} &\le \|\nabla \bar{q}\|_{W^{-2,3}(\Omega)} \\ &\le \|\Delta \bar{\boldsymbol{u}}\|_{\mathbf{W}^{-2,3}(\Omega)} + C_2 \|\bar{\boldsymbol{u}}\|^2_{\mathbf{L}^3(\Omega)} + C_1 \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\ &\le C_1 \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + 2(1 + C_2) C\, \beta, \end{aligned}$$

where C_1 is the continuity constant of the Sobolev embedding $[\mathbf{X}_{3,3/2}(\Omega)]' \hookrightarrow \mathbf{W}^{-2,3}(\Omega)$ and C_2 is the continuity constant of the Sobolev embedding $\mathbf{W}^{1,3/2}_0(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$, which is (5.48) and the proof of existence is completed.

ii) **Uniqueness**. We shall next prove uniqueness. Let us denote by $(\boldsymbol{u}_1, q_1)$ the solution obtained in step i) and by $(\boldsymbol{u}_2, q_2)$ any other very weak solution corresponding to the same data. Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $q = q_1 - q_2$. We find that

$$-\Delta \boldsymbol{u} + \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \ \ \Omega, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \ \ \Gamma.$$

As $\boldsymbol{u} \cdot \nabla \boldsymbol{u}_1$ belongs to $\mathbf{W}^{-1,3/2}(\Omega)$, using uniqueness argument and Proposition 4.5, the function $\boldsymbol{u}$ belongs to $\mathbf{W}^{1,3/2}(\Omega)$ and we have the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \le C_1 \, \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)} \left(1 + \|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\right),$$

where $C_1 > 0$ is given by (4.39). Thanks to Theorem 4.10, we have also:

$$\|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \le C(1 + \|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)})(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}),$$

where $C > 0$ is the constant given in (4.43). We deduce then

$$\|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \le \frac{C(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)})}{1 - C(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)})} \le 2\, \beta C,$$

provided that $\beta \leq \alpha_1$. Using finally the embedding $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, we obtain the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \leq 2CC_1C_2\beta(1+2C\beta)\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)},$$

where C_2 is the continuity constant of the above embedding. Consequently

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \leq 0,$$

provided that

$$\beta < \frac{-C_1C_2 + \sqrt{C_1C_2(4+C_1C_2)}}{4CC_1C_2}.$$

We deduce that $\boldsymbol{u} = \mathbf{0}$ and the proof of uniqueness is completed. ∎

Corollary 5.3 *Let $\boldsymbol{f}, h, \boldsymbol{g}$ satisfy (3.11), (5.46) and*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \ \ h \in L^r(\Omega), \ \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \ \text{with } \frac{1}{r} \leq \frac{1}{p} + \frac{1}{s}\ , \tag{5.55}$$

where $\max\{r,3\} \leq p$ and s is defined by (4.42) . Then, the solution $(\boldsymbol{u}, q)$ given by Theorem 5.2 point i) belongs to $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$. If moreover $\boldsymbol{f}, h$ and $\boldsymbol{g}$ satisfy the condition (5.49), then this solution is unique, up to a constant for q.

Proof.[Sketch of the proof] First, we observe that the assumptions (5.55) imply that the assumptions of Theorem 5.2 are verified. Let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ then the solution given by Theorem 5.2 and satisfying the estimate

$$\| \ \boldsymbol{u} \ \|_{\mathbf{L}^3(\Omega)} \leq C \left(\| \ \boldsymbol{f} \ \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| \ h \ \|_{L^{3/2}(\Omega)} + \| \ \boldsymbol{g} \ \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right).$$

Observe then that $(\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow (\mathbf{X}_{r_0',p'}(\Omega))'$ and $L^r(\Omega) \hookrightarrow L^{r_0}(\Omega)$ where $1/r_0 = 1/p + 1/3$. Using Theorem 4.10, there exist a unique $(\boldsymbol{w}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying $-\Delta \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \pi = \boldsymbol{f} = -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q$, div $\boldsymbol{w} = h$ in Ω and $\boldsymbol{w} = \boldsymbol{g}$ on Γ. Setting $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $\theta = \pi - q$, that means that

$$-\Delta \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \nabla \theta = \mathbf{0}, \quad \text{div } \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

and thanks to Theorem 4.10 and uniqueness argument, we deduce that $\boldsymbol{z} = \mathbf{0}$, $\nabla \pi = \nabla q$ and then $\boldsymbol{w} = \boldsymbol{u}$. The uniqueness of $(\boldsymbol{u}, q)$, up to a constant for q, is immediate. ∎

Theorem 5.4 (Very weak solution of Navier-Stokes equations, arbitrary external forces) *Let $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ satisfy the compatibility condition* (3.11). *There exists a constant $\delta > 0$ depending only on Ω such that if*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq \delta, \tag{5.56}$$

then the problem (NS) has a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$.

Proof.[Sketch of the proof] We decompose the problem into two parts. First, we are looking to find a pair $(\boldsymbol{v}_\varepsilon, q^1_\varepsilon)$ solution of the problem:

$$(NS_1)\begin{cases} -\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla q^1_\varepsilon &= \boldsymbol{f} - \boldsymbol{f}_\varepsilon \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v}_\varepsilon &= h - h_\varepsilon \quad \text{in } \Omega, \\ \boldsymbol{v}_\varepsilon &= \boldsymbol{g} - \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma, \end{cases}$$

and then to find $(\boldsymbol{z}_\varepsilon, q^2_\varepsilon)$ solution of the problem:

$$(NS_2)\begin{cases} -\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla q^2_\varepsilon &= \boldsymbol{f}_\varepsilon \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{z}_\varepsilon &= h_\varepsilon \quad \text{in } \Omega, \\ \boldsymbol{z}_\varepsilon &= \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega),\ h_\varepsilon \in L^2(\Omega)$ and $\boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$ satisfy

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \varepsilon$$

and

$$\|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_j}| \leq 2\delta$$

(see Lemma 2.7 and Lemma 2.12). The pair $(\boldsymbol{u}, q) = (\boldsymbol{v}_\varepsilon + \boldsymbol{z}_\varepsilon, q^1_\varepsilon + q^2_\varepsilon)$ is then solution to problem (NS).

The existence of solution for (NS_1) follows from Theorem 5.2 and solution of (NS_2) is based on the classical theory and the use of Hopf's Lemma (see [13], Remark VIII.4.4 for instance). ∎

Theorem 5.5 (Regularity for Navier-Stokes equations) *Let* $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *be the solution given by Theorem 5.4. Then, the following regularity results hold:*
i) Suppose that

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))',\ h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ *and* $\max\{r, 3\} \leq p$. *Then* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$.
ii) Let $r \geq 3/2$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega),\ h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma). \tag{5.57}$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.
iii) Let $1 < r < \infty$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{L}^r(\Omega),\ h \in W^{1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma). \tag{5.58}$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.
iv) Suppose that $3/2 \leq p \leq 3$, $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$ *and*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega), \; h \in W^{\sigma,r}(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),$$

with $\sigma = \frac{3}{p} - 1$, $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ *and* $r \leq p$. *Then* $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.
v) Let σ *such that* $1/p < \sigma \leq 1$ *and* $\sigma \geq 3/p - 1$. *Suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \; h \in W^{\sigma-1,p}(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

Proof.[Sketch of the proof] First, we remark that under the assumptions in i) ii) and iii), we have that $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$.
i) Let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 5.4. Using Theorem 4.10, there exist a unique $(\boldsymbol{w}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying $-\Delta \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \pi = \boldsymbol{f} = -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q$, div $\boldsymbol{w} = h$ in Ω and $\boldsymbol{w} = \boldsymbol{g}$ on Γ. Setting $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $\theta = \pi - q$, that means that

$$-\Delta \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \nabla \theta = \mathbf{0}, \quad \text{div } \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

and thanks to Theorem 4.10 and uniqueness argument, we deduce that $\boldsymbol{z} = \nabla \theta = \mathbf{0}$ and then $\boldsymbol{w} = \boldsymbol{u}$ and $\pi = q + c$, with c constant. The point i) is proved.

ii) Let $r \geq 3/2$ and $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfy (5.57). Let $p \geq 3$ be defined by $1/p = 1/r - 1/3$. Then $\mathbf{W}^{1-1/r,r}(\Gamma) \hookrightarrow \mathbf{W}^{-1/p,p}(\Gamma)$ and $\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))'$. If $r \leq 3$, by point i), we deduce that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ and then $\boldsymbol{u} \otimes \boldsymbol{u} \in \mathbf{L}^r(\Omega)$. But $-\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} - \text{div}\,(\boldsymbol{u} \otimes \boldsymbol{u}) \in \mathbf{W}^{-1,r}(\Omega)$ and by Stokes regularity, we obtain that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$. If now $r > 3$, we know that $u \in \mathbf{W}^{1,3}(\Omega)$ and thanks to Sobolev embeddings, $\boldsymbol{u} \otimes \boldsymbol{u} \in \mathbf{L}^r(\Omega)$ and again as above, we deduce that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

iii) Let $1 < r < \infty$ and $\boldsymbol{f}, h, \boldsymbol{g}$ satisfy (5.58). We observe first that $\mathbf{L}^r(\Omega) \hookrightarrow \mathbf{W}^{-1,3/2}(\Omega)$, $W^{1,r}(\Omega) \hookrightarrow L^{3/2}(\Omega)$ and $\mathbf{W}^{2-1/r,r}(\Gamma) \hookrightarrow \mathbf{W}^{1/3,3/2}(\Gamma)$ and then by step ii), we obtain that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,3/2}(\Omega) \times L^{3/2}(\Omega)$. If $r < 3$, we deduce thanks to Theorem 4.8 that $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$. If now $r \geq 3$, then $\boldsymbol{u} \in \mathbf{L}^\infty(\Omega)$ and using again Theorem 4.8, we obtain the same conclusion.

iv) Follows from Corollary 4.11 point i).

v) Is consequence of Theorem 4.4 (for $\sigma = 1$) and Theorem 4.12 (for $\sigma < 1$). ∎

Remark 5.6 *i) In particular, when* $p = 2$ *and* $r = 6/5$, *if*

$$\boldsymbol{f} \in \mathbf{W}^{-1/2,6/5}(\Omega), \; h \in W^{1/2,6/5}(\Omega), \; \boldsymbol{g} \in \mathbf{L}^2(\Gamma),$$

then the solution given by the previous theorem point iv) satisfies $(\boldsymbol{u}, q) \in \mathbf{H}^{1/2}(\Omega) \times H^{-1/2}(\Omega)$.

ii) Point i) shows in particular that for any $p \geq 3$*, if*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma), \quad \text{with } \frac{3p}{3+p} \leq r \leq p,$$

and $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ *for any* $i = 1, \ldots, I$ *and* $h = 0$*, then Problem (NS) has a solution* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$*. In [25], D. Serre proves that for any* $3/2 < r < 2$ *(and then for any* $r > 3/2$*), if*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma),$$

with $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ *for any* $i = 0, \ldots, I$ *and* $h = 0$*, then (NS) has a solution* $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$*. Our point ii) proves that this result holds if* $r = 3/2$ *without supposing* h *or the flux* $\boldsymbol{g}$ *through* Γ_i *to be equal to 0, more precisely it suffices to assume the condition of smallness:*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq \delta.$$

iii) Because of the relation (3.11)*, the condition (5.56) is automatically fullfiled when the norm* $\|h\|_{L^{3/2}(\Omega)}$ *is sufficiently small and* $I = 0$*, that means that the boundary* Γ *is connected, which is the case considered by Kim [17].*

iv) Marusič-Paloka in [20] proves Theorem 5.4 with $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ *(which is included in the dual space* $(\mathbf{X}_{3,3/2}(\Omega))'$*),* $h = 0$ *and* $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ *(which is included in* $\mathbf{W}^{-1/3,3}(\Gamma)$*) with* $\|\boldsymbol{g}\|_{\mathbf{L}^2(\Gamma)}$ *small. Moreover, the domain* Ω *is assumed simply-connected. In fact, the solution* $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$ *is more regular and belongs to* $\mathbf{H}^{1/2}(\Omega)$ *as pointed in the point i) of this remark.*

v) Galdi et al. in [14] prove Theorem 5.4 and Theorem 5.5 point i) with $\boldsymbol{f} = \text{div}\, \mathbb{F}_0$*, where* $\mathbb{F}_0 \in \mathbb{L}^r(\Omega)$*,* $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ *with* $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ *and* $\max\{2r, 3\} \leq p$*. They assume the domain* Ω *is of class* $\mathcal{C}^{2,1}$*. Moreover they suppose* $\boldsymbol{f}$*,* h *and* $\boldsymbol{g}$ *sufficiently small with respect to their norms. The small condition on the external forces is in fact unnecessary.*

Acknowledgments

The second author thanks the Laboratoire de Mathematiques Appliquées, University of Pau et des Pays de l'Adour, by the financial support during her stays in Pau (France).

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