Weak time regularity and uniqueness for a Q-Tensor model

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Abstract

The coupled Navier-Stokes and Q-Tensor system is one of the models used to describe the behavior of the nematic liquid crystals. The existence of weak solutions and a uniqueness criteria have been already studied (see [11] for a Cauchy problem in the whole $\mathbb{R}^3$ and [7] for a initial-boundary problem in a bounded domain Ω). Nevertheless, results on strong regularity have only been treated in [11] for a Cauchy problem in the whole $\mathbb{R}^3$.

In this paper, imposing Dirichlet or Neumann boundary conditions, we show the existence and uniqueness of a local in time weak solution with weak regularity for the time derivative of the velocity and the tensor variables $(\boldsymbol{u}, Q)$. Moreover, we gives a regularity criteria implying that this solution is global in time. Note that the regularity furnished by the weak regularity for $(\boldsymbol{u}, Q)$ and the weak regularity for $(\partial_t \boldsymbol{u}, \partial_t Q)$ is not equivalent to the strong regularity.

Finally, when large enough viscosity is imposed, we obtain the existence (and uniqueness) of global in time strong solution. In fact, if non-homogeneous Dirichlet condition for Q is imposed, the strong regularity needs to be obtained together with the weak regularity for $(\partial_t \boldsymbol{u}, \partial_t Q)$.

Key words: Q-Tensor, Navier-Stokes equations, regularity, uniqueness.

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1 The Q-Tensor model and main results

1.1 The model

Liquid crystals can be seen as an intermediate phase of matter between crystalline solids and isotropic fluids. Nematic liquid crystals (N) are liquids where their rod-like molecules have center of mass isotropically distributed but with anisotropic direction (almost constant on average over small regions). The optic of the liquid crystals can be uniaxial (points have one single refractive index) or biaxial (points have two refraction indices). Moreover, it can exist isotropic zones (so-called "defect patterns") within the liquid crystal material.

Dynamics of (N) can be described by the Ericksen-Leslie formulation, through velocity and pressure variables $(\boldsymbol{u}, p)$ jointly to an order parameter which is a director vector $\boldsymbol{d}$ related to the orientation of molecules (cf. [9, 5]). In this case, only uniaxial liquid crystals can be modeled by $(\boldsymbol{u}, p, \boldsymbol{d})$-systems. The Q-Tensor model (cf. [2, 6, 14]) proposes a new formulation in order to describe the three types of optic for (N). The director vector field $\boldsymbol{d} \in \mathbb{R}^3$ is replaced by the tensor $Q \in \mathbb{R}^{3\times 3}$, which is related to the second moment of a probability measure $\mu(\boldsymbol{x}, \cdot) : \mathcal{L}(\mathbb{S}^2) \to [0,1]$ for each $\boldsymbol{x} \in \Omega$ describing the orientation of the molecules, being $\mathcal{L}(\mathbb{S}^2)$ the family of Lebesgue measure sets on the unit sphere. In such a way, $\mu(\boldsymbol{x}, A)$ is the probability that the molecules with centre of mass in a very small neighborhood of the point $\boldsymbol{x} \in \Omega$ are pointing in a direction contained in $A \subset \mathbb{S}^2$. This probability must satisfy $\mu(\boldsymbol{x}, A) = \mu(\boldsymbol{x}, -A)$ in order to reproduce the "head-to-tail" symmetry. As a consequence, the first moment of the probability measure vanishes, $\langle p \rangle = \int_{\mathbb{S}^2} p_i \, d\mu(p) = 0$, hence the main information on μ comes from the second moment $M(\mu)_{ij} = \int_{\mathbb{S}^2} p_i \, p_j \, d\mu(p)$, $i, j = 1, 2, 3$. If the orientation of the molecules is equally distributed, then the distribution is isotropic and $\mu = \mu_0$, $d\mu_0(p) = \frac{1}{4\pi} \, dA$ and $M(\mu_0) = \frac{1}{3} Id$. The de Gennes order-parameter tensor Q measures the deviation of the second moment tensor from its isotropic value and is defined as:

$$Q = M(\mu) - M(\mu_0) = \int_{\mathbb{S}^2} \left(p \otimes p - \frac{1}{3} Id \right) \, d\mu(p) \tag{1}$$

Definition (1) implies that Q is symmetric and traceless.

We focus on the study of a Q-tensor model (QT) for a nematic liquid crystal filling a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ. The unknowns are

$$(\boldsymbol{u}, p, Q) : (0,T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{3\times 3}.$$

Velocity and pressure $(\boldsymbol{u}, p)$ satisfy the PDE-system:

$$\begin{cases} D_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \nabla \cdot \tau(Q) + \nabla \cdot \sigma(H,Q) & \text{in } \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \end{cases} \tag{2}$$

where $D_t = \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is the material derivative, $\nu > 0$ is the viscosity coefficient and the tensors $\tau = \tau(Q)$ and $\sigma = \sigma(H,Q)$ are defined by

$$\begin{cases} \tau_{ij}(Q) &=& -\varepsilon\, (\partial_j Q : \partial_i Q) = -\varepsilon\, \partial_j Q_{kl}\, \partial_i Q_{kl}, \quad \varepsilon > 0 \quad \text{(symmetric tensor)}, \\ \sigma(H,Q) &=& H\,Q - Q\,H \quad \text{(antisymmetric tensor if } Q \text{ and } H \text{ are symmetric)}, \end{cases}$$

with $H = H(Q) = -\varepsilon\, \Delta Q + f(Q)$ and

$$f(Q) = a\,Q - \frac{b}{3}\left(Q^2 + QQ^t + Q^tQ\right) + c\,|Q|^2\,Q \quad \text{with } a,\, b \in \mathbb{R} \text{ and } c > 0. \tag{3}$$

Here, we denote by $|Q|^2 = Q : Q$ the tensor euclidean norm.

The tensor Q is governed by the PDE-system:

$$D_t Q - S(\nabla \boldsymbol{u}, Q) = -\gamma\, H(Q) \quad \text{in } \Omega \times (0,T), \tag{4}$$

where

$$S(\nabla \boldsymbol{u}, Q) = \nabla \boldsymbol{u}\, Q^t - Q^t\, \nabla \boldsymbol{u} \tag{5}$$

(which is called the stretching term) and $\gamma > 0$ is a material-dependent elastic constant. Note that tensor H is the variational derivative in $L^2(\Omega)$ of a free energy functional $\mathcal{E}(Q)$, because

$$H = \frac{\delta \mathcal{E}(Q)}{\delta Q}, \qquad \mathcal{E}(Q) = \frac{\varepsilon}{2}|\nabla Q|^2 + F(Q)$$

where functional $F(Q)$ is defined as

$$F(Q) = \frac{a}{2}\,|Q|^2 - \frac{b}{3}\,(Q^2 : Q) + \frac{c}{4}\,|Q|^4. \tag{6}$$

It is easy to check $F'(Q) = f(Q)$ (see [7]).

Previous system should be enclosed with the following initial and boundary conditions:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad Q|_{t=0} = Q_0 \quad \text{in } \Omega, \tag{7}$$

$$\boldsymbol{u}|_{\Gamma} = \mathbf{0} \quad \text{in } (0,T), \tag{8}$$

and

$$\text{either } \partial_{\mathbf{n}} Q|_{\Gamma} = 0 \quad \text{or} \quad Q|_{\Gamma} = Q_0|_{\Gamma} \quad \text{in } (0,T), \tag{9}$$

where $\mathbf{n}$ is the outward normal vector to the boundary Γ and $\partial_{\mathbf{n}}$ represents the normal derivative of Q. For the Neumann condition, the compatibility condition $\partial_{\mathbf{n}} Q_0|_{\Gamma} = 0$ must be satisfied.

Remark 1.1 *If we impose time-dependent Dirichlet boundary conditions for Q:*

$$Q|_{\Gamma} = Q_{\Gamma} \;\; \textit{with } Q_{\Gamma} = Q_{\Gamma}(t), \tag{10}$$

the same type of results can also be obtained, although for technical reasons, this case will be presented separately. In this case, the compatibility condition $Q_{\Gamma}(0) = Q_0|_{\Gamma}$ must be satisfied.

Remark 1.2 *The case of considering space-periodic boundary conditions for $(\boldsymbol{u}, Q)$, since all boundary integrals vanish, is easier to deal with than the case of considering (8) and (9) boundary conditions, see Remark 3.2.*

In what follows, the vector fields and matrix fields (and the corresponding spaces) will be denoted by boldface Roman and special Roman, respectively.

1.2 Some previous results

Existence of global in time weak solutions, uniqueness criteria and maximum principle for Q were already obtained in [7]. Recall that a weak solution in $(0,T)$ for the (QT)-model (2)-(4) has the weak-regularity:

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)), \\ Q \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)), \end{cases} \tag{11}$$

satisfies the $\boldsymbol{u}$-system (2) in a variational form and satisfies the Q-system (4) point-wisely.

The uniqueness of weak solution was proved [7] under the regularity hypothesis:

$$\begin{cases} \nabla \boldsymbol{u} \in L^{\frac{2q}{2q-3}}(0,T;\mathbf{L}^q(\Omega)), & \text{for } 2 \le q \le 3 \\ \Delta Q \in L^{\frac{2s}{2s-3}}(0,T;\mathbb{L}^s(\Omega)) & \text{for } 2 \le s \le 3. \end{cases} \tag{12}$$

In [7], properties of symmetry and traceless of Q were deduced a posteriori whether the stretching term $S(\nabla \boldsymbol{u}, Q)$ and the function $f(Q)$ respectively were chosen in the following adequate form:

$$S(\nabla \boldsymbol{u}, Q) = \mathbf{W}\, Q^t - Q^t\, \mathbf{W}, \tag{13}$$

where $\mathbf{W} = W(\nabla \boldsymbol{u}) = \dfrac{1}{2}\left(\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^t\right)$ is the antisymmetric part of $\nabla \boldsymbol{u}$, and

$$f(Q) = a\,Q - \frac{b}{3}\left(Q^2 + QQ^t + Q^t Q\right) + c\,|Q|^2\,Q + \alpha(Q)\,\mathbb{I} \quad \text{with } a,\, b \in \mathbb{R} \text{ and } c > 0, \tag{14}$$

where $\mathbb{I}$ denotes the identity matrix and $\alpha(Q)$ is a suitable scalar function allowing to deduce traceless for Q, given in (75) and (76) below.

On the other hand, when the Cauchy problem in the whole $\mathbb{R}^3$ is considered, one has [11] the existence of weak solutions, uniqueness and regularity for system (2)-(4), with the stretching term and the functional defined by (13) and (14), respectively. Moreover, this type of results are extended in [10] to a more complete model.

1.3 Some regularity definitions

In this work, we are going to study two possibilities to improve the weak regularity of the (QT)-model:

- The weak regularity for $(\partial_t \boldsymbol{u}, \partial_t Q)$, what we will call "weak-t" solution, that is:

$$\text{(w-t)} \quad \begin{cases} \partial_t \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)),\ \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \\ \partial_t Q \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)),\ Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)). \end{cases} \tag{15}$$

- The strong regularity (as in the Navier-Stokes framework):

$$(St) \quad \begin{cases} \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t \boldsymbol{u} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^3(\Omega)), \\ \partial_t Q \in L^\infty(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1(\Omega)). \end{cases} \tag{16}$$

We are obtaining each one of these regularities using different arguments:

- Ladyszhenskaya's estimates allow to consider all the boundary conditions studied in this work (and without imposing large viscosity ν). They consist in taking adequate test function for both (QT) and $\partial_t(QT)$ models, where the boundary terms always vanish after integrating by parts.

  Some results by using this way will be stated in Theorem 1.1, Theorem 1.3 and Corollary 1.2.

- Prodi's estimates can be applied to obtain strong regularity for the (QT)-model but imposing homogeneous Dirichlet condition for Q, either local in time for any data or global in time under additional regularity hypothesis on $\nabla \boldsymbol{u}$ (see Theorem 1.4). When other boundary conditions for Q (Neumann or Dirichlet) are considered, some boundary terms do not vanish, and these terms will be only controlled taking large enough viscosity ν (see Theorem 1.5).

1.4 The main results

In general, as we are going to study strong regularity for the (QT)-model, the following regularity and compatibility conditions must be imposed for the initial data: $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^1_0(\Omega) \times \mathbb{H}^2(\Omega)$ with $\nabla \cdot \boldsymbol{u}_0 = 0$, $\boldsymbol{u}_0|_\Gamma = 0$ and, either $\partial_{\mathbf{n}} Q_0|_\Gamma = 0$ or $Q_0|_\Gamma = Q_\Gamma(0)$ for Neumann or Dirichlet, respectively. However, in order to prove weak-t regularity we will need the higher regularity $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$.

Moreover, for the case of time-dependent Dirichlet data for Q, that is $Q|_\Gamma = Q_\Gamma$ with $Q_\Gamma = Q_\Gamma(t)$, we will use the lifting function $\widetilde{Q}$ solving:

$$\partial_t \widetilde{Q} - \gamma\, \varepsilon\, \Delta \widetilde{Q} = 0 \quad \text{in } (0,T) \times \Omega, \quad \widetilde{Q}|_\Gamma = Q_\Gamma, \quad \widetilde{Q}|_{t=0} = Q_0. \tag{17}$$

Ladyzhensakaya's estimates will be used to prove the following three results, where the two first ones correspond to the existence of local in time weak solution for $(\partial_t \boldsymbol{u}, \partial_t Q)$, which is proved for any boundary data.

Theorem 1.1 (Local in time weak-t regularity for time-independent b.c.) *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0,T)$ *of the (QT) problem (2)-(4) and (7)-(8)-(9). Assume* $(\boldsymbol{u}_0, Q_0) \in$

$\mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$. *Then, there exists a time* $T^*(\leq T)$ *such that* $(\boldsymbol{u}, Q)$ *is the unique weak-t solution of (QT) in* $(0, T^*)$, *i.e. satisfying the (weak-t)-regularity (15) for* $T = T^*$.

Recently, we have known the work [1] where the authors have analyzed the weak-t regularity for a more complete Q-tensor model appearing in [10] in the same sense as in Theorem 1.1. In this model, Q is already symmetric and traceless (see Subsection 5.1) and a mixed Neumann and Dirichlet condition for Q is considered. The argument made in [1] is different from our proof of Theorem 1.1.

Corollary 1.2 (Local in time weak-t solution for time-dependent b.c.) *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0, T)$ *of the (QT) problem (2)-(4), (7)-(8), with time-dependent Dirichlet boundary condition (10) for Q. Assume* $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$ *and* $\partial_t \widetilde{Q} \in L^\infty(0, T; \mathbb{H}^1(\Omega)) \cap L^4(0, T; \mathbb{H}^2(\Omega))$. *Then there exists a time* T^* *such that* $(\boldsymbol{u}, Q)$ *is the unique (local in time) weak-t solution of (QT) in* $(0, T^*)$.

Remark 1.3 *Observe that regularity for time derivatives* $(\partial_t \boldsymbol{u}, \partial_t Q)$ *needs the initial regularity* $\partial_t \boldsymbol{u}(0) \in \mathbf{L}^2(\Omega)$ *and* $\partial_t Q(0) \in \mathbb{H}^1(\Omega)$. *Using systems (2) and (4) at time* $t = 0$, *the initial condition* $(\boldsymbol{u}, Q)(0) = (\boldsymbol{u}_0, Q_0)$, *and taking into account that* $\nabla \cdot (\partial_t \boldsymbol{u}(0)) = 0$ *and* $\partial_t \boldsymbol{u}(0)|_\Gamma = 0$ *(which implies that the regularity for* $\boldsymbol{u}$ *and* $\partial_t \boldsymbol{u}$ *does not depend on the pressure), it holds:*

$$\begin{aligned} \|\partial_t \boldsymbol{u}(0)\|_{\mathbf{L}^2(\Omega)} &\leq C \left((1 + \|\boldsymbol{u}_0\|^{1/2}_{\mathbf{H}^1(\Omega)}) \, \|\boldsymbol{u}_0\|_{\mathbf{H}^2(\Omega)} + (1 + \|Q_0\|^{1/2}_{\mathbb{H}^2(\Omega)}) \, \|Q_0\|_{\mathbb{H}^3(\Omega)} \right), \\ \|\partial_t Q(0)\|_{\mathbb{H}^1(\Omega)} &\leq C \left((1 + \|\boldsymbol{u}_0\|^{1/2}_{\mathbf{H}^1(\Omega)}) \, \|Q_0\|_{\mathbb{H}^3(\Omega)} + \|\boldsymbol{u}_0\|_{\mathbf{H}^2(\Omega)} \|Q_0\|^{1/2}_{\mathbb{H}^2(\Omega)} + 1 \right). \end{aligned}$$

This is the reason why one imposes $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$ *in Theorem 1.1 and Corollary 1.2.*

Assuming regularity criteria over $\nabla \boldsymbol{u}$ and ΔQ, global in time "weak-t" regularity can be proved for the (QT)-system:

Theorem 1.3 (Regularity criteria for global in time weak-t regularity) *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0, T)$ *of the (QT) problem (2)-(4) with initial and boundary data (7)-(9) or (10), having the additional regularity:*

$$\begin{cases} \nabla \boldsymbol{u} \in L^{2q/(2q-3)}(0, T; \mathbf{L}^q(\Omega)), \quad 3/2 \leq q \leq 3, \\ \Delta Q \in L^{2s/(2s-3)}(0, T; \mathbb{L}^s(\Omega)), \quad 3/2 \leq s \leq 3. \end{cases} \tag{18}$$

Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$ *and, for the case of (10),* $\partial_t \widetilde{Q} \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^4(0,T;\mathbb{H}^2(\Omega))$. *Then,* $(\boldsymbol{u}, Q)$ *is the unique weak-t solution of the system in the whole time interval* $(0,T)$.

The proof of Theorem 1.1, Corollary 1.2 and Theorem 1.3 are done along Section 2. The time-dependent Dirichlet boundary conditions follows from Subsection 2.5.

The next result gives local in time strong solution, which is global when a regularity criterium on $\nabla \boldsymbol{u}$ is assumed (but without imposing additional regularity for ΔQ).

Theorem 1.4 (Strong solution for $Q|_\Gamma = \mathbf{0}$ and $S(\cdot,\cdot)$ given in (13)) *Let us consider the (QT) problem (2)-(4) with homogeneous-Dirichlet conditions for Q and $S(\cdot,\cdot)$ given in (13). Then, there exists a unique strong solution $(\boldsymbol{u}, Q)$ in $(0,T^*)$ where either T^* is small enough or $T^* = T$ ($T > 0$ fixed) whether:*

$$\nabla \boldsymbol{u} \in L^{2q/(2q-3)}(0,T;\mathbf{L}^q(\Omega)), \quad 2 \le q \le 3. \tag{19}$$

Remark 1.4 *Note that the choice of $S(\cdot,\cdot)$ given in (13) implies the symmetry of Q and H, which in particular deals to the antisymmetry of tensor $\sigma(\cdot,\cdot)$. For the general stretching term $S(\cdot,\cdot)$ given in (5), Theorem 1.4 is also true considering space-periodic boundary conditions for $(\boldsymbol{u}, Q)$, see Remark 3.2 below.*

Remark 1.5 *In Theorem 1.4, Berselli's criteria (19) can be replaced by Serrin's criteria:*

$$\boldsymbol{u} \in L^{2s/(s-3)}(0,T;\mathbf{L}^s(\Omega)) \quad \text{for } s \ge 3. \tag{20}$$

See Remark 3.3 below.

On the other hand, assuming large enough viscosity $\nu > 0$, any boundary conditions can be treated.

Theorem 1.5 (Global in time regularity for large enough viscosity) *Let us consider the Q-Tensor system (2)-(4) for large enough viscosity $\nu > 0$. Then:*

i) There exists a unique global in time strong solution when Neumann or homogeneous Dirichlet conditions for Q are considered.

ii) There exists a unique global in time strong and weak-t solution when non-homogeneous Dirichlet condition for Q is considered.

Remark 1.6 *The existence of a unique global in time strong solution for a $(\boldsymbol{u}, p, \boldsymbol{d})$-nematic model with stretching term and space-periodic boundary conditions for $(\boldsymbol{u}, \boldsymbol{d})$ is studied in [13] and [15].*

The rest of the paper is organized as follows. Ladyzhenskaya's estimates and the weak-t regularity will be studied in Section 2. Section 3 will be devoted to obtain strong regularity by using Prodi's estimates. The case of large enough viscosity will be treated in Section 4. Other models will be analyzed in Section 5; the modified (QT) model with traceless and symmetry for Q studied in [11] and [7] is treated in Subsection 5.1, and the nematic problem with stretching terms of [13] in Subsection 5.2.

Remark 1.7 (General remark) *In what follows, we will bound the terms depending on $\|\boldsymbol{u}(t)\|_{\mathbf{L}^2(\Omega)}$ and $\|Q(t)\|_{\mathbb{H}^1(\Omega)}$ by a constant, because they are bounded in $L^\infty(0,T)$ due to weak regularity (11). Moreover, all estimates in this paper will be made without assuming a $L^\infty(0,T;L^\infty(\Omega))$-bound of Q as consequence of a maximum principle for the Q-system [7]. This fact will allows us to extend the arguments to a nematic $(\boldsymbol{u}, p, \boldsymbol{d})$-system with stretching terms which does not satisfy the maximum principle, see Subsection 5.2*

2 Ladyzhenskaya's estimates

In this section, we will obtain the so-called Ladyzhenskaya's estimates for $(\boldsymbol{u}, Q)$. For brevity, we only show a formal argument, but a rigorous proof by means of a Faedo-Galerkin method can be done following ideas given in [7].

Different types of regularity will be obtained:

- We will call *intermediate strong regularity* to the part of strong regularity for $(\boldsymbol{u}, Q)$ given in (16) concerning to L^∞-norm in time, but no in L^2-norm in time. Concretely,

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \quad \partial_t \boldsymbol{u} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)), \quad \partial_t Q \in L^\infty(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1(\Omega)). \end{cases} \tag{21}$$

- On the other hand, we will call *weak-t regularity* to the weak regularity for $(\partial_t \boldsymbol{u}, \partial_t Q)$ given in (15). Concretely,

$$\left\{ \begin{array}{ll} \partial_t \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)), & \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \\ \partial_t Q \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)), & Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)). \end{array} \right. \tag{22}$$

2.1 Intermediate strong differential inequalities

2.1.1 Intermediate strong inequalities for Q

First of all, taking $\gamma \partial_t H$ as test function in the Q-system (4),

$$\gamma\,(\partial_t Q, \partial_t H) + \gamma\,((\boldsymbol{u}\cdot\nabla)Q, \partial_t H) - \gamma\,(S(\nabla \boldsymbol{u}, Q), \partial_t H) + \frac{\gamma^2}{2}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} = 0. \tag{23}$$

Using that $\partial_t H = \partial_t(-\varepsilon\Delta Q + f(Q))$ and integrating by parts in the first term (the boundary term vanishes when either non-homogeneous time-independent Dirichlet or homogeneous Neumann boundary condition for Q are considered):

$$\gamma\,(\partial_t Q, \partial_t H) = \gamma\,\varepsilon\,\|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} + \gamma\,\varepsilon \int_\Omega \partial_t Q : \partial_t(f(Q))\,d\boldsymbol{x} \tag{24}$$

Using $|f'(Q)| \le C\,(1+|Q|^2)$ (since $f(Q)$ is a cubic polynomial) and taking into account Remark 1.7, we obtain:

$$\|Q\|^2_{\mathbb{L}^\infty(\Omega)} \le C\,\|Q\|_{\mathbb{H}^1(\Omega)}\|Q\|_{\mathbb{H}^2(\Omega)} \le \widetilde{C}\,\|Q\|_{\mathbb{H}^2(\Omega)}, \tag{25}$$

for $\widetilde{C}$ a new constant dependent on $\|Q\|_{H^1(\Omega)}$ but independent on the time, it is easy to bound:

$$\int_\Omega \partial_t Q : \partial_t(f(Q))\,d\boldsymbol{x} \le C\,\left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right)\,\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}. \tag{26}$$

The convective term can be bounded as (in what follows, $\delta > 0$ will be a small enough constant):

$$\begin{array}{rcl} ((\boldsymbol{u}\cdot\nabla)Q, \partial_t H) & \le & \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla Q\|_{\mathbb{L}^3(\Omega)}\|\partial_t H\|_{\mathbb{L}^2(\Omega)} \\ & \le & C\,\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^1(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^2(\Omega)}\|\partial_t H\|_{\mathbb{L}^2(\Omega)} \\ & \le & \delta\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{array}$$

From (25), we bound the stretching term as:

$$\begin{aligned}(S(\nabla \boldsymbol{u}, Q), \partial_t H) &\leq \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{L}^\infty(\Omega)} \|\partial_t H\|_{\mathbb{L}^2(\Omega)} \\ &\leq \delta \|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + C_\delta \|Q\|_{\mathbb{H}^2(\Omega)} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

In summary, from (23) we obtain:

$$\begin{aligned}\gamma\,\varepsilon \|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} &+ \frac{\gamma^2}{2}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} \leq \gamma\, C \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} \\ &+ \delta\gamma \|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + \gamma\, C_\delta \|Q\|_{\mathbb{H}^2(\Omega)} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{27}$$

Second, taking $\partial_t Q$ as test function in $\partial_t(4)$, one has

$$\frac{1}{2}\frac{d}{dt}\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} + (\partial_t \boldsymbol{u} \cdot \nabla Q, \partial_t Q) - (S(\nabla(\partial_t \boldsymbol{u}), Q), \partial_t Q) + \gamma\,(\partial_t H, \partial_t Q) = 0, \tag{28}$$

where we have used that $(\boldsymbol{u} \cdot \nabla(\partial_t Q), \partial_t Q) = 0$ and $(S(\nabla \mathbf{u}, \partial_t Q), \partial_t Q) = 0$. Observe that $(\partial_t H, \partial_t Q)$ can be decomposed like in (24) also using (26). We bound the second term of (28) as:

$$\begin{aligned}|(\partial_t \boldsymbol{u} \cdot \nabla Q, \partial_t Q)| &\leq \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbb{L}^3(\Omega)} \|\partial_t Q\|_{\mathbb{L}^6(\Omega)} \\ &\leq C \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|^{1/2}_{\mathbb{H}^1(\Omega)} \|Q\|^{1/2}_{\mathbb{H}^2(\Omega)} \|\nabla(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \\ &\leq \delta\,\gamma\varepsilon \|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega} + C_{\delta,\gamma,\varepsilon} \|Q\|_{\mathbb{H}^2(\Omega)} \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

For the third term of (28), taking into account that

$$-(S(\nabla(\partial_t \boldsymbol{u}), Q), \partial_t Q) = (S(\partial_t \boldsymbol{u}, \nabla Q), \partial_t Q) + (S(\partial_t \boldsymbol{u}, Q), \nabla(\partial_t Q)) := K_1 + K_2,$$

we can bound K_1 and K_2 as:

$$\begin{aligned}K_1 &\leq \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbb{L}^3(\Omega)} \|\partial_t Q\|_{\mathbb{L}^6(\Omega)} \\ &\leq \delta\,\gamma\varepsilon \|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} + C_{\delta,\gamma,\varepsilon} \|Q\|_{\mathbb{H}^2(\Omega)} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}, \\ K_2 &\leq \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{L}^\infty(\Omega)} \|\nabla(\partial_t Q)\|_{\mathbb{L}^6(\Omega)} \\ &\leq \delta\,\gamma\varepsilon \|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} + C_{\delta,\gamma,\varepsilon} \|Q\|_{\mathbb{H}^2(\Omega)} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)},\end{aligned}$$

where (25) has been used. Therefore, supposing homogeneous Neumann or time-independent Dirichlet boundary conditions and choosing δ small enough, (28) becomes:

$$\begin{aligned}\frac{d}{dt}\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} &+ \gamma\,\varepsilon \|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} \\ &\leq C_{\gamma,\varepsilon} \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\right).\end{aligned} \tag{29}$$

Adding (27) to (29), we obtain:

$$
\begin{aligned}
\frac{d}{dt}\Big(\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} &+ \frac{\gamma^2}{2}\,\|H\|^2_{\mathbb{L}^2(\Omega)}\Big) + \gamma\,\varepsilon\,\|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} \\
&\le C_{\gamma,\varepsilon}\,\big(1+\|Q\|_{\mathbb{H}^2(\Omega)}\big)\,\Big(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\Big) \\
&+ \gamma\,\delta\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + \gamma\,C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}
\tag{30}
$$

2.1.2 Intermediate strong inequalities for $\boldsymbol{u}$

Taking $\partial_t \boldsymbol{u}$ as test function in $\boldsymbol{u}$-system (2), we obtain:

$$
\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \partial_t \boldsymbol{u}) + \int_\Omega (\partial_t \boldsymbol{u}\cdot\nabla)Q : H\,d\boldsymbol{x} + (\sigma(H,Q), \nabla(\partial_t \boldsymbol{u})),
$$

where we can bound the right hand side as follows:

$$
\begin{aligned}
((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \partial_t \boldsymbol{u}) &\le \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\|\partial_t \boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \\
&\le C\,\|\boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|\nabla \boldsymbol{u}\|^{3/2}_{\mathbf{L}^2(\Omega)}\|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)} \\
&\le \delta\,\|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} + C_\delta\,\|\nabla \boldsymbol{u}\|^3_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega (\partial_t \boldsymbol{u}\cdot\nabla)Q : H\,d\boldsymbol{x} &\le \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla Q\|_{\mathbb{L}^3(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\
&\le C\,\|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^1(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^2(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\
&\le \delta\,\|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} + C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)}.
\end{aligned}
$$

Using (25),

$$
\begin{aligned}
(\sigma(H,Q), \nabla(\partial_t \boldsymbol{u})) &\le \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)}\|Q\|_{\mathbb{L}^\infty(\Omega)} \\
&\le \delta\,\|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} + C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)}.
\end{aligned}
$$

In such a way, we arrive at the inequality:

$$
\begin{aligned}
\frac{\nu}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} &+ \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \le \delta\,\|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} \\
&+ C_\delta\Big(\|\nabla \boldsymbol{u}\|^3_{\mathbf{L}^2(\Omega)} + \|Q\|_{\mathbb{H}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)}\Big).
\end{aligned}
\tag{31}
$$

2.2 Weak-t differential inequalities

Taking $(\partial_t(2), \partial_t \boldsymbol{u})$ and $(\partial_t(4), -\varepsilon\,\Delta(\partial_t Q))$, we obtain respectively:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} + (\partial_t \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \partial_t\boldsymbol{u}) \\
&\quad -(\partial_t\boldsymbol{u}\cdot\nabla Q, -\varepsilon\,\Delta(\partial_t Q) + \partial_t(f(Q))) - (\partial_t\boldsymbol{u}\cdot\nabla(\partial_t Q), H) \\
&\quad +(\sigma(-\varepsilon\,\Delta(\partial_t Q) + \partial_t(f(Q)), Q), \nabla(\partial_t\boldsymbol{u})) + (\sigma(H, \partial_t Q), \nabla(\partial_t\boldsymbol{u})) = 0,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} - \varepsilon\int_\Gamma \partial^2_{tt}Q : \partial_{\mathbf{n}}(\partial_t Q)\,d\sigma + (\partial_t\boldsymbol{u}\cdot\nabla Q, -\varepsilon\,\Delta(\partial_t Q)) \\
&\quad +(\boldsymbol{u}\cdot\nabla(\partial_t Q), -\varepsilon\,\Delta(\partial_t Q)) - (S(\nabla(\partial_t\mathbf{u}), Q), -\varepsilon\Delta(\partial_t Q)) \\
&\quad -(S(\nabla\boldsymbol{u}, \partial_t Q), -\varepsilon\Delta(\partial_t Q)) + \gamma\varepsilon^2\,\|\Delta(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad +\gamma\,(\partial_t(f(Q)), -\varepsilon\Delta(\partial_t Q)) = 0.
\end{aligned}
\tag{33}
$$

Now, the ideas given in [7] to deduce weak estimates will be used here in order to vanish the terms:

$$
(\sigma(-\varepsilon\,\Delta\partial_t Q, Q), \nabla(\partial_t\boldsymbol{u})) \quad \text{with} \quad -(S(\nabla(\partial_t\mathbf{u}), Q), -\varepsilon\Delta(\partial_t Q))
$$

and

$$
-(\partial_t\boldsymbol{u}\cdot\nabla Q, -\varepsilon\,\Delta(\partial_t Q)) \text{ of (32)} \quad \text{with} \quad (\partial_t\boldsymbol{u}\cdot\nabla Q, -\varepsilon\,\Delta(\partial_t Q)) \text{ of (33)}.
$$

Remark 2.1 (The boundary term in (33)) *Assuming homogeneous Neumann boundary condition for Q, that is $\partial_{\mathbf{n}} Q|_\Gamma = 0$, then its derivative with respect to the time satisfies $\partial_{\mathbf{n}}(\partial_t Q)|_\Gamma = 0$ and the boundary term in (33) vanishes. In the case of a time-independent Dirichlet boundary condition, that is $Q|_\Gamma = Q_\Gamma$ with Q_Γ independent on the time, then its derivatives with respect to the time $\partial_t Q|_\Gamma$ and $\partial^2_{tt} Q|_\Gamma$ vanish, and in consequence the boundary term does.*

Adding (32) and (33) and supposing that the boundary term $\varepsilon\int_\Gamma \partial^2_{tt}Q : \partial_{\mathbf{n}}(\partial_t Q)\,d\sigma$ vanishes (which is true for either homogeneous Neumann or time-independent Dirichlet boundary conditions), we obtain:

$$
\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,\Big(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)}\Big) + \nu\|\nabla(\partial_t \mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} + \gamma\,\varepsilon^2\,\|\Delta(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad = -(\partial_t \boldsymbol{u}\cdot\nabla \boldsymbol{u}, \partial_t \boldsymbol{u}) + (\partial_t \mathbf{u}\cdot\nabla Q, \partial_t(f(Q))) + (\partial_t \boldsymbol{u}\cdot\nabla(\partial_t Q), H) \\
&\quad -(\sigma\,(\partial_t(f(Q)), Q)\,, \nabla(\partial_t \boldsymbol{u})) - (\sigma(H, \partial_t Q), \nabla(\partial_t \boldsymbol{u})) - (\boldsymbol{u}\cdot\nabla(\partial_t Q), -\varepsilon\,\Delta(\partial_t Q)) \\
&\quad +(S(\nabla \boldsymbol{u}, \partial_t Q), -\varepsilon\Delta(\partial_t Q)) - \gamma\,(\partial_t(f(Q)), -\varepsilon\Delta(\partial_t Q)) := \sum_{i=1}^{8} I_i.
\end{aligned}
\tag{34}
$$

Using the Poincaré's inequality for $\boldsymbol{u}$ and the $H^2(\Omega)$-regularity of the elliptic system (4), there exist two constants C_1 and C_2 such that:

$$
C_1\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} \le \|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)}, \quad C_2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \le \|\Delta(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)}.
$$

Therefore adding ε(29) to (34),

$$
\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,\Big(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\Big) + C_1\,\nu\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\gamma\,\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \\
&\quad \le C_{\gamma,\varepsilon}\,\Big(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\Big)\,\Big(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\Big) + \sum_{i=1}^{8} I_i.
\end{aligned}
\tag{35}
$$

In what follows, every I_i-term will be bounded in both ways: either to obtain local in time weak-t estimates (without regularity hypothesis) or global in time estimates (assuming additional regularity hypothesis for $\nabla\boldsymbol{u}$). Sometimes, both estimates will coincide.

The I_1-term corresponds to the nonlinear Navier-Stokes system:

$$
\begin{aligned}
I_1 &\le \int_\Omega |\nabla \boldsymbol{u}||\partial_t \boldsymbol{u}|^2\,d\boldsymbol{x} \\
&\le \begin{cases} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^4(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \le \|\partial_t \boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|\nabla(\partial_t \boldsymbol{u})\|^{3/2}_{\mathbf{L}^2(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \\ \text{or} \\ \le \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^{\frac{2q}{q-1}}(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le \|\partial_t \boldsymbol{u}\|^{(2q-3)/q}_{\mathbf{L}^2(\Omega)}\|\nabla(\partial_t \boldsymbol{u})\|^{3/q}_{\mathbf{L}^2(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \end{cases} \\
&\le \frac{\nu}{10}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_\nu \begin{cases} \|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} & \text{in any case} \\ \|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} & \text{for } \frac{3}{2} \le q \le 3. \end{cases}
\end{aligned}
$$

The more difficult terms to estimate correspond to:

$$
I_3 = \int_\Omega |\partial_t \boldsymbol{u}||\nabla(\partial_t Q)||H|\,d\boldsymbol{x}, \quad I_4 = \varepsilon\int_\Omega |\boldsymbol{u}||\nabla(\partial_t Q)||\Delta(\partial_t Q)|\,d\boldsymbol{x},
$$

$$I_6 = \int_\Omega |H||\partial_t Q||\nabla(\partial_t \boldsymbol{u})|\,d\boldsymbol{x}, \quad I_7 = \varepsilon \int_\Omega |\nabla \boldsymbol{u}||\partial_t Q||\Delta(\partial_t Q)|\,d\boldsymbol{x}.$$

Two different bounds for each one are presented:

$$I_3 \le \begin{cases} \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)}\|\nabla(\partial_t Q)\|_{\mathbb{L}^6(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \dfrac{\nu}{10}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \dfrac{\gamma\varepsilon^2}{20}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\nu,\gamma,\varepsilon}\|H\|^4_{\mathbb{L}^2(\Omega)}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \\ \text{or} \\ \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla(\partial_t Q)\|_{\mathbb{L}^{6s/(5s-6)}(\Omega)}\|H\|_{\mathbb{L}^s(\Omega)} \qquad (\text{for } 3/2 \le s \le 3) \\ \qquad \le \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\nabla(\partial_t Q)\|^{(2s-3)/s}_{\mathbb{L}^2(\Omega)}\|\nabla(\partial_t Q)\|^{(3-s)/s}_{\mathbb{H}^1(\Omega)}\|H\|_{\mathbb{L}^s(\Omega)} \\ \qquad \le \dfrac{\nu}{10}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \dfrac{\gamma\,\varepsilon^2}{20}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\nu,\gamma,\varepsilon}\|H\|^{2s/(2s-3)}_{\mathbb{L}^s(\Omega)}\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \end{cases}$$

$$I_4 \le \begin{cases} \varepsilon\,\|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla(\partial_t Q)\|_{\mathbb{L}^3(\Omega)}\|\Delta(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \varepsilon\,\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla(\partial_t Q)\|^{1/2}_{\mathbb{L}^2(\Omega)}\|\nabla(\partial_t Q)\|^{3/2}_{\mathbb{H}^1(\Omega)} \\ \qquad \le \dfrac{\gamma\varepsilon^2}{6}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\gamma,\nu,\varepsilon}\|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \quad (\text{in any case}) \\ \text{or} \\ \varepsilon\,\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}\|\nabla(\partial_t Q)\|_{\mathbb{L}^{2p/(p-2)}(\Omega)}\|\Delta(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \varepsilon\,C\,\|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|\nabla(\partial_t Q)\|_{\mathbb{L}^{2p/(p-2)}(\Omega)}\|\partial_t Q\|_{\mathbb{H}^2(\Omega)} \quad (\text{for } p = 3q/(3-q)) \\ \qquad \le \dfrac{\gamma\varepsilon^2}{20}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\gamma,\varepsilon}\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \quad (\text{if } 3/2 < q < 3) \end{cases}$$

$$I_6 \le \begin{cases} \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|_{\mathbb{L}^\infty(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|^{1/2}_{\mathbb{H}^1(\Omega)}\|\partial_t Q\|^{1/2}_{\mathbb{H}^2(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \dfrac{\nu}{10}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \dfrac{\gamma\varepsilon^2}{20}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\nu,\gamma,\varepsilon}\,\|H\|^4_{\mathbb{L}^2(\Omega)}\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \\ \text{or} \\ \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|_{\mathbb{L}^{2s/(s-2)}(\Omega)}\|H\|_{\mathbb{L}^s(\Omega)} \quad (\text{for } 3/2 \le s \le 3) \\ \qquad \le \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|_{\mathbb{W}^{1,6s/(5s-6)}(\Omega)}\|H\|_{\mathbb{L}^s(\Omega)} \\ \qquad \le \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|\partial_t Q\|^{(2s-3)/s}_{\mathbb{H}^1(\Omega)}\|\partial_t Q\|^{(3-s)/s}_{\mathbb{H}^2(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \\ \qquad \le \dfrac{\nu}{10}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \dfrac{\gamma\varepsilon^2}{20}\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\nu,\gamma,\varepsilon}\|H\|^{2s/(2s-3)}_{\mathbb{L}^s(\Omega)}\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \end{cases}$$

$$I_7 \leq \begin{cases} \varepsilon \, \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\partial_t Q\|_{\mathbb{L}^\infty(\Omega)} \|\Delta(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \\ \qquad \leq \dfrac{\gamma\varepsilon^2}{6} \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\gamma,\varepsilon} \|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)} \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \quad \text{(in any case)} \\ \text{or} \\ \varepsilon \, \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \|\partial_t Q\|_{\mathbb{L}^{2q/(q-2)}(\Omega)} \|\Delta(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \\ \qquad \leq \varepsilon \, \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \|\partial_t Q\|_{\mathbb{W}^{1,6q/(5q-6)}(\Omega)} \|\partial_t Q\|_{\mathbb{H}^2(\Omega)} \\ \qquad \leq \varepsilon \, \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \|\partial_t Q\|^{(2q-3)/q}_{\mathbb{H}^1(\Omega)} \|\partial_t Q\|^{3/q}_{\mathbb{H}^2(\Omega)} \\ \qquad \leq \dfrac{\gamma\varepsilon^2}{20} \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C_{\gamma,\varepsilon} \|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)} \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \quad \text{(for regularity criteria).} \end{cases}$$

To estimate the term $I_8 = \varepsilon \displaystyle\int_\Omega |\Delta(\partial_t Q)||\partial_t(f(Q))| \, d\boldsymbol{x}$, we can use the fact that:

$$\|\partial_t(f(Q))\|^2_{\mathbb{L}^2(\Omega)} \leq C \left(1 + \|Q\|^2_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}, \tag{36}$$

and therefore:

$$\begin{aligned} I_8 &\leq \varepsilon \, \|\Delta(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \|\partial_t(f(Q))\|_{\mathbb{L}^2(\Omega)} \\ &\leq \frac{\gamma\varepsilon^2}{6} \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C \left(1 + \|Q\|^2_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

It remains to bound the terms:

$$I_2 = \int_\Omega |\partial_t \boldsymbol{u}||\nabla Q||\partial_t(f(Q))| \, d\boldsymbol{x} \quad \text{and} \quad I_5 = \int_\Omega |\nabla(\partial_t \boldsymbol{u})||Q||\partial_t(f(Q))| \, d\boldsymbol{x},$$

for which we will use the estimate:

$$\|\partial_t(f(Q))\|_{\mathbb{L}^3(\Omega)} \leq C \left(1 + \|Q\|^{1/2}_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|_{\mathbb{H}^1(\Omega)},$$

obtaining, respectively:

$$\begin{aligned} I_2 &\leq \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \|\nabla Q\|_{\mathbb{L}^2(\Omega)} \|\partial_t(f(Q))\|_{\mathbb{L}^3(\Omega)} \\ &\leq \frac{\nu}{10} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_\nu \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}, \end{aligned}$$

$$\begin{aligned} I_5 &\leq \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{L}^6(\Omega)} \|\partial_t(f(Q))\|_{\mathbb{L}^3(\Omega)} \\ &\leq \frac{\nu}{10} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_\nu \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}. \end{aligned}$$

In summary, from (35):

$$\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right) + C_1\,\nu\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\gamma\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}$$
$$\leq a(t)\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + C_{\gamma,\nu,\varepsilon}\left(\|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)} + \|H\|^4_{\mathbb{L}^2(\Omega)}\right)\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right), \tag{37}$$

where $a \in L^1(0,T)$ is defined as:

$$a(t) = C_{\gamma,\varepsilon,\nu}\left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \tag{38}$$

Observe that from (37) we cannot obtain any estimate in time yet. But, adding (37) to the intermediate strong inequality for Q (30) and the intermediate strong inequality for $\boldsymbol{u}$ (31) we could obtain a local in time solution (see Subsection 2.3).

2.3 The global in time weak-t estimates

Adding ε(30) and (31) to (37), we obtain the following inequality:

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\
&+C_1\,\nu\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\gamma\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}\\
&\leq \delta\left(\gamma\,\varepsilon\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)}\right)\\
&+C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\left(\gamma\,\varepsilon\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|H\|^2_{\mathbb{L}^2(\Omega)}\right)\\
&+C_\delta\,\|\nabla \boldsymbol{u}\|^3_{\mathbf{L}^2(\Omega)} + a(t)\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right)\\
&+C_{\gamma,\nu,\varepsilon}\left(\|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)} + \|H\|^4_{\mathbb{L}^2(\Omega)}\right)\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right).
\end{aligned} \tag{39}$$

The expression $\partial_t H = -\varepsilon\,\Delta(\partial_t Q) + \partial_t(f(Q))$ together with (36) implies that:

$$\begin{aligned}
\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} &\leq 2\,\varepsilon^2\,\|\Delta(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} + 2\,\|\partial_t(f(Q))\|^2_{\mathbb{L}^2(\Omega)}\\
&\leq C\,\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + C\left(1 + \|Q\|^2_{H^2(\Omega)}\right)\|\partial_t Q\|^2_{L^2(\Omega)}.
\end{aligned}$$

Under an adequate choice of δ and C_δ depending on γ, ν, ε ($\delta = \min\{\frac{\nu}{2}, \frac{1}{2\varepsilon}\}$), (39) yields to:

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &\quad + C_1\,\frac{\nu}{2}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\frac{\gamma\varepsilon^2}{2}\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}\\ &\quad \le \widetilde{a}(t)\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &\quad + C_{\nu,\gamma,\varepsilon}\,\left(\|H\|^4_{\mathbb{L}^2(\Omega)} + \|\nabla \boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)}\right)\,\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right),\end{aligned} \tag{40}$$

where

$$\widetilde{a}(t) = a(t) + C_{\gamma,\varepsilon}\,\|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \equiv C_{\gamma,\varepsilon,\nu}\,\left(1 + \|Q\|^2_{\mathbb{H}^2(\Omega)} + \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\right).$$

We have:

$$y'(t) + z(t) \le \widetilde{a}(t)\,y(t) + C_{\nu,\gamma,\varepsilon}\,y(t)^3 \tag{41}$$

where

$$y(t) = \|\partial_t \boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q(t)\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H(t)\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)}$$

and

$$z(t) = C_1\,\frac{\nu}{2}\|\partial_t \boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\frac{\gamma\varepsilon^2}{2}\,\|\partial_t Q(t)\|^2_{\mathbb{H}^2(\Omega)}.$$

Integrating in time, and choosing regular enough initial data, see Remark 1.3 above, we obtain the existence of a small enough time T^* such that:

$$\begin{cases}\partial_t \boldsymbol{u} \in L^\infty(0,T^*;\mathbf{L}^2(\Omega)) \cap L^2(0,T^*;\mathbf{H}^1(\Omega)),\\ \partial_t Q \in L^\infty(0,T^*;\mathbb{H}^1(\Omega)) \cap L^2(0,T^*;\mathbb{H}^2(\Omega)),\\ \boldsymbol{u} \in L^\infty(0,T^*;\mathbf{H}^1(\Omega)), \quad H \in L^\infty(0,T^*;\mathbb{L}^2(\Omega)),\end{cases} \tag{42}$$

proving the statement of Theorem 1.1.

On the other hand, global in time regularity will be deduced whenever $\nabla \boldsymbol{u}$ and H be regular enough. Following the same procedure as before, but taking the alternative estimates for the terms I_3, I_4, I_6 and I_7, we can obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &\quad + C_1\,\frac{\nu}{2}\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\frac{\gamma\varepsilon^2}{2}\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}\\ &\quad \le \widetilde{a}(t)\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &\quad + C_{\gamma,\varepsilon,\nu}\,\left(\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)} + \|H\|^{2s/(2s-3)}_{\mathbb{L}^s(\Omega)}\right)\,\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right).\end{aligned} \tag{43}$$

We have to use additional regularity in order to bound the last term at the right hand side of (43). In fact, assuming (18), we conclude:

$$\begin{cases} \partial_t \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)), & \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \\ \partial_t Q \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)), & H \in L^\infty(0,T;\mathbb{L}^2(\Omega)), \end{cases} \tag{44}$$

and the statement of existence of Theorem 1.3 holds. The uniqueness will be seen in Subsection 2.4. Observe that $\Delta Q \in L^{2s/(2s-3)}(0,T;\mathbb{L}^s(\Omega))$ for $3/2 \le s \le 3$ implies the same regularity for H, because $f(Q) \in L^{2s/(2s-3)}(0,T;\mathbb{L}^s(\Omega))$. In fact, $f(Q)$ is more regular. Concretely,

$$\|f(Q)\|_{\mathbb{L}^s(\Omega)}^{2s/(2s-3)} \le \|Q\|_{\mathbb{L}^2(\Omega)}^{(2-s)/(2s-3)} \|Q\|_{\mathbb{H}^1(\Omega)}^{(3s-2)/(2s-3)}, \quad \text{for } 3/2 \le s \le 2,$$

and

$$\|f(Q)\|_{\mathbb{L}^s(\Omega)}^{2s/(2s-3)} \le \|Q\|_{\mathbb{H}^1(\Omega)}^{(s+2)/(2s-3)} \|Q\|_{\mathbb{H}^2(\Omega)}^{(s-2)/(2s-3)}, \quad \text{for } 2 \le s \le 3,$$

which implies that $f(Q) \in L^\infty(0,T;\mathbb{L}^s(\Omega))$ for $3/2 \le s \le 2$ and $f(Q) \in L^{2s/(s-2)}(0,T;\mathbb{L}^s(\Omega))$ for $2 \le s \le 3$.

2.4 The uniqueness of weak solution

In [7], it is proved that a uniqueness criteria for the weak solution is:

$$\begin{cases} \nabla \boldsymbol{u} \in L^{2q/(2q-3)}(0,T;\mathbf{L}^q(\Omega)), & \text{for } 2 \le q \le 3 \\ \Delta Q \in L^{2s/(2s-3)}(0,T;\mathbb{L}^s(\Omega)) & \text{for } 2 \le s \le 3. \end{cases} \tag{45}$$

On the other hand, weak-t regularity implies in particular $\nabla \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\Delta Q \in L^\infty(0,T;\mathbb{L}^2(\Omega))$, hence regularity (45) for $q = s = 2$ in $(0,T^*)$ is obtained. Thus, the local in time uniqueness of weak-t solution is proved.

2.5 Time-dependent Dirichlet boundary conditions for Q

Now, we consider the case of time-dependent Dirichlet boundary data for Q, that is $Q|_\Gamma = Q_\Gamma$ and $Q_\Gamma = Q_\Gamma(t)$, and try to reproduce the Ladyzhenskaya estimates in this case. In order to cancel the boundary integral terms appearing in (24) and (33), we use the lifting function defined in (17). Then, instead of system (4) for Q, we treat the following problem for $\widehat{Q} = Q - \widetilde{Q}$:

$$\partial_t \widehat{Q} + (\boldsymbol{u} \cdot \nabla) Q - S(\nabla \boldsymbol{u}, Q) + \gamma \widehat{H} = 0 \quad \text{in } (0,T) \times \Omega, \quad \widehat{Q}|_{\partial\Omega} = \mathbf{0}, \quad \widehat{Q}|_{t=0} = \mathbf{0} \tag{46}$$

where $\widehat{H} = -\varepsilon \, \Delta \widehat{Q} + f(Q)$ and $Q = \widehat{Q} + \widetilde{Q}$. In this sense, reproducing the intermediate strong inequalities for $\widehat{Q}$ following (46), we obtain instead of (27) and (29) the following inequalities (estimates for $\boldsymbol{u}$ remain unchanged):

$$\begin{aligned} \gamma \, \varepsilon \, \|\partial_t (\nabla \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} \;&+\; \frac{\gamma^2}{2} \frac{d}{dt} \|\widehat{H}\|^2_{\mathbb{L}^2(\Omega)} \leq \gamma \, \delta \, \|\partial_t \widehat{H}\|^2_{\mathbb{L}^2(\Omega)} \\ &+\; \gamma \, C_\varepsilon \, \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \left(\|\partial_t \widehat{Q}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \widetilde{Q}\|^2_{\mathbb{L}^2(\Omega)} \right) \\ &+\; \gamma \, C_\delta \, \|Q\|_{\mathbb{H}^2(\Omega)} \, \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\frac{d}{dt} \|\partial_t \widehat{Q}\|^2_{\mathbb{L}^2(\Omega)} + \gamma \, \varepsilon \, \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad \leq C_{\gamma,\varepsilon} \, \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right) \left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t \widehat{Q}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \widetilde{Q}\|^2_{\mathbb{L}^2(\Omega)} \right) \\ &\quad + C_{\gamma,\varepsilon} \, \|\partial_t \widetilde{Q}\|_{\mathbb{L}^2(\Omega)} \|\partial_t \widetilde{Q}\|_{\mathbb{H}^1(\Omega)} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{48}$$

In the latter inequality (obtained deriving (46) with respect to t and taking $\partial_t \widehat{Q}$ as test function) the new term appearing on the right hand side of (48) comes from the fact that:

$$(\boldsymbol{u} \cdot \nabla(\partial_t Q), \partial_t \widehat{Q}) = (\boldsymbol{u} \cdot \nabla(\partial_t \widetilde{Q}), \partial_t \widehat{Q}) \neq 0, \quad (S(\nabla \boldsymbol{u}, \partial_t Q), \partial_t \widehat{Q}) = (S(\nabla \boldsymbol{u}, \partial_t \widetilde{Q}), \partial_t \widehat{Q}) \neq 0$$

and therefore:

$$\begin{aligned} (\boldsymbol{u} \cdot \nabla(\partial_t \widetilde{Q}), \partial_t \widehat{Q}) \;&\leq\; \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \|\nabla(\partial_t \widehat{Q})\|_{\mathbb{L}^2(\Omega)} \|\partial_t \widetilde{Q}\|_{\mathbb{L}^3(\Omega)} \\ &\leq\; \frac{\gamma\varepsilon}{2} \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} + C_{\gamma,\varepsilon} \|\partial_t \widetilde{Q}\|_{\mathbb{L}^2(\Omega)} \|\partial_t \widetilde{Q}\|_{\mathbb{H}^1(\Omega)} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}, \\ (S(\nabla \boldsymbol{u}, \partial_t \widetilde{Q}), \partial_t \widehat{Q}) \;&\leq\; \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\partial_t \widehat{Q}\|_{\mathbb{L}^6(\Omega)} \|\partial_t \widetilde{Q}\|_{\mathbb{L}^3(\Omega)} \\ &\leq\; \frac{\gamma\varepsilon}{2} \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} + C_{\gamma,\varepsilon} \|\partial_t \widetilde{Q}\|_{\mathbb{L}^2(\Omega)} \|\partial_t \widetilde{Q}\|_{\mathbb{H}^1(\Omega)} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

With respect to the weak-t differential inequalities, this time instead of (33), we obtain:

$$\begin{aligned} &\frac{\varepsilon}{2} \frac{d}{dt} \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} - \varepsilon \int_\Gamma \partial^2_{tt} \widehat{Q} : \partial_{\mathbf{n}}(\partial_t \widehat{Q}) \, d\sigma + (\partial_t \boldsymbol{u} \cdot \nabla Q, -\varepsilon \, \Delta(\partial_t \widehat{Q})) \\ &\quad + (\boldsymbol{u} \cdot \nabla(\partial_t Q), -\varepsilon \, \Delta(\partial_t \widehat{Q})) - (S(\nabla(\partial_t \mathbf{u}), Q), -\varepsilon \Delta(\partial_t \widehat{Q})) \\ &\quad - (S(\nabla \boldsymbol{u}, \partial_t Q), -\varepsilon \Delta(\partial_t \widehat{Q})) + \gamma \varepsilon^2 \, \|\Delta(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \gamma \, (\partial_t (f(Q)), -\varepsilon \Delta(\partial_t \widehat{Q})) = 0, \end{aligned}$$

which added to (32) (taking into account that $Q=\widehat{Q}+\widetilde{Q}$) leads to:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega)}+\varepsilon\,\|\nabla(\partial_t\widehat{Q})\|^2_{L^2(\Omega)}\right)+\nu\|\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(\Omega)}+\gamma\,\varepsilon^2\,\|\Delta(\partial_t\widehat{Q})\|^2_{L^2(\Omega)}\\
&=\sum_{i=1}^{5}J_i+\sum_{i=1}^{2}\widetilde{J}_i+\sum_{i=1}^{3}\widehat{J}_i=-(\partial_t\boldsymbol{u}\cdot\nabla\boldsymbol{u},\partial_t\boldsymbol{u})+(\partial_t\boldsymbol{u}\cdot\nabla Q,\partial_t(f(Q)))\\
&\quad+(\partial_t\boldsymbol{u}\cdot\nabla(\partial_t Q),H)-\Big(\sigma\left(\partial_t(f(Q)),Q\right),\nabla(\partial_t\boldsymbol{u})\Big)-(\sigma(H,\partial_t Q),\nabla(\partial_t\boldsymbol{u}))\\
&\quad+(\partial_t\boldsymbol{u}\cdot\nabla Q,-\varepsilon\,\Delta(\partial_t\widetilde{Q}))-\Big(\sigma\Big(-\varepsilon\,\Delta(\partial_t\widetilde{Q}),Q\Big),\nabla(\partial_t\boldsymbol{u})\Big)\\
&\quad-(\boldsymbol{u}\cdot\nabla(\partial_t Q),-\varepsilon\,\Delta(\partial_t\widehat{Q}))\\
&\quad+(S(\nabla\boldsymbol{u},\partial_t Q),-\varepsilon\Delta(\partial_t\widehat{Q}))-\gamma\,(\partial_t(f(Q)),-\varepsilon\Delta(\partial_t\widehat{Q})).
\end{aligned}$$

Observe that $J_i=I_i$ for $i=1,\,2,\,3$, $J_4=I_5$; $J_5=I_6$, $\widehat{J}_1$ and $\widehat{J}_2$ are similar to I_4 and I_7, respectively; and $\widehat{J}_3$ is similiar to I_8. The "completely" new terms are $\widetilde{J}_1$ and $\widetilde{J}_2$ that can be bounded as:

$$\begin{aligned}
\widetilde{J}_1 &\le \varepsilon\,\|\partial_t\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla Q\|_{\mathbb{L}^3(\Omega)}\|\Delta(\partial_t\widetilde{Q})\|_{\mathbb{L}^2(\Omega)}\\
&\le \varepsilon\,\|\partial_t\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^1(\Omega)}\|Q\|^{1/2}_{\mathbb{H}^2(\Omega)}\|\partial_t\widetilde{Q}\|_{\mathbb{H}^2(\Omega)}\\
&\le \frac{\nu}{14}\,\|\partial_t\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)}+C_{\varepsilon,\nu}\,\|Q\|_{\mathbb{H}^2(\Omega)}\|\partial_t\widetilde{Q}\|^2_{\mathbb{H}^2(\Omega)}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\widetilde{J}_2 &\le \varepsilon\,\|\nabla(\partial_t\boldsymbol{u})\|_{\mathbf{L}^2(\Omega)}\|Q\|_{\mathbb{L}^\infty(\Omega)}\|\Delta(\partial_t\widetilde{Q})\|_{\mathbb{L}^2(\Omega)}\\
&\le \frac{\nu}{14}\,\|\partial_t\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)}+C_{\varepsilon,\nu}\,\|Q\|_{\mathbb{H}^2(\Omega)}\|\partial_t\widetilde{Q}\|^2_{\mathbb{H}^2(\Omega)}.
\end{aligned}$$

In summary, instead of (37) we have:

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\varepsilon\,\|\partial_t\widehat{Q}\|^2_{\mathbb{H}^1(\Omega)}\right)+\nu\|\nabla(\partial_t\boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)}+\gamma\varepsilon^2\,\|\nabla(\partial_t\widehat{Q})\|^2_{\mathbb{H}^1(\Omega)}\\
&\le C_{\gamma,\varepsilon,\nu}\,\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\left(\|\partial_t\widehat{Q}\|^2_{\mathbb{H}^1(\Omega)}+\|\partial_t\widetilde{Q}\|^2_{\mathbb{H}^1(\Omega)}\right)\\
&+C_{\nu,\gamma,\varepsilon}\left(\|\nabla\boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)}+\|\widehat{H}\|^4_{\mathbb{L}^2(\Omega)}+\|\widetilde{Q}\|^4_{\mathbb{H}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t\widehat{Q}\|^2_{\mathbb{H}^1(\Omega)}+\|\partial_t\widetilde{Q}\|^2_{\mathbb{H}^1(\Omega)}\right)\\
&+C_{\gamma,\varepsilon,\nu}\,\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\,\|\partial_t\widetilde{Q}\|^2_{\mathbb{H}^2(\Omega)}.
\end{aligned}\tag{49}$$

Adding the previous estimate to (30) multiplied by ε and (31), we obtain:

$$y'(t)+Y(t)\le\widehat{a}(t)\,y(t)+C_1\,y(t)^3+C_2\,y(t)^2+b(t),\tag{50}$$

where

$$\begin{cases} y(t) &= \|\partial_t \boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t \widehat{Q}(t)\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|\widehat{H}(t)\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla \boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)}, \\ Y(t) &= \nu\|\nabla(\partial_t \boldsymbol{u})(t)\|^2_{\mathbf{L}^2(\Omega)} + \gamma\varepsilon^2\,\|\nabla(\partial_t \widehat{Q})(t)\|^2_{\mathbb{H}^1(\Omega)}, \end{cases} \tag{51}$$

and

$$\begin{cases} \widehat{a}(t) &= C_{\gamma,\varepsilon,\nu}\ \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)} + \|\partial_t \widetilde{Q}\|^4_{\mathbb{H}^2(\Omega)}\right) \\ b(t) &= C_{\gamma,\varepsilon,\nu}\ \left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right)\ \|\partial_t \widetilde{Q}\|^2_{\mathbb{H}^2(\Omega)}. \end{cases}$$

Since $\widehat{a} \in L^1(0,T)$ and $b \in L^1(0,T)$ if $\partial_t \widetilde{Q} \in L^4(0,T;\mathbb{H}^2(\Omega))$, then the statement of Corollary 1.2 for the time-dependent Dirichlet case can be easily obtained.

Remark 2.2 *Regularity criteria (18) are also valid to deduce the existence of a global in time weak-t regular solution in the case of time-dependent boundary conditions for Q.*

3 Strong regularity (proof of Theorem 1.4)

Now, we consider the case of boundary conditions $Q|_\Gamma = 0$ and the stretching term $S(\cdot,\cdot)$ given by (13). Strong regularity for the (QT)-system will be obtained locally in time for any data and global in time under regularity criteria (19) for $\nabla \boldsymbol{u}$.

By using that $\boldsymbol{u}|_\Gamma = 0$, $Q|_\Gamma = 0$ and the Q-system (4), one has

$$H(Q)|_\Gamma = S(\nabla \boldsymbol{u}, Q)|_\Gamma = 0. \tag{52}$$

3.1 Prodi's strong estimates

We multiply $\boldsymbol{u}$-system (2) by $A\boldsymbol{u} := P_H(-\Delta \boldsymbol{u})$ and Q-system (4) by $-\Delta H$, obtaining:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} + \nu\,\|A\boldsymbol{u}\|^2_{L^2(\Omega)} &= -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, A\boldsymbol{u}) + ((A\boldsymbol{u}\cdot\nabla)Q, H) \\ &- \ (\sigma(H,Q), \nabla(A\boldsymbol{u})) := K_1 + K_2 + K_3 \end{aligned} \tag{53}$$

and

$$(\nabla(\partial_t Q), \nabla H) + \gamma\|\nabla H\|^2_{L^2(\Omega)} = -(\nabla(\boldsymbol{u}\cdot\nabla Q), \nabla H) + (\nabla S(\nabla \boldsymbol{u}, Q), \nabla H) := K_4 + K_5. \tag{54}$$

We want to bound (53) and (54) jointly. Firstly, we rewrites the term $(\nabla\partial_t Q, \nabla H)$ as:

$$\begin{aligned}(\nabla\partial_t Q, \nabla H) &= (\partial_t(-\Delta Q), H) + \int_\Gamma \partial_{\mathbf{n}}(\partial_t Q)\, H\, d\sigma \\ &= \frac{1}{2\varepsilon}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} - \frac{1}{\varepsilon}(\partial_t(f(Q)), H) := \frac{1}{2\varepsilon}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} - K_0,\end{aligned} \tag{55}$$

where the boundary term vanish owing to (52). Secondly, we bound $K_1 = (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, A\boldsymbol{u})$:

$$K_1 \leq \begin{cases} \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)}\|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \leq C\,\|\nabla\boldsymbol{u}\|^{3/2}_{\mathbf{L}^2(\Omega)}\|A\boldsymbol{u}\|^{3/2}_{\mathbf{L}^2(\Omega)} \\ \qquad \leq \frac{\nu}{10}\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_\nu\,\|\nabla\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)} \\ \text{or} \\ \|\boldsymbol{u}\|_{\mathbf{L}^{2q/(q-2)}(\Omega)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \leq C\,\|\nabla\boldsymbol{u}\|^{(2q-3)/q}_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}\|_{\mathbf{L}^q(\Omega)}\|A\boldsymbol{u}\|^{3/q}_{\mathbf{L}^2(\Omega)} \\ \qquad \leq \frac{\nu}{10}\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_\nu\,\|\nabla\boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{cases}$$

Now, we try to control the terms $K_3 + K_5 = (\nabla S(\nabla\boldsymbol{u}, Q), \nabla H) - (\sigma(H, Q), \nabla(A\boldsymbol{u}))$. On the basis that:

$$S(\nabla(\Delta\boldsymbol{u}), Q) : H = \sigma(H, Q) : \nabla(\Delta\boldsymbol{u}), \tag{56}$$

we apply the integration by parts

$$(\nabla S(\nabla\boldsymbol{u}, Q), \nabla H) = -(\Delta S(\nabla\boldsymbol{u}, Q), H) + \int_\Gamma \partial_{\mathbf{n}} S(\nabla\boldsymbol{u}, Q)\, H\, d\sigma = -(\Delta S(\nabla\boldsymbol{u}, Q), H).$$

The boundary integral

$$\int_\Gamma \partial_{\mathbf{n}} S(\nabla\boldsymbol{u}, Q)\, H\, d\sigma \tag{57}$$

has vanished due to (52). Observe that considering non-homogeneous Dirichlet boundary condition (i.e. $Q_\Gamma \neq 0$), the boundary term (57) does not vanish, because $H|_\Gamma \neq 0$.

Since $S(\cdot,\cdot)$ is quadratic,

$$\Delta S(\nabla\boldsymbol{u}, Q) = S(\nabla(\Delta\boldsymbol{u}), Q) + 2S(\nabla(\nabla\boldsymbol{u}), \nabla Q) + S(\nabla\boldsymbol{u}, \Delta Q),$$

hence using (56),

$$K_3 + K_5 = -(\sigma(H, Q), \nabla(\Delta\boldsymbol{u} + A\boldsymbol{u})) - 2S(\nabla(\nabla\boldsymbol{u}), \nabla Q) - S(\nabla\boldsymbol{u}, \Delta Q).$$

The worse term to manage with is:

$$-(\sigma(H, Q), \nabla(\Delta\boldsymbol{u} + A\boldsymbol{u})). \tag{58}$$

Now, we are going to prove that, under homogeneous Dirichlet boundary data for Q, (58) vanishes although $A\,\boldsymbol{u} \neq -\Delta \boldsymbol{u}$. For this, we use the Helmholtz decomposition $-\Delta \boldsymbol{u} = A\boldsymbol{u} + \nabla \pi$, with $\int_\Omega \pi\, d\boldsymbol{x} = 0$. Therefore, owing to the symmetry of $\nabla(\nabla \pi)$, (58) is rewritten as:

$$-(\sigma(H,Q), \nabla(\Delta \boldsymbol{u} + A\boldsymbol{u})) = (\sigma(H,Q), \nabla(\nabla\pi)) = (\sigma_s(H,Q), \nabla(\nabla\pi)),$$

where $\sigma_s = (\sigma + \sigma^t)/2$ denotes the symmetric part of σ, hence the previous term vanishes if the tensor σ is antisymmetric (i.e. $\sigma_s = 0$). This fact occurs considering a modified Q-tensor model as in [7] (see also Section 5.1 later), where $S(\nabla \boldsymbol{u}, Q)$ is defined in (13), which implies that $(\sigma_s(H,Q), \nabla(\nabla\pi)) = 0$ and (58) vanishes.

Thus $K_3 + K_5 = -2(S(\nabla(\nabla \boldsymbol{u}), \nabla Q), H)) - (S(\nabla \boldsymbol{u}, \Delta Q), H)$, hence:

$$K_3 + K_5 \leq \int_\Omega |D^2\boldsymbol{u}||\nabla Q||H|\, d\boldsymbol{x} + \int_\Omega |\nabla \boldsymbol{u}||D^2 Q||H|\, d\boldsymbol{x} := K_6 + K_7.$$

Term $K_2 = ((A\boldsymbol{u}\cdot\nabla)Q), \nabla H)$ is bounded by K_6. Term $K_4 = -(\nabla(\boldsymbol{u}\cdot\nabla Q), \nabla H)$ is bounded as

$$K_4 \leq \int_\Omega |\nabla \boldsymbol{u}||\nabla Q||\nabla H|\, d\boldsymbol{x} - \int_\Omega (\boldsymbol{u}\cdot\nabla)\nabla Q : \nabla H\, d\boldsymbol{x} := K_8 + K_9.$$

Integrating by parts, the treatment of K_9 reduces to that of K_7 because of:

$$\begin{aligned} K_9 &= \int_\Omega (\nabla \boldsymbol{u}\cdot\nabla(\nabla Q)) : H\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}\cdot\nabla)(\Delta Q) : H\, d\boldsymbol{x} \\ &\leq K_7 - \frac{1}{\varepsilon}\int_\Omega (\boldsymbol{u}\cdot\nabla)H : H\, d\boldsymbol{x} + \frac{1}{\varepsilon}\int_\Omega (\boldsymbol{u}\cdot\nabla) f(Q) : H\, d\boldsymbol{x} \\ &= K_7 - \frac{1}{\varepsilon}\int_\Omega (\boldsymbol{u}\cdot\nabla)H : f(Q)\, d\boldsymbol{x} := K_7 + K_{10}. \end{aligned}$$

In summary, adding (53), (54) and the following energy law (see [7]):

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla Q\|^2_{\mathbb{L}^2(\Omega)} + 2\int_\Omega F_\mu(Q)\, d\boldsymbol{x} \right) + \nu\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H\|^2_{\mathbb{L}^2(\Omega)} \leq 0,$$

where $F_\mu(Q) = F(Q) + \mu$ with $\mu = \mu(a,b,c) \geq 0$ such that

$$\int_\Omega F_\mu(Q)(t)\, d\boldsymbol{x} \geq \frac{c}{4}\|Q(t)\|^4_{\mathbb{L}^4(\Omega)} \geq 0,$$

we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla Q\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{\varepsilon}\,\|H\|^2_{\mathbb{L}^2(\Omega)} + 2\int_\Omega F_\mu(Q)\,d\boldsymbol{x}\right) \\ +\nu\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H\|^2_{\mathbb{H}^1(\Omega)} \le C\,\left(K_0 + K_1 + K_6 + K_7 + K_8 + K_{10}\right). \tag{59}$$

Now, we try to get adequate estimates for the terms depending on $f(Q)$, that is K_0 and K_{10}. Using that $Q \in L^\infty(0,T;\mathbb{H}^1(\Omega))$, we will remark that:

$$\|f(Q)\|_{\mathbb{L}^2(\Omega)} \le C, \tag{60}$$

$$\|\nabla(f(Q))\|_{\mathbb{L}^2(\Omega)} \le C\,\|Q\|_{\mathbb{H}^2(\Omega)} \in L^2(0,T). \tag{61}$$

Lemma 3.1 *Using (60)-(61) and $Q \in L^\infty(0,T;\mathbb{H}^1(\Omega))$, we can obtain the following chain of equivalencies (recall that $Q|_\Gamma = 0$):*

$$\|Q\|_{\mathbb{H}^2(\Omega)} \approx \|\Delta Q\|_{\mathbb{L}^2(\Omega)} \approx \|H\|_{\mathbb{L}^2(\Omega)} + \|f(Q)\|_{\mathbb{L}^2(\Omega)} \approx \|H\|_{\mathbb{L}^2(\Omega)} + 1 \tag{62}$$

$$\|Q\|_{\mathbb{H}^3(\Omega)} \approx \|\Delta Q\|_{\mathbb{H}^1(\Omega)} \approx \|H\|_{\mathbb{H}^1(\Omega)} + \|f(Q)\|_{\mathbb{H}^1(\Omega)} \approx \|H\|_{\mathbb{H}^1(\Omega)} + 1. \tag{63}$$

In all these estimates, we will split into two ways: one to obtain local in time strong solution, and another one to obtain global in time strong solution supposing additional regularity hypothesis over $\nabla\boldsymbol{u}$ (and not extra-regularity for ΔQ).

The bounds obtained for the K_i-terms, $i = 0,\dots,10$, were also obtained in [3] and in [13] for the nematic liquid crystal model with stretching terms.

The K_0 term can be bounded, using that $|\partial_t(f(Q))| \le C\,|\partial_t Q|\,(1+|Q|+|Q|^2) \le C\,|\partial_t Q|\,(1+|Q|^2)$ as follows,

$$K_0 = \frac{1}{\varepsilon}\,(\partial_t(f(Q)), H) \le C_\varepsilon \int_\Omega |\partial_t Q|\,\left(1+|Q|^2\right)\,|H|\,d\boldsymbol{x} := K_{01} + K_{02},$$

where

$$K_{01} \le C_\varepsilon \int_\Omega |\partial_t Q||H|\,d\boldsymbol{x} \le C_\varepsilon \|\partial_t Q\|_{\mathbb{L}^2(\Omega)}\|H\|_{\mathbb{L}^2(\Omega)} \le C_\varepsilon\left(\|\partial_t Q\|^{2/3}_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)}\right)$$

(note that applying weak regularity (11) in the Q-system, then $\partial_t Q \in L^{4/3}(0,T;\mathbb{L}^2(\Omega))$) and using (62),

$$\begin{aligned} K_{02} &\leq C_\varepsilon \int_\Omega |Q|^2 |\partial_t Q||H|\,d\boldsymbol{x} \leq C_\varepsilon \|Q\|^2_{\mathbb{L}^{12}(\Omega)} \|\partial_t Q\|_{\mathbb{L}^2(\Omega)} \|H\|_{\mathbb{L}^3(\Omega)} \\ &\leq C_\varepsilon \|Q\|^2_{\mathbb{W}^{1,12/5}(\Omega)} \|\partial_t Q\|_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{H}^1(\Omega)} \\ &\leq C_\varepsilon \|Q\|^{3/2}_{\mathbb{H}^1(\Omega)} \|Q\|^{1/2}_{\mathbb{H}^2(\Omega)} \|\partial_t Q\|_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{H}^1(\Omega)} \\ &\leq \frac{\gamma}{10} \|H\|^2_{\mathbb{H}^1(\Omega)} + C_{\gamma,\varepsilon} \Big(\|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} \Big). \end{aligned}$$

Therefore,

$$K_0 \leq \frac{\gamma}{10} \|H\|^2_{\mathbb{H}^1(\Omega)} + C_{\gamma,\varepsilon} \left(\|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} \right). \tag{64}$$

Using (60) we bound K_{10} as follows:

$$\begin{aligned} K_{10} &\leq \frac{1}{\varepsilon} \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \|\nabla H\|_{\mathbb{L}^2(\Omega)} \|f(Q)\|_{\mathbb{L}^2(\Omega)} \\ &\leq \frac{\nu}{10} \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{10} \|H\|^2_{\mathbb{H}^1(\Omega)} + C_{\nu,\gamma,\varepsilon} \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

On the other hand, using (62)-(63) conveniently:

$$K_6 \leq \begin{cases} \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbb{L}^6(\Omega)} \|H\|_{\mathbb{L}^3(\Omega)} \leq \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbb{L}^6(\Omega)} \|H\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{H}^1(\Omega)} \\ \qquad \leq \frac{\gamma}{10} \|H\|^2_{\mathbb{H}^1(\Omega)} + \frac{\nu}{10} \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_{\gamma,\nu} \|H\|^6_{\mathbb{L}^2(\Omega)} + \|H\|^2_{\mathbb{L}^2(\Omega)} \\ \text{or} \\ \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbb{L}^{2q/(q-2)}(\Omega)} \|H\|_{\mathbb{L}^q(\Omega)} \leq \|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{W}^{2,6q/(5q-6)}(\Omega)} \|H\|_{\mathbb{L}^q(\Omega)} \\ \qquad \leq \|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|^{(2q-3)/q}_{\mathbb{H}^2(\Omega)} \|Q\|^{(3-q)/q}_{\mathbb{H}^3(\Omega)} \|H\|_{\mathbb{L}^q(\Omega)} \\ \qquad \leq \frac{\gamma}{10} \|H\|^2_{\mathbb{H}^1(\Omega)} + \frac{\nu}{10} \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_{\gamma,\nu} \left\{ \left(\|H\|^{2q/(2q-3)}_{\mathbb{L}^q(\Omega)} + 1 \right) \|H\|^2_{\mathbb{L}^2(\Omega)} + 1 \right\} \end{cases}$$

Remark 3.1 (Other treatment of the K_6-term) *Integrating by parts, the K_6-term can be rewritten as a combination of the K_7 and K_8 terms. Concretely,*

$$-2\int_\Omega S(\nabla(\nabla \boldsymbol{u}), \nabla Q) : H\,d\boldsymbol{x} = 2\int_\Omega S(\nabla \boldsymbol{u}, \nabla(\nabla Q)) : H\,d\boldsymbol{x} + 2\int_\Omega S(\nabla \boldsymbol{u}, \nabla Q) : \nabla H\,d\boldsymbol{x}$$
$$-2\int_\Gamma S(\nabla \boldsymbol{u}, \partial_{\mathbf{n}} Q) : H\,d\sigma,$$

where the boundary integral term vanishes (the boundary term would always vanish if homogeneous Neumann boundary condition for Q were considered). As a consequence, no regularity criteria for either H or ∇Q is necessary to bound the K_6-term.

Terms K_7 and K_8 can be bounded as:

$$K_7 \leq \begin{cases} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)}\|D^2Q\|_{\mathbb{L}^2(\Omega)}\|H\|_{\mathbb{L}^6(\Omega)} \leq \|\nabla \boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|A\boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|Q\|_{\mathbb{H}^2(\Omega)}\|H\|_{\mathbb{H}^1(\Omega)} \\ \qquad \leq \dfrac{\gamma}{10}\|H\|^2_{\mathbb{H}^1(\Omega)} + \dfrac{\nu}{10}\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_{\gamma,\nu}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\left(\|H\|^4_{\mathbb{L}^2(\Omega)}+1\right) \\ \text{or} \\ \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|D^2Q\|_{\mathbb{L}^{6q/(5q-6)}(\Omega)}\|H\|_{\mathbb{L}^6(\Omega)} \\ \qquad \leq \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|Q\|^{(2q-3)/q}_{\mathbb{H}^2(\Omega)}\|Q\|^{(3-q)/q}_{\mathbb{H}^3(\Omega)}\|H\|_{\mathbb{H}^1(\Omega)} \\ \qquad \leq \dfrac{\gamma}{10}\|H\|^2_{\mathbb{H}^1(\Omega)} + C_\gamma\left\{\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\left(\|H\|^2_{\mathbb{L}^2(\Omega)}+1\right)+1\right\}, \quad 2\leq q\leq 3 \end{cases}$$

$$K_8 \leq \begin{cases} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)}\|\nabla Q\|_{\mathbb{L}^6(\Omega)}\|\nabla H\|_{\mathbb{L}^2(\Omega)} \leq \|\nabla \boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|A\boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)}\|Q\|_{\mathbb{H}^2(\Omega)}\|H\|_{\mathbb{H}^1(\Omega)} \\ \qquad \leq \dfrac{\gamma}{10}\|H\|^2_{\mathbb{H}^1(\Omega)} + \dfrac{\nu}{10}\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_{\gamma,\nu}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\left(\|H\|^4_{L^2(\Omega)}+1\right) \\ \text{or} \\ \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|\nabla Q\|_{\mathbb{L}^{\frac{2q}{q-2}}(\Omega)}\|\nabla H\|_{\mathbb{L}^2(\Omega)} \leq \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\|Q\|_{\mathbb{W}^{2,6q/(5q-6)}(\Omega)}\|H\|_{\mathbb{H}^1(\Omega)} \\ \qquad \leq \dfrac{\gamma}{10}\|H\|^2_{\mathbb{H}^1(\Omega)} + C_\gamma\left\{\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\left(\|H\|^2_{\mathbb{L}^2(\Omega)}+1\right)+1\right\}, \quad 2\leq q\leq 3. \end{cases}$$

Considering all previous estimates in (59), we can obtain one of the two following inequalities:

$$\begin{aligned} &\frac{d}{dt}\left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla Q\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{\varepsilon}\|H\|^2_{\mathbb{L}^2(\Omega)} + \frac{c}{4}\|Q\|^4_{\mathbb{L}^4(\Omega)}\right) + \nu\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H\|^2_{\mathbb{H}^1(\Omega)} \\ &\quad \leq C_{\gamma,\varepsilon}\left(\|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)}\right) + C_\nu\|\nabla \boldsymbol{u}\|^6_{\mathbf{L}^2(\Omega)} \\ &\quad +C_{\gamma,\nu,\varepsilon}\left(\|H\|^4_{\mathbb{L}^2(\Omega)}+1\right)\left(\|H\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right) \end{aligned} \tag{65}$$

or

$$\begin{aligned} &\frac{d}{dt}\left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla Q\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{\varepsilon}\|H\|^2_{\mathbb{L}^2(\Omega)} + \frac{c}{4}\|Q\|^4_{\mathbb{L}^4(\Omega)}\right) + \nu\|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H\|^2_{\mathbb{H}^1(\Omega)} \\ &\quad \leq C_{\gamma,\varepsilon}\left(\|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)}\right) + C_{\nu,\gamma,\varepsilon}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad +C_\gamma\left\{\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\left(\|H\|^2_{\mathbb{L}^2(\Omega)}+1\right)+1\right\} + C_\nu\|\nabla \boldsymbol{u}\|^{2q/(2q-3)}_{\mathbf{L}^q(\Omega)}\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{66}$$

Therefore, calling

$$\begin{cases} y(t) &=& \|\boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \|H(t)\|^2_{\mathbb{L}^2(\Omega)}, \\ Y(t) &=& \nu\,\|A\boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \gamma\,\|H(t)\|^2_{\mathbb{H}^1(\Omega)}, \end{cases} \tag{67}$$

we obtain for one hand from (65) an estimate of kind:

$$y'(t) + Y(t) \le C\,\left(y(t)^3 + y(t) + 1\right), \tag{68}$$

which leads to a local in time estimate, and for other hand, from (66) we get:

$$y'(t) + Y(t) \le a(t)\,y(t) + b(t), \tag{69}$$

where $a \in L^1(0,T)$ supposing the additional regularity hypothesis (19), which leads to a global in time estimate.

Remark 3.2 (Space-periodic boundary conditions for $(\boldsymbol{u}, Q)$) *In this case, $-\Delta\boldsymbol{u} = A\boldsymbol{u}$ and (57) and (58) vanish. Moreover, Remark 3.1 is also true hence no additional regularity hypothesis on ΔQ has to be imposed. Therefore, the statement to obtain strong solutions in Theorem 1.4 (local in time for any data and global in time assuming additional regularity hypothesis (19)) is true.*

Remark 3.3 (Serrin's regularity criteria for $\boldsymbol{u}$) *The regularity hypothesis (19) on $\nabla\boldsymbol{u}$ appearing in Theorem 1.4 can be changed by the most known regularity criteria on $\boldsymbol{u}$ (20), called the* Serrin's condition. *In fact, condition (20) is replaced by (19) in [8] for a nematic model without stretching terms. Thus, in order to proceed similarly with the Q-tensor system, we should analyze the stretching terms which correspond to the K_7 term:*

$$\begin{aligned} \left|\int_\Omega \nabla\boldsymbol{u}\,D^2Q : H\,d\boldsymbol{x}\right| &= \left|-\int_\Omega \boldsymbol{u}\,\nabla(D^2Q) : H\,d\boldsymbol{x} - \int_\Omega \boldsymbol{u}\,D^2Q : \nabla H\,d\boldsymbol{x}\right| \\ &\le C\int_\Omega |\boldsymbol{u}|\,|H|\,|\nabla H|\,d\boldsymbol{x} \le \|\boldsymbol{u}\|_{\mathbf{L}^s(\Omega)}\|\nabla H\|_{\mathbb{H}^1(\Omega)}\|H\|_{\mathbf{L}^{2s/(s-2)}(\Omega)} \\ &\le \|\boldsymbol{u}\|_{\mathbf{L}^s(\Omega)}\|\nabla H\|^{1+3/s}_{\mathbb{H}^1(\Omega)}\|H\|^{(s-3)/s}_{\mathbf{L}^2(\Omega)} \le \frac{\gamma}{10}\|\nabla H\|^2_{\mathbb{H}^1(\Omega)} + C_{\delta,\gamma}\,\|\boldsymbol{u}\|^{2s/(s-3)}_{\mathbf{L}^s(\Omega)}\,\|H\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Finally, K_7-term can be bounded in terms of Serrin's criteria (20).

4 Global regularity with ν large enough (Proof of Theorem 1.5)

4.1 Prodi's estimates for Neumann or homogeneous Dirichlet conditions

Under these boundary conditions, the boundary term in (55) vanishes, but $K_3 + K_5$ must be bounded in an alternative way, obtaining terms of type K_6 and K_8 and the following new term:

$$K_{11} = \int_\Omega |D^2 \boldsymbol{u}||Q||\nabla H|\, d\boldsymbol{x}. \tag{70}$$

Hence, the statement of Theorem 1.5 follows bounding the terms K_0, K_1, K_6, K_7, K_8, K_{10} and K_{11}. The estimates for K_0 and K_{10} are the same than in Subsection 3.1. For K_1, $K_6 - K_8$ and K_{11}, we bound as follows:

$$\begin{aligned} K_1 &\le \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \le C\, \|\boldsymbol{u}\|^{3/2}_{\mathbf{H}^1(\Omega)} \|A\boldsymbol{u}\|^{3/2}_{\mathbf{L}^2(\Omega)} \\ &\le \delta \|\boldsymbol{u}\|^{2/3}_{\mathbf{H}^1(\Omega)} \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C_\delta \|\boldsymbol{u}\|^4_{\mathbf{H}^1(\Omega)}, \end{aligned}$$

$$\begin{aligned} K_6 &\le \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\nabla Q\|_{\mathbf{L}^6(\Omega)} \|H\|_{\mathbf{L}^3(\Omega)} \\ &\le \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{H}^2(\Omega)} \|H\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{H}^1(\Omega)} \\ &\le \delta \left(\|Q\|_{\mathbb{H}^2(\Omega)} \|D^2\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|H\|^2_{\mathbb{H}^1(\Omega)} \right) + C_\delta \|Q\|^2_{\mathbb{H}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} K_7 &\le \|\nabla\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \|D^2 Q\|_{\mathbf{L}^2(\Omega)} \|H\|_{\mathbf{L}^3(\Omega)} \\ &\le \|D^2\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{H}^2(\Omega)} \|H\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H\|^{1/2}_{\mathbb{H}^1(\Omega)} \\ &\le \delta \left(\|Q\|_{\mathbb{H}^2(\Omega)} \|D^2\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|H\|^2_{\mathbb{H}^1(\Omega)} \right) + C_\delta \|Q\|^2_{\mathbb{H}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} K_8 &\le \|\nabla\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \|\nabla Q\|_{\mathbf{L}^6(\Omega)} \|\nabla H\|_{\mathbf{L}^2(\Omega)} \\ &\le \|\boldsymbol{u}\|^{1/2}_{\mathbf{H}^1(\Omega)} \|\boldsymbol{u}\|^{1/2}_{\mathbf{H}^2(\Omega)} \|Q\|_{\mathbb{H}^2(\Omega)} \|\nabla H\|_{\mathbb{L}^2(\Omega)} \\ &\le \delta \left(\|Q\|^2_{\mathbb{H}^2(\Omega)} \|D^2\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla H\|^2_{\mathbb{L}^2(\Omega)} \right) + C_\delta \|Q\|^2_{\mathbb{H}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} K_{11} &\le C\, \|A\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|Q\|_{\mathbb{L}^\infty(\Omega)} \|\nabla H\|_{\mathbb{L}^2(\Omega)} \\ &\le \delta \|\nabla H\|^2_{\mathbb{L}^2(\Omega)} + C_\delta \|Q\|^2_{\mathbb{L}^\infty(\Omega)} \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

From previous estimates, we obtain a time differential inequality of type:

$$y'(t) + (\nu - C\, y(t))\; Y(t) \le a(t)\, y(t),$$

where $y(t)$ and $Y(t)$ are defined in (67) and

$$\begin{cases} a(t) &=& C\, \left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|Q\|^2_{\mathbb{H}^2(\Omega)} + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} + 1 \right), \\ b(t) &=& C\, \left(1 + \|\partial_t Q\|^{4/3}_{\mathbb{L}^2(\Omega)} \right). \end{cases}$$

Since $a, b \in L^1(0,T)$, imposing ν large enough and using the argument done in [4] we can deduce the existence of global in time strong solution.

4.2 Strong and weak-t regularity for non-homogeneous Dirichlet conditions

The argument of large enough viscosity ν used for the homogeneous Neumann or Dirichlet conditions (9) (with $Q_\Gamma = 0$) cannot be directly used in the case of non-homogeneous Dirichlet boundary conditions (10), because the boundary term appearing in (55) does not vanish. Using the lifting function $\widetilde{Q}$ defined in (17), the problem for the Q-tensor is to find $\widehat{Q}$ ($\widehat{Q} = Q - \widetilde{Q}$) such that:

$$\partial_t \widehat{Q} + (\boldsymbol{u}\cdot\nabla) Q - S(\nabla \boldsymbol{u}, Q) + \gamma\, \widehat{H} = 0, \quad \widehat{Q}|_\Gamma = 0, \quad \widehat{Q}(0) = 0, \tag{71}$$

where $Q = \widehat{Q} + \widetilde{Q}$ and $\widehat{H} = -\varepsilon\, \Delta \widehat{Q} + f(Q)$. Then, the following formulation (similar to (54)) is satisfied:

$$(\nabla(\partial_t \widehat{Q}), \nabla \widehat{H}) + (\nabla(\boldsymbol{u}\cdot\nabla Q), \nabla \widehat{H}) - (\nabla S(\nabla \boldsymbol{u}, \widehat{Q}), \nabla \widehat{H}) + \gamma \|\nabla \widehat{H}\|^2_{L^2(\Omega)} = 0.$$

Estimate (55) is now replaced by:

$$(\nabla \partial_t \widehat{Q}, \nabla \widehat{H}) = \frac{1}{2\varepsilon}\frac{d}{dt}\|\widehat{H}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \partial_{\mathbf{n}}(\partial_t \widehat{Q})\, \widehat{H}\, d\sigma - \frac{1}{\varepsilon}\left(\partial_t(f(Q)), \widehat{H} \right) \tag{72}$$

Observe that, although $\widehat{Q}|_{\partial\Omega} = 0$, the boundary term in (72) does not vanish because from (71) we can deduce:

$$\gamma \widehat{H}|_\Gamma = S(\nabla \boldsymbol{u}, \widetilde{Q}|_\Gamma) - \partial_t \widehat{Q}|_\Gamma \neq 0.$$

But it can be bounded as follows:

$$
\begin{aligned}
&\int_\Gamma \partial_{\mathbf{n}}(\partial_t Q)\, H(Q)\, d\sigma \le \|\partial_{\mathbf{n}}(\partial_t Q)|_\Gamma\|_{\mathbb{L}^2(\Gamma)} \|H(Q)|_\Gamma\|_{\mathbb{L}^2(\Gamma)} \\
&\le C\, \|\nabla(\partial_t Q)\|_{\mathbb{H}^{1/2}(\Omega)} \|H(Q)\|_{\mathbb{H}^{1/2}(\Omega)} \\
&\le C\, \|\partial_t Q\|^{1/2}_{\mathbb{H}^1(\Omega)} \|\partial_t Q\|^{1/2}_{\mathbb{H}^2(\Omega)} \|H(Q)\|^{1/2}_{\mathbb{L}^2(\Omega)} \|H(Q)\|^{1/2}_{\mathbb{H}^1(\Omega)} \\
&\le \varepsilon \left(\|H(Q)\|^2_{\mathbb{H}^1(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \right) + C_\varepsilon \left(\|H(Q)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} \right).
\end{aligned} \tag{73}
$$

In this case, we would need to handle with the weak-t estimates now obtained using that ν is large enough. This new hypothesis need to rewritte the terms J_1, J_3, J_5, $\widehat{J}_1$ and $\widehat{J}_2$ of Subsection 2.5 as follows:

$$
\begin{aligned}
J_1 &\le \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \\
&\le \delta \|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} \|\nabla \boldsymbol{u}\|^{2/3}_{\mathbf{L}^2(\Omega)} + C_\delta \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
J_3 &\le \|\partial_t \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \|\nabla(\partial_t Q)\|_{\mathbb{L}^6(\Omega)} \|H\|_{\mathbb{L}^2(\Omega)} \\
&\le \delta \left(\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} + \|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)} \right) + C_\delta \|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
J_5 &\le \|H\|_{\mathbb{L}^2(\Omega)} \|\partial_t Q\|_{\mathbb{L}^\infty(\Omega)} \|\nabla(\partial_t \boldsymbol{u})\|_{\mathbf{L}^2(\Omega)} \\
&\le \delta \left(\|H\|_{\mathbb{L}^2(\Omega)} \|\nabla(\partial_t \boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \right) + C_\delta \|H\|^2_{\mathbb{L}^2(\Omega)} \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{J}_1 &\le \|\boldsymbol{u}\|_{\mathbf{L}^\infty} \|\nabla(\partial_t Q)\|_{\mathbb{L}^2(\Omega)} \|\Delta(\partial_t \widehat{Q})\|_{\mathbb{L}^2(\Omega)} \\
&\le \delta \left(\|\Delta(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} + \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \left(1 + \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} \right) \right) + C_\delta \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \|\nabla(\partial_t \widetilde{Q})\|^4_{\mathbb{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{J}_2 &\le \|\nabla \boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \|\partial_t Q\|_{\mathbb{L}^6(\Omega)} \|\Delta(\partial_t \widehat{Q})\|_{\mathbb{L}^2(\Omega)} \\
&\le \delta \left(\|\Delta(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} + \|A\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} \right) \\
&+ C_\delta \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \left(\|\nabla(\partial_t \widehat{Q})\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla(\partial_t \widetilde{Q})\|^2_{\mathbb{L}^2(\Omega)} \right).
\end{aligned}
$$

Instead of (49), we obtain:

$$
y'(t) + (\nu - C\, y(t))\; Y(t) \le a(t)\, y(t) + b(t),
$$

where $y(t)$ and $Y(t)$ are defined in (51) and

$$
\begin{cases}
a(t) = C \left(\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|H\|^2_{\mathbb{L}^2(\Omega)} \right), \\
b(t) = C \left(\left(1 + \|Q\|^2_{\mathbb{H}^2(\Omega)} \right) \|\partial_t \widetilde{Q}\|^2_{\mathbb{H}^1(\Omega)} + \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \|\partial_t \widetilde{Q}\|^4_{\mathbb{H}^1(\Omega)} + \|Q\|_{\mathbb{H}^2(\Omega)} \|\partial_t \widetilde{Q}\|^2_{\mathbb{H}^2(\Omega)} \right).
\end{cases}
$$

Hence we can deduce the existence of global in time strong solution.

5 Application to other models

5.1 The traceless and symmetric Q-tensor model

In [7] the authors analyze a modified model of the Q-tensor maintaining the constraints of traceless and symmetry of the tensor Q. The symmetry property needs that the model considers the stretching defined in (13), which implies that the stretching term is symmetric, and thus Q is also symmetric (and therefore $H(Q) = -\varepsilon\,\Delta Q + f(Q)$). As a consequence, $\sigma(H,Q)$ is antisymmetric in such a way that:

$$(\sigma(H,Q),\nabla\boldsymbol{u}) = (\sigma(H,Q),\mathbf{W}) \quad \text{with} \quad \mathbf{W} = \frac{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^t}{2} \quad \text{(vorticity tensor).}$$

The global in time weak regularity and uniqueness criteria were also proved for this modified model in [7]. The argument of strong regularity can also be reproduced for this model. The key point of application appears in the argument of local in time strong regularity for the system when space-periodic boundary conditions are considered (in the other boundary cases the explicit new form of the stretching term and the tensor $\sigma(H,Q)$ is not essential). Concretely, (56) now reads:

$$S(\Delta\mathbf{W},Q) : H = \sigma(H,Q) : \Delta\mathbf{W}$$

and the conclusion by Prodi's estimates remains true.

The effect of the traceless property is the use of a modified expression from $H(Q)$ that is written as:

$$\widetilde{H} = H + \alpha(Q)\,\mathbb{I} \quad (\mathbb{I} \text{ is the identity matrix}) \tag{74}$$

where

$$\widetilde{f}(Q) = f(Q) + \alpha(Q)\,\mathbb{I}$$

that allows to obtain $tr(Q) = 0$ for a certain scalar function $\alpha(Q)$. Two choices are analyzed:

$$\alpha_1(Q) = \frac{a}{3}\,tr(Q) + \frac{b}{9}\,tr(Q^2 + QQ^t + Q^tQ) \tag{75}$$

and

$$\alpha_2(Q) = -\frac{tr(f(Q))}{3}. \tag{76}$$

Observe that the terms appearing in $\widetilde{f}(Q)$ are of the same type of those in $f(Q)$, and therefore the bounds for $\widetilde{f}(Q)$ in the different Sobolev spaces are similar to those for $f(Q)$. The proof of the strong regularity with this new $\widetilde{f}(Q)$ is therefore easily obtained.

5.2 The complete nematic model of [12]

We consider the EDP system appearing in [12, system (1.9), p. 1187], that reads as:

$$(LC)\left\{\begin{array}{rcl} D_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p - \lambda\nabla\cdot\sigma^e &=& \mathbf{0} \quad \text{in } Q, \\ \nabla\cdot\boldsymbol{u} &=& 0 \quad \text{in } Q, \\ \mathcal{D}_t\boldsymbol{d} + \gamma\boldsymbol{w} &=& \mathbf{0} \quad \text{in } Q, \end{array}\right.$$

where $\nu > 0, \lambda > 0, \gamma > 0$ are the fluid viscosity, a elasticity and a relaxation in time constants, respectively. The elastic stress tensor σ^e is defined as:

$$\begin{aligned} \sigma^e = \sigma^{e,1} + \sigma^{e,2}, \quad & \text{with } \sigma^{e,1} = -(\nabla\boldsymbol{d})^t\nabla\boldsymbol{d} \quad \text{(Korteweg tensor)} \\ & \text{and } \sigma^{e,2} = -\beta\boldsymbol{w}\,\boldsymbol{d}^t - (1+\beta)\boldsymbol{d}\,\boldsymbol{w}^t \quad \text{(stretching term)} \end{aligned} \tag{77}$$

where $\beta \in \mathbb{R}$ and

$$\boldsymbol{w} = -\Delta\boldsymbol{d} + \frac{1}{\varepsilon^2}\mathbf{f}(\boldsymbol{d}) \quad \text{with } \mathbf{f}(\boldsymbol{d}) = \left(|\boldsymbol{d}|^2 - 1\right)\,\boldsymbol{d}.$$

Here $|\boldsymbol{d}|$ denotes the euclidean norm in $\mathbb{R}^3$ and $\varepsilon > 0$ is a penalization parameter. This penalization function has a potential structure, i. e. there exists the function $F(\boldsymbol{d}) = \frac{1}{4}\left(|\boldsymbol{d}|^2 - 1\right)^2$ such that $\mathbf{f}(\boldsymbol{d}) = \nabla_{\boldsymbol{d}}(F(\boldsymbol{d}))$ for all $\boldsymbol{d} \in \mathbb{R}^3$. Observe that $\boldsymbol{w} = \dfrac{\delta E_e(\boldsymbol{d})}{\delta\boldsymbol{d}}$ is the variational derivative of the elastic energy

$$E_e(\boldsymbol{d}) = \frac{1}{2}\int_\Omega |\nabla\boldsymbol{d}|^2 + \frac{1}{\varepsilon^2}\int_\Omega \mathbf{F}(\boldsymbol{d}). \tag{78}$$

The time derivative

$$\mathcal{D}_t\boldsymbol{d} = D_t\boldsymbol{d} + C(\boldsymbol{d}, \nabla\boldsymbol{u})$$

contains the material derivative $D_t\boldsymbol{d} = \partial_t\boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d}$ and the quadratic term

$$C(\boldsymbol{d}, \nabla\boldsymbol{u}) = \beta(\nabla\boldsymbol{u})\boldsymbol{d} + (1+\beta)(\nabla\boldsymbol{u})^t\boldsymbol{d}$$

modeling the so-called stretching effects, depending on the form of the molecules [12]. In fact, the constant $\beta = -\alpha$ is associated with the aspect ratio r of the ellipsoid particles.

The case of α near to 1 corresponds to rod like particles (then the transport is purely covariant stretching), the case of α near to 0 corresponds to disc like particles (then the transport is anti-stretching) and the case of α near to $1/2$ corresponds to the spherical shape (the transport is the rigid rotation of the center of the mass). Some numerical results for the (LC)-model can be seen in [16] and [17].

Up to the constant β, the structure of $\sigma^e(\boldsymbol{d},\boldsymbol{w})$ is similar to $\tau(Q)+\sigma(Q,H)$, where $\boldsymbol{d}$ and $\boldsymbol{w}$ have similar roles to Q and H. The structure of $C(\boldsymbol{d},\nabla\boldsymbol{u})$ is almost similar to $S(\nabla\boldsymbol{u},Q)$. The main difference with respect to the QT-model is that:

$$C(\boldsymbol{d},\nabla\boldsymbol{u})\cdot\boldsymbol{d}=(1+2\,\beta)\,(\boldsymbol{d}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{d}\neq 0$$

and therefore the maximum principle is not longer satisfied for $\boldsymbol{d}$ (except for the spherical molecules when $\beta=-1/2$). However, this property has not been used in the proofs of regularity in this work.

The existence of global in time weak solution [13] is based on the fact that:

$$-(\nabla\cdot\sigma^{e,2},\boldsymbol{u})=(\sigma^{e,2},\nabla\boldsymbol{u})=-(C(\boldsymbol{d},\nabla\boldsymbol{u}),\boldsymbol{w}).$$

Therefore, we can easily extrapolate the regularity results to this (LC)-model, obtaining all results of this paper depending on the boundary conditions for $\boldsymbol{d}$.

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