STABILITY AND UNIQUENESS FOR COOPERATIVE DEGENERATE LOTKA-VOLTERRA MODEL [1]

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Abstract

In this work we deal with the existence, stability and uniqueness of positive solution of the symbiotic Lotka-Volterra degenerate model. We study and characterize the existence of the principal eigenvalue for weakly coupled elliptic cooperative singular systems. We use it, monotony methods and blowing up arguments to get our results and to show the change of behaviour between the cases of weak and strong mutualism and between non-degenerate and degenerate model.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with smooth boundary $\partial\Omega$. Consider the parabolic degenerate problem

$$\begin{cases} w_t - \Delta(w^m) = \lambda w - w^2 & \text{in } (0,+\infty)\times\Omega, \\ w(t,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ w(0,x) = w_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1}$$

where $m > 1, \lambda \in \mathbb{R}$ and $w_0(x)$ is a bounded nonnegative function. This kind of equations was introduced by Gurtin and MacCamy [1] to model the evolution of a biological population whose density is w.

It is well known, see for example [2], that problem (1) possesses a unique nonnegative solution $w \in \mathcal{C}(\mathbb{R}_+; L^1(\Omega)) \cap L^\infty(D_T)$ for any $T > 0$ where $D_T = (0,T]\times\Omega$ and $\mathbb{R}_+ = [0,+\infty)$.

In [3] and [4] the large time behaviour of the nonnegative solutions of (1) was studied. It was shown that if $\lambda > 0$ the unique positive steady-state solution of (1), w_λ, attracts in $L^p(\Omega)$ norm ($p = +\infty$ if $N = 1$ and $p \in [1,+\infty)$ if $N \geq 2$) any solution of (1) for any w_0 in a suitable subset of $L^\infty(\Omega)$.

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Later, Bertsch and Rostamian [5] studied the stability of the positive steady-state of (1) by linearizing (1) at the steady-state. As consequence of their results, it can be proved that if $\lambda > 0$ then w_λ is exponentially stable in the $L^\infty(\Omega)$ norm for any $w_0 \geq 0$ and nontrivial.

This study is more difficult when one treats with systems instead of equations. Hence, we consider the degenerate Lotka-Volterra model

$$\begin{cases} w_t - \Delta(w^m) = \lambda w - w^2 + bwz & \text{in } (0,+\infty)\times\Omega, \\ z_t - \Delta(z^m) = \mu z - z^2 + cwz & \text{in } (0,+\infty)\times\Omega, \\ w(t,x) = z(t,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ w(0,x) = w_0(x), z(0,x) = z_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{2}$$

where $m > 1; \lambda, \mu, b, c \in \mathbb{R}$ and w_0, z_0 are bounded and nonnegative functions. Here w and z represent the densities of two species inhabiting in Ω, λ and μ are the growth rates of the species, b and c are the interactions rates between the species. Since $m > 1$ the diffusion, the rate the moving of these species from high density regions to low density ones, is slow. In the prey-predator ($b < 0$ and $c > 0$) and competition ($b < 0$ and $c < 0$) cases, Pozio and Tesei [6] proved the existence of a unique nonnegative global solution (w, z) of (2) with $(w, z) \in (\mathcal{C}(\mathbb{R}_+; L^1(\Omega)) \cap L^\infty(D_T))^2$ for any $T > 0$. In both cases, they proved that if we have a pair of sub-supersolutions $(\underline{w}, \underline{z}), (\overline{w}, \overline{z})$ of the stationary problem associated to (2), then the interval $I = [(\underline{w}, \underline{z}), (\overline{w}, \overline{z})]$ is stable in $L^p(\Omega)$ norm in the following sense: there exists a set $\mathcal{K}$ containing a neighbourhood of I such that for any $(w_0, z_0) \in \mathcal{K}$, the distance from I to (w, z) goes to zero in the $L^p(\Omega)$ norm as t diverges.

In [7] it was proved that under the assumptions of the existence of a pair of sub-supersolutions $(\underline{w}, \underline{z}), (\overline{w}, \overline{z})$ of the stationary problem associated to (2), this problem possesses a unique non-negative global solution. In the symbiotic case ($b > 0$ and $c > 0$) the authors showed that for $(w_0, z_0) = (\underline{w}, \underline{z})$ (resp. $(w_0, z_0) = (\overline{w}, \overline{z})$) the corresponding solution (w, z) of (2) converges monotonically increasing in t (resp. decreasing) to the minimal (w_*, z_*) (resp. maximal (w^*, z^*)) solution of the stationary problem of (2) in the $L^p(\Omega)$ norm. Moreover, (w_*, z_*) (resp. (w^*, z^*)) is stable for any $(w_0, z_0) \in [(\underline{w}, \underline{z}), (w_*, z_*)]$ (resp.$(w_0, z_0) \in [(w^*, z^*), (\overline{w}, \overline{z})]$).

In this work we will adopt a different way. Specifically, we consider the symbiotic Lotka-Volterra model

$$\begin{cases} u_t - \Delta u = \lambda u^{1/m} - u^{2/m} + bu^{1/m}v^{1/m} & \text{in } (0,+\infty)\times\Omega, \\ v_t - \Delta v = \mu v^{1/m} - v^{2/m} + cu^{1/m}v^{1/m} & \text{in } (0,+\infty)\times\Omega, \\ u(t,x) = v(t,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), v(0,x) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{3}$$

where $m > 1$; $b, c > 0$; $\lambda, \mu \in \mathbb{R}$ and u_0 and v_0 bounded positive functions. Observe that (3) is a parabolic problem with linear diffusion but the second terms are not Lipschitz continuous or

$\mathcal{C}^1$ functions in $\overline{\Omega} \times \mathbb{R}^2_+$ and so we can not apply the classical results about semilinear parabolic systems (see for example [8]). In fact, even in the scalar case, there are examples with the second term Hölder continuous functions $\overline{\Omega} \times \mathbb{R}_+$ and possessing infinitely many nonnegative solutions (see pag. 27 and Theorem 1.6.1 in [9]). Moreover, in our knowledge, there is not a general theory of the sub-supersolutions method (an iteration method which starts at a pair of sub-supersolution and leads to a solution in between) with Hölder but not Lipschitz continuous nonlinearities for parabolic problem. In Section 3 we study the parabolic problem

$$\begin{cases} u_t + L_1 u = f(x,u,v) & \text{in } (0,+\infty)\times\Omega, \\ v_t + L_2 v = g(x,u,v) & \text{in } (0,+\infty)\times\Omega, \\ u(t,x) = v(t,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), v(0,x) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{4}$$

where L_1 and L_2 are uniformly elliptic operators and $f, g \in \mathcal{C}^\beta(\overline{\Omega} \times \mathbb{R}^2_+)$ for some $\beta \in (0,1)$ (see [10] pp. 52-53 for the definition of Hölder continuous spaces in not necessarily bounded set). We prove that the sub-supersolution method is valid for problem (4). For that, we use the sequences built in [11] in the case of elliptic systems and some results of Chapter 8 in [9]. Although we have already indicated that nonuniqueness can occur in this case, when $f, g \in \mathcal{C}^2(\Omega \times (0,+\infty)^2)$ and satisfy a technical assumption we will prove that there exists a unique positive solution, where by positive solution we will denote a solution of (4) which belongs to the interior of the positive cone of $\mathcal{C}^1(\overline{\Omega})$ for any $t \geq 0$.

Observe that even though the parabolic problems (2) and (3) are different, we will show that their linearizing at a steady-state are equivalent and so, the stability of a positive steady-state is also equivalent (see Proposition 4). But, the study of the linearized around a steady-state solution of (3) leads us to consider the spectrum of a linear singular eigenvalue problem for weakly coupled elliptic systems. In the scalar case this kind of problems has been studied by Bertsch and Rostamian [5] when the operator is in divergence form and by Hernández, Mancebo and Vega [12] for more general operator, but it has not been yet studied for weakly coupled systems. So, in Section 2 we analyze the problem

$$\begin{cases} \mathcal{L}U = M(x)U + \lambda U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\mathcal{L}$ is a diagonal 2×2 matrix of uniformly elliptic operators and M is a 2×2 cooperative matrix where the coefficients are allowed to blow up near $\partial\Omega$ at a controlled rate. We prove the existence of a real eigenvalue of (5) denoted $\sigma_1(\mathcal{L} - M)$. The positivity of $\sigma_1(\mathcal{L} - M)$, which provides us with the stability of the steady-state solution of (3), will be characterized by means of the existence of a positive strict supersolution of $\mathcal{L} - M$ and also like that $\mathcal{L} - M$ satisfies the strong maximum principle following the results in [13], [14] and [15].

We use these results to show that if we denote

$$M(x) = \begin{pmatrix} f_u(\cdot, u_s, v_s) & f_v(\cdot, u_s, v_s) \\ g_u(\cdot, u_s, v_s) & g_v(\cdot, u_s, v_s) \end{pmatrix}$$

where (u_s, v_s) is a positive steady-state of (4), then (u_s, v_s) is exponentially stable if $\sigma_1(\mathcal{L}-M) > 0$ and unstable if $\sigma_1(\mathcal{L}-M) < 0$ in the Lyapunov sense with the $\mathcal{C}^1(\overline{\Omega})$ norm. This result generalizes to system the obtained for the scalar case in [12].

In Section 4 we apply the above result to study (3) and its corresponding stationary problem

$$\begin{cases} -\Delta u = \lambda u^{1/m} - u^{2/m} + b u^{1/m} v^{1/m} & \text{in } \Omega, \\ -\Delta v = \mu v^{1/m} - v^{2/m} + c u^{1/m} v^{1/m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

When $m = 1$ these problems have been studied in [16], [17], [18], [19] and references therein and when $m > 1$ in [20] and [21], where results of existence of positive solutions of (6) are given. This model has attracted much less attention than the cases prey-predator and competition, even in the case $m = 1$, due basically to the absence of a priori bounds. Specifically, if $b, c > 0$, $bc \le 1$ (weak mutualism between the species) and $m > 1$ we prove, as Korman did in [17] when $m = 1$, that the positive solution of (3) exists for all time and is bounded in $L^\infty(\Omega)$. However, if $bc > 1$ (strong mutualism) and $1 < m < 2$ the positive solution of (3) blows up in finite time. If $m = 2$ the positive solution of (3) exists for all time but it is not bounded. But if $m > 2$ again the positive solution of (3) exists for all time and is bounded in $L^\infty(\Omega)$ in contrast with the case $m = 1$, see Theorem 3.2 in [17]. This can have a biological interpretation as follows: if the mutualism interaction is large ($bc > 1$) but the diffusion is very slow ($m > 2$), the species can coexist in Ω unlike the linear diffusion ($m = 1$).

Our main results in this section can be summarized as follows:

- If $bc < 1$ and $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ there exists a unique positive solution of (3). Moreover, if either $m < 2$, b and c are small; or $m = 2$; or $m > 2$ and (λ, μ) belongs to a suitable subset, then there exists a unique and globally stable steady-state solution. Finally, if $\lambda, \mu \le 0$ then the trivial solution is globally stable.

- If $bc = 1$, $m > 1$ and $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ there exists a unique positive solution of (3) and at least a positive solution of (6). Moreover under condition (26), the steady-state solution is asymptotically stable.

- Assume $bc > 1$. If $1 < m < 2$, λ and μ are sufficiently large, then the positive solution of (3) blows up in finite time and if λ and μ are positive and sufficiently small, then there exists a unique positive global solution of (3) and at least a positive solution of (6). If

$m = 2$ there exists a unique positive global solution of (3) but not bounded. If $m > 2$ and $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ there exists a unique positive solution of (3) and at least a positive solution of (6). Again, in any case, under condition (26), the steady-state solution is asymptotically stable.

Our results improve the results previously mentioned of [7] and [6] and generalizes to systems the results of [5], see Remarks 4 and 5.

Finally, in Remark 6 we show that some of these results are optimal.

2 The maximum principle for singular cooperative systems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $\mathcal{C}^{3,\gamma}$ boundary $\partial\Omega$ for some $\gamma > 0$ if $N > 1$ and a bounded open interval for $N = 1$. We denote $d(x)$ the distance from x to $\partial\Omega$. It is known that $d \in \mathcal{C}^{2,\gamma}(\overline{\Omega}_1)$, with $\Omega_1 = \{x \in \Omega : d(x) < \rho_1\}$ for some $\rho_1 > 0$. Suppose that we have a second order elliptic operator of the form

$$L = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i}, \tag{7}$$

where the coefficients satisfy:

- L is an uniformly elliptic operator, i.e., there exists $\kappa > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2, \qquad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^N.$$

- There exists $\beta \in (0,1)$ such that for all $i,j = 1,...,N$;

$$a_{ij} = a_{ji} \in \mathcal{C}^{\beta}(\overline{\Omega}) \cap \mathcal{C}^3(\Omega); \quad b_i \in \mathcal{C}^{\beta}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega).$$

- There exist $\alpha \in (-1,1)$ and $C > 0$ such that for all $i,j,l = 1,...,N$;

$$\left|\frac{\partial a_{ij}}{\partial x_l}\right| + |b_i| < C(1 + d(x)^{\alpha}), \qquad \left|\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}\right| + \left|\frac{\partial b_i}{\partial x_i}\right| < Cd(x)^{\alpha-1} \quad \forall x \in \Omega.$$

We denote these assumptions by **(HE)**.

Let $M(x) = (m_{ij}(x))$ and $N(x) = (n_{ij}(x))$ be two 2×2 matrices whose elements belong to the Fréchet space $\mathcal{C}^1(\Omega)$ and such that there exists $K > 0$ satisfying:
(HM) $m_{ij} > 0 \;\; i \neq j$;

$$\left|\frac{\partial m_{ij}(x)}{\partial x_l}\right| d(x)^{2-\alpha} \leq K \quad i,j = 1,2; l = 1,...,N; \tag{8}$$

(HN) $n_{ij} \geq 0 \;\; i \neq j; \;\; n_{ii}(x) > 0$ for all $x \in \Omega$ and satisfy assumption (8).

Along this paper, we denote $X = X_0 \times X_0, Y = Y_0 \times Y_0$ where $X_0 = \mathcal{C}_0^1(\overline{\Omega})$ and $Y_0 = \mathcal{C}^\tau(\overline{\Omega})$, for some $\tau \in (0,1)$. As usually, we define $\text{int}(P_X)$, $\text{int}(P_Y)$, with

$$\text{int}(P_{X_0}) = \Big\{u \in \mathcal{C}_0^1(\overline{\Omega}) : u(x) > 0 \text{ in } \Omega \text{ and } \partial u/\partial\nu < 0 \text{ on } \partial\Omega\Big\}$$

where ν denotes the outward unit normal on $\partial\Omega$. Finally, we say that a function $u \in X_0$ is positive if $u \in \text{int}(P_{X_0})$.

On the other hand, for each $T > 0$ we denote

$$D_T = (0,T] \times \Omega, \quad S_T = (0,T] \times \partial\Omega, \quad D = (0,+\infty) \times \Omega, \quad S = (0,+\infty) \times \partial\Omega.$$

The object of this section is to analyze the following singular eigenvalue problems:

$$\begin{cases} \mathcal{L}U = M(x)U + \lambda U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

$$\begin{cases} \mathcal{L}U = M(x)U + \lambda N(x)U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where

$$\mathcal{L} = diag(L_1, L_2), \quad U = (u_1, u_2)^t,$$

and L_i, $i = 1, 2$ are operators as (7). We say that $\mathcal{L}$ satisfies (HE) if L_1 and L_2 satisfy it. Similar problems for the scalar case were studied in [12]. The authors proved the next result which we include by the sake of completeness.

Theorem 1 *Let L be an operator like (7) whose coefficients satisfy (HE) and $m \in \mathcal{C}^1(\Omega)$ a function verifying (8). Then the spectrum of the problem*

$$\begin{cases} Lu = m(x)u + \lambda u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

consists of an infinite, countable set of eigenvalues which are isolated and the corresponding eigenfunctions are in $\mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\delta}(\overline{\Omega})$ for $\delta \in (0, \delta_0)$ with $\delta_0 = \min\{\gamma, \alpha + 1\}$. Moreover, there exists a unique real eigenvalue, denoted by $\sigma_1(L - m)$, which is simple with an associated eigenfunction $\varphi_1 \in$int(P_{X_0}).

On the other hand, if $m \leq 0$ in Ω and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\delta}(\overline{\Omega})$ is such that

$$Lu - mu \geq 0 \textit{ in } \Omega \textit{ and } u \geq 0 \textit{ on } \partial\Omega,$$

and $u(x_0) = 0$ for some $x_0 \in \overline{\Omega}$, then if $x_0 \in \Omega$, $u = 0$ in $\overline{\Omega}$ and if $x_0 \in \partial\Omega$ and $u > 0$ in Ω, then $\partial u/\partial\nu < 0$ at x_0.

Along this work, the next result plays an essential role because it lets us obtain each positive solution of (10) as a positive solution of a new problem similar to (10) with the signs of the sums $m_{ii}+m_{ij}$ controlled in a neighbourhood of $\partial\Omega$. This result was proved in [12] in the scalar case.

Lemma 1 *Let $U \in \mathcal{Z} = (\mathcal{C}^2(\Omega)\cap\mathcal{C}_0^{1,\delta}(\overline{\Omega}))^2$, for some $\delta\in(0,\delta_0)$, be a positive solution of (10). There exist functions $\varphi^+,\varphi^-\in\mathcal{C}^{2,\delta}(\overline{\Omega})$ with $\varphi^\pm>0$ in $\overline{\Omega}$ such that $W^\pm=\varphi^\pm U\in\mathcal{Z}$ is solution of*

$$\begin{cases} \mathcal{L}^\pm W^\pm = \mathcal{M}^\pm(x)W^\pm + \lambda N(x)W^\pm & \text{in } \Omega,\\ W^\pm = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where

$$\mathcal{L}^\pm = diag(L_1^\pm, L_2^\pm) \quad \text{and} \quad \mathcal{M}^\pm = \begin{pmatrix} m_{11}^\pm & m_{12}\\ m_{21} & m_{22}^\pm \end{pmatrix},$$

$L_k^\pm$ are of the type (7) with coefficients verifying (HE), $m_{ii}^\pm$ satisfy (8) and

$$\pm(m_{ii}^\pm + m_{ij}) > 0 \text{ in a neighbourhood of } \partial\Omega.$$

Remark 1 *This result implies that if there exists the positive principal eigenvalue of (10), denoted by $\sigma_1(\mathcal{L}-M;N)$, then $\sigma_1(\mathcal{L}-M;N)=\sigma_1(\mathcal{L}^\pm-\mathcal{M}^\pm;N)$, and conversely.*

Proof: Let α and ρ_1 be as in (HE). Suppose $\alpha\neq 0$. Without lost of generality we can assume that $\alpha<0$. We consider $\varphi^\pm(x)=e^{\pm\psi(d(x))}$ where ψ is a regular and nonnegative function such that

$$\psi(\rho) = \begin{cases} 0 & \text{if } \rho>\rho_1,\\ \frac{1}{\varepsilon}\rho^{\alpha+1} & \text{if } \rho\in[0,\rho_1/2], \end{cases}$$

where ε is a positive constant to be chosen. Now, if U is solution of (10), then $W^\pm=\varphi^\pm U$ satisfies (11) with $m_{kk}^\pm = m_{kk} + \frac{\alpha+1}{\varepsilon}d(x)^{\alpha-1}A_k^\pm(\varepsilon,x)$, where

$$A_k^\pm(\varepsilon,x) = \sum_{i,j=1}^N a_{ij}^k(x)\left(\frac{\partial d}{\partial x_i}\frac{\partial d}{\partial x_j}\left(\frac{\alpha+1}{\varepsilon}d(x)^{\alpha+1}\mp\alpha\right)\mp d(x)\frac{\partial^2 d}{\partial x_i\partial x_j}\right) \pm d(x)\sum_{i=1}^N b_i^k(x)\frac{\partial d}{\partial x_i},$$

and a_{ij}^k, b_i^k are the coefficients of the operator L_k. Then $(m_{kk}^\pm + m_{kj})d(x)^{1-\alpha} = (m_{kk} + m_{kj})d(x)^{1-\alpha} + \frac{\alpha+1}{\varepsilon}A_k^\pm(\varepsilon,x)$, here the first term of the second member is bounded but it has no definite sign. It is clear that $\pm A_k^\pm(\varepsilon,x)>0$ near $\partial\Omega$, and so, if ε is sufficiently small $\pm(m_{kk}^\pm+m_{kj})>0$ in a neighbourhood of $\partial\Omega$.

When $\alpha=0$ we take

$$\psi(\rho) = \frac{1}{\varepsilon}\int_0^\rho \ln(s)ds \quad \text{if } \rho\in[0,\rho_1/2].$$

Now, we can reason similarly to the above case taking $\varphi^\pm(x)=e^{\mp\psi(d(x))}$. $\diamond$

The following result was shown in [22] when the coefficients are bounded.

Definition 1 *A function $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\delta}(\overline{\Omega})$ is said a supersolution of $L - m$ if $L\varphi \geq m\varphi$ in Ω and $\varphi \geq 0$ on $\partial\Omega$. If in addition, $L\varphi > m\varphi$ in Ω or $\varphi > 0$ on $\partial\Omega$, then it is said that φ is a strict supersolution.*

Theorem 2 *Assume that L is an operator of the form (7) whose coefficients satisfy (HE) and $m \in \mathcal{C}^1(\Omega)$ a function verifying (8). The following assertions are equivalent:*

(a) $\sigma_1(L - m) > 0$;

(b) *$L - m$ admits a positive strict supersolution;*

(c) *The problem*

$$\begin{cases} Lu = m(x)u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in Y_0$, satisfies the strong maximum principle, i.e., if $f \in P_{Y_0}\backslash\{0\}$ then $u \in$int(P_{X_0}).

Proof: (a) implies (b) taking the supersolution $\varphi = \varphi_1$, the principal eigenfunction associated to $\sigma_1(L - m)$. From Krein-Rutman theorem, see Theorem 3.2 in [23], and the fact that

$$\sigma_1(L - m) = \frac{1}{\operatorname{spr}([L - m]^{-1})}$$

where $\operatorname{spr}(T)$ denotes the spectral radius of the operator T, it follows that (c) implies (a). We only have to prove that (b) implies (c). Assume $f \in P_{Y_0}\backslash\{0\}$ and suppose that $\min_{\overline{\Omega}} u < 0$. We define

$$\hat{s} = \min\{s \in \mathbb{R} : u + s\varphi \geq 0 \text{ in } \Omega\},$$

where φ is the positive strict supersolution of $L - m$. Observe that $\hat{s} > 0$ and is well defined. Indeed, if $\varphi > 0$ on $\partial\Omega$, then it is clear that $\hat{s}$ is well defined. Assume that $\varphi = 0$ on $\partial\Omega$. By Lemma 1 we can suppose that $m < 0$ in a neighbourhood of $\partial\Omega$. Thus, we can take a positive constant K such that $m - K < 0$ in Ω. Then

$$(L - (m - K))\varphi > 0,$$

and by Theorem 1, $\varphi \in \operatorname{int}(P_{X_0})$. Thus, there exists $k > 0$ such that $kd(x) \leq \varphi(x)$. Given $x \in \Omega$, we take $y \in \partial\Omega$ such that $d(x) = |x - y|$. Then,

$$-u(x) = u(y) - u(x) \leq k_1 |x - y| = k_1 d(x)$$

for some $k_1 > 0$, and so,

$$\frac{-u(x)}{\varphi(x)} \leq \frac{k_1}{k}.$$

Therefore,

$$L(u+\hat{s}\varphi)-(m-K)(u+\hat{s}\varphi)>0,$$

and by Theorem 1, $u+\hat{s}\varphi\in$int(P_{X_0}) which is a contradiction with the definition of $\hat{s}$. This implies that $u\geq 0$, again by Theorem 1 it follows that $u\in$int(P_{X_0}). ◇

We are ready now to generalize the above results to systems with singular coefficients.

Definition 2 *We say that $\Phi\in(\mathcal{C}^2(\Omega)\cap\mathcal{C}^{1,\delta}(\overline{\Omega}))^2$ is a supersolution of $\mathcal{L}-M$ if $\mathcal{L}\Phi\geq M\Phi$ in Ω and $\Phi\geq 0$ on $\partial\Omega$. If in addition, $\mathcal{L}\Phi>M\Phi$ in Ω or $\Phi>0$ on $\partial\Omega$, then it is said that Φ is a strict supersolution.*

Theorem 3 *Under the assumptions $(HE)-(HM)$, the following conditions are equivalent:*

(a) $\mathcal{L}-M$ admits a positive strict supersolution;

(b) The operator $[\mathcal{L}-M]^{-1}:X\longmapsto X$ is well defined, compact and strongly positive;

(c) The problem

$$\begin{cases}\mathcal{L}U=M(x)U+F & \text{in } \Omega,\\ U=0 & \text{on } \partial\Omega,\end{cases}$$

where $F\in Y$, satisfies the strong maximum principle, i.e., if $F\in P_Y\setminus\{0\}$ then $U\in$int(P_X);

(d) The operator $[\mathcal{L}-M]:X\longmapsto Y$ possesses a strictly positive eigenvalue, denoted by $\sigma_1(\mathcal{L}-M)$. This eigenvalue is simple and it is the only eigenvalue of (9) possessing a positive eigenfunction $\Phi_1\in$int(P_X).

Proof: (b) implies (c) trivially. (c) implies (d) by Krein-Rutman theorem. Taking Φ_1 as a positive strict supersolution it follows that (d) implies (a). We must prove that (a) implies (b) to conclude the proof. It is not hard to find $h_{ii}\in\mathcal{C}^1(\Omega)$ satisfying (HM), with $h_{ii}>0$ and $h_{ii}+m_{ii}>0$. Since any positive constant is a strict supersolution of L_i, it follows from Theorem 2 that $\sigma_1(L_i)>0$, and so,

$$\sigma_1(L_i+h_{ii})>0. \tag{12}$$

We consider the map $\mathcal{K}:(\mathcal{C}^1(\overline{\Omega}))^2\mapsto X$ defined by $\mathcal{K}F=U$, where $F=(f_1,f_2)^t$ and $U=(u_1,u_2)^t$ is the unique solution of $(L_i+h_{ii})u_i=f_i$ in Ω, $u_i=0$ on $\partial\Omega$. From (12), $\mathcal{K}$ is well defined and compact. Now, we define

$$H=diag(h_{11},h_{22}).$$

and the map $AF=(M+H)F$. Finally, we consider $\mathcal{A}=\mathcal{K}A:X\mapsto X$. The operator $\mathcal{A}$ is compact and from (12) strongly positive. So, by Krein-Rutman theorem it follows that the spectral radius of $\mathcal{A}$ is positive. It is sufficient now to continue the proof as Theorem 1.1 in [13] (see also Theorem 2.1 in [15]). $\diamond$

The next result shows the existence of $\sigma_1(\mathcal{L}{-}M)$ independently of its positivity.

Theorem 4 *Assume (HE)-(HM). There exists one real eigenvalue of (9), denoted* $\sigma_1(\mathcal{L}{-}M)$ *associated with a positive eigenfunction* $\Phi_1 \in int(P_X)$*. The eigenvalue is simple and there is no other eigenvalue to a positive eigenfunction.*

Proof: Firstly, we claim there exists $K > 0$ such that

$$\sigma_1(\mathcal{L}{-}M + KI) > 0,$$

and so, $\sigma_1(\mathcal{L}{-}M) = \sigma_1(\mathcal{L}{-}M + KI) - K$. Note that $\sigma_1(\mathcal{L}{-}M) < 0$ is not excluded. From Remark 1, it is sufficient to prove that $\sigma_1(\mathcal{L}^- - \mathcal{M}^- + KI) > 0$, where $\mathcal{L}^-$ and $\mathcal{M}^-$ are defined in Lemma 1, for which we will find a positive strict supersolution of $\mathcal{L}^- - \mathcal{M}^- + K$ and apply Theorem 3. Let R be a positive constant. The pair (R, R) is a strict positive supersolution of $\mathcal{L}^- - \mathcal{M}^- + K$ if

$$K > m_{11}^- + m_{12} \qquad \text{and} \qquad K > m_{22}^- + m_{21}. \tag{13}$$

By Lemma 1, (13) is true near $\partial\Omega$, so it is enough to take K sufficiently large. $\diamond$

Respect to (10), we obtain:

Theorem 5 *Assume (HE), (HM) and (HN). There exists exactly one positive principal eigenvalue of (10) denoted by* $\sigma_1(\mathcal{L}{-}M; N)$*, it is algebraically simple and the associated eigenfunction lies in* $int(P_X)$*.*

Proof: As in the proof of Theorem 4, there exists $K > 0$ such that $\sigma_1(\mathcal{L}{-}M + KN) > 0$. Then, by Theorem 3 the operator $[\mathcal{L}{-}M + KN]^{-1}$ is compact and strongly positive. Now, the result follows as an application of Krein-Rutman theorem. $\diamond$

The following result will be used to compare principal eigenvalues of different matrices. Its proof is similar to Theorem 3.2 in [15], and so we omit it.

Lemma 2 *Assume (HE). Let* $A(x) = (a_{ij}(x))$ *and* $B(x) = (b_{ij}(x))$ *be two matrices with* a_{ij}, b_{ij} *satisfying (HM),* $a_{ij} \geq b_{ij}$ *and* $a_{ij}(x_0) > b_{ij}(x_0)$ *for some* $x_0 \in \Omega$ *and some* $i, j \in \{1, 2\}$*. Then,* $\sigma_1(\mathcal{L}{-}A) < \sigma_1(\mathcal{L}{-}B)$*.*

The following result will be used in the next section:

Proposition 1 *Assume (HE), (HM) and (HN). If* $\sigma_1(\mathcal{L}{-}M) > 0$ *(resp.* < 0*), then* $\sigma_1(\mathcal{L}{-}M; N) > 0$ *(resp.* < 0*).*

Proof: Observe that $\overline{\mu} = \sigma_1(\mathcal{L}{-}M; N)$ if and only if $\sigma_1(\mathcal{L}{-}M - \overline{\mu}N) = 0$. It is known that the map $\mu \mapsto \sigma_1(\mathcal{L}{-}M - \mu N)$, defined whenever $M + \mu N$ verifies (HM), is analytic, see [24], and by Lemma 2 strictly decreasing. The existence of $\overline{\mu}$ by Theorem 5 directs to the assertation. $\diamond$

Remark 2 *The results of this section can be generalized when $M(x) = (m_{ij}(x))$ is a $n \times n$ matrix such that $m_{ij} \in \mathcal{C}^1(\Omega)$ satisfy (8) and M is irreducible (see [13], [14] and [15]).*

3 Parabolic problems with Hölder continuous reactions terms

In this section we study the following cooperative parabolic problem

$$\begin{cases} u_t + L_1 u = f(x,u,v) & \text{in } D_T, \\ v_t + L_2 v = g(x,u,v) & \text{in } D_T, \\ u(t,x) = v(t,x) = 0 & \text{on } S_T, \\ u(0,x) = u_0(x); v(0,x) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{14}$$

where $L_k, k = 1,2$, are of the form (7), $u_0, v_0 \in \mathcal{C}^\beta(\overline{\Omega}), \beta \in (0,1)$, and satisfy the compatibility condition $u_0(x) = v_0(x) = 0$ on $\partial\Omega$.

Here, for every $\vartheta \in (0,1)\,, k \in \mathbb{N}$ and every bounded open subset D of $\mathbb{R} \times \mathbb{R}^N$, we denote $\mathcal{C}^{k+\vartheta}(\overline{D})$ the Banach space of all continuous functions $u : \overline{D} \mapsto \mathbb{R}$ for which all the derivatives $D_t^r D_x^s u$, $0 < 2r + s \leq k$, exist and are continuous in $\overline{D}$ and with norm $\|u\|_{k+\vartheta}$ finite (see [8] for the definiton of $\|\cdot\|_{k+\vartheta}$ and $\mathcal{C}^{k+\vartheta}(\Sigma)$ for arbitrary subset Σ).

Respect to the reactions terms, we suppose
(HR) $f, g : \overline{\Omega} \times \mathbb{R}_+^2 \mapsto \mathbb{R}$ $f, g \in \mathcal{C}^\beta(\overline{\Omega} \times \mathbb{R}_+^2) \cap \mathcal{C}^2(\Omega \times (0,+\infty)^2)$, f increasing in v, g increasing in u.

Definition 3 *The functions $\underline{w} = (\underline{u}, \underline{v}), \overline{w} = (\overline{u}, \overline{v}) \in \mathcal{F} = (\mathcal{C}^\beta(\overline{D}_T) \cap \mathcal{C}^2(D_T))^2$ are called a pair of sub-supersolutions of (14) if $\underline{w} \leq \overline{w}$ and*

$$\begin{array}{lll} (i) & \overline{u}_t + L_1\overline{u} - f(x,\overline{u},\overline{v}) \geq 0 \geq \underline{u}_t + L_1\underline{u} - f(x,\underline{u},\underline{v}) & \text{in } D_T, \\ (ii) & \overline{v}_t + L_2\overline{v} - g(x,\overline{u},\overline{v}) \geq 0 \geq \underline{v}_t + L_2\underline{v} - g(x,\underline{u},\underline{v}) & \text{in } D_T, \\ (iii) & \overline{u}(t,x) \geq 0 \geq \underline{u}(t,x);\ \overline{v}(t,x) \geq 0 \geq \underline{v}(t,x) & \text{on } S_T, \\ (iv) & \overline{u}(0,x) \geq u_0(x) \geq \underline{u}(0,x);\ \overline{v}(0,x) \geq v_0(x) \geq \underline{v}(0,x) & \text{in } \overline{\Omega}. \end{array}$$

Theorem 6 *Assume (HR). Let $\underline{w}$ and $\overline{w}$ be a pair of sub-supersolutions of (14). Then there exist $w_* = (u_*, v_*)$ and $w^* = (u^*, v^*) \in (\mathcal{C}^{1+\beta}(\overline{D}_T) \cap \mathcal{C}^{2+\beta}(D_T))^2$ minimal and maximal solution of (14) such that for every $w = (u,v) \in (\mathcal{C}^{1+\beta}(\overline{D}_T) \cap \mathcal{C}^{2+\beta}(D_T))^2$ solution of (14) with $\underline{w} \leq w \leq \overline{w}$, we have*

$$\underline{w} \leq w_* \leq w \leq w^* \leq \overline{w}.$$

Proof: We can define two sequences which converge to the minimal and maximal solutions of (14). These sequences can be built as in [11] for the case of elliptic systems. The convergence of these sequences to the minimal and maximal solution of (14) follows as in Sections 8.2 and 8.3 in [9]. $\diamond$

For the uniqueness of positive solution of (14), we need the following maximum principle for parabolic singular cooperative systems.

Proposition 2 *Let* $U = (u_1, u_2)^t \in \mathcal{F}$ *be such that*

$$U_t + \mathcal{L}U \leq M(t,x)U \quad \textit{in } D_T, \tag{15}$$

where $M(t,x) = (m_{ij}(t,x))$, m_{ij} *satisfy (HM) for all* $t \geq 0$. *If* $u_k(0,x) \leq R$ *in* Ω *and* $u_k \leq R$ *on* S_T, *then* $u_k \leq R$ *in* D_T *for* $k = 1,2$. *If* $u_k(t_0,x_0) = R$ *for some* $(t_0,x_0) \in D_T$, *then* $u_k = R$ *in* $[0,t_0] \times \Omega$. *Moreover, if* $u_k(t_0,x_0) = R$ *for some* $(t_0,x_0) \in (0,T) \times \partial\Omega$, *then* $\partial u_k/\partial\nu > 0$ *at* (t_0,x_0).

Proof: We realize the change of variable of Lemma 1, $W^- = \varphi^- U$, and we obtain from (15) that W^- verifies an inequality as

$$W_t^- + \mathcal{L}^- W^- \leq \mathcal{M}^-(t,x)W^- \quad \text{in } D_T,$$

where in particular

$$m_{11}^-(t,x) + m_{12}(t,x) < 0 \quad \text{and} \quad m_{22}^-(t,x) + m_{21}(t,x) < 0 \qquad \text{for } t \in [0,T] \text{ and } x \text{ near } \partial\Omega.$$

It is sufficient to apply Theorem 15 of Chapter 3 in [25] to obtain the outcome. $\diamond$

Proposition 3 *Assume (HR) and that for any* $(u,v) \in$*int*(P_X) *the matrix*

$$M(x) = \begin{pmatrix} f_u(x,u(x),v(x)) & f_v(x,u(x),v(x)) \\ g_u(x,u(x),v(x)) & g_v(x,u(x),v(x)) \end{pmatrix}$$

satisfies (HM). Then there exists at most a positive solution (u,v) *of (14), where by positive solution we denote that* $(u(t,x),v(t,x)) \in$*int*(P_X) *for any* $t \geq 0$.

Proof: We suppose that there exist two positive solutions (u_i,v_i), $i = 1,2$, and we define $U = (u,v)^t$ where $u = u_1 - u_2$ and $v = v_1 - v_2$. Then, U holds

$$\begin{cases} U_t + \mathcal{L}U = M(t,x)U & \text{in } D_T, \\ U = 0 & \text{on } S_T, \\ U(0,x) = 0 & \text{in } \overline{\Omega}, \end{cases}$$

where

$$M(t,x) = \begin{pmatrix} f_u(x,\xi_1,\xi_2) & f_v(x,\xi_1,\xi_2) \\ g_u(x,\eta_1,\eta_2) & g_v(x,\eta_1,\eta_2) \end{pmatrix},$$

where $\min\{u_1,u_2\} \le \xi_1,\eta_1 \le \max\{u_1,u_2\}$ and $\min\{v_1,v_2\} \le \xi_2,\eta_2 \le \max\{v_1,v_2\}$. By (HR) $m_{ij} > 0$ $i \neq j$, and since (u_i,v_i), $i = 1,2$ are positive there exists $\hat{d} > 0$ such that $\hat{d}d(x) \le \min\{u_1,u_2\}$, $\hat{d}d(x) \le \min\{v_1,v_2\}$. By hypothesis, it is clear now that m_{ij} satisfy (HM) for all $t \ge 0$. Now, it is sufficient to apply Proposition 2 and we obtain the result. $\diamond$

The previous results can be used to study the asymptotic behaviour of the time-dependent solution and stability or instability of the steady-state in the Lyapunov sense, see for example Definition 10.1.1 in [9]. We consider the system

$$\begin{cases} u_t + L_1 u = f(x,u,v) & \text{in } D, \\ v_t + L_2 v = g(x,u,v) & \text{in } D, \\ u(t,x) = v(t,x) = 0 & \text{on } S, \\ u(0,x) = u_0(x); v(0,x) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{16}$$

and the corresponding steady-state system

$$\begin{cases} L_1 u = f(x,u,v) & \text{in } \Omega, \\ L_2 v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Theorem 7 *Let $(u_s,v_s) \in$int(P_X) be a solution of (17) and*

$$M(x) = \begin{pmatrix} f_u(\cdot,u_s,v_s) & f_v(\cdot,u_s,v_s) \\ g_u(\cdot,u_s,v_s) & g_v(\cdot,u_s,v_s) \end{pmatrix}.$$

If $\sigma_1(\mathcal{L}{-}M) > 0$ then (u_s,v_s) is exponentially stable and if $\sigma_1(\mathcal{L}{-}M) < 0$, (u_s,v_s) is unstable with the norm of $\mathcal{C}^1(\overline{\Omega})$.

Proof: Since $(u_s,v_s) \in$int(P_X), there exist $K_i, i = 1,2$ such that

$$K_1 d(x) \le u_s(x) \le K_2 d(x) \qquad \text{and} \qquad K_1 d(x) \le v_s(x) \le K_2 d(x) \qquad \text{for all } x \in \overline{\Omega}.$$

Denote $I = [K_1(d(x),d(x)), K_2(d(x),d(x))]$. As in the proof of the Theorem 3, there exist functions $n_{ij} \in \mathcal{C}^1(\Omega)$ that satisfy (HN) and are such that for all $(u,v) \in I$

$$\begin{aligned} 1 + d(x)\left(|f_{uu}(\cdot,u,v)| + |f_{uv}(\cdot,u,v)|\right) &< n_{11}, \\ 1 + d(x)(|g_{vu}(\cdot,u,v)| + |g_{vv}(\cdot,u,v)|) &< n_{22}, \\ d(x)(|f_{vu}(\cdot,u,v)| + |f_{vv}(\cdot,u,v)|) &< n_{12}, \\ d(x)(|g_{uu}(\cdot,u,v)| + |g_{uv}(\cdot,u,v)|) &< n_{21}. \end{aligned} \tag{18}$$

Firstly, assume $\sigma_1(\mathcal{L}-M) > 0$. Then by Proposition 1, $\sigma_1(\mathcal{L}-M; N) > 0$ where $N(x) = (n_{ij}(x))$. We take $(\psi_1, \psi_2) \in$int(P_X) an eigenfunction of (10) associated with $\sigma_1(\mathcal{L}-M; N)$. We will prove that

$$(\underline{u}, \underline{v}) = (u_s - \rho e^{-\sigma t}\psi_1, v_s - \rho e^{-\sigma t}\psi_2) \qquad (\overline{u}, \overline{v}) = (u_s + \rho e^{-\sigma t}\psi_1, v_s + \rho e^{-\sigma t}\psi_2),$$

is a pair of sub-supersolutions of (16) where $\sigma = \sigma_1(\mathcal{L}-M; N)$ and ρ is a positive constant such that $(\overline{u}, \overline{v}), (\underline{u}, \underline{v}) \in I$ and

$$\rho < \sigma \min \left\{ \frac{d(x)}{\psi_1(x)}, \frac{d(x)}{\psi_2(x)} \right\}. \tag{19}$$

Indeed, for example, $\overline{u}$ is a supersolution if $u_s + \rho\psi_1 \geq u_0$ and, after to apply the mean value theorem twice,

$$\begin{aligned} n_{11}\psi_1 + n_{12}\psi_2 \geq \psi_1 + \tfrac{1}{\sigma}\{(f_{uu}(\cdot, \eta_1, \eta_2)(\xi_1 - u_s) + f_{uv}(\cdot, \eta_1, \eta_2)(\xi_2 - u_s))\psi_1 + \\ (f_{vu}(\cdot, \hat{\eta}_1, \hat{\eta}_2)(\xi_1 - u_s) + f_{vv}(\cdot, \hat{\eta}_1, \hat{\eta}_2)(\xi_2 - u_s))\psi_2\} \end{aligned} \tag{20}$$

where

$$\begin{aligned} u_s \leq \eta_1, \hat{\eta}_1 \leq \xi_1 \leq u_s + \rho e^{-\sigma t}\psi_1 \leq u_s + \rho\psi_1, \\ v_s \leq \eta_2, \hat{\eta}_2 \leq \xi_2 \leq v_s + \rho e^{-\sigma t}\psi_2 \leq v_s + \rho\psi_2. \end{aligned}$$

But (20) follows from (18) choosing ρ as (19). Thus,

$$|u(t,x) - u_s| \leq \rho e^{-\sigma t}\psi_1 \text{ and } |v(t,x) - v_s| \leq \rho e^{-\sigma t}\psi_2$$

for any (u_0, v_0) such that $|u_0 - u_s| \leq \rho\psi_1$ and $|v_0 - v_s| \leq \rho\psi_2$. The stability follows directly and a standard boot-strapping argument shows the stability in $\mathcal{C}^1(\overline{\Omega})$ norm.

Assume now that $\sigma_1(\mathcal{L}-M) < 0$. In this case, we take $n_{ij} \in \mathcal{C}^1(\Omega)$ that satisfy (HN) and are such that for all $(u, v) \in I$

$$\begin{aligned} J + d(x)(|f_{uu}(\cdot, u, v)| + |f_{uv}(\cdot, u, v)|) < n_{11}, \\ J + d(x)(|g_{vu}(\cdot, u, v)| + |g_{vv}(\cdot, u, v)|) < n_{22}, \\ d(x)(|f_{vu}(\cdot, u, v)| + |f_{vv}(\cdot, u, v)|) < n_{12}, \\ d(x)(|g_{uu}(\cdot, u, v)| + |g_{uv}(\cdot, u, v)|) < n_{21}, \end{aligned}$$

where $J = \frac{\omega}{(\omega - 1)}$ for some $w \in (0, 1)$. Now, we define

$$(\underline{u}, \underline{v}) = (u_s + \rho(1 - \omega e^{-\beta t})\psi_1, v_s + \rho(1 - \omega e^{-\beta t})\psi_2)$$

with $w \in (0, 1), \beta = -\sigma_1(\mathcal{L}-M; N) > 0$ and $\rho > 0$ to be selected. Again, it can be proved that $(\underline{u}, \underline{v})$ is a subsolution of (16). So,

$$u(t,x) \geq u_s + \rho(1 - \omega e^{-\beta t})\psi_1 \qquad \text{and} \qquad v(t,x) \geq v_s + \rho(1 - \omega e^{-\beta t})\psi_2$$

for any $u_0 \geq u_s + \rho\psi_1$ and $v_0 \geq v_s + \rho\psi_2$. Now, the instability is an easy consequence. $\diamond$

Now, we connect the stability of steady-state solution with its uniqueness between a sub and a supersolution of (17). The following results generalize other ones when f, g are $\mathcal{C}^1$ or Lipschitz continuous. The proofs follow from Theorem 6 and Proposition 2 as in Section 10.4 and 10.5 in [9].

Theorem 8 *Let $\underline{w} = (\underline{u}, \underline{v})$ and $\overline{w} = (\overline{u}, \overline{v})$ be a pair of sub-supersolutions of (17) and we denote $w(t; \underline{w})$ (resp. $w(t; \overline{w})$) the corresponding solution of (16) with $(u_0, v_0) = (\underline{u}, \underline{v})$ (resp. $(u_0, v_0) = (\overline{u}, \overline{v})$). Then:*

1. *$w(t; \underline{w})$, $w(t; \overline{w})$ are monotone nondecreasing and nonincreasing in t, respectively and $w(t; \overline{w}) \geq w(t; \underline{w})$ in D. Moreover,*

$$\lim_{t \to +\infty} w(t; \overline{w}) = w^*, \qquad \lim_{t \to +\infty} w(t; \underline{w}) = w_*$$

   and $w^* \geq w_*$.

2. *The functions w^* and w_* are the maximal and the minimal solutions of (17) in $[\underline{w}, \overline{w}]$.*

Theorem 9 *Assume $\underline{w} = (\underline{u}, \underline{v})$ and $\overline{w} = (\overline{u}, \overline{v})$ be a pair of sub-supersolutions of (17). Then (u_s, v_s) is the unique solution of (17) in $[\underline{w}, \overline{w}]$ if and only if (u_s, v_s) is asymptotically stable for any $(u_0, v_0) \in [\underline{w}, \overline{w}]$.*

4 Application

In this section we apply the above results to the symbiotic degenerate Lotka-Volterra model

$$\begin{cases} u_t - \Delta u = u^{1/m}(\lambda - u^{1/m} + bv^{1/m}) & \text{in } D, \\ v_t - \Delta v = v^{1/m}(\mu - v^{1/m} + cu^{1/m}) & \text{in } D, \\ u(t,x) = v(t,x) = 0 & \text{on } S, \\ u(0,x) = u_0(x); v(0,x) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{21}$$

and the corresponding steady-state system

$$\begin{cases} -\Delta u = u^{1/m}(\lambda - u^{1/m} + bv^{1/m}) & \text{in } \Omega, \\ -\Delta v = v^{1/m}(\mu - v^{1/m} + cu^{1/m}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where $m > 1; b, c > 0$; $\lambda, \mu \in \mathbb{R}$ and $(u_0, v_0) \in$int(P_X). We refer to [21] to a biological interpretation of (21) and (22). Observe that the second terms of (21) and (22) satisfy the assumptions imposed in the previous sections. To state our results we need some notations. We will denote θ_γ the unique positive solution (see [21]) of the degenerate logistic equation, i.e.,

$$\begin{cases} -\Delta w = w^{1/m}(\gamma - w^{1/m}) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists if and only if $\gamma > 0$, is increasing in γ and satisfies

$$\theta_\gamma < \gamma^m. \tag{23}$$

Also, we define when they exist

$$M = \left(\frac{\lambda + b\mu}{1 - bc}\right)^m, N = \left(\frac{\mu + c\lambda}{1 - bc}\right)^m, \delta = \inf_{\overline{\Omega}} \frac{\theta_\mu}{\theta_\lambda}$$
$$P = \left(\frac{1 + b}{1 - bc}\right)^m, Q = \left(\frac{1 + c}{1 - bc}\right)^m \text{ and } \quad R = \max\left\{P^{(m-1)/(2-m)}, Q^{(m-1)/(2-m)}\right\}$$

Note that if $\lambda \geq \mu > 0$, then $\delta \leq 1$. Moreover, since $\theta_\mu, \theta_\lambda \in \text{int}(P_{X_0})$, then $\delta > 0$.

The main result of this section is:

Theorem 10 *1. Assume $bc < 1$. If $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, there exists a unique global positive solution $(u(t,x), v(t,x))$ of (21) and at least a positive solution (u_s, v_s) of (22). In addition, if some of the following options occurs:*

(i) Either $1 < m < 2$ and

$$b^m < \frac{\delta}{RQ}, \qquad c^m < \frac{\delta}{RP}, \tag{24}$$

(ii) or $m = 2$,

(iii) or $m > 2$, $1 - bc(m-1)^2 > 0$ and

$$\lambda\left(1 - bc(m-1)\right) \geq b\mu(m-2), \qquad \mu\left(1 - bc(m-1)\right) \geq c\lambda(m-2), \tag{25}$$

then (u_s, v_s) is the unique positive solution of (22) and it is globally asymptotically stable.

If $\lambda, \mu \leq 0$, then the trivial solution $(0,0)$ is globally asymptotically stable and so (22) does not possess positive solution.

2. Assume $bc = 1$. If $m > 1$ and $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, then there exists a unique global positive solution $(u(t,x), v(t,x))$ of (21) and at least a positive solution (u_s, v_s) of (22). Moreover, a positive solution (u_s, v_s) of (22) is asymptotically stable if

$$\lambda(m-1) + (m-2)(bv_s^{1/m} - u_s^{1/m}) > 0 \quad \text{and} \quad \mu(m-1) + (m-2)(cu_s^{1/m} - v_s^{1/m}) > 0. \tag{26}$$

3. *Assume* $bc > 1$. *If* $1 < m < 2$, *then there exist* $0 < \lambda_0 < \lambda_1$ *and* $0 < \mu_0 < \mu_1$ *such that if* $\lambda > \lambda_1$ *and* $\mu > \mu_1$ *the solution of (21) blows up in finite time and if* $0 < \lambda < \lambda_0$ *and* $0 < \mu < \mu_0$ *there exists a unique global positive solution* $(u(t,x), v(t,x))$ *of (21) and at least a positive solution* (u_s, v_s) *of (22). If* $m > 2$ *and* $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ *there exists a unique global positive solution* $(u(t,x), v(t,x))$ *of (21) and at least a positive solution* (u_s, v_s) *of (22). If* $m = 2$ *there exists a unique global positive solution* $(u(t,x), v(t,x))$ *of (21) and not bounded.*

   Moreover, in any case, a positive solution (u_s, v_s) *of (22) is asymptotically stable if it satisfies (26).*

To obtain this result we need a new version of the Serrin-McKenna-Walter sweeping principle for systems with nonlinearities Hölder continuous. The proof follows, by the strong maximum principle for L_1 and L_2, as Theorem 4 in [26].

Lemma 3 *Assume the* f, g *satisfy the assumptions of Proposition 3. Let* (u, v) *be a positive solution of (17) and consider the family of positive functions* $W_r := (u_r, v_r)$ *with* $r \in (r_0, r_1]$ *satisfying:*

1. $L_1 u_r \geq f(x, u_r, v_r), L_2 v_r \geq g(x, u_r, v_r)$ *in* Ω *and* $u_r, v_r \geq 0$ *on* $\partial\Omega$.
2. $W_r(x)$ *depends continuously on* r *and is nondecreasing in* r *for all* $x \in \Omega$.
3. $(u, v) \leq (u_{r_1}, v_{r_1})$ *in* Ω.
4. *Either* $W_r(x)$ *is increasing in* r *or* $\partial W_r(x)/\partial\nu$ *changes continuously with* r *for* $x \in \partial\Omega$.
5. $u_r \neq u$ *and* $v_r \neq v$ *for all* r.

Then

$$(u, v) \leq \inf_{r \in (r_0, r_1]} (u_r, v_r).$$

Remark 3 *This theorem has its counterpart for a family of subsolutions with the corresponding changes in the inequalities, see Remark below Theorem 4 in [26].*

The following result will be used to prove the main result of this section. It provides us a priori bounds of the positive solution of (22) and extends Theorem 3.2. in [16].

Theorem 11 *Assume* $bc < 1$ *and* $\lambda \geq \mu > 0$. *Let* (u, v) *be a positive solution of (22). Then*

1. *If* $1 < m < 2$

$$\theta_\mu \leq u \leq RP\theta_\lambda \qquad \theta_\mu \leq v \leq RQ\theta_\lambda. \tag{27}$$

2. If $m \geq 2$

$$\theta_\mu \leq u \leq P\theta_\lambda \qquad \theta_\mu \leq v \leq Q\theta_\lambda. \tag{28}$$

Proof: It is not hard to prove that

$$\theta_\mu \leq \theta_\lambda \leq u, \qquad \theta_\mu \leq v.$$

We consider the family

$$(u_r, v_r) = (r^m P\theta_\lambda, r^m Q\theta_\mu), \qquad r > 1.$$

By the choice of P and Q, we have $bQ^{1/m} - P^{1/m} = cP^{1/m} - Q^{1/m} = -1$. Since $\lambda \geq \mu > 0$, the above family satisfies the conditions of Lemma 3 provided that

$$\begin{array}{l} \theta_\lambda^{1/m}\left(-r + r^{m-1}P^{(m-1)/m}\right) + \lambda\left(1 - r^{m-1}P^{(m-1)/m}\right) \leq 0, \\ \theta_\lambda^{1/m}\left(-r + r^{m-1}Q^{(m-1)/m}\right) + \lambda\left(1 - r^{m-1}Q^{(m-1)/m}\right) \leq 0. \end{array} \tag{29}$$

On the other hand, since $r > 1, P > 1$ and $Q > 1$ we have

$$1 - r^{m-1}P^{(m-1)/m} < 0 \qquad \text{and} \qquad 1 - r^{m-1}Q^{(m-1)/m} < 0.$$

Now we study two cases:
Case 1: Assume $1 < m < 2$. In this case if $r \geq R^{1/m}$ then

$$-r + r^{m-1}P^{(m-1)/m} \leq 0 \qquad \text{and} \qquad -r + r^{m-1}Q^{(m-1)/m} \leq 0,$$

and so it follows (29). Now taking $r_0 = R^{1/m}$ and applying Lemma 3 we obtain (27).
Case 2: Assume $m \geq 2$. In this case,

$$-r + r^{m-1}P^{(m-1)/m} \geq 0 \qquad \text{and} \qquad -r + r^{m-1}Q^{(m-1)/m} \geq 0.$$

Using now (23), we get that a sufficient condition for (28) is $\lambda(1 - r) \leq 0$, which is obvious for $r \geq 1$. An application of Lemma 3 concludes the proof. ◇

Proof of Theorem 10: 1. Assume $\lambda, \mu > 0$. If $\lambda = 0$ or $\mu = 0$ it can reason similarly (see [21]). Firstly, it is known (see [21]) that for any positive solution (u, v) of (22), we have

$$\theta_\lambda \leq u \leq M \qquad\qquad \theta_\mu \leq v \leq N. \tag{30}$$

We take $(u_0, v_0) \in \text{int}(P_X)$. Then there exist $\rho_1 > 0, \rho_2 \geq 1$ such that $(\rho_1\theta_\lambda, \rho_1\theta_\mu) \leq (u_0, v_0) \leq (\rho_2 M, \rho_2 N)$. Now, it is not hard to prove that

$$(\underline{u}, \underline{v}) = (\rho_1\theta_\lambda, \rho_1\theta_\mu) \qquad \text{and} \qquad (\overline{u}, \overline{v}) = (\rho_2 M, \rho_2 N)$$

is a pair of sub-supersolutions of (21) and (22). From Theorem 6 follows the existence of $(u(t,x), v(t,x))$ positive solution of (21) in $I = [(\underline{u}, \underline{v}), (\overline{u}, \overline{v})]$. Proposition 3 gives us the uniqueness of positive solution of (21). The existence of at least a positive solution (u_s, v_s) and a minimal positive solution (u_*, v_*) of (22) in I was proved in [11]. Now, we show the uniqueness of positive solution of (22) for which it is sufficient to prove that $u_s \le u_*$ and $v_s \le v_*$. We take the family

$$W_r = (ru_s, rv_s)$$

with $r \in [r_0, 1)$ where r_0 is a sufficiently small constant such that $r_0 u_s \le u_*$ and $r_0 v_s \le v_*$. The family W_r satisfies the conditions of Remark 3 if

$$\begin{aligned} \lambda(1 - r^{1-1/m}) + (bv_s^{1/m} - u_s^{1/m})(r^{1/m} - r^{1-1/m}) \ge 0, \\ \mu(1 - r^{1-1/m}) + (cu_s^{1/m} - v_s^{1/m})(r^{1/m} - r^{1-1/m}) \ge 0. \end{aligned} \tag{31}$$

If $1 < m < 2$, since $r < 1$ then $r^{1-1/m} < 1$ and $r^{1/m} < r^{1-1/m}$ and so, (31) is true if

$$bv_s^{1/m} \le u_s^{1/m} \qquad \text{and} \qquad cu_s^{1/m} \le v_s^{1/m} \tag{32}$$

It is clear that (24) implies (32) by the bound (27).

If $m = 2$, (31) is true trivially.

If $m > 2$, then $r^{1/m} > r^{1-1/m}$. Since $u_s \le M$ and $v_s \le N$, then (31) holds if

$$\lambda(1 - r^{1-1/m}) \ge M^{1/m}(r^{1/m} - r^{1-1/m}), \quad \mu(1 - r^{1-1/m}) \ge N^{1/m}(r^{1/m} - r^{1-1/m}).$$

It is not hard to prove that

$$f(r) = \frac{1 - r^{1-1/m}}{r^{1/m} - r^{1-1/m}}$$

is decreasing if $r \le 1$, so (25) implies (31). This shows the uniqueness of positive solution of (22) in I. But, by (30), any positive solution of (22) belongs to I. Now, Theorem 9 completes the proof of the global stability.

We assume now that $\lambda, \mu \le 0$. The pair

$$(\underline{u}, \underline{v}) = (0, 0) \qquad \text{and} \qquad (\overline{u}, \overline{v}) = (\rho_1 e^{-\sigma t}\varphi_1, \rho_2 e^{-\sigma t}\varphi_1)$$

is a pair of sub-supersolutions of (21) if $\rho_1\varphi_1 \ge u_0 \ge 0, \rho_2\varphi_1 \ge v_0 \ge 0$ and

$$\begin{aligned} (\rho_1 e^{-\sigma t}\varphi_1)^{1-1/m}(\sigma_1 - \sigma) + (e^{-\sigma t}\varphi_1)^{1/m}(\rho_1^{1/m} - b\rho_2^{1/m}) \ge \lambda, \\ (\rho_2 e^{-\sigma t}\varphi_1)^{1-1/m}(\sigma_1 - \sigma) + (e^{-\sigma t}\varphi_1)^{1/m}(\rho_2^{1/m} - c\rho_1^{1/m}) \ge \mu, \end{aligned} \tag{33}$$

where σ_1 and φ_1 denote the principal eigenvalue and eigenfunction of $-\Delta$ under homogeneous Dirichlet condition, respectively; and $\sigma, \rho_1, \rho_2 > 0$ to be chosen. Given $(u_0, v_0) \in \text{int}(P_X)$, we can choose $0 < \sigma < \sigma_1, \rho_1, \rho_2 > 0$ such that $\rho_1\varphi_1 \ge u_0$ and $\rho_2\varphi_1 \ge v_0$ and satisfying (33). So applying Proposition 3, for any positive solution (u, v) of (21) we have $\overline{u} \geq u$ and $\overline{v} \geq v$. This completes the first part of the Theorem.

2. Assume $bc = 1$ and $m > 1$. As in the first part of the proof, there exists $\rho_1 > 0$ sufficiently small such that $(\underline{u}, \underline{v}) = (\rho_1\theta_\lambda, \rho_1\theta_\mu)$ is a subsolution of (21) and (22). As supersolution we take

$$(\overline{u}, \overline{v}) = (A\varphi(x), dA\varphi(x)),$$

with $d = c^m, A > 0$ and $\varphi(x) = R^2 - |x|^2$ with R so large that $\varphi > 0$ in $\overline{\Omega}$. It is not hard to prove that for A large $(\overline{u}, \overline{v}) > (u_0, v_0)$ in $\overline{\Omega}$ and $(\overline{u}, \overline{v})$ is a supersolution of (21) and (22). In fact, this function provides us an upper bound of the positive solutions, see Theorem 32.VI in [27]. A similar argument to the first part completes the proof of existence and uniqueness.
To complete the proof of the second part, it remains to show the stability of (u_s, v_s). By Theorem 7, the stability of (u_s, v_s) is given by the sign of the principal eigenvalue of

$$\mathcal{L}U = M(x)U + \sigma U \tag{34}$$

where $U = (u, v)^t$ and

$$\mathcal{L} = diag(-\Delta, -\Delta),\ M = \frac{1}{m}\begin{pmatrix} u_s^{1/m-1}(\lambda - 2u_s^{1/m} + bv_s^{1/m}) & bu_s^{1/m}v_s^{1/m-1} \\ cu_s^{1/m-1}v_s^{1/m} & v_s^{1/m-1}(\mu - 2v_s^{1/m} + cu_s^{1/m}) \end{pmatrix}.$$

By (26), (u_s, v_s) is a positive strict supersolution of (34). Theorems 3 and 7 complete the proof.

3. Now we assume $bc > 1$ and $1 < m < 2$. We consider the map $h(\alpha) = c\alpha + \alpha^{1/m} - \alpha^{1-1/m}, \alpha \geq 0$ and let $\overline{\alpha}$ be the least root of $h(\alpha) = b$. Since $bc > 1$, we deduce $\overline{\alpha} < b^m$.
Case 1: Assume $\lambda = \overline{\alpha}^{1-1/m}\mu$ and $u_0 = \overline{\alpha}v_0$. Then the positive solution of (21) is of the form $(u, \frac{1}{\overline{\alpha}}u)$ and

$$\begin{cases} u_t - \Delta u = u^{1/m}(\lambda + \Lambda u^{1/m}) & \text{in } D, \\ u(t, x) = 0 & \text{on } S, \\ u(0, x) = u_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{35}$$

where $\Lambda = b\overline{\alpha}^{-1/m} - 1 > 0$. Let φ_1 choose such that $\int_\Omega \varphi_1 dx = 1$. We consider

$$q(t) = \int_\Omega u(t, x)\varphi_1 dx,$$

and so $q(0) > 0$. Since $m < 2$, we can fix $\tau \in (1, 2/m)$. Thus, there exists $\lambda_1(\tau) > 0$ such that if $\lambda > \lambda_1$

$$\lambda u^{1/m} - \sigma_1 u + \Lambda u^{2/m} > u^\tau, \qquad u > 0.$$

Thus, by the Hölder inequality

$$q'(t) > \int_\Omega u^\tau \varphi_1 dx \geq q(t)^\tau.$$

The blow up of the equation $y'(t) = y^\tau(t)$, $\tau > 1$; $y(0) > 0$ shows that $\|u(t,x)\|_\infty \to +\infty$ if $t \uparrow T_0$ for some $T_0 < +\infty$.

Case 2: Assume now $\lambda > \overline{\alpha}^{1-1/m}\mu$. Since $(u_0, v_0) \in$int(P_X), we can choose $w_0 \in$int(P_{X_0}), such that $u_0 \geq \overline{\alpha} w_0$ and $v_0 \geq w_0$. Now, we consider the problem

$$\begin{cases} w_t - \Delta w = w^{1/m}(\overline{\alpha}^{1-1/m}\mu - w^{1/m} + bz^{1/m}) & \text{in } D, \\ z_t - \Delta z = z^{1/n}(\mu - z^{1/m} + cw^{1/m}) & \text{in } D, \\ w(t,x) = z(t,x) = 0 & \text{on } S, \\ w(0,x) = \overline{\alpha} w_0(x); z(0,x) = w_0(x) & \text{in } \overline{\Omega}. \end{cases} \tag{36}$$

By a classical monotony argument, it is easy to prove that $w(t,x) \leq u(x,t)$ and $z(t,x) \leq v(x,t)$ where (u,v) (resp. (w,z)) is the unique positive solution of (21) (resp. (36)). The Case 1 finishes the proof.

In the particular case $m = 2$, we take

$$(\underline{u}, \underline{v}) = (\rho_1\theta_\lambda, \rho_1\theta_\mu) \quad \text{and} \quad (\overline{u}, \overline{v}) = (Ae^{\gamma t}, Ae^{\gamma t})$$

where $\rho_1 > 0$, $\gamma > 0$, and A large. It is easy to prove that $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ is a pair of sub-supersolution of (21), i.e. we obtain global existence but not boundedness.

Finally, we take

$$(\underline{u}, \underline{v}) = (\rho_1\theta_\lambda, \rho_2\theta_\mu) \qquad \text{and} \qquad (\overline{u}, \overline{v}) = (Se, Se),$$

where ρ_1,ρ_2 and S are positive constants to be chosen and e is the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

The pair $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ are sub-supersolutions of (21) and (22) if

$$S^{1-1/m} + S^{1/m}e^{2/m}(1-b) \geq \lambda e^{1/m}, \qquad S^{1-1/m} + S^{1/m}e^{2/m}(1-c) \geq \mu e^{1/m}. \tag{37}$$

If $m > 2$, for any $\lambda, \mu > 0$ there exists $S > 0$ such that (37) holds. If $m < 2$, we can find $\lambda_0, \mu_0 > 0$ such that for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$ there exists $S > 0$ satisfying (37).

The stability follows similarly to the second part of the proof. $\diamond$

Remark 4 *In our knowledge, our results about existence, uniqueness and blow-up of positive solution of (21) are new. Moreover, for the elliptic system (22), it has been only studied the existence of nonnegative and positive solution in [20] and [21] when $bc < 1$ (weak mutualism). The results of existence when $bc \geq 1$ (strong mutualism), uniqueness and stability are also new.*

Now, we will prove the equivalence between the linearized of (2) and (3).

Proposition 4 *Let (w_s, z_s) be a positive steady-state of (2) and $(u_s, v_s) = (w_s^m, z_s^m)$ the corresponding positive steady-state of (3) Then, (u_s, v_s) is linearly stable (resp. unstable) if, and only if, (w_s, z_s) is linearly stable (resp. unstable).*

Proof: We have just seen that the stability of (u_s, v_s), a positive steady-state of (3), depends on the spectrum of (34). Let (w_s, z_s) be a positive steady-state of (2), its stability depends on the spectrum of

$$\begin{pmatrix} -\Delta(mw_s^{m-1}\cdot) & 0 \\ 0 & -\Delta(mz_s^{m-1}\cdot) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} \lambda - 2w_s + bz_s & bw_s \\ cz_s & \mu - 2z_s + cw_s \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}+\sigma\begin{pmatrix} w \\ z \end{pmatrix}. \tag{38}$$

Making the change of variables $u_s = w_s^m, v_s = z_s^m$ and $\xi = mw_s^{m-1}w, \eta = mz_s^{m-1}z$, (38) is equivalent to

$$\mathcal{L}\Theta = M(x)\Theta + \sigma N(x)\Theta \tag{39}$$

where $\Theta = (\xi, \eta)^t$ and

$$N(x) = \begin{pmatrix} \frac{1}{m}u_s^{1/m-1} & 0 \\ 0 & \frac{1}{m}v_s^{1/m-1} \end{pmatrix}.$$

So, if the principal eigenvalue of (34) is positive (resp. negative), Proposition 1 and the equivalence between (38) and (39) imply that the principal eigenvalue of (38) is also positive (resp. negative). ◇

Remark 5 *Since the stability (resp. unstability) of (u_s, v_s) implies the stability (resp. unstability) of (w_s, z_s), our results are improvement on previous ones. Indeed, in [6] the authors showed the stability in the $L^p(\Omega)$ norm of the interval formed by the sub and the supersolution, in the sense explained in the Introduction. In [7] the authors proved the stability of the minimal and maximal steady-state solution when the initial date belongs to a suitable subset. Both results are weaker than ours.*

On the other hand, Theorem 7 shows that the principle of linearized stability holds for singular cooperative systems, generalizing Theorem 4.1 of [5].

Remark 6 *As a consequence from Theorem 10, it follows that if the interaction coefficients (b and c) are small, then (22) possesses a unique positive solution. Moreover, this is optimal because if $bc > 1$ and $1 < m < 2$, we can choose λ and μ, see (35), such that system (22) is equivalent to*

$$\begin{cases} -\Delta w = \lambda w^{1/m} + \Lambda w^{2/m} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

By Theorem 2.3 in [28], there exists at least two positive solutions of (40) if $\lambda \in (0, \Lambda_*)$ for some $\Lambda_* > 0$ and $2(N-2)/(N+2) \leq m < 2$.

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References

[1] M. E. Gurtin, R. C. MacCamy, On the diffusion of biological populations, Math. Biosci., 33 (1977) 35-49.

[2] D. G. Aronson, M. G. Crandall, L. A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Anal., 8 (1982) 1001-1022.

[3] P. de Mottoni, A. Schiaffino, A. Tesei, Attractivity properties of nonnegative solutions for a class of nonlinear degenerate parabolic problems, Ann. Mat. Pura Appl. (4),136 (1984) 35-48.

[4] M. A. Pozio, A. Tesei, Support properties of solutions for a class of degenerate parabolic problems, Comm. Partial Differential Equations, 12 (1987) 47-75.

[5] M. Bertsch, R. Rostamian, The principle of linearized stability for a class of degenerate diffusion equations, J. Differential Equations, 57 (1985) 373-405.

[6] M. A. Pozio, A. Tesei, Degenerate parabolic problems in population dynamics, Japan J. Appl. Math., 2 (1985) 351-380.

[7] R. dal Passo, P. de Mottoni, Some existence, uniqueness and stability results for a class of semilinear degenerate elliptic systems, Boll. Un. Mat. Ital. C (6), 3 (1975) 203-231.

[8] H. Amann, Invariant sets and existence theorems for semilinear parabolic and elliptic systems, J. Math. Anal. Appl., 65 (1978) 432-467.

[9] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[10] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1983.

[11] M. Delgado, A. Suárez, Existence of solutions for elliptic systems with Hölder continuous nonlinearities, Differential Integral Equations, 13 (2000) 453-477.

[12] J. Hernández, F. Mancebo, J. M. Vega de Prada, On the linearization of some singular nonlinear elliptic problems and applications, submitted.

[13] G. Sweers, Strong positivity in $\mathcal{C}(\overline{\Omega})$ for elliptic systems, Math. Z., 209 (1992) 251-271.

[14] W. Walter, The minimum principle for elliptic systems, Appl. Anal., 47 (1992) 1-6.

[15] J. López-Gómez, M. Molina-Meyer, The maximum principle for cooperative weakly coupled elliptic systems and some applications, Differential Integral Equations, 7 (1994) 383-398.

[16] P. Korman, A. Leung, On the existence and uniqueness of positive steady state in the Volterra-Lotka ecological models with diffusion, Appl. Anal., 26 (1987) 145-160.

[17] P. Korman, Dynamics of the Lotka-Volterra systems with diffusion, Appl. Anal., 44 (1992) 191-207.

[18] Y. Lou, Necessary and sufficient condition for the existence of positive solutions of certain cooperative system, Nonlinear Anal., 26 (1996) 1079-1095.

[19] M. Delgado, J. López-Gómez, A. Suárez, On the symbiotic Lotka-Volterra model with diffusion and transport effects, J. Differential Equations, 160 (2000) 175-262.

[20] A. Cañada, J. L. Gámez, Elliptic systems with nonlinear diffusion in population dynamics, Differential Equations Dynam. Systems, 3 (1995) 189-204.

[21] M. Delgado, A. Suárez, On the existence of dead cores for degenerate Lotka-Volterra models, Proc. Roy. Soc. Edinburgh Sect. A., 130 (2000) 743-766.

[22] J. López-Gómez, The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems, J. Differential Equations, 127 (1996) 263-294.

[23] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., 18 (1976) 620-709.

[24] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1975.

[25] M. H. Protter, H. F. Weinberger, Maximum Principles in Differential Equations, Prentice Hall, New Jersey, 1967.

[26] P. J. McKenna, W. Walter, On the Dirichlet problem for elliptic systems, Appl. Anal., 21 (1986) 207-224.

[27] W. Walter, Differential and Integral Inequalities, Springer, 1970.

[28] A. Ambrosetti, H. Brézis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal., 122 (1994) 519-543.