Nonnegative solutions for the degenerate logistic indefinite sublinear equation

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Abstract

The goal of this paper is to study the nonnegative steady-states solutions of the degenerate logistic indefinite sublinear problem. We combine bifurcation method and linking local sub-supersolution technique to show the existence and multiplicity of nonnegative solutions. We employ a change of variable already used in a different context and the spectral singular theory to prove uniqueness results.

Key Words. Degenerate logistic indefinite equation, Singular eigenvalue problems, Indefinite sublinear problems, Multiplicity results.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded and regular domain of $\mathbb{R}^N$ and we consider the degenerate logistic indefinite sublinear model

$$\begin{cases} \mathcal{L}w^m = \lambda w + a(x)w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $m > 1$; $\lambda \in \mathbb{R}$ that it will be regarded as a parameter, $a \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, changes sign and $\mathcal{L}$ is a second order operator of the form

$$\mathcal{L}u := -\sum_{i,j=1}^{N} D_i(a_{ij}D_j u) + \sum_{i=1}^{N} b_i(x)D_i u, \tag{1.2}$$

with $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and uniformly elliptic in the sense that

$$\exists\theta > 0 \quad \text{such that} \quad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \quad \forall \xi \in \mathbb{R}^N,\ \forall x \in \Omega. \tag{1.3}$$

We write $a = a^+ + a^-$ where $a^+(x) := \max\{a, 0\}$ and $a^- := \min\{a, 0\}$. We define the sets:

$$A_+ := \{x \in \Omega : a^+(x) > 0\}, \quad A_- := \{x \in \Omega : a^-(x) < 0\},$$

$$A_0 := \Omega \setminus (\overline{A}_+ \cup \overline{A}_-)$$

and assume that A_+ is open and sufficiently smooth, that is, the finite number of connected components A_+^k, $k = 1, \ldots, r$, are sufficiently smooth.

Equation (1.1) has been proposed as a model for population density of a steady-state single species $w(x)$ inhabiting in a heterogeneous environment Ω. Here we are assuming that Ω is fully surrounded by inhospitable areas, since the population density is subject to homogeneous Dirichlet boundary conditions. In fact, the term $m > 1$ was introduced in [18], see also [25], by describing the dynamics of biological population whose mobility depends upon their density. The parameter λ represents the growth rate of the species and $a(x)$ describes the limiting effects of crowding in the species in A_- and the intraspecific cooperation in A_+. Observe that in A_0 the population is free from crowding and symbiosis effects. Finally, $\mathcal{L}$ measures the diffusivity and the external transport effects of the species. In this context, $m > 1$ means that the diffusion, the rate of movement of the species from high density regions to low density ones, is slower than in the linear case ($m = 1$), which seems give more realistic models, see [18].

The change of variable $u := w^m$ transforms (1.1) into

$$\begin{cases} \mathcal{L}u = \lambda u^q + a(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with $q = 1/m$ and $p = 2/m$. Along this work we suppose

(H) $\quad\quad 0 < q < p \leq 1$

so, we are assuming that $m \geq 2$, that includes the "very slow diffusion"(i.e. $m > 2$) and the self-diffusion ($m = 2$), see [23].

In the last years the case $m = 1$ ($q = 1$ and $p = 2$) has attracted much attention, see [2], [3], [9], [10], [17], [22], [26] and references therein.

When $1 < m < 2$ ($q < 1 < p$) and $a(x) \equiv a_0$ with a_0 a positive constant, (1.4) was studied in [4] in the particular case $\mathcal{L} = -\Delta$ and in [6] when $\mathcal{L}$ is a quasilinear operator. When a changes sign, (1.4) was analyzed in [24] in the particular case $\lambda \leq 0$. Recently, in [15] the authors have studied (1.4) when a changes sign and $\mathcal{L}$ is an operator as (1.2). In this work it was shown that from the trivial solution $u = 0$ bifurcates supercritically at value $\lambda = 0$ a continuum of nonnegative solutions of (1.4). Assuming some restrictions on a^+ and p in order to obtain a priori bounds of the solutions, it was proved that there exists a value $\lambda^* > 0$ such that (1.4) possesses a nonnegative and nontrivial solution if, and only if, $\lambda \in (-\infty, \lambda^*]$. Moreover, there exist at least two solutions for $\lambda \in (0, \lambda^*)$ and a unique linearly asymptotically stable in such interval.

When $m \geq 2$ ($q < p \leq 1$), only partial results are known about (1.4). When $\lambda \geq 0$, the existence of nonnegative solutions was proved in [8], see Theorem II.1. When $\lambda = 0$, $A_- = \emptyset$ and $\mathcal{L} = -\Delta$ the existence and uniqueness of positive solution was proved in [20], see also [30]. When $\lambda = 0$, $\mathcal{L} = -\Delta$ and a changes sign, (1.4) was studied in detail in [7]. In this work, the authors proved the existence of nonnegative solutions of (1.4). Moreover, they showed that when $\|a^-\|_\infty$ is small, (1.4) possesses a unique nontrivial solution, see Theorem 2.4 in [7]. However, when $\|a^-\|_\infty$ is large they showed multiplicity results and the existence of *dead cores* for the solutions, i.e., regions in Ω where the solutions vanish identically.

We are going to improve and generalize these results and show that a drastic change occurs when $m \geq 2$ with respect to the case $m < 2$. Indeed, we show that, as in the case $1 < m < 2$, from the trivial solution $u = 0$ bifurcates a continuum of nonnegative solutions at $\lambda = 0$. When $m > 2$ this bifurcation is subcritical and when $m = 2$ the bifurcation direction depends on the sign of $\sigma_1[\mathcal{L} - a(x)]$, where $\sigma_1[\mathcal{L} - a(x)]$ stands for the principal eigenvalue of the operator $\mathcal{L} - a(x)$ subject to homogeneous Dirichlet boundary conditions. Specifically, when $m > 2$ we prove that there exist two values $-\infty < \lambda_* \leq \lambda_{**} < 0$ such that, (1.4) admits a nonnegative solution if, and only if, $\lambda \geq \lambda_*$; a unique and linearly asymptotically stable if $\lambda > 0$ and at least two nonnegative solutions in $\lambda \in (\lambda_{**}, 0)$. When $m = 2$, we prove that if $\sigma_1[\mathcal{L} - a(x)] = 0$ then

(1.4) has positive solutions if, and only if, $\lambda = 0$ (vertical bifurcation). In this case, infinitely positive solutions exist. If $\sigma_1[\mathcal{L} - a(x)] > 0$, (1.4) has positive solutions if, and only if, $\lambda > 0$, moreover the solution is unique and linearly asymptotically stable. Finally, $\sigma_1[\mathcal{L} - a(x)] < 0$, (1.4) has positive solutions if, and only if, $\lambda < 0$.

An outline of the work is as follows: in Section 2 we collect results of a linear eigenvalue problem with singular potential. These results will be used in the next sections. In Section 3 we apply the Leray-Schauder degree and bifurcation theory to show the existence of an unbounded continuum of nonnegative solution emanating at $\lambda = 0$ from the trivial solution $u = 0$. In Section 4 we study the case $p < 1$. Finally, in Section 5, the case $p = 1$ is analyzed.

2 Singular eigenvalue problem

Let $M \in C^1(\Omega)$ be such that there exist two constants $K > 0$ and $\gamma \in [0, 2)$ for which

$$|M(x)|[\mathrm{dist}(x, \partial\Omega)]^\gamma \leq K \quad x \in \Omega. \tag{2.1}$$

We consider the following singular linear eigenvalue problem

$$\begin{cases} (\mathcal{L} + M(x))u = \sigma u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\sigma \in \mathbb{R}$ and M verifies (2.1). The next result was proved in [19], except (2.3), which follows by Theorem 7, Chapter 2 of [28].

Theorem 2.1 *Suppose $M \in C^1(\Omega)$ satisfies* (2.1). *Then, there exists a unique value of σ, denoted by $\sigma_1[\mathcal{L} + M]$ and called principal eigenvalue of* (2.2), *for which* (2.2) *possesses positive solution $\varphi_1 \in C_0^1(\overline{\Omega})$, unique up to multiplicative constants, and called principal eigenfunction of* (2.2). *Moreover,*

$$\frac{\partial \varphi_1}{\partial \nu}(x) < 0 \tag{2.3}$$

for each $x \in \partial\Omega$ and where ν stands for any outward direction to Ω at x.

Furthermore, $\sigma_1[\mathcal{L} + M]$ is increasing with respect to M and decreasing with respect to Ω, and if $\sigma_1[\mathcal{L} + M] > 0$ then $u = 0$ is the unique solution of

$$(\mathcal{L} + M(x))u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Hereafter, we denote the space $C_0^0(\overline{\Omega}) := \{u \in C^0(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. Moreover $B_\rho := \{u \in C_0^0(\overline{\Omega}) : \|u\|_\infty < \rho\}$. and for any $f \in C^0(\overline{\Omega})$ we denote

$$f_M := \sup_{x\in\overline{\Omega}} f(x).$$

Finally, $\mathcal{L}^*$ stands for the adjoint of $\mathcal{L}$ with respect to the inner product of $L^2(\Omega)$. Recall that $\sigma_1[\mathcal{L}^*] = \sigma_1[\mathcal{L}]$.

The following characterization of the positivity of $\sigma_1[\mathcal{L} + M]$ was shown in [21] when $M \in L^\infty(\Omega)$, and in [14] when M satisfies (2.1).

Definition 2.2 *A function $\varphi \in C^2(\Omega)\cap C^1(\overline{\Omega})$ is said a supersolution of $\mathcal{L}+M$ if $(\mathcal{L}+M)\varphi \geq 0$ in Ω and $\varphi \geq 0$ on $\partial\Omega$. If in addition, $(\mathcal{L}+M)\varphi > 0$ in Ω or $\varphi > 0$ on $\partial\Omega$, then it is said that φ is a strict supersolution.*

Proposition 2.3 *Assume that M satisfies* (2.1). *Then: $\sigma_1[\mathcal{L} + M] > 0$ if, and only if, $\mathcal{L} + M$ admits a positive strict supersolution.*

Along this work, we need to apply this result assuming less regularity to the strict supersolution.

Proposition 2.4 *Assume that M satisfies* (2.1). *Then: $\sigma_1[\mathcal{L} + M] > 0$ if, and only if, there exists $\varphi \in C^2(\Omega) \cap C_0^0(\overline{\Omega})$ such that $\varphi > 0$ in Ω and $(\mathcal{L} + M(x))\varphi > 0$ in Ω.*

Proof: If $\sigma_1[\mathcal{L} + M] > 0$, then we can take $\varphi = \varphi_1$. Now, assume that there exists a positive function $\varphi \in C^2(\Omega) \cap C_0^0(\overline{\Omega})$ such that

$$(\mathcal{L} + M(x))\varphi := F > 0 \quad \text{in } \Omega.$$

It is well-known, see Lemma 2.7 in [19], that $\sigma_1[\mathcal{L} + M] > 0$ is equivalent to prove that given $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $v \neq 0$, and

$$(\mathcal{L} + M(x))v \geq 0 \quad \text{in } \Omega, \qquad v \geq 0 \quad \text{on } \partial\Omega,$$

then $v > 0$ in Ω and $\partial v/\partial n < 0$ for all $x \in \partial\Omega$ such that $v(x) = 0$, where n stands for the outward unit normal to Ω in x.

By an adequate change of variable, see Lemma 2.1 in [19] or Lemma 1 in [14], we can suppose that $M \geq 0$ in a neighborhood of $\partial\Omega$. For each $\varepsilon > 0$ and $K > 0$, we define

$$w := v + \varepsilon + \varepsilon K\varphi \in C^2(\Omega) \cap C^0(\overline{\Omega}),$$

and so,

$$(\mathcal{L} + M(x))w \geq \varepsilon(M + KF) > 0 \quad \text{in } \Omega, \tag{2.4}$$

for K sufficiently large. Moreover, for any $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that $w > 0$ in $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \gamma(\varepsilon)\}$. By (2.4), we can apply the generalized maximum principle and we get that $w > 0$ in $\Omega \backslash \Omega_\varepsilon$. Thus, $w > 0$ in Ω for all $\varepsilon > 0$, and we obtain that $v \geq 0$ in Ω. Hence, taking $M_1 := \max\{M, 0\}$, we get

$$(\mathcal{L} + M_1)v \geq (\mathcal{L} + M)v \geq 0,$$

and the result follows by the strong maximum principle. □

3 Bifurcation from the trivial solution

In this section we adapt the results of [5], see also [6] and [15], to show that a bifurcation from the trivial solution of (1.4) occurs at $\lambda = 0$. We include them for the reader's convenience and send to [15] for details. Observe that by elliptic regularity a solution $u \in C_0^0(\overline{\Omega})$ of (1.4), it belongs to $C^{1+\mu}(\Omega) \cap C_0^1(\overline{\Omega})$ for $\mu := \min\{\alpha, q\}$.

We extend the function

$$f(\lambda, x, s) := \begin{cases} \lambda s^q + a(x)s^p & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Note that f can take negative values. Finally, we define the map

$$\mathcal{K}_\lambda : C_0^0(\overline{\Omega}) \mapsto C_0^0(\overline{\Omega}); \qquad \mathcal{K}_\lambda(u) := u - \mathcal{L}^{-1}(f(\lambda, x, u))$$

where $\mathcal{L}^{-1}$ is the inverse of the operator $\mathcal{L}$ under homogeneous Dirichlet boundary conditions, which is well-defined since $\sigma_1[\mathcal{L}] > 0$. Indeed, observe that positive constants are strict supersolutions of $\mathcal{L}$, and so, by Proposition 2.3, $\sigma_1[\mathcal{L}] > 0$. Now, we can prove that u is a nonnegative solution of (1.4) if, and only if, u is a zero of the map $\mathcal{K}_\lambda$. It is clear that every nonnegative solution is a zero of $\mathcal{K}_\lambda$. Conversely, let u be a zero of $\mathcal{K}_\lambda$ and assume that the set

$$\Omega_- := \{x \in \Omega : u(x) < 0\} \neq \emptyset.$$

Then,

$$\mathcal{L}u = 0 \quad \text{in } \Omega_- \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega_-.$$

Since $\sigma_1[\mathcal{L}] > 0$ and $\Omega_- \subset \Omega$, then

$$0 < \sigma_1[\mathcal{L}] < \lambda_1(\mathcal{L}, \Omega_-)$$

where $\lambda_1(\mathcal{L}, \Omega_-)$ denotes the principal eigenvalue of $\mathcal{L}$ in Ω_- defined in (1.10) of [11]. Now, by Theorem 1.1 of [11], the maximum principle holds in Ω_- and so $u = 0$ in Ω_-, which leads us to a contradiction.

In order to prove the main result of this section we use the Leray-Schauder degree of $\mathcal{K}_\lambda$ on B_ρ with respect to zero, denoted by $\deg(\mathcal{K}_\lambda, B_\rho)$, and the index of the isolated zero u of $\mathcal{K}_\lambda$, denoted by $i(\mathcal{K}_\lambda, u)$.

Theorem 3.1 *The value $\lambda = 0$ is the only bifurcation point from the trivial solutions for* (1.4)*. Moreover, there exists a continuum $\mathcal{C}_0$ of nonnegative solutions of* (1.4) *unbounded and connected in $\mathbb{R} \times C^0_0(\overline{\Omega})$ emanating from* $(0, 0)$.

Proof: We divide the proof in several steps.

Step 1: If $\lambda < 0$, then $i(\mathcal{K}_\lambda, 0) = 1$.

Define the family of maps

$$\mathcal{H}_1 : [0, 1] \times C^0_0(\overline{\Omega}) \mapsto C^0_0(\overline{\Omega}); \quad \text{by} \quad \mathcal{H}_1(t, u) := \mathcal{L}^{-1}(t(\lambda u^q + a(x)u^p)).$$

It is not hard to prove that there exists $\delta > 0$ such that $u \neq \mathcal{H}_1(t, u)$ for $u \in \overline{B}_\delta$, $u \neq 0$ and $t \in [0, 1]$. Hence, the homotopy defined by $\mathcal{H}_1$ is admissible and so, taking $\varepsilon \in (0, \delta]$, we have

$$\begin{aligned} i(\mathcal{K}_\lambda, 0) &= \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_1(1, \cdot), B_\varepsilon) = \deg(I - \mathcal{H}_1(0, \cdot), B_\varepsilon) \\ &= \deg(I, B_\varepsilon) = 1. \end{aligned}$$

Step 2: If $\lambda > 0$, then $i(\mathcal{K}_\lambda, 0) = 0$.

Fix $\phi \in C^0_0(\overline{\Omega})$, $\phi > 0$. We define the map

$$\mathcal{H}_2 : [0, 1] \times C^0_0(\overline{\Omega}) \mapsto C^0_0(\overline{\Omega}); \quad \text{by} \quad \mathcal{H}_2(t, u) := \mathcal{L}^{-1}(\lambda u^q + a(x)u^p + t\phi).$$

Again it can be proved that there exists $\delta > 0$ such that

$$u \neq \mathcal{H}_2(t, u) \quad \text{for all } u \in \overline{B}_\delta,\ u \neq 0 \text{ and } t \in [0, 1]. \tag{3.1}$$

So, the homotopy defined by $\mathcal{H}_2$ is admissible. Then, taking $\varepsilon \in (0, \delta]$ we have

$$i(\mathcal{K}_\lambda, 0) = \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_2(0, \cdot), B_\varepsilon) = \deg(I - \mathcal{H}_2(1, \cdot), B_\varepsilon) = 0.$$

The last equality follows because $\mathcal{L}u = \lambda u^q + a(x)u^p + \phi$ has no solution in $\overline{B}_\varepsilon$, see (3.1).

Step 3: $\lambda = 0$ is the unique bifurcation point from the trivial solution.

That $\lambda = 0$ is a bifurcation point from the trivial solution follows directly by Steps 1 and 2. We will show that there is not any other bifurcation point in $\mathbb{R}\setminus\{0\}$. Suppose there exists a sequence of solutions (λ_n, u_n) of (1.4) such that $\lambda_n \to \lambda_0 < 0$ and $\|u_n\|_\infty \to 0$. With a similar argument to the one used at the beginning of this section, we can prove that $u_n \geq 0$. Since $\|u_n\|_\infty \to 0$ and $\lambda_n \to \lambda_0 < 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, it holds

$$\mathcal{L}u_n = \lambda_n u_n^q + a(x)u_n^p \leq 0 \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega,$$

which implies that $u_n = 0$.

Now, assume that there exists a sequence of solutions (λ_n, u_n) of (1.4) such that $\lambda_n \to \lambda_0 > 0$ and $\|u_n\|_\infty \to 0$. Observe that, by the strong maximum principle, $u_n > 0$. We take $K \geq \sigma_1[\mathcal{L}]$, so there exists $n_0 \in \mathbb{N}$ such that

$$\lambda_n u_n^q + a(x)u_n^p > K u_n \qquad \text{for all } n \geq n_0,$$

and so,

$$(\mathcal{L} - K)u_n > 0 \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Hence, u_n is a positive strict supersolution of $\mathcal{L}-K$, and by Proposition 2.3, we get $\sigma_1[\mathcal{L}-K] > 0$, and so $K < \sigma_1[\mathcal{L}]$, which leads us to a contradiction.

Finally, the existence of an unbounded and connected continuum of nonnegative solutions of (1.4) follows from a slight modification of the proof of Theorem 1.3 in [29], see also Theorem 3.1 in [1] and Theorem 4.4 in [6]. □

4 The very slow diffusion case: $p < 1$.

Along this section we assume $p < 1$, that is $m > 2$ in the original equation (1.1). The main result in this case is the following:

Theorem 4.1 *Assume $p < 1$. There exist $-\infty < \lambda_* \leq \lambda_{**} < 0$ such that:*

a) (1.4) has a nonnegative and nontrivial solution if, and only if, $\lambda \in [\lambda_*, \infty)$,

b) If $\lambda \in (0, \infty)$, (1.4) possesses exactly a solution, which is positive and linearly asymptotically stable, *c) If $\lambda \in (\lambda_{**}, 0)$, (1.4) possesses at least two nonnegative and nontrivial solutions.*

This result will be an easy consequence of the following ones.

Proposition 4.2 *Assume $p < 1$. Then, there exists $-\infty < \underline{\lambda} < 0$ such that for $\lambda < \underline{\lambda}$,* (1.4) *has no solution.*

Proof: It is not hard to prove that

$$\lambda s^q + a_M s^p - \sigma_1[\mathcal{L}]s < 0 \quad \forall s \in \mathbb{R}_+,\ \forall \lambda < \underline{\lambda} \tag{4.1}$$

where

$$\underline{\lambda} := (\sigma_1[\mathcal{L}])^{(p-q)/(p-1)} a_M^{(1-q)/(1-p)} \left(\frac{p-q}{1-q}\right)^{(p-q)/(1-p)} \frac{p-1}{1-q}. \tag{4.2}$$

Now, let (λ, u) be a nonnegative solution of (1.4) for $\lambda < \underline{\lambda}$. Multiplying (1.4) by φ_1^*, the eigenfunction associated to $\mathcal{L}^*$ and taking account (4.1), we obtain

$$0 = \int_\Omega (\lambda u^q + a(x)u^p - \sigma_1[\mathcal{L}^*]u)\varphi_1^* \le \int_\Omega (\lambda u^q + a_M u^p - \sigma_1[\mathcal{L}]u)\varphi_1^* < 0,$$

which is a contradiction. This completes the proof. □

The following result is well-known when that $\mathcal{L} = -\Delta$. It will be very useful in this work.

Proposition 4.3 *Assume $p < 1$ and let $b \in C^\alpha(\overline{\Omega})$ be such that $b \ge 0$ and $b \ne 0$. Consider the following problem*

$$\begin{cases} \mathcal{L}u = b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then, (4.3) *possesses a unique positive solution, denoted by $z_{[b,p]}$.*

Proof: Firstly, we are going to use the linking local sub-supersolution method to prove the existence of nonnegative solution of (4.3). Since $b \ge 0$, $b \ne 0$, there exists $x_0 \in \Omega$ and $r_0 > 0$ such that

$$b(x) \ge b_0 > 0 \quad \text{for all } x \in B := B(x_0, r_0) \text{ and } \quad \overline{B} \subset \Omega$$

for some constant $b_0 > 0$ and where $B(x_0, r_0)$ is the ball of radius r_0 centered at x_0. We define

$$\Psi := \begin{cases} \varphi_1^B & \text{in } \overline{B}, \\ 0 & \text{in } \Omega \setminus B, \end{cases} \tag{4.4}$$

where φ_1^B is the principal eigenfunction of $\mathcal{L}$ in B associated to the principal eigenvalue, $\sigma_1^B[\mathcal{L}]$, and normalized so that $\sup_{x\in B}\varphi_1^B = 1$. Observe that $\Psi \in H^1(\Omega)$ and that

$$\frac{\partial \varphi_1^B}{\partial n_{\mathcal{L}}} < 0 \quad \text{on } \partial B, \tag{4.5}$$

where $n_{\mathcal{L}}$ denotes the conormal associated with $\mathcal{L}$, i.e., $(n_{\mathcal{L}})_i := \sum_{j=1}^N a_{ij} n_j$, being $n := (n_1, \ldots, n_N)$ the outward unit normal to B. Indeed, (4.5) follows by (2.3) and the fact that $n_{\mathcal{L}}$ is an outward direction because by (1.3), it follows

$$n \cdot n_{\mathcal{L}} = \sum_{i,j=1}^{N} a_{ij} n_i n_j > 0.$$

We define $e \in C^2(\overline{\Omega})$ the unique positive solution of

$$\begin{cases} \mathcal{L}e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Now, thanks to (4.5) we can apply Lemma I.1 in [8] to show that the pair $(\underline{u}, \overline{u}) := (\varepsilon\Psi, Ke)$ is a sub-supersolution of (1.4) provided of $\varepsilon > 0$ and $K > 0$ satisfy

$$\underline{u} \leq \overline{u}, \quad \varepsilon \leq \left(\frac{b_0}{\sigma_1^B[\mathcal{L}]}\right)^{1/(1-p)}, \quad K \geq (b_M \|e\|_\infty^p)^{1/(1-p)}.$$

This proves the existence of at least a nonnegative solution of (4.3). By the strong maximum principle, any nonnegative solution of (4.3) is positive.

For the uniqueness, we assume that (4.3) possesses two positive solutions $v \neq u$. By the integral mean value theorem, we get

$$\mathcal{L}(u - v) = b(x)(u^p - v^p) = b(x)p \int_0^1 [tu + (1-t)v]^{p-1} dt\, (u - v) \quad \text{in } \Omega.$$

Hence,

$$\begin{cases} (\mathcal{L} - b(x)pM(x))(u - v) = 0 & \text{in } \Omega, \\ u - v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$M(x) := \int_0^1 [tu + (1-t)v]^{p-1} dt.$$

Since u and v are strictly positive, there exist positive constants $C_u, C_v > 0$ such that

$$C_u \text{dist}(x, \partial\Omega) \leq u(x), \qquad C_v \text{dist}(x, \partial\Omega) \leq v(x) \qquad \forall x \in \Omega,$$

and so,

$$|M(x)|[\text{dist}(x,\partial\Omega)]^{1-p} \leq K, \tag{4.7}$$

for some $K > 0$. Hence M verifies (2.1). Moreover, it satisfies the following estimate

$$pM < u^{p-1} \qquad \text{in } \Omega.$$

Thus, according to Theorem 2.1

$$\sigma_1[\mathcal{L} - b(x)pM] > \sigma_1[\mathcal{L} - b(x)u^{p-1}] = 0.$$

Therefore, $u - v = 0$. This shows the uniqueness of positive solution of (4.3). $\square$

The next result shows the existence of a nonnegative maximal solution of (1.4) for $\lambda = 0$. Related results were proved in [7] (Theorem 2.2) and in [27] (Theorem 4) when $\mathcal{L} = -\Delta$.

Proposition 4.4 *Assume $p < 1$ and $\lambda = 0$. Then* (1.4) *admits a maximal nonnegative solution U^0. Moreover,*

$$U^0 > 0 \quad \textit{in } \overline{A}_+. \tag{4.8}$$

Proof: Observe that any nonegative solution u of (1.4) for $\lambda = 0$ is a subsolution of (4.3) with $b(x) \equiv a_M$. Since for K sufficiently large, $\overline{u} := Ke$ is a supersolution of (4.3) and $u \leq \overline{u}$, from the uniqueness of positive solution of (4.3), we obtain that

$$u \leq z_{[a_M,p]}$$

for any nonnegative solution u of (1.4) for $\lambda = 0$. Moreover, $z_{[a_M,p]}$ is a supersolution of (1.4) for $\lambda = 0$. Thus, we deduce the existence of a maximal nonnegative solution of (1.4) for $\lambda = 0$, which we call U^0. Finally, we will prove (4.8). For that, again we use the linking local sub-supersolution method. For any $k = 1, \dots, r$, we consider $x_k \in A_+^k$ and $r_k > 0$ such that $\overline{B}_k := \overline{B}(x_k, r_k) \subset A_+^k$. We define

$$\Psi := \begin{cases} \varphi_1^{B_k} & \text{in } \overline{B}_k, \text{ for all } k = 1, \dots, n, \\ 0 & \text{in } \Omega \backslash (\cup_{k=1}^r B_k), \end{cases}$$

where $\varphi_1^{B_k}$ is the principal eigenfunction of $\mathcal{L}$ in B_k. By a similar reasoning to the used in the Proposition 4.3, it can be proved that we can apply Lemma I.1 in [8] to show that the pair $(\underline{u}, \overline{u}) := (\varepsilon\Psi, Ke)$ is a sub-supersolution of (1.4), provided that ε and K are sufficiently small and large, respectively. Now, the strong maximum principle shows (4.8), see Lemma 2.1 in [7]. This completes the proof. □

The next result shows the uniqueness and stability of the positive solution when $\lambda > 0$. The existence will be shown in Theorem 4.1. For the uniqueness we would like to point out that we use a change of variable already used in a different context in [30], see also [7] and [12].

Proposition 4.5 *Assume $p < 1$ and $\lambda > 0$. Then, there exists at most a unique positive solution of* (1.4)*, say u_λ. Moreover,*

$$\sigma_1[\mathcal{L} - \lambda q u_\lambda^{q-1} + pa(x)u_\lambda^{p-1}] > 0,$$

that is, u_λ is linearly asymptotically stable.

Proof: Firstly, observe that since $\lambda > 0$ then, by the strong maximum principle any nonnegative and nontrivial solution u is in fact strictly positive. So, we can define the change of variable

$$w := \frac{u^{1-p}}{1-p}$$

which transforms (1.4) into

$$\begin{cases} \mathcal{L}w - \dfrac{p}{(1-p)w}\displaystyle\sum_{i,j=1}^{N} a_{ij} D_i w D_j w = \lambda(1-p)^{\frac{q-p}{1-p}} w^{\frac{q-p}{1-p}} + a(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Assume that there exist two positive solution $u_1 \neq u_2$ of (1.4). Let $x_0 \in \Omega$ be such that

$$\Phi := u_1 - u_2$$

attains its positive maximum. If such positive maximum does not exist, we will reason similarly with the function $\Phi := u_2 - u_1$. Since $x_0 \in \Omega$, there exists $r > 0$ such that

$$u_1(x) > u_2(x) \geq \rho > 0 \quad \text{for all } x \in B(x_0, r),$$

for some $\rho > 0$. Now, we define

$$\Psi := w_1 - w_2$$

where $w_i := u_i^{1-p}/(1-p)$. So by (4.9), we get

$$\mathcal{L}\Psi - \frac{p}{1-p}(\sum_{i,j=1}^{N} a_{ij}[\frac{1}{w_1} D_i w_1 D_j w_1 - \frac{1}{w_2} D_i w_2 D_j w_2]) =$$

$$= \lambda(1-p)^{(q-p)/(1-p)}(w_1^{(q-p)/(1-p)} - w_2^{(q-p)/(1-p)}).$$

On the other hand, it can be proved that

$$\sum_{i,j=1}^{N} a_{ij}[\frac{1}{w_1} D_i w_1 D_j w_1 - \frac{1}{w_2} D_i w_2 D_j w_2] = \sum_{i=1}^{N} c_i D_i \Psi - c(x)\Psi$$

where

$$c_i = \sum_{j=1}^{N} a_{ij} \frac{1}{w_1}(D_j w_1 + D_j w_2), \qquad c(x) = \frac{1}{w_1 w_2} \sum_{i,j=1}^{N} a_{ij} D_i w_2 D_j w_2.$$

So, Ψ verifies in $B(x_0, r)$

$$\mathcal{L}_1 \Psi + \frac{p}{1-p} c(x)\Psi = \lambda(1-p)^{(q-p)/(1-p)}(w_1^{(q-p)/(1-p)} - w_2^{(q-p)/(1-p)}), \tag{4.10}$$

being

$$\mathcal{L}_1 = -\sum_{i,j=1}^{N} D_i(a_{ij} D_j) + \sum_{i=1}^{N}(b_i - \frac{p}{1-p} c_i) D_i.$$

By (1.3), $c(x) \geq 0$ in $B(x_0, r)$, and from (H) we have that

$$w_2^{(q-p)/(1-p)} > w_1^{(q-p)/(1-p)} \qquad \text{in } B(x_0, r),$$

and so by the strong maximun principle of Hopf, see for example Theorem 3.5 in [16], $\Psi = C > 0$ in $B(x_0, r)$ with C constant. Thus, the left hand side of (4.10) is non-negative and right one negative. This gives a contradiction and completes the proof of the uniqueness.

Now, we prove the stability of the positive solution. Let (λ, u_λ) be a positive solution of $\lambda > 0$. By the strong maximum principle, it can be shown, as we did in (4.7), that the function

$$M(x) := -\lambda q u_\lambda^{q-1} - p a(x) u_\lambda^{p-1}$$

satisfies (2.1). Thus, $\sigma_1[\mathcal{L} - \lambda q u_\lambda^{q-1} - p a(x) u_\lambda^{p-1}]$ is well defined. Now, it is not difficult to prove that

$$(\mathcal{L} - \lambda q u_\lambda^{q-1} - p a(x) u_\lambda^{p-1}) u_\lambda^p =$$
$$p(1-p) u_\lambda^{p-2} \sum_{i,j=1}^{N} a_{ij} D_i u D_j u + \lambda(p-q) u_\lambda^{p+q-1} > 0.$$

Hence, $u_\lambda^p \in C^2(\Omega) \cap C_0^0(\overline{\Omega})$ is a positive strict supersolution of the operator $\mathcal{L} - \lambda q u_\lambda^{q-1} - p a(x) u_\lambda^{p-1}$. The result is a consequence of Proposition 2.4 □

Proof of Theorem 4.1: Firstly, we are going to show that the bifurcation from the trivial solution $u = 0$ is subcritical. Suppose the contrary: there exists a sequence of nonnegative and nontrivial solutions (λ_n, u_n) verifying $\lambda_n \geq 0$, $\lambda_n \to 0$ and $\|u_n\|_\infty \to 0$. We distinguish two cases:

Case 1: $\lambda_n > 0$. In this case, by Proposition 4.5, we have that $u_n = u_{\lambda_n}$. Now, it is clear that for each $n \in \mathbb{N}$ there exists a positive constant $K_n > 0$ such that the pair $(U^0, K_n e)$ is a sub-supersolution of (1.4) for $\lambda = \lambda_n$, and so by the uniqueness of positive solution for $\lambda_n > 0$, we have

$$U^0 \leq u_n \leq K_n e. \tag{4.11}$$

Case 2: $\lambda_n = 0$. Since u_n is nonnegative, there exists $\rho_n > 0$ sufficiently small such that the function $u_n - \rho_n e$ attains a positive maximum in Ω. Let $x_n \in \Omega$ be such that $(u_n - \rho_n e)(x_n) := \max_{x \in \overline{\Omega}} (u_n - \rho_n e)(x) > 0$. Then,

$$0 \leq \mathcal{L}(u_n - \rho_n e)(x_n) = a(x_n) u_n^p(x_n) - \rho_n$$

and so,

$$0 < \rho_n \leq a(x_n) u_n^p(x_n). \tag{4.12}$$

Therefore, $x_n \in A_+$. Assume, that $x_n \in A_+^{k_0}$ for some $k_0 \in \{1, \dots, r\}$. By (4.12), it follows that $u_n \geq 0$, $u_n \neq 0$ in $A_+^{k_0}$. From the strong maximum principle, see again Lemma 2.1 in [7], it follows that

$$u_n > 0 \quad \text{in } \overline{A_+^{k_0}}.$$

Hence, u_n is a supersolution of (4.3) in A_{k_0} with $b(x) \equiv a(x)$. We can build a subsolution as (4.4), and we conclude by Proposition 4.3 that

$$z_{[a,p]} \leq u_n \quad \text{in } A_+^{k_0}. \tag{4.13}$$

Hence, in any case by (4.11) and (4.13) it follows that $\|u_n\|_\infty$ does not approach to 0.

Now, we define

$$\lambda_* := \inf\{\lambda \in \mathbb{R} : (1.4) \text{ has a nonnegative and nontrivial solution.}\}$$

We have just proved that $-\infty < \lambda_* < 0$. Take $\lambda_0 \in (\lambda_*, 0)$. So, there exists u_μ with $\mu \in [\lambda_*, \lambda_0)$ solution of (1.4). Then, the pair $(\underline{u}, \overline{u}) := (u_\mu, U^0)$ is a sub-supersolution of (1.4) for $\lambda = \lambda_0$, and so there exists a solution of (1.4) for $\lambda = \lambda_0$. Observe that $u_\mu \leq U^0$ due to the maximality of U^0. The existence of solution for $\lambda = \lambda_*$ follows by a standard compactness argument.

Finally, the subcritical bifurcation at $\lambda = 0$, the connectivity of the continuum $\mathcal{C}_0$ of nonnegative solutions, (4.11) and (4.13) imply the existence of λ_{**} such that for $\lambda \in (\lambda_{**}, 0)$, (1.4) admits at least two nonnegative solutions. This completes the proof. □

The next result shows that λ_* goes 0 as $\|a^+\|_\infty \to 0$. This result is consistent with that when $a^+ \equiv 0$, (1.4) has positive solution if, and only if, $\lambda > 0$, see [13].

Lemma 4.6 *Assume $p < 1$. Then $\lambda_* \uparrow 0$ as $\|a^+\|_\infty \to 0$.*

Proof: If $\|a^+\|_\infty \to 0$, then $a_M \to 0$. The result follows by (4.2). □

5 The self-diffusion case: $p = 1$.

In the particular case $p = 1$, the bifurcation direction of the continuum $\mathcal{C}_0$ depends on the sign of $\sigma_1[\mathcal{L} - a(x)]$.

Theorem 5.1 *Assume $p = 1$. Then,*

a) If $\sigma_1[\mathcal{L} - a(x)] = 0$, then (1.4) admits nonnegative and nontrivial solutions if, and only if, $\lambda = 0$. Moreover, in this case (1.4) has infinitely many positive solutions.

b) If $\sigma_1[\mathcal{L} - a(x)] > 0$, then (1.4) admits nonnegative and nontrivial solutions if, and only if, $\lambda > 0$. In this case (1.4) has a unique positive solution which is linearly asymptotically stable.

c) If $\sigma_1[\mathcal{L} - a(x)] < 0$, then (1.4) admits nonnegative and nontrivial solutions if, and only if, $\lambda < 0$.

Proof: In the case $p = 1$, observe that (1.4) can be written as

$$(\mathcal{L} - a(x))u = \lambda u^q \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

In the first paragraph, $\sigma_1[\mathcal{L} - a(x)] = 0$, the Fredholm alternative provides us the result.

Assume that $\sigma_1[\mathcal{L} - a(x)] > 0$. The maximum principle applied to (5.1) implies that if $\lambda \le 0$, (1.4) does not admit nonnegative solution.

Assume that $\sigma_1[\mathcal{L} - a(x)] < 0$ and that there exists a nonnegative solution u of (5.1). Then, multiplying (5.1) by ψ_1^*, principal eigenfunction of $\mathcal{L}^* - a(x)$ and taking account that $\sigma_1[\mathcal{L}^* - a(x)] = \sigma_1[\mathcal{L} - a(x)] < 0$, we obtain

$$\sigma_1[\mathcal{L}^* - a(x)] \int_\Omega \psi_1^* u = \lambda \int_\Omega u^q \psi_1^*,$$

and so $\lambda < 0$.

We claim that for $\lambda \in I$, a compact interval in $\mathbb{R}$, there exists $C > 0$ such that (1.4) does not possess positive solution u with $\|u\|_\infty > C$. Indeed, we suppose the contrary: there exists a sequence (λ_n, u_n) of solutions of (1.4) with $\lambda_n \to \lambda_0 \in \mathbb{R}$ and $\|u_n\|_\infty \to +\infty$. Let $v_n := u_n/\|u_n\|_\infty$ be, so

$$(\mathcal{L} - a(x))v_n = \lambda_n \frac{u_n^q}{\|u_n\|_\infty},$$

hence $v_n \to v \geq 0$ with $\|v\|_\infty = 1$ and

$$(\mathcal{L} - a(x))v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{5.2}$$

If $\sigma_1[\mathcal{L} - a(x)] > 0$, by the maximum principle we obtain that $v \equiv 0$. If $\sigma_1[\mathcal{L} - a(x)] < 0$, multiplying (5.2) by ψ_1^*, we obtain

$$\sigma_1[\mathcal{L}^* - a(x)] \int_\Omega \psi_1^* v = 0,$$

and so $v \equiv 0$.

Now, Theorem 3.1 provides us the existence of nonnegative solution for $\lambda > 0$ (resp. $\lambda < 0$) if $\sigma_1[\mathcal{L} - a(x)] > 0$ (resp. $\sigma_1[\mathcal{L} - a(x)] < 0$.)

For the uniqueness in the case $\lambda > 0$, we can repeat exactly the argument used in Proposition 4.3 to show that (4.3) possesses a unique positive solution.

On the other hand, let (λ, u) be a positive solution of (5.1) with $\lambda > 0$, we have

$$\sigma_1[\mathcal{L} - a(x) - q\lambda u^{q-1}] > \sigma_1[\mathcal{L} - a(x) - \lambda u^{q-1}] = 0.$$

This shows the stability and completes the proof. □

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