Study of a nonlinear Kirchhoff equation with non-homogeneous material

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Abstract

In this paper we study a non-homogeneous elliptic Kirchhoff equation with nonlinear reaction term. We analyze the existence and uniqueness of positive solution. The main novelty is the inclusion of non-homogeneous term making the problem without a variational structure. We use mainly bifurcation arguments to get the results.

Key Words. Kirchhoff equation, non-homogeneous material, bifurcation methods.
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1 Introduction

In this paper we study the following nonlinear Kirchhoff equation with non-homogeneous material

$$\begin{cases} -M(x,\|u\|^2)\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $0 < q \leq 1$, $\lambda \in \mathbb{R}$ and

$$M(x,s) := a(x) + b(x)s, \qquad \|u\|^2 = \int_\Omega |\nabla u|^2 dx,$$

with $a, b \in C^{\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$ and $a(x) \geq a_0 > 0, b \geq 0$. Equation (1.1) is the steady-state problem associated to the time dependent problem

$$\begin{cases} u_{tt} - M(x, \|u\|^2)\Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

which models small vertical vibrations of an elastic string with fixed ends when the density of the material is not constant. The problem (1.2) was proposed by J. L. Lions [17] (see also [16] and [24]). The elliptic version of (1.2) was studied in [22] and [26] for bounded and unbounded domains, respectively. In these papers a fixed point argument and the Galerkin method are used to prove the existence of a solution.

However, in contrast with the non-homogeneous case, when a and b are positive constants the problem has a variational structure and has been investigated extensively during last years. In [2], [8], [9], [11], [15], [19] and [25] the following problem was studied

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for different properties on f. In [2] and [19] the Mountain Pass Theorem and a truncation argument is applied to prove the existence of a solution when f is subcritical (see also [9] for the critical case). In [8] the Mountain Pass Theorem and the Ekeland's Principle were used to show the existence of multiple non-trivial solutions of (1.3) with a concave nonlinearity. In [25] variational results were employed for nonlinearities f which are resonant at an eigenvalue. In [15] existence of positive solutions was showed using topological degree arguments and variational method for functions f asymptotically linear at zero and asymptotically 3-linear at infinity. When $\Omega = \mathbb{R}^N$ the problem

$$\begin{cases} M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + V(x)u\right)[-\Delta u + V(x)u] = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

has been analyzed under appropriate assumptions on V and f. In [3] it is shown existence of solution for f subcritical and critical. Multiplicity of solutions were showed in [11], [14],

[19], [27], [28] and [29] using genus or category theory. The case in which the Laplace operator is replaced by the p-Laplacian or the $p(x)$-Laplacian has been considered in [6] and [5] respectively. The case where M is the identity and $V(x) = b > 0$ is studied in [4] via minimization and in [13] by a monotonicity trick. For sign changing solutions see the papers [20], [21] and [30].

The purpose of this paper is to take a first step to study the problem

$$-M(x, \|u\|^2)\Delta u = f(x, u)$$

for general non-linearities f. For that, we have chosen the sublinear case $f(x, s) = \lambda s^q$ for $0 < q \leq 1$. We employ two different techniques to study our problem. For the case $q < 1$ we use the bifurcation method to show that there exists a positive solution for all $\lambda > 0$ and no positive solution for $\lambda \leq 0$. This result is similar to the one obtained in the homogeneous case, although the techniques that we apply are different. In order to apply the bifurcation method, we need to rewrite (1.1) as a fixed point equation of a compact operator. For that, we have to deal with the case $f = f(x)$.

For the case $q = 1$ we have used an argument based on the eigenvalue problems of elliptic equations and their properties. In this case, we can see the consequences of the non-homogeneous term b. In this case we show the existence of positive solution for $\lambda \in (\lambda_0, \lambda_1)$ where λ_0 and λ_1 are positive constants; while for the homogeneous case the existence holds for $\lambda > \lambda_0$, see Section 4 for details.

In all the cases (lineal, $q = 1$ and $q < 1$) we have proved the uniqueness of positive solution of (1.1). In our knowledge, the results are new even in the homogeneous case.

An outline of the paper is as follows: In Section 2 we give a motivation of the non-homogeneous problem, Section 3 is devoted to the linear case, in Sections 4 and 5 we study the cases $q = 1$ and $q < 1$, respectively.

2 Motivation of the problem

In this section we would like to deduce (1.2). We point out that problem (1.2) appears, for example, when one considers small transversal vibrations of an elastic string with fixed ends which is composed by a non-homogeneous material. In this case, distinct points can have distinct densities and tensions. Let us consider an elastic string of length L composed of a non-homogeneous material, resting on the horizontal axis x and with fixed ends at the points $\{0, L\}$. We denote by $u(x,t)$ and $\tau(x,t)$, respectively, the displacement and the tension of point x at the time t.

Since we will submit the string to small vibrations, perpendicular to the axis x, we can consider only the vertical component $\tau(x,t)\sin\theta$ of $\tau(x,t)$, where θ is an angle such that $\sin\theta \approx \dfrac{\partial u}{\partial x}$. So, by using the Newton's second law of dynamics, we deduce

$$\frac{\partial}{\partial x}(\tau(x,t)\sin\theta) = d(x,t)\frac{\partial^2 u}{\partial t^2}, \tag{2.1}$$

where $d(x,t)$ denotes the density at x in the instant t. Again, since the vibrations are small, we can consider that the variation of $\tau(x,t)$ is small, therefore

$$\frac{\partial}{\partial x}(\tau(x,t)\sin\theta) = \frac{\partial \tau}{\partial x}\sin\theta + \tau\frac{\partial \sin\theta}{\partial x} \approx \tau(x,t)\frac{\partial^2 u}{\partial x^2}. \tag{2.2}$$

From (2.1) and (2.2),

$$\tau(x,t)\frac{\partial^2 u}{\partial x^2} = d(x,t)\frac{\partial^2 u}{\partial t^2}. \tag{2.3}$$

On the other hand, if we denote by h the area of the cross-section (which we consider constant) and by $E(x)$ the Young modulus of the material that makes up the point x, it follows from Hooke's law and from straightforward computations that

$$\tau(x,t) = \tau(x,0) + \frac{hE(x)}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx. \tag{2.4}$$

Replacing (2.4) in (2.3), we obtain

$$\frac{\partial^2 u}{\partial t^2} - \left[\frac{\tau(x,0)}{d(x,t)} + \frac{hE(x)}{2Ld(x,t)}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{\partial^2 u}{\partial x^2} = 0. \tag{2.5}$$

This last equation is the time-dependent uni-dimensional version of the problem (1.2).

3 The linear case

In this section we analyze the case when f does not depend on u:

$$\begin{cases} -M(x, \|u\|^2)\Delta u = f(x) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

when $f \in L^\infty(\Omega)$. Let us introduce some notation. Let $P := \{u \in L^\infty(\Omega) : u(x) \geq 0 \text{ a.e. } x \in \Omega\}$ be the positive cone in $L^\infty(\Omega)$ and the set $U := P \cup (-P)$. Given $f \in C(\overline{\Omega})$ we denote

$$f_L := \min_{x \in \overline{\Omega}} f(x), \qquad f_M := \max_{x \in \overline{\Omega}} f(x).$$

In the following result, we prove the existence of a classical positive solution of (3.1) as well as the compactness of the solution operator.

Proposition 3.1. *For each $f \in L^\infty(\Omega)$ there exists a solution u of (3.1). Moreover, if $f \in U$, there exists a unique solution. Furthermore, the operator solution $T : U \mapsto U$ defined by*

$$T(f) := u$$

is compact.

Proof. When $f \equiv 0$ the result is trivial. So, assume that $f \neq 0$. We use a fixed point argument. For any $R \geq 0$, u_R stands for the unique solution of

$$\begin{cases} -(a(x) + b(x)R)\Delta u_R = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Observe that u_R is the solution of a linear equation, therefore, by the elliptic regularity and the continuity in R of

$$h(R) = \frac{f}{a(x) + b(x)R}$$

we deduce that the map $R \mapsto u_R$ is continuous. Moreover,

$$-\frac{\|f\|_\infty}{a_L} \leq -\Delta u_R \leq \frac{\|f\|_\infty}{a_L},$$

and then $\|u_R\|_\infty \leq C$.

Now, we have to find R such that

$$R = \int_\Omega |\nabla u_R|^2 dx = \int_\Omega \frac{f(x)u_R}{a(x) + b(x)R} dx.$$

Define

$$g(R) := \int_\Omega \frac{f(x)u_R}{a(x) + b(x)R} dx,$$

we have to find a fixed point of $R = g(R)$. Observe that $g(0) > 0$. Indeed,

$$-\Delta u_0 = \frac{f(x)}{a(x)}$$

and so

$$g(0) = \int_\Omega \frac{f(x)u_0}{a(x)}dx = \int_\Omega |\nabla u_0|^2 dx > 0.$$

On the other hand,

$$|g(R)| \leq \int_\Omega \frac{|f(x)||u_R|}{a(x)+b(x)R}dx \leq \frac{1}{a_L}\|f\|_\infty \|u_R\|_\infty \leq C.$$

This concludes the existence.

We prove the uniqueness. Assume that $f \in U$, for example $f \in P$, and that there exist two positive solutions $u \neq v$. If $\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla v|^2 dx$ then it follows that $u = v$.

Then, assume that $\int_\Omega |\nabla u|^2 dx > \int_\Omega |\nabla v|^2 dx$. Since $f \geq 0$, we infer by the maximum principle that $v > u$ in Ω. But,

$$0 < \int_\Omega (|\nabla u|^2 - |\nabla v|^2)dx = \int_\Omega \nabla(u-v)\cdot\nabla(u+v)dx = \int_\Omega (u-v)(-\Delta(u+v))dx < 0,$$

an absurdum.

Let $T : U \to H_0^1(\Omega)\cap L^\infty(\Omega)$ be the operator defined by $T(f) = u$, where u is a weak solution to

$$-\Delta u = \frac{f(x)}{a(x)+b(x)\|u\|^2} \quad \text{in } \Omega \ \text{ and } \ u = 0 \text{ on } \partial\Omega. \tag{3.3}$$

We are going to show that T is compact. Let $(f_n) \subset L^\infty(\Omega)$ be a bounded sequence. Then there is $\alpha > 0$ such that

$$\|f_n\|_\infty \leq \alpha, \ \forall\, n \in \mathbb{N}. \tag{3.4}$$

So, if $u_n = T(f_n) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ we have

$$-\Delta u_n = \frac{f_n(x)}{a(x)+b(x)\|u_n\|^2}. \tag{3.5}$$

The elliptic regularity [10] asserts that there is a positive constant C_s that does not depend on n such that

$$\|u_n\|_{W^{2,s}} \leq C_s \left\| \frac{f_n}{a+b\|u_n\|^2} \right\|_s \leq C_s \left(\frac{\alpha}{a_L}\right)|\Omega|^{\frac{1}{s}}, \tag{3.6}$$

for all $n \in \mathbb{N}$ and $s \in (1,\infty)$. We pick s sufficiently large, by the compact embedding, there exists a subsequence such that

$$u_n \to u \text{ in } L^\infty(\Omega).$$

In the sequel, we prove the continuity of T. Let $\{f_n\} \subset L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$ be such that

$$f_n \to f \text{ in } L^\infty(\Omega). \tag{3.7}$$

From (3.7) we conclude $\{f_n\}$ is bounded in $L^\infty(\Omega)$. So, denoting $u_n = T(f_n)$ and arguing as in (3.6) we conclude that $\{u_n\}$ is bounded in $W^{2,s}(\Omega)$ for all $s \in (1, \infty)$. By the compact embedding we have

$$u_n \to u \text{ in } L^\infty(\Omega) \cap H_0^1(\Omega). \tag{3.8}$$

From (3.8), it follows that $u = T(f)$. Therefore T is continuous and the proof is complete.

□

4 Non-linear eigenvalue problem

In this section we study the equation

$$\begin{cases} -M(x, \|u\|^2)\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In order to show our results, we need to introduce some notation. Given a domain $D \subset \Omega$ and a strictly positive function $A \in C^\gamma(\overline{\Omega})$ for some $\gamma \in (0, 1)$, we denote by $\lambda_1(-A\Delta; D)$ the principal eigenvalue of the problem

$$\begin{cases} -A(x)\Delta\varphi = \lambda\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{4.2}$$

In the following result we show some properties of $\lambda_1(-A\Delta; D)$.

Proposition 4.1. *a) If $D_1 \subset D_2 \subset \Omega$, then*

$$\lambda_1(-A\Delta; D_2) \le \lambda_1(-A\Delta; D_1).$$

b) If $A_i \in C^\gamma(\overline{\Omega})$, $i = 1, 2$ are positive functions such that $A_1 \le A_2$ in Ω, then $\lambda_1(-A_1\Delta; D) \le \lambda_1(-A_2\Delta; D)$.

c) Let $A, B \in C^{\gamma}(\overline{\Omega})$ be two functions such that A is strictly positive, $B \geq 0$, the set

$$B_0 := int(\{x \in D : B(x) = 0\})$$

is a connected subset of D, and consider the map

$$\lambda_1(\mu) := \lambda_1(-(A + \mu B)\Delta, D), \quad \mu \geq 0.$$

Then, $\lambda_1(\mu)$ is a continuous and increasing function and

$$\lim_{\mu \to 0} \lambda_1(\mu) = \lambda_1(-A\Delta, D), \qquad \lim_{\mu \to +\infty} \lambda_1(\mu) = \lambda_1(-A\Delta, B_0).$$

If $B_L > 0$ then $\lambda_1(-(A + \mu B)\Delta, D) \to +\infty$ as $\mu \to \infty$.

Proof. Paragraph a) follows, for instance, by [18, Proposition 3.2].

The proof of b) is as follows. Observe that (4.2) is equivalent to

$$-\Delta \varphi = \lambda \frac{1}{A(x)} \varphi.$$

If $A_1 \leq A_2$, then $1/A_1 \geq 1/A_2$. Hence, $\lambda_1(-A_1\Delta; D) \leq \lambda_1(-A_2\Delta; D)$.

We prove now paragraph c). First, $\lambda_1(\mu)$ is continuous from classical results, see [12]. That $\lambda_1(\mu)$ is increasing follows by paragraph b).

Assume that $B_L > 0$, then

$$\lambda_1(\mu) = \lambda_1(-(A + \mu B)\Delta, D) \geq \lambda_1(-(A_L + \mu B_L)\Delta, D) = \lambda_1(-\Delta, D)(A_L + \mu B_L),$$

and we conclude that $\lambda_1(\mu) \to \infty$ as $\mu \to \infty$.

On the other hand, assume that $B_0 \neq \emptyset$, then by a)

$$\lambda_1(\mu) = \lambda_1(-(A + \mu B)\Delta, D) \leq \lambda_1(-(A + \mu B)\Delta, B_0) = \lambda_1(-A\Delta, B_0),$$

and so, $\lim_{\mu \to \infty} \lambda_1(\mu) := \lambda_0 \leq \lambda_1(-A\Delta, B_0)$. Consider now φ_μ the positive eigenfunction associated to $\lambda_1(\mu)$ such that $\int_\Omega \varphi_\mu^2 dx = 1$. Then,

$$\int_\Omega |\nabla \varphi_\mu|^2 dx = \lambda_1(\mu) \int_\Omega \frac{\varphi_\mu^2}{A + \mu B} dx \leq \frac{\lambda_1(-A\Delta, B_0)}{A_L},$$

and so $\{\varphi_\mu\}$ is bounded in $H_0^1(\Omega)$. We can conclude that there exists φ_∞ such that $\|\varphi_\infty\|_2 = 1$ and

$$\varphi_\mu \to \varphi_\infty \quad \text{in } L^2(\Omega), \quad \varphi_\mu \rightharpoonup \varphi_\infty \quad \text{in } H_0^1(\Omega) \text{ as } \mu \to \infty.$$

Take $\varphi \in C^1_c(\overline{B}_0)$, then

$$\int_\Omega \nabla\varphi_\mu \cdot \nabla\varphi dx = \lambda_1(\mu)\int_\Omega \frac{\varphi_\mu\varphi}{A+\mu B}dx = \lambda_1(\mu)\int_{B_0}\frac{\varphi_\mu\varphi}{A}dx,$$

and passing to the limit

$$\int_{B_0} \nabla\varphi_\infty \cdot \nabla\varphi dx = \lambda_0\int_{B_0}\frac{\varphi_\infty\varphi}{A}dx, \quad \forall\varphi \in C^1_c(\overline{B}_0).$$

We denote by D any domain such that $D \subset \Omega \setminus B_0$ and $\varphi \in C^1_c(D)$ and assume that $\varphi_\infty > 0$ in D. Then,

$$\int_\Omega \nabla\varphi_\mu \cdot \nabla\varphi dx = \lambda_1(\mu)\int_\Omega \frac{\varphi_\mu\varphi}{A+\mu B}dx = \lambda_1(\mu)\int_D\frac{\varphi_\mu\varphi}{A+\mu B}dx,$$

and passing to the limit

$$\int_D \nabla\varphi_\infty \cdot \nabla\varphi dx = 0, \quad \forall\varphi \in C^1_c(D).$$

This implies that φ_∞ is constant, that is, $\varphi_\infty = 0$ in D. This implies that $\varphi_\infty \in H^1_0(B_0)$, and then $\lambda_0 = \lambda_1(-A\Delta, B_0)$. □

The main result in this section is:

Theorem 4.2. *Assume that $\Omega_0 := int(\{x \in \Omega : b(x) = 0\})$ is a regular sub-domain of Ω. Then, (4.1) possesses a positive solution if and only if*

$$\lambda \in (\lambda_1(-a\Delta;\Omega), \lambda_1(-a\Delta;\Omega_0)), \tag{4.3}$$

where we denote by $\lambda_1(-a\Delta;\Omega_0) = \infty$ when $b_L > 0$. Moreover, in such case, the solution is unique and it will be denoted by u_λ. Furthermore,

$$\begin{aligned} \lim_{\lambda\downarrow\lambda_1(-a\Delta;\Omega)} \|u_\lambda\| &= \lim_{\lambda\downarrow\lambda_1(-a\Delta;\Omega)} \|u_\lambda\|_\infty = 0, \\ \lim_{\lambda\uparrow\lambda_1(-a\Delta;\Omega_0)} \|u_\lambda\| &= \lim_{\lambda\uparrow\lambda_1(-a\Delta;\Omega_0)} \|u_\lambda\|_\infty = \infty. \end{aligned} \tag{4.4}$$

Proof. Assume that u is a positive solution of (4.1), then

$$\begin{aligned} \lambda = \ & \lambda_1(-(a(x) + b(x)\textstyle\int_\Omega |\nabla u|^2 dx)\Delta;\Omega) \\ < \ & \lambda_1(-(a(x) + b(x)\textstyle\int_\Omega |\nabla u|^2 dx)\Delta;\Omega_0) = \lambda_1(-a\Delta;\Omega_0). \end{aligned}$$

On the other hand, by Proposition 4.1

$$\lambda = \lambda_1(-(a(x) + b(x)\int_\Omega |\nabla u|^2 dx)\Delta; \Omega) > \lambda_1(-a\Delta; \Omega).$$

Now, we fix any $\lambda \in (\lambda_1(-a\Delta; \Omega), \lambda_1(-a\Delta; \Omega_0))$. By Proposition 4.1 there exists a unique $t_0(\lambda)$ such that

$$\lambda_1(-(a + bt_0)\Delta; \Omega) = \lambda.$$

Moreover, by Proposition 4.1 we have that

$$\lim_{\lambda \downarrow \lambda_1(-a\Delta;\Omega)} t_0(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \uparrow \lambda_1(-a\Delta;\Omega_0)} t_0(\lambda) = +\infty. \tag{4.5}$$

For a fixed t_0 take $\varphi_0 > 0$ the positive eigenfunction associated to $\lambda_1(-(a+bt_0)\Delta; \Omega)$ such that

$$\int_\Omega |\nabla \varphi_0|^2 dx = t_0. \tag{4.6}$$

Then, it is not hard to show that φ_0 is solution of (4.1).

We prove now the uniqueness. Assume that there exist two positive solutions $u \neq v$. Then,

$$\lambda = \lambda_1(-(a(x) + b(x)\int_\Omega |\nabla u|^2 dx)\Delta; \Omega) = \lambda_1(-(a(x) + b(x)\int_\Omega |\nabla v|^2 dx)\Delta; \Omega).$$

Therefore $\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla v|^2 dx$, and u is proportional to v, which implies that $u = v$. This concludes the uniqueness.

Finally, we show (4.4). Observe that by (4.6) we have that $t_0(\lambda) = \|\varphi_0\|^2$. When $\lambda \downarrow \lambda_1(-a\Delta; \Omega)$ we get by (4.5) that $\|\varphi_0(\lambda)\| \to 0$ and by a boot-strapping argument we conclude that $\|\varphi_0\|_\infty \to 0$. Indeed, we can show that

$$\|\varphi_0\|_{W^{2,s}} \leq C\|\varphi_0\|_s \quad \text{for } s > 1.$$

Hence, since $\|\varphi_0(\lambda)\| \to 0$ we have that $\|\varphi_0(\lambda)\|_{2^*} \to 0$ and then $\|\varphi_0(\lambda)\|_{W^{2,2^*}} \to 0$. A boot-strapping argument concludes that $\|\varphi_0(\lambda)\|_\infty \to 0$.

On the other hand, when $\lambda \uparrow \lambda_1(-a\Delta; \Omega_0)$ by (4.5) then $\|\varphi_0(\lambda)\| \to \infty$ and hence by the equation

$$\|\varphi_0\| \leq C\|\varphi_0\|_\infty.$$

This concludes the proof. □

5 The case $q < 1$.

During this section we will analyze the problem (1.1) for $0 < q < 1$ and $\lambda \in \mathbb{R}$. We are going to use the bifurcation method. To this aim, we consider the Banach space $X := C_0(\overline{\Omega})$, denote $B_\rho := \{u \in X : \|u\|_\infty < \rho\}$. Define the map

$$\mathcal{K}_\lambda : X \mapsto X; \qquad \mathcal{K}_\lambda(u) := u - T(\lambda(u^+)^q),$$

where $u^+ := \max\{u, 0\}$ and T is the operator defined in Section 3. It is clear that u is a non-negative solution of (1.1) if, and only if, u is a zero of the map $\mathcal{K}_\lambda$. Observe that $\mathcal{K}_\lambda$ is compact. Indeed, the map from X into $U := P \cup (-P)$ defined by $u \mapsto \lambda(u^+)^q$ is continuous, and T from U to X is compact from Proposition 3.1.

In order to prove the main result of this section we use the Leray-Schauder degree of $\mathcal{K}_\lambda$ on B_ρ with respect to zero, denoted by $\deg(\mathcal{K}_\lambda, B_\rho)$, and the index of the isolated zero u of $\mathcal{K}_\lambda$, denoted by $i(\mathcal{K}_\lambda, u)$.

In the following result, we show that from the trivial solution emanates an unbounded continuum of positive solution.

Theorem 5.1. *The value $\lambda = 0$ is the only bifurcation point from the trivial solution for* (1.1)*. Moreover, there exists a continuum $\mathcal{C}_0$ of positive solutions of* (1.1) *unbounded in $\mathbb{R} \times X$ emanating from* $(0, 0)$*.*

The following lemmas play a fundamental rolle in the proof of the result.

Lemma 5.2. *If $\lambda < 0$, then $i(\mathcal{K}_\lambda, 0) = 1$.*

Proof. Fix $\lambda < 0$ and define the map

$$\mathcal{H}_1 : [0, 1] \times X \mapsto X; \qquad \mathcal{H}_1(t, u) := T(t\lambda(u^+)^q).$$

We claim that there exists $\delta > 0$ such that

$$u \neq \mathcal{H}_1(t, u) \quad \text{for all } u \in \overline{B}_\delta,\ u \neq 0 \text{ and } t \in [0, 1].$$

Indeed, suppose that there exist sequences $u_n \in X\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0, 1]$ such that

$$u_n = \mathcal{H}_1(t_n, u_n),$$

that is

$$-M(x,\|u_n\|^2)\Delta u_n = t_n\lambda(u_n^+)^q \le 0,$$

and so $u_n \le 0$, and going back to the equation, $u_n \equiv 0$, an absurdum.

Taking now $\varepsilon \in (0,\delta]$, the homotopy defined by $\mathcal{H}_1$ is admissible and so,

$$\begin{aligned} i(\mathcal{K}_\lambda,0) = &\ \deg(\mathcal{K}_\lambda,B_\varepsilon) = \deg(I-\mathcal{H}_1(1,\cdot),B_\varepsilon) = \deg(I-\mathcal{H}_1(0,\cdot),B_\varepsilon) = \\ = &\ \deg(I,B_\varepsilon) = 1. \end{aligned}$$

□

Lemma 5.3. *If $\lambda > 0$, then $i(\mathcal{K}_\lambda,0) = 0$.*

Proof. Fix $\lambda > 0$ and $\phi \in X$, $\phi > 0$. We define the map

$$\mathcal{H}_2 : [0,1]\times X \mapsto X; \qquad \mathcal{H}_2(t,u) := T(\lambda(u^+)^q + t\phi).$$

We will show that there exists $\delta > 0$ such that $u \neq \mathcal{H}_2(t,u)$ for all $u \in \overline{B}_\delta$, $u \neq 0$ and $t \in [0,1]$. Indeed, suppose the contrary: there exist sequences $u_n \in X\setminus\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$ such that $u_n = \mathcal{H}_2(t_n,u_n)$, that is

$$-M(x,\|u_n\|^2)\Delta u_n = \lambda(u_n^+)^q + t_n\phi.$$

Since $t_n\phi \ge 0$, from the maximum principle we have that $u_n > 0$.

On the other hand, since $\|u_n\|_\infty \to 0$ we get

$$\int_\Omega |\nabla u_n|^2dx = \int_\Omega \frac{\lambda u_n^{q+1} + t_n\phi u_n}{a(x)+b(x)\int_\Omega|\nabla u_n|^2dx}dx \le C(\|u_n\|_\infty^{q+1} + \|u_n\|_\infty) \le C,$$

for some positive constant C. Hence, by Proposition 4.1 we have that

$$\lambda_1(-M(x,\|u_n\|^2)\Delta,\Omega) \le \lambda_1(-(a_M+b_MC^2)\Delta,\Omega) := \Lambda.$$

Fix this value of Λ, then for n large we have that $\lambda u_n^q > \Lambda u_n$ and then

$$-(a_M+b_MC^2)\Delta u_n \ge -M(x,\|u_n\|)\Delta u_n = \lambda(u_n^+)^q + t_n\phi > \Lambda u_n,$$

and so $\lambda_1(-(a_M+b_MC^2)\Delta,\Omega) > \Lambda$, an absurdum.

This proves that the homotopy defined by $\mathcal{H}_2$ is admissible. Then, if we take $\varepsilon \in (0,\delta]$ we have

$$i(\mathcal{K}_\lambda,0) = \deg(\mathcal{K}_\lambda,B_\varepsilon) = \deg(I-\mathcal{H}_2(0,\cdot),B_\varepsilon) = \deg(I-\mathcal{H}_2(1,\cdot),B_\varepsilon) = 0.$$

□

Proof of Theorem 5.1: First, we would like to point out that we can not apply directly Theorem 1.3 in [23] because our equation

$$u = T(\lambda(u^+)^q) \tag{5.1}$$

can be not written in the form (0.1) of [23], we have not differentiability at $u = 0$ nor $\lambda = 0$ is an eigenvalue with odd multiplicity of the "linearized" problem around $u = 0$. However, we can prove our result following the main lines of the cited result (see [1] for a similar problem).

We denote by $\mathcal{S}$ the closure of the set on non-trivial solutions of (5.1) and $\mathcal{C}_0$ the maximal connected subset of $\mathcal{S} \cup \{(0,0)\}$ to which $(0,0)$ belongs. We are going to show that $\mathcal{C}_0$ is unbounded in $\mathbb{R} \times X$. Assume that $\mathcal{C}_0$ is bounded. First, we show that $\mathcal{C}_0$ can not meet $(\lambda, 0)$ for any $\lambda \neq 0$ showing that $(\lambda, 0)$ is an isolated solution of (5.1), or equivalently of (1.1), for $\lambda \neq 0$. It is clear that for $\lambda \leq 0$ problem (1.1) does not possess a positive solution. Assume now that there exist $\lambda_0 > 0$ and a sequence of positive solutions of (1.1) such that $\lambda_n \to \lambda_0$ and $\|u_n\|_\infty \to 0$. Then, fixed $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we get

$$-M(x, \|u_n\|)\Delta u_n = \lambda_n u_n^q \geq (\lambda_0 - \varepsilon) u_n^q > \Lambda u_n,$$

where Λ is defined in Lemma 5.3. Now, we arrive at a contradiction in a similar way that in the proof of Lemma 5.3.

Then, $\mathcal{C}_0$ verifies the hypotheses of Lemma 1.2 in [23] and so, there exist a bounded set $\mathcal{O} \subset \mathbb{R} \times X$ such that $(0,0) \in \mathcal{O}$, $\partial\mathcal{O} \cap \mathcal{S} = \emptyset$, and $\mathcal{O}$ contains no trivial solutions others than those in open ball B_ε of $\mathbb{R} \times X$ where $\varepsilon > 0$ small.

Now, we can follow Theorem 1.3 in [23] to conclude that the existence of $\varepsilon > 0$ and values $\underline{\lambda}$ and $\overline{\lambda}$ such that $-\varepsilon < \underline{\lambda} < 0 < \overline{\lambda} < \varepsilon$ and (see (1.11) in [23])

$$i(\mathcal{K}_{\underline{\lambda}}, 0) = i(\mathcal{K}_{\overline{\lambda}}, 0).$$

This is an absurdum with Lemmas 5.2 and 5.3. Hence, we conclude the existence of an unbounded continuum $\mathcal{C}_0$ of solutions of (1.1) bifurcating from $(0,0)$. $\square$

We are ready to prove the main result of this section:

Theorem 5.4. *The equation (1.1) has a positive solution if and only if $\lambda > 0$. Moreover, the solution is unique.*

Proof. By Theorem 5.1 we know the existence of an unbounded continuum $\mathcal{C}_0$ of positive solution of (1.1).

Observe that if u is a positive solution of (1.1) then for $\lambda > 0$

$$-\Delta u \leq \frac{\lambda}{a_L} u^q,$$

and hence $\|u\|_\infty$ is bounded for all $\lambda > 0$. As consequence of Theorem 5.1 we obtain positive solution for all $\lambda > 0$.

We show now the uniqueness of positive solution of (1.1). Assume that there exist two positive solutions $u \neq v$. If $\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla v|^2 dx$, then, u and v are positive solutions of

$$-\Delta w = \lambda \frac{1}{a(x) + b(x) \int_\Omega |\nabla u|^2 dx} w^q. \tag{5.2}$$

Since this equation has only one positive solution, we can deduce that $u = v$.

On the other hand, if $\int_\Omega |\nabla u|^2 dx > \int_\Omega |\nabla v|^2 dx$, then, v is supersolution of (5.2), and so $v > u$. Hence,

$$0 < \int_\Omega (|\nabla u|^2 - |\nabla v|^2) dx = \int_\Omega \nabla(u - v) \cdot \nabla(u + v) dx = \int_\Omega (u - v)(-\Delta(u + v)) dx < 0.$$

This concludes the proof.

□

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