Nonlinear age-dependent diffusive equations: a bifurcation approach

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Abstract

The main goal of this paper is the study of the existence and uniqueness of positive solutions of some nonlinear age-dependent diffusive models, arising from dynamic populations. We use bifurcation method, for which it has been necessary to study in detail the linear and eigenvalue problems associated to the nonlinear problem in an appropriate space.

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1 Introduction

In this paper we present some results of existence and uniqueness of positive solutions of the following nonlinear problem

$$\begin{cases} u_a - \Delta u + \mu(x,a)u = \lambda u + g(u) & \text{in } Q := \Omega \times (0,A), \\ u = 0 & \text{on } \Sigma := \partial\Omega \times (0,A), \\ u(x,0) = \displaystyle\int_0^A \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $N \leq 3$, $A > 0$, β a regular and positive function, μ is a measurable, non-negative and non-trivial function blowing up at $a = A$ and

$$g(u) = \pm \frac{u}{1+u}.$$

Equation (1.1) models the steady-state solutions of its time-dependent counterpart. Hence $u(x,a)$ represents the density population of a species localized in $x \in \Omega$, its habitat, of age $a < A$, its maximal age. Here μ denotes the mortality rate of the species, so that μ blows up at $a = A_\dagger$, β is the fertility rate function and λ will be considered a parameter. Finally, the reaction function $g(u)$ is a Holling-Tanner type. This kind of was introduced because the classical Lotka-Volterra model has the defect that the predators must be capable of consuming an infinite number of preys. For the Holling-Tanner model this difficulty disappears, and a fixed predator population, the prey follows an equation similar to (1.1), see [6], for which it is interesting its study.

Observe that (1.1) presents three difficulties roughly speaking: the nonlinearity g, the blowing-up of μ and the non-local initial condition. In fact, the most of works related with this problem analyze the time-dependent problem, see for instance [10], [17], [18] and references therein. However, very little is known about of stationary problems. Only partial results for specific examples are shown in [18]. In [11] it is proved that the sub-supersolution method works for equations as (1.1) and it was applied to the logistic equation, i.e. $g(u) = -u^p$, $p > 1$ basically, showing that a unique bounded positive solution exists. Of course, when applicable, the sub-supersolution method gives us more information relating to the solution. However, in some equations it is difficult to find, or simply it does not exist, the sub or supersolution. So, it is interesting to have different methods to prove the existence of solution.

In this work, we apply the bifurcation method to (1.1). In our knowledge this is the first time in which this method is applied in this kind of problem, and of course we think that this is the first step towards more general nonlinearities. For that, we need the compactness of certain operators involving the study of the following linear problem

$$\begin{cases} u_a - \Delta u + \mu(x,a)u = f(x,a) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \displaystyle\int_0^{A_\dagger} \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $f \in L^2(Q)$, $f \geq 0$.

Previously, we have to study in detail the eigenvalue problem

$$\begin{cases} u_a - \Delta u + \mu(x,a)u = \lambda u & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \displaystyle\int_0^{A_\dagger} \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{1.3}$$

Problem (1.2) was studied in [13] and the authors proved the existence and uniqueness of positive solution for large μ. Moreover, in [13], see also [11], and using some indirect reasoning, the authors showed the existence of a principal eigenvalue of (1.3), denoted by $\lambda_0(\mu)$, that is, an eigenvalue with a positive eigenfunction associated to. In [13], the authors assume a restrictive condition on μ, specifically for any $0 < A_0 < A$

$$\sup_{a\in[0,A_0]} \int_\Omega \mu^2(a,x)dx \quad \text{is continuous with respect to } A_0 \in [0,A].$$

This condition has been removed in [11] and in this paper, where we also show some properties of the principal eigenvalue and its associated eigenfunction.

With respect to (1.2) we show that there exists a unique positive solution provided $\lambda_0(\mu) > 0$, improving one of the main results of [13], (see Theorem 1 and pages 194-195).

Now, we want to apply a bifurcation method to the nonlinear problem (1.1); we have chosen the natural space $L^\infty(Q)$, so that we need the compactness of the operator $\mathcal{T} : f \in L^\infty(Q) \mapsto u \in L^\infty(Q)$, solution of (1.2). For that, we have to impose some restriction on the growth of μ (see condition (5.2)), similar to the used previously by other authors, see for instance [2], and the use the Sobolev spaces. Finally, we apply the bifurcation method to (1.1) and conclude:

- If $g(u) = u/(1+u)$, then (1.1) possesses a positive solution if, and only if, $\lambda \in (\lambda_0(\mu) - 1, \lambda_0(\mu))$. When the solution exists, this is unique.

- If $g(u) = -u/(1+u)$, then (1.1) possesses a positive solution if, and only if, $\lambda \in (\lambda_0(\mu), \lambda_0(\mu) + 1)$.

This paper is arranged as follows. In Section 2 we study problem (1.2). Section 3 is dedicated to establish some useful properties of the principal eigenvalue, in Section 4 we analyze problem (1.3), in Section 5 we show the compactness of the map $\mathcal{T}$, and finally the bifurcation results and the study of (1.1) is made in Section 6. We include an appendix stating some important known results for reader's convenience.

2 The eigenvalue problem

Along this work we assume the following hypotheses:

$(\mathcal{H}_\mu)$ μ is a function such that $\mu \in L^\infty(\overline{\Omega} \times (0, r))$ for $r < A$ and

$$\int_0^r \mu_M(a)da < \infty, \qquad \int_0^A \mu_L(a)da = +\infty, \tag{2.1}$$

being $\mu_L(a) := \text{essinf}_{x\in\overline{\Omega}}\mu(x,a)$ and $\mu_M(a) := \text{esssup}_{x\in\overline{\Omega}}\mu(x,a)$ and

$$\nabla_x \mu \in (L^\infty(Q))^N. \tag{2.2}$$

$(\mathcal{H}_\beta)$ $\beta \in C^2(\overline{Q})$ and

$$mes\{a \in [0, A] : \beta_L(a) := \inf_{x\in\overline{\Omega}} \beta(x, a) > 0\} > 0.$$

Definition 2.1. *Denote by $L^2_+(Q) := \{f \in L^2(Q) : f(x, a) \geq 0$ a. e. $(x, a) \in Q\}$. We say that $u \in L^2_+(Q)$ is a quasi-interior point of $L^2_+(Q)$, and we write $u \gg 0$, if*

$$\iint_Q u(x, a)f(x, a)da\, dx > 0, \qquad \text{for all } f \in L^2_+(Q) \text{ and non-trivial.}$$

Along the paper we are going to use the following notations: the norm of the spaces $L^q(Q)$, $1 < q \leq +\infty$, it will denoted by $\|\cdot\|_q$; and we denote by

$$\mathcal{V} := L^2(0, A; H_0^1(\Omega)), \qquad \mathcal{W} := L^2(0, A; H_0^1(\Omega) \cap H^2(\Omega)),$$

with its respective norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$.

Definition 2.2. *λ is a principal eigenvalue of* (1.3) *if there exists $u \gg 0$, $u \in \mathcal{V}$, $u_a + \mu u \in L^2(0, A; H^{-1}(\Omega))$ solution of* (1.3) *in the sense that $\forall v \in \mathcal{V}$:*

$$\int_0^A < u_a + \mu u, v > \, da + \iint_Q \nabla u \cdot \nabla v \, da\, dx = \lambda \iint_Q uv \, da\, dx,$$

$$u(x, 0) = \int_0^A \beta(x, a) u(x, a) \, da, \quad \text{in } \Omega,$$

where $<,>$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Our main results in this section is:

Theorem 2.3. *There exists a unique principal eigenvalue of* (1.3)*, denoted by $\lambda_0(\mu)$. This eigenvalue is simple and it is the only having a positive eigenfunction, denoted by φ_1, in fact $\varphi_1 \gg 0$. Moreover, $\varphi_1 \in L^\infty(Q)$ and if $\mu \in L^\infty(Q)$ we have that $\varphi_1 \in W_q^{2,1}(Q)$, for some $q > N + 2$. Furthermore, if λ is any other real eigenvalue of* (1.3)*, then*

$$\lambda_0(\mu) < \lambda. \tag{2.3}$$

Remark 2.4. *Conditions (2.2) and $N \leq 3$ are only used to prove that $\varphi_1 \in L^\infty(Q)$.*

Before giving the main result in this section, we need the following one, some part yet showed in [11]. Consider the equation

$$\begin{cases} u_a - \Delta u + \mu(x, a) u = f(x, a) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

Proposition 2.5. *Suppose that $\phi \in L^2(\Omega)$ and $f \in L^2(Q)$. Then, there exists a unique solution u of* (2.4) *such that $u \in \mathcal{V}$ and $u_a + \mu u \in L^2(0, A; H^{-1}(\Omega))$.*

With respect to the regularity of the solution:

a) For each $0 < A_0 < A$ *we have that* $u \in C([0, A_0]; L^2(\Omega))$.

b) If $\phi \in H_0^1(\Omega)$, *then* $u \in \mathcal{W}$ *and*

$$\|u\|_{\mathcal{W}} \le C\{\|f\|_2, \|\phi\|_{H_0^1(\Omega)}, \|\nabla_x \mu\|_\infty\}. \tag{2.5}$$

c) If $\phi \in L^\infty(\Omega)$ *and* $f \in L^\infty(Q)$, *then* $u \in L^\infty(Q)$ *and*

$$\|u\|_\infty \le C\{\|f\|_\infty, \|\phi\|_\infty\}. \tag{2.6}$$

Furthermore, we have the following comparison principles:

d) If $f \ge 0$ *and* $\phi \ge 0$, *then* $u \ge 0$. *If* $\phi \ge 0$ *and non-trivial, we deduce that* $u \gg 0$.

e) If $f_1 \ge f_2 \ge 0$, $\phi_1 \ge \phi_2 \ge 0$ *and* $\mu_1 \le \mu_2$ *in their respective domains, then* $u_1 \ge u_2$, *where* u_i, $i = 1, 2$, *is the solution of* (2.4) *with* $f = f_i$, $\phi = \phi_i$ *and* $\mu = \mu_i$.

f) If $\overline{u}$ *is supersolution (resp.* $\underline{u}$ *is a subsolution) of* (2.4), *then* $\overline{u} \ge u$ *(resp.* $\underline{u} \le u$*)*.

Proof. Under the change of variable

$$w = e^{-ka}u, \qquad k > 0,$$

w satisfies

$$\begin{cases} w_a - \Delta w + (\mu + k)w = g := fe^{-ka} & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x, 0) = \phi(x) & \text{in } \Omega, \end{cases} \tag{2.7}$$

and so by $(\mathcal{H}_\mu)$, we can take k large such that $\mu + k/3 \ge 0$. We study now (2.7) instead of (2.4).

Define

$$\mu_n := \min\{\mu, n\}, \qquad n \in \mathbb{N},$$

and consider the problem

$$\begin{cases} w_a - \Delta w + (\mu_n(x, a) + k)w = g(x, a) & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x, 0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{2.8}$$

Now, for each $n \in \mathbb{N}$, since $\mu_n + k$ is bounded, there exists a unique w_n solution of (2.8) with $w_n \in \mathcal{V}$ and $(w_n)_a \in L^2(0, A; H^{-1}(\Omega))$. Multiplying (2.8) by w_n and integrating we obtain

$$\frac{1}{2}\int_\Omega |w_n(x, A)|^2 + \iint_Q |\nabla w_n|^2 + \iint_Q (\mu_n + k)w_n^2 = \iint_Q g w_n + \frac{1}{2}\int_\Omega \phi^2. \tag{2.9}$$

Applying that $2ab \le (\varepsilon^2 a^2 + (1/\varepsilon^2)b^2)$ we get

$$\iint_Q |\nabla w_n|^2 + \iint_Q [(\mu_n + k/3)w_n^2 + (k/3)w_n^2] \le C. \tag{2.10}$$

Now, we can extract a sequence (w_n) such that

$$\begin{aligned} &w_n \rightharpoonup w && \text{in } \mathcal{V}, \\ &\sqrt{\mu_n + (k/3)}w_n \rightharpoonup h && \text{in } L^2(Q), \\ &(w_n)_a + (\mu_n + k/3)w_n \rightharpoonup z && \text{in } L^2(0, A; H^{-1}(\Omega)). \end{aligned}$$

On the other hand, for $\varphi \in C_c^\infty(0, A; H_0^1(\Omega))$, and for n large enough, we get

$$\int_0^A < (w_n)_a + (\mu_n + k/3)w_n, \varphi >= \iint_Q (-w_n\varphi_a + (\mu + k/3)w_n\varphi) \to$$

$$\to \iint_Q (-w\varphi_a + (\mu + k/3)w\varphi), \quad \text{as } n \to \infty,$$

and so

$$z = w_a + (\mu + k/3)w.$$

This shows that w is solution of (2.7), so that u is solution of (2.4).

On the other hand, from (2.9) and taking k sufficiently large such that $\mu_n + k > 0$, we get

$$\|w_n\|_2 \le C(\|g\|_2, \|\phi\|_{L^2(\Omega)}) \tag{2.11}$$

and

$$\|w_n\|_{\mathcal{V}} \le C(\|g\|_{L^2(Q)}, \|\phi\|_{L^2(\Omega)}). \tag{2.12}$$

Hence, since $\|w\|_{\mathcal{V}} \le \liminf \|w_n\|_{\mathcal{V}}$ it follows that

$$\|w\|_{\mathcal{V}} \le C(\|g\|_{L^2(Q)}, \|\phi\|_{L^2(\Omega)}). \tag{2.13}$$

The regularity $u \in C([0, A_0]; L^2(\Omega))$, $A_0 < A$, follows considering the equation (2.4) in $Q_0 := \Omega \times (0, A_0)$ where μ is bounded, see for example Theorem X.11 of [7].

Assume now that $\phi \in H_0^1(\Omega)$ and multiply the equation of w_n by $-\Delta w_n$, we obtain

$$\frac{1}{2}\int_\Omega |\nabla w_n(A)|^2 + \iint_Q (\Delta w_n)^2 + \iint_Q \nabla[(\mu_n + k)w_n]\cdot \nabla w_n = -\iint_Q g\Delta w_n + \frac{1}{2}\int_\Omega |\nabla \phi|^2.$$

Now,

$$\iint_Q \nabla[(\mu_n + k)w_n]\cdot \nabla w_n = \iint_Q w_n \nabla(\mu_n + k)\cdot \nabla w_n + \iint_Q (\mu_n + k)|\nabla w_n|^2,$$

and so taking k large such that $\mu_n + k > 0$, we get

$$\|\Delta w_n\|_2^2 \le C(\|g\|_2^2 + \|\phi\|^2_{H_0^1(\Omega)} + \|\nabla(\mu_n + k)\|_\infty(\|w_n\|_2^2 + \|w_n\|^2_{\mathcal{V}})).$$

Now, using (2.11) and (2.12) we obtain that

$$\|w_n\|_{\mathcal{W}} \le C\|\Delta w_n\|_2 \le C\{\|g\|_2, \|\phi\|_{H_0^1(\Omega)}, \|\nabla_x \mu\|_\infty\},$$

whence (2.5) follows .

For the uniqueness, take two different solutions u_1 and u_2. Then, $z = u_1 - u_2$ satisfies that

$$z_a - \Delta z + \mu(x,a)z = 0, \qquad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = 0 \quad \text{in } \Omega.$$

It suffices to multiply this problem by z and obtain that $z \equiv 0$.

Now, assume that $f \ge 0$ and $\phi \ge 0$ and let u the solution of (2.4). By the classical maximum principle (observe that the potential μ_n is bounded) applied to (2.8) it follows that $w_n \ge 0$, and so that $u \ge 0$. This shows first part of paragraph d). Now e) and f) are direct consequences.

Now, let us show second part of paragraph d). Observe that if u is solution of (2.4), then $u \ge z$ being z the solution of

$$\begin{cases} z_a - \Delta z + \mu_M(a)z = f \ge 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{2.14}$$

But, if z is the solution of (2.14) then

$$z = exp(-\int_0^a \mu_M(s)ds)q(x,t)$$

where q is solution of

$$\begin{cases} q_a - \Delta q = f exp(\int_0^a \mu_M(s)ds) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{2.15}$$

Hence, since $q \gg 0$ it follows that $u \geq z \gg 0$.

Finally, we show c). Take

$$\overline{u} := K exp(-\int_0^a g(s)ds),$$

where $g \in C[0, A]$ such that $\mu - g \geq \delta > 0$ for some $\delta > 0$ and K is a positive constant. It is easy to see that $\overline{u}$ is a supersolution of (2.4) provided of

$$K \geq \frac{1}{\delta} f exp(\int_0^a g(s)ds) \quad \text{and} \quad K \geq \|\phi\|_\infty.$$

So, it is clear that we can choose a positive K such that

$$u \leq C \sup\{\|f\|_\infty, \|\phi\|_\infty\} exp(-\int_0^a g(s)ds).$$

A similar lower bound can be found, and thus we conclude that $u \in L^\infty(Q)$ and (2.6). $\square$

Now, define the operator $\mathcal{B}_\lambda : L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{B}_\lambda(\phi) = \int_0^A \beta(x,a) e^{\lambda a} z_\phi(x,a) da,$$

where z_ϕ is the unique solution of (2.4) with $f \equiv 0$. Then, it is shown in [11] that $\mathcal{B}_\lambda$ is a well-defined, compact, positive and irreducible operator. (Recall that a operator T is irreducible if there exists $\lambda > r(T)$ such that $(\lambda - T)^{-1}$ is strongly positive operator. Moreover, for a linear operator T is irreducible if T itself or a power of T is strongly positive, see pag. 118 in [8]).

Hence, we can apply Theorem 12.3 in [8] and conclude that $r(\mathcal{B}_\lambda) > 0$, is an algebraically simple eigenvalue of $\mathcal{B}_\lambda$ and $\mathcal{B}_\lambda^*$ (the adjoint), with a quasi-interior eigenfunction and a strictly positive eigenfunctional, respectively.

Moreover, in [11] it is proved that λ is a eigenvalue of (1.3) if and only if $\mathcal{B}_\lambda$ has a fixed point. In fact, λ_0 is a principal eigenvalue of (1.3) if, and only if, $r(\mathcal{B}_{\lambda_0}) = 1$.

The following result was proved in [11], but there we used a version of Krein-Rutman Theorem which assumed the existence of a cone with non-empty interior. We give now an alternative proof.

Lemma 2.6. *The map $\lambda \mapsto r(\mathcal{B}_\lambda)$ is continuous and increasing.*

Proof. First, we show that the map $\lambda \mapsto r(\mathcal{B}_\lambda)$ is increasing. Take $\lambda_1 < \lambda_2$ and consider $\varphi_i \gg 0$ and φ_i^* the eigenfunction and the eigenfunctional associated to $r(\mathcal{B}_{\lambda_i})$, of $\mathcal{B}_{\lambda_i}$ and $\mathcal{B}_{\lambda_i}^*$, $i = 1, 2$, respectively. Then,

$$r(\mathcal{B}_{\lambda_2}) < \varphi_2^*, \varphi_1 >=< \mathcal{B}_{\lambda_2}^* \varphi_2^*, \varphi_1 >=< \varphi_2^*, \mathcal{B}_{\lambda_2}\varphi_1 >>< \varphi_2^*, \mathcal{B}_{\lambda_1}\varphi_1 >= r(\mathcal{B}_{\lambda_1}) < \varphi_2^*, \varphi_1 >,$$

whence we deduce that $r(\mathcal{B}_{\lambda_2}) > r(\mathcal{B}_{\lambda_1})$.

Now, we deduce the continuity of the map $\lambda \mapsto r(\mathcal{B}_\lambda)$. Indeed, take a sequence $\lambda_n \to \lambda_0$, then for all $\varepsilon > 0$ we have that $\lambda_0 - \varepsilon \leq \lambda_n \leq \lambda_0 + \varepsilon$ for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. We can show that

$$e^{\varepsilon A} r(\mathcal{B}_{\lambda_0})\varphi_0 - \mathcal{B}_{\lambda_0+\varepsilon}(\varphi_0) \geq 0,$$

and so by Corollary 12.4 in [8] we deduce that $e^{\varepsilon A} r(\mathcal{B}_{\lambda_0}) > r(\mathcal{B}_{\lambda_0+\varepsilon})$, and thus

$$e^{-\varepsilon A} r(\mathcal{B}_{\lambda_0}) \leq r(\mathcal{B}_{\lambda_0-\varepsilon}) \leq r(\mathcal{B}_{\lambda_n}) \leq r(\mathcal{B}_{\lambda_0+\varepsilon}) \leq e^{\varepsilon A} r(\mathcal{B}_{\lambda_0}),$$

whence the continuity follows. □

Proof of Theorem 2.3: With the previous notation and considerations, we need to prove that there exists a real value λ_0 such that $r(\mathcal{B}_{\lambda_0}) = 1$.

Finally, in [11] we construct two operators $\mathcal{A}_\lambda$ and $\mathcal{C}_\lambda$ such that $\mathcal{A}_\lambda \leq \mathcal{B}_\lambda \leq \mathcal{C}_\lambda$. With a similar argument to the used in Lemma 2.6, we can deduce that

$$r(\mathcal{A}_\lambda) \leq r(\mathcal{B}_\lambda) \leq r(\mathcal{C}_\lambda).$$

Finally, in [11] we show that there exist $\lambda_A > \lambda_C$ such that $1 = r(\mathcal{A}_{\lambda_A}) = r(\mathcal{C}_{\lambda_C})$. Hence, we get that there exists a unique real value λ_0 such that $r(\mathcal{B}_{\lambda_0}) = 1$.

Let λ be a real eigenvalue of (1.3) such that $\lambda \neq \lambda_0$. Then, 1 is an eigenvalue of $\mathcal{B}_\lambda$, and so $r(\mathcal{B}_\lambda) \geq 1$, and so $\lambda > \lambda_0$.

Now we show that $\varphi_1 \in L^\infty(\Omega)$. Indeed, $\varphi_1 \in L^2(Q)$ and so

$$\varphi_1(x, 0) = \phi(x) := \int_0^{A_\dagger} \beta(x, a)\varphi_1(x, a)da \in L^2(\Omega),$$

and then $\varphi_1 \in \mathcal{V}$. So, φ_1 is a solution of an equation as (1.3) with $\mu - \lambda_0(\mu)$ instead of μ and $\phi(x) \in H_0^1(\Omega)$, and then by Proposition 2.5 b), $\varphi_1 \in \mathcal{W}$. So, since $N \leq 3$, it follows that $\phi \in L^\infty(\Omega)$, and so $\varphi_1 \in L^\infty(Q)$ again by Proposition 2.5 c).

Finally, assume that μ is bounded. Denote by φ_1 the positive eigenfunction associated to $\lambda_0(\mu)$. Then, $(\varphi_1)_a - \Delta\varphi_1 = F := (-\mu + \lambda_0)\varphi_1 \in L^2(Q)$ and $\varphi_1(x,0) = \int_0^A \beta(x,a)\varphi_1(x,a)da \in L^2(\Omega)$, and then $\varphi_1 \in \mathcal{V}$. So, $\varphi_1(x,0) \in H_0^1(\Omega)$ and by Theorem X.11 of [7] it follows that $\varphi_1 \in \mathcal{W}$. Then,

$$\varphi_1(x,0) = \int_0^{A_\dagger} \beta(x,a)\varphi_1(x,a)da \in H^2(\Omega) \hookrightarrow W^{2-2/q_0,q_0}(\Omega),$$

by Lemma 7.1, with $q_0 = 2(N+2)/N$. So, by Theorem 7.3, $\varphi_1 \in W^{2,1}_{q_0}(Q)$. Hence

$$\varphi_1(x,0) \in W^{2,q_0}(\Omega) \hookrightarrow W^{2-2/q_1,q_1}(\Omega)$$

with $q_1 = q_0(N+2)/N = 2((N+2)/N)^2$. Again, by Theorem 7.3, $\varphi_1 \in W^{2,1}_{q_1}(Q)$. Repeating this argument, we obtain that

$$\varphi_1 \in W^{2,1}_{q_n}(Q), \qquad q_n = 2((N+2)/N)^{n+1},$$

where we have used that $q_{n+1} > q_n$ and $2 - N/q_n + N/q_{n+1} \geq 0$. It suffices to take n sufficiently large such that $q_n > N+2$, and by Lemma 7.2 it follows that $\varphi_1 \in C^1(\overline{Q})$. $\square$

3 Some properties of the principal eigenvalue

Our first result provides us with some properties of the principal eigenvalue with respect to the potential:

Proposition 3.1. *a) The map $\mu \mapsto \lambda_0(\mu)$ is increasing. Specifically, if $\mu_1 \leq \mu_2$ and $\mu_1 < \mu_2$ in a subset of positive measure, then $\lambda_0(\mu_1) < \lambda_0(\mu_2)$.*

b) Denote by $\mu_n := \min\{\mu, n\}$. Then,

$$\lambda_0(\mu_n) \uparrow \lambda_0(\mu) \quad \textit{as } n \to \infty.$$

Proof. Take $\mu_1 < \mu_2$ in a set of positive measure. Denote by $\mathcal{B}^1_\lambda$ and $\mathcal{B}^2_\lambda$ the operators defined previously for μ_1 and μ_2, respectively, and $\varphi_i \gg 0$ the eigenfunctions associated to

$r(\mathcal{B}^i_\lambda) =$ for $i = 1, 2$. It is easy to show that $\mathcal{B}^2_\lambda\varphi_1 > \mathcal{B}^1_\lambda\varphi_1$ and so, with a similar argument to the proof of the increasing of $r(\mathcal{B}_\lambda)$ in Lemma 2.6, we deduce that

$$r(\mathcal{B}^2_\lambda) > r(\mathcal{B}^1_\lambda).$$

This concludes that $\lambda_0(\mu_1) < \lambda_0(\mu_2)$.

For the second paragraph, denote by $\lambda_n := \lambda_0(\mu_n)$. Thanks to the first part of this result, we have that λ_n is increasing and bounded by $\lambda_0(\mu)$ and $\lambda_n \uparrow \lambda_* \leq \lambda_0(\mu)$. Denote by φ_n the positive eigenfunction associated to λ_n normalized such that $\|\varphi_n\|_\infty = 1$. Then, multiplying by φ_n, we obtain

$$\frac{1}{2}\int_\Omega |\varphi_n(x, A_\dagger)|^2 + \iint_Q |\nabla\varphi_n|^2 + \iint_Q (\mu_n + R)\varphi_n^2 = (\lambda_n + R)\iint_Q \varphi_n^2 + \frac{1}{2}\int_\Omega [\int_0^{A_\dagger} \beta\varphi_n]^2 < C,$$

for some $C > 0$. Then, for $R > 0$ large, we can apply the same argument used previously and conclude $\varphi_n \rightharpoonup \varphi_1$ in $\mathcal{V}$ as $n \to +\infty$ being φ_1 a positive solution of

$$u_a - \Delta u + \mu u = \lambda_* u, \qquad u = 0, \qquad u(x, 0) = \int_0^{A_\dagger} \beta(x, a)u(x, a)da.$$

Moreover, by Proposition 2.5 we deduce that $\varphi_1 \gg 0$, and so Theorem 2.3 assures that $\lambda_* = \lambda_0(\mu)$. $\square$

Now, we prove the monotonicity of the principal eigenvalue with respect to the domain. For that, we need the following result, whose idea of the proof comes from [22] (see also [19]).

Lemma 3.2. *Assume that* $\mu \in L^\infty(Q)$ *and there exists* $0 < \phi \in W^{2,1}_q(Q)$, $q > N + 2$, *such that*

$$\begin{cases} \phi_a - \Delta\phi + \mu(x, a)\phi \geq 0 & \text{in } Q, \\ \phi > 0 & \text{on } \Sigma, \\ \phi(x, 0) \geq \displaystyle\int_0^{A_\dagger} \beta(x, a)\phi(x, a)da & \text{in } \Omega. \end{cases} \tag{3.1}$$

Then,

$$\lambda_0(\mu) > 0.$$

Proof. Assume that $\lambda_0(\mu) \leq 0$, denote by φ_1 the principal eigenfunction associated to $\lambda_0(\mu)$ normalized such that $\|\varphi_1\|_\infty = 1$ and consider the set

$$\Gamma := \{\varepsilon \in \mathbb{R} : \phi + \varepsilon\varphi_1 \geq 0 \quad \text{in } \overline{Q}\}.$$

Denote by $\varepsilon_0 = \min \Gamma$ and $u_0 = \phi + \varepsilon_0 \varphi_1$. It is clear that $\varepsilon_0 < 0$ and that $u_0 \neq 0$, $u_0 > 0$. Then, using (3.1) we get

$$\begin{cases} (u_0)_a - \Delta u_0 + \mu(x,a) u_0 \geq \varepsilon_0 \lambda_0(\mu) \varphi_1 \geq 0, & \text{in } Q, \\ u_0 > 0 & \text{on } \Sigma, \\ u_0(x,0) \geq \displaystyle\int_0^{A_\dagger} \beta(x,a) u_0(x,a) da & \text{in } \Omega. \end{cases}$$

Observe that $u_0(x,0) > 0$, and so by the strong maximum principle, see for instance Theorem 13.5 in [9], u_0 is strictly positive, in the sense that it is positive and its normal derivative at $\partial\Omega$ is negative. This contradicts that ε_0 is the infimum of Γ. This concludes the proof. $\square$

The proof of the following result is routine, and so we omit it.

Lemma 3.3. *Consider two regular domains $\Omega_1 \subset \Omega_2$, $Q_i := \Omega_i \times (0, A_\dagger)$, $i = 1, 2$ and $\mu \in L^\infty(Q_2)$, β a regular (OJO CON ESTO SI ES POSIBLE AHORA) and non-negative function in Q_2 verifying $(\mathcal{H}_\beta)$. If we denote by $\lambda_0^i(\mu)$, $i = 1, 2$ the corresponding principal eigenvalues of* (1.3) *in Q_i, then:*

$$\lambda_0^2(\mu) < \lambda_0^1(\mu).$$

The next result shows that the principal eigenvalue is continuous with respect to the domain. Its proof follows the lines of Theorem 3.1 in [8]. First, we introduce a particular definition of convergence of domains (see [8]):

Definition 3.4. *We say that a sequence of domain Ω_n, with $\Omega \subset \Omega_{n+1} \subset \Omega_n$, converges to Ω, we write $\Omega_n \downarrow \Omega$, if:*

a) For all Ω' such that $\overline{\Omega}' \subset \Omega$ implies $\overline{\Omega}' \subset int(\Omega_n)$ for large $n \in \mathbb{N}$.

b) For any open set U with $\overline{\Omega} \subset U$ we have $\Omega_n \subset U$ for large $n \in \mathbb{N}$.

Lemma 3.5. *Assume that $\mu \in L^\infty(Q)$. Take a sequence Ω_n of domain such that $\Omega_n \downarrow \Omega$ and $\Omega_{n+1} \subset \Omega_n$. If we denote by $\lambda_0^n(\mu)$ the corresponding principal eigenvalues on Ω_n, with μ prolonged by zero and β in a regular way in $\Omega_n \setminus \Omega$ and verifying $(\mathcal{H}_\beta)$, then*

$$\lambda_0^n(\mu) \uparrow \lambda_0(\mu) \quad \text{as } n \to \infty.$$

Proof. Denote by $\tilde{\mu}$, $\tilde{\beta}$ the prolongations of μ and β to Ω_n, respectively, and $\lambda_n := \lambda_0^n(\mu)$. Then, by Lemma 3.3 the sequence λ_n is increasing and bounded, and so $\lambda_n \uparrow \lambda_* \leq \lambda_0(\mu)$. Denote by φ_n the eigenfunction associated to λ_n, normalized such that $\|\varphi_n\|_\infty = 1$. Then,

$$(\varphi_n)_a - \Delta\varphi_n + (\tilde{\mu} - \lambda_n)\varphi_n = 0, \text{ in } Q_n, \quad \varphi_n = 0 \text{ in } \partial\Omega_n \times (0, A_\dagger),$$

$$\varphi_n(x, 0) := \varphi_{0n} = \int_0^{A_\dagger} \tilde{\beta}\varphi_n \quad \text{in } \Omega_n.$$

Multiplying this equation by φ_n and integrating, and using a similar argument to the used in the second paragraph of Proposition 3.1, we obtain that

$$\|\varphi_n\|_{L^2(0,A_\dagger;H_0^1(\Omega_n))}, \|\varphi_n\|_{L^2(Q_n)} \leq C,$$

If we extend φ_n outside Ω_n by zero, we have that φ_n is uniformly bounded in $L^2(0, A_\dagger; W^{1,2}(\mathbb{R}^N))$, and so there exists φ such that

$$\varphi_n \rightharpoonup \varphi \quad \text{in } L^2(0, A_\dagger; W^{1,2}(\mathbb{R}^N)),$$

and so

$$\varphi_{0n}\rfloor_\Omega \rightharpoonup \varphi_0 := \int_0^{A_\dagger} \beta\varphi, \quad \text{in } L^2(\Omega).$$

On the other hand, since Ω is regular, Ω satisfies condition (3.6) of [8], see Theorem 3.7 of [23]. Then, we can apply Theorem 3.1 in [8], and conclude that $\varphi_n \to \varphi$, where φ is a positive function verifying

$$\varphi_a - \Delta\varphi + (\mu - \lambda_*)\varphi = 0, \text{ in } Q, \quad \varphi = 0 \text{ in } \partial\Omega, \quad \varphi(x, 0) = \int_0^{A_\dagger} \beta\varphi \quad \text{in } \Omega,$$

whence we can deduce that $\lambda_* = \lambda_0(\mu)$. □

4 The linear problem

In a similar way to Definition ??, it can be defined the concepts of solution and sub-supersolution of (1.2). The main result in this section is:

Theorem 4.1. *Assume that $f \in L^2(Q)$ and $\lambda_0(\mu) > 0$. Then, there exists a unique solution u of* (1.2). *Moreover, $u \in \mathcal{W}$. Finally, if $f \geq 0$ and non-trivial, then $u > 0$.*

Proof. Consider the equation

$$\varphi - \mathcal{B}_0(\varphi) = F(x), \tag{4.1}$$

with

$$F(x) := \int_0^A \beta(x,a) z_0^f(x,a)da,$$

being z_0^f the unique solution of (2.4) with $\psi = 0$. If the homogeneous equation $\varphi - \mathcal{B}_0(\varphi) = 0$ has a non-trivial solution, then 1 is an eigenvalue of $\mathcal{B}_0$, and so $r(\mathcal{B}_0) \geq 1$, an contradiction because $\lambda_0(\mu) > 0$. Hence, by the Fredholm alternative Theorem we can assure the existence of a unique solution of (4.1). Now, it is not difficult to show that if φ is solution of (4.1) then z_φ^f is solution of (1.2).

On the other hand, if $f \geq 0$ and non-trivial, then $F > 0$. Now, we can apply Corollary 12.4 in [8] to conclude that the solution φ of (4.1) is positive. So, $z_\varphi^f > 0$. □

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Remark 4.2. *a) Observe that if μ verifies $(\mathcal{H}_\mu)$ then $\lim_{a\uparrow A} \varphi_1(x,a) = 0$. Indeed, it suffices to check that φ_1 is a subsolution of (2.4) with $\mu = \mu_L - \lambda_0(\mu)$, $f \equiv 0$ and $\phi = \overline{\beta}A\|\varphi_1\|_\infty$, and so*

$$\varphi_1 \leq exp(\lambda_0(\mu)a - \int_0^a \mu_L(s)ds)q(x,a),$$

where q is the solution of (2.15) with $f = 0$ and $\phi = \overline{\beta}A\|\varphi_1\|_\infty$. Since q is bounded, it is clear that, by $(\mathcal{H}_\mu)$, $\lim_{a\uparrow A} \varphi_1(x,a) = 0$.

b) However, if μ is bounded by the strong maximum principle there exists $\phi(x) > 0$ such that $\lim_{a\uparrow A} \varphi_1(x,a) \geq \phi \gg 0$.

5 Compactness of the operator $\mathcal{T}$ in L^∞

Our aim now is to prove that the operator $\mathcal{T} : L^\infty(Q) \mapsto L^\infty(Q)$, $f \mapsto \mathcal{T}(f) := u$ solution of (1.2) is well defined and compact.

Proposition 5.1. *Assume that $\lambda_0(\mu) > 0$.*

a) The operator $\mathcal{T}$ is well-defined. Specifically, if $f \in L^\infty(Q)$, and u is solution of (1.2), then $u \in L^\infty(Q)$, and moreover

$$\|u\|_\infty \leq C\|f\|_\infty, \quad \textit{for some positive constant } C > 0. \tag{5.1}$$

b) Moreover, assume that

$$\mu(x,a)\int_0^a e^{-\int_\sigma^a \mu_L(s)ds}d\sigma \in L^q(Q), \quad \textit{for some } q > 2(N+2)/N. \tag{5.2}$$

The map $\mathcal{T}$ is compact.

Proof. Take $\mu_{n_0}=\min\{\mu,n_0\}$ such that $\lambda_0(\mu_{n_0})>0$ and fix n_0. This choice is possible by Proposition 3.1 b). Consider now a sequence of domains $\Omega_n \downarrow \Omega$ as Lemma 3.5. For $\mu_{n_0}\in L^\infty(Q)$ fixed, using Lemma 3.5 there exists another $n_1\in\mathbb{N}$ such that $\lambda_0^{n_1}(\mu_{n_0})>0$, where this number denotes the principal eigenvalue in Ω_{n_1}. It is not hard to show that if ξ is the positive eigenfunction associated to $\lambda_0^{n_1}(\mu_{n_0})$, then $\overline{u}:=K\xi$ is supersolution of (1.2) for $K>0$ a positive constant. Indeed, it must hold that

$$K(\mu-\mu_{n_0}+\lambda_0^{n_1}(\mu_{n_0}))\xi \geq f.$$

Taking into account that $\mu-\mu_{n_0}\geq 0$ in Ω, it suffices that

$$K\geq \frac{1}{\lambda_0^{n_1}(\mu_{n_0})\xi_L}\|f\|_\infty,$$

where $\xi_L:=\inf_{\overline{\Omega}\times[0,A_\dagger]}\xi>0$ because μ_{n_0} is bounded, see Remark 4.2 b). This concludes the proof of the first paragraph.

On the other hand, by (5.1), u, solution of (1.2), is subsolution of (2.4) with $\phi = C\|f\|_\infty$, for some positive constant C, denote by z this solution. Now, it can be shown that

$$\overline{z}:=exp(-\int_0^a \mu_L(s)ds)g(a),$$

where

$$g(a):=\|f\|_\infty(C+\int_0^a e^{\int_0^\sigma \mu_L(s)ds}d\sigma)$$

is supersolution of z. Hence, $u\leq\overline{z}$ and so

$$u\leq \|f\|_\infty exp(-\int_0^a \mu_L(s)ds)(C+\int_0^a e^{\int_0^\sigma \mu_L(s)ds}d\sigma). \tag{5.3}$$

Take a sequence f_n with $\|f_n\|_\infty\leq K$ and $u_n:=\mathcal{T}(f_n)$. Then, thanks to (5.2) and (5.3) we get

$$|\mu u_n|\leq K|\mu| exp(-\int_0^a \mu_L(s)ds)(C+\int_0^a e^{\int_0^\sigma \mu_L(s)ds}d\sigma)\in L^q(Q), \quad q>2(N+2)/N. \tag{5.4}$$

So, u_n is solution of a parabolic equation with

$$(u_n)_a - \Delta u_n = -\mu u_n + f_n, \quad u_n(0,x) = \int_0^A \beta(a,x) u_n(a,x) da := \phi_n(x).$$

By Theorem 4.1, it follows that $u_n \in \mathcal{W}$, and so $\phi_n \in H^2(\Omega) \cap H_0^1(\Omega)$. By Lemma 7.1, it follows that

$$H^2(\Omega) \hookrightarrow W^{2-2/q_0, q_0}(\Omega)$$

if $q_0 = 2(N+2)/N$ and so that $\phi_n \in W^{2-2/q_0,q_0}(\Omega)$. Thanks to (5.4), $-\mu u_n + f_n \in L^{q_0}(\Omega)$. On the other hand, by Theorem 7.3 with $m \equiv 0$, $u_n \in W^{2,1}_{q_0}(Q)$, and

$$\|u_n\|_{W^{2,1}_{q_0}(Q)} \leq C(\| - \mu u_n + f_n\|_{q_0}, \|\phi_n\|_{W^{2-2/q_0,q_0}(\Omega)}). \tag{5.5}$$

First, observe that $\| - \mu u_n + f_n\|_{q_0}$ is uniformly bounded by (5.4). Second, observe that by (5.3)

$$\|u_n\|_2 \leq C \quad \text{(independent of } n.)$$

Now, multiplying the equation of u_n by u_n, integrating and taking into account that $q_0 > 2$, we get that

$$\frac{1}{2}\int_\Omega |u_n(A)|^2 + \iint_Q |\nabla u_n|^2 = \iint_Q (-\mu u_n + f_n) u_n + \frac{1}{2}\int_\Omega \left[\int_0^A \beta u_n\right]^2 \leq C(\beta)\|u_n\|_2^2,$$

and so

$$\|u_n\|_{\mathcal{V}} \leq C(\beta).$$

Finally, multiplying by $-\Delta u_n$, and with a similar argument to the used in Proposition 2.5 b), we get

$$\|u_n\|_{\mathcal{W}} \leq C(\| - \mu u_n + f_n\|_2, \|u_n(x,0)\|_{H_0^1(\Omega)}) \leq C(\| - \mu u_n + f_n\|_{q_0}, \|u_n\|_{\mathcal{V}}) \leq C(\beta).$$

Hence

$$\|\phi_n\|_{W^{2-2/q_0,q_0}(\Omega)} \leq C\|\phi_n\|_{H^2(\Omega)} \leq C(\beta)\|u_n\|_{\mathcal{W}} \leq C(\beta,\mu),$$

and so by (5.5),

$$\|u_n\|_{W^{2,1}_{q_0}(Q)} \leq C(\beta,\mu).$$

Since $q_0 > (N+2)/2$, by Lemma 7.2 we have that $W^{2,1}_{q_0}(Q) \hookrightarrow L^\infty(Q)$ compactly. The proof is accomplished. $\square$

Remark 5.2. *If we consider, for instance, the function*

$$\mu(a) = \frac{1}{A_\dagger - a},$$

then the condition (5.2) *is equivalent to*

$$\ln(1 - \frac{a}{A_\dagger}) \in L^q(0, A_\dagger),$$

which it is true for all $1 \leq q < \infty$*.*

6 A bifurcation result

Along this section we assume (5.2). We can use the above results to study the following nonlinear problem, a Holling-Tanner model,

$$\begin{cases} u_a - \Delta u + \mu(x,a)u = \lambda u \pm \dfrac{u}{1+u} & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \displaystyle\int_0^{A_\dagger} \beta(x,a)u(x,a)da & \text{in } \Omega. \end{cases} \tag{$HT_\pm$}$$

We look for positive solutions, by physical meaning, in $L^\infty(Q)$. We prolong the nonlinearity $\lambda u \pm u/(1+u)$ by zero for $u \leq 0$.

In order to apply the above results, we need:

Proposition 6.1. *If* $u \in \mathcal{V}$ *is solution of* $(HT_\pm)$*, then* $u \in L^\infty(Q)$*. Moreover, if* $u > 0$ *then* $u \gg 0$*.*

Proof. Observe that if u is a solution of $(HT_\pm)$, then

$$u_a - \Delta u + (\mu - \lambda)u = \pm\frac{u}{1+u} \in L^\infty(Q), \qquad u(x,0) \in H_0^1(\Omega),$$

and so $u \in \mathcal{W}$. Hence, since $N \leq 3$ it follows that $u(x,0) \in L^\infty(\Omega)$ and so by Proposition 2.5 c) that $u \in L^\infty(Q)$. Assume now that $u > 0$. If $u(x,0) = 0$, then the unique solution of $(HT_\pm)$ is $u = 0$. Hence, we have that $u(x,0) > 0$, and then by Proposition 2.5 d) we get that $u \gg 0$. This proves the result. □

Our main result is:

Theorem 6.2. *a) From the trivial solution $u = 0$ bifurcates an unbounded continuum $\mathcal{C}_\pm$ of positive solutions of $(HT_\pm)$ at $\lambda = \lambda_0(\mu) \mp 1$, respectively.*

b) In the case (HT_+), there exists a positive solution if, and only if, $\lambda \in (\lambda_0(\mu) - 1, \lambda_0(\mu))$. When the solution exists, this is unique.

c) In the case (HT_-), there exists a positive solution if, and only if, $\lambda \in (\lambda_0(\mu), \lambda_0(\mu)+ 1)$.

Proof. Observe that since

$$\frac{u}{1+u} = u - \frac{u^2}{1+u},$$

$(HT_\pm)$ can be written as

$$u = (\lambda \pm 1)\mathcal{T}u + \mathcal{N}u \quad \text{in } L^\infty(Q),$$

where $\mathcal{N}u := \mathcal{T}(\mp u^2/(1+u))$. It is clear that $\lambda = \lambda_0(\mu) \mp 1$ are simple eigenvalues of $\mathcal{T}$ in $L^2(Q)$, and so also in $L^\infty(Q)$. Now, taking into account that $\mathcal{N}(u) = o(\|u\|_\infty)$ for $\|u\|_\infty \sim 0$, it suffices to apply the classical Rabinowitz's Theorem [21] and conclude the first paragraph.

For the second one, observe that if u is a positive solution of (HT_+), then

$$u_a - \Delta u + (\mu - \frac{1}{1+u})u = \lambda u,$$

and so, thanks to that $u \gg 0$, $\lambda = \lambda_0(\mu - 1/(1+u))$. Then, by Theorem 2.3 we have

$$\lambda_0(\mu) - 1 = \lambda_0(\mu - 1) < \lambda = \lambda_0(\mu - \frac{1}{1+u}) < \lambda_0(\mu),$$

where we have used that $1/(1+u) < 1$.

Now, it remains to show that if (λ_n, u_n) solution of (HT_+) with $\lambda_n \to \lambda_1$ and $\|u_n\|_\infty \to \infty$ then $\lambda_1 = \lambda_0(\mu)$. Indeed, if we denote

$$w_n = \frac{u_n}{\|u_n\|_\infty},$$

then w_n verifies

$$(w_n)_a - \Delta w_n + \mu w_n = \lambda_n w_n + w_n\frac{1}{1+u_n}, \qquad \varphi_n(x) := w_n(x,0) = \int_0^A \beta(x,a)w_n(x,a)da.$$

Now, observe that $1/(1+u_n)$ is bounded in $L^\infty(Q)$ and so there exists $h \in L^2(Q)$ such that

$$\frac{1}{1+u_n} \rightharpoonup h \quad \text{in } L^2(Q).$$

We multiply the equation of w_n by w_n and integrating, we get

$$\frac{1}{2}\int_\Omega |w_n(x,A)|^2 + \iint_Q |\nabla w_n|^2 + \iint_Q (\mu+k)w_n^2 = \iint_Q ((\lambda_n+k)w_n^2 + w_n^2\frac{1}{1+u_n}) + \frac{1}{2}\int_\Omega [\int_0^A \beta w_n]^2.$$

Since $\|w_n\|_\infty = 1$, it follows that

$$\iint_Q |\nabla w_n|^2 + \iint_Q (\mu + k/3)w_n^2 \leq C,$$

and so,

$$\|w_n\|_{\mathcal{V}} \leq C. \tag{6.1}$$

Moreover, with a similar argument to the used in Proposition 2.5 we have that

$$\begin{aligned}
&w_n \rightharpoonup w && \text{in } \mathcal{V},\\
&\sqrt{\mu+k/3}\,w_n \rightharpoonup \sqrt{\mu+k/3}\,w && \text{in } L^2(Q),\\
&(w_n)_a + (\mu+k/3)w_n \rightharpoonup w_a + (\mu+k/3)w && \text{in } L^2(0,A;H^{-1}(\Omega)),
\end{aligned}$$

for some $k > 0$.

Finally, we can repeat exactly the proof of Proposition 5.1 b) with $f_n = (\lambda_n + k)w_n + 1/(1+u_n)w_n \in L^\infty(\Omega)$ and conclude that w_n is bounded in $W^{2,1}_{q0}(Q)$. Since this space is compactly imbedded in L^∞, it follows that

$$w_n\frac{1}{1+u_n} \rightharpoonup wh \quad \text{in } L^2(Q).$$

Therefore, we can conclude that $w \geq 0$ and $\|w\|_\infty = 1$ and verifies

$$w_a - \Delta w + \mu w = \lambda_1 w + wh,$$

By the strong maximum principle we can deduce $w \gg 0$ and so

$$\frac{1}{1+u_n} = \frac{1}{1+\|u_n\|_\infty w_n} \to 0, \quad \text{in } L^2(Q),$$

and hence

$$w_a - \Delta v + \mu w = \lambda_1 w,$$

whence we obtain that $\lambda_1 = \lambda_0(\mu)$. Hence, the unbounded continuum $\mathcal{C}_+$ of positive solution verifies that $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_+) = (\lambda_0(\mu) - 1, \lambda_0(\mu))$. This concludes the result.

For the uniqueness we follow the idea of [15]. Take two positive solutions, u_1 and u_2. Then, $\lambda = \lambda_0(\mu - 1/(1+u_1))$ and so

$$\lambda_0(\mu - \lambda - \frac{1}{1+u_1}) = 0. \tag{6.2}$$

Denote $w := u_1 - u_2 \neq 0$, then

$$w_a - \Delta w + (\mu - \lambda - \frac{1}{(1+u_1)(1+u_2)})w = 0,$$

and so, for some $j \in \mathbb{N}$,

$$\lambda_j(\mu - \lambda - \frac{1}{(1+u_1)(1+u_2)}) = 0,$$

for some λ_j eigenvalue of (1.3) with $\mu - \lambda - \frac{1}{(1+u_1)(1+u_2)}$ instead of μ. Using now Theorem 2.3 we have

$$0 = \lambda_j(\mu - \lambda - \frac{1}{(1+u_1)(1+u_2)}) \geq \lambda_0(\mu - \lambda - \frac{1}{(1+u_1)(1+u_2)}) >$$
$$\lambda_0(\mu - \lambda - \frac{1}{(1+u_1)}) = 0,$$

a contradiction with (6.2).

The third paragraph follows by the same lines that the second one. $\square$

7 Appendix

We have employed along the work the following results and notations.

Lemma 7.1. *([1]) Assume that Ω is regular. Let $s > 0$, $1 < p < q < +\infty$ and denote $\chi := s - N/p + N/q$. If $\chi \geq 0$ we have*

$$W^{s,p}(\Omega) \hookrightarrow W^{\chi,q}(\Omega).$$

For $q \geq 1$ define

$$W^{2,1}_q(Q) := \{u \in L^q(Q) : \partial_a u, \partial^\alpha_x u \in L^q(Q) \quad \text{for } |\alpha| \leq 2\}.$$

Lemma 7.2. *([5]) We have that $W^{2,1}_q(Q) \hookrightarrow C(\overline{Q})$ if $q > (N+2)/2$ and $W^{2,1}_q(Q) \hookrightarrow C^1(\overline{Q})$ if $q > N+2$.*

Consider the linear problem

$$\begin{cases} u_a - \Delta u + m(x,a)u = f(x,a) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \varphi(x) & \text{in } \Omega. \end{cases} \tag{7.3}$$

Theorem 7.3. *([16]) Assume that $f \in L^q(Q)$, $m \in L^\infty(Q)$, $\varphi \in W^{2-2/q,q}(\Omega)$ and $\varphi = 0$ on $\partial\Omega$. Then, there exists a unique solution $u \in W^{2,1}_q(Q)$ of (7.3) and*

$$\|u\|_{W^{2,1}_q(Q)} \leq C\{\|m\|_\infty, \|f\|_q, \|\varphi\|_{W^{2-2/q,q}(\Omega)}\}.$$

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