Pullback permanence in a non-autonomous competitive Lotka-Volterra model

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Abstract

The goal of this work is to study in some detail the asymptotic behaviour of a non-autonomous Lotka-Volterra model, both in the conventional sense (as $t \to \infty$) and in the "pullback" sense (starting a fixed initial condition further and further back in time). The non-autonomous terms in our model are chosen such that one species will eventually die out, ruling out any conventional type of permanence. In contrast we introduce the notion of "pullback permanence" and show that this property is enjoyed by our model. This is not just a mathematical artifice, but rather shows that if we come across an ecology that has been evolving for a very long time we still expect that both species are represented (and their numbers are bounded below), even if the final fate of one of them is less happy. The main tools in the paper are the theory of attractors for non-autonomous differential equations, the sub-supersolution method and the spectral theory for linear elliptic equations.

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1 Introduction

In this paper we analyze the long-time behaviour of the non-autonomous competitive Lotka-Volterra system

$$\begin{cases} u_t - \Delta u = u(\lambda - a(t)u - bv) & \text{in } \Omega \times (s, +\infty), \\ v_t - \Delta v = v(\mu - cv - du) & \text{in } \Omega \times (s, +\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (s, +\infty), \\ u(s,x) = u_0(x), \quad v(s,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, b, c, d are positive constants, $\lambda, \mu \in \mathbb{R}$ and $0 < a(t) \leq A$. Problem (1) models the interactions between two competing species inhabiting a region Ω: $u(x,t)$ and $v(x,t)$ represent the population densities at location $x \in \Omega$ and time t. Moreover, we are assuming that Ω is completely surrounded by inhospitable areas, because both population densities are subject to homogeneous Dirichlet boundary conditions. Here, the operator $-\Delta$ takes into account the diffusivity of the species, λ and μ are the growth rates of the species, b and d describe the interaction rates between the species and finally, $a(t)$ and c are the limiting effects of crowding in each population.

The starting point of this paper is the following observation, which forms the basis of the relatively recent theory of non-autonomous attractors as developed by Crauel et al. [10], Kloeden & Schmalfuss [19], and Schmalfuss [30]. Suppose that $x(t; s, x_0)$ denotes the solution of some system at time t that is equal to x_0 at time s. For an autonomous system we always have

$$x(t; s, x_0) = x(t - s; 0, x_0)$$

and so considering the time asymptotic behaviour as $t \to +\infty$ is exactly the same as considering what happens as $s \to -\infty$. However, in a non-autonomous system the initial time is as important as the final time, and these two different types of "time asymptotic behaviour" are not equivalent. We do not aim here to assert the primacy of one of these approaches over the other, but rather to demonstrate that the "pullback" procedure (considering the behaviour as $s \to -\infty$) is a useful tool that can add to our understanding of non-autonomous systems. Similar ideas are applied to the ordinary differential equation version of (1) in Langa et al. [21] for which more detailed results are possible.

In population dynamics, a basic question is to determine whether the two species will survive in the long term. This has been formalized as the criterion of *permanence* (see Hale and Waltman [12], Hutson and Schmitt [16] and references therein). The system (1) is said to be *permanent* if for any positive initial data u_0 and v_0, the solution $(u(t,s;u_0,v_0), v(t,s;u_0,v_0))$ enters in finite time into a compact set strictly bounded away from zero in each component.

In the autonomous case, that is when $a(t) = a > 0$, results about permanence have been obtained using various techniques. These results depend on the value of λ and μ with respect to certain principal eigenvalues of associated linear elliptic problems. We need some notation in order to state these results. Given $f \in L^\infty(\Omega)$, we denote by $\lambda_1(f)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta w + f(x)w = \sigma w \text{ in } \Omega, \\ w = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

We write $\lambda_1 := \lambda_1(0)$. On the other hand, given $\gamma, e \in \mathbb{R}$ and $e > 0$, we denote by $w_{[\gamma,e]}$ the unique positive solution of

$$\begin{cases} -\Delta w = \gamma w - e w^2 \text{ in } \Omega, \\ w = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Observe that w is related to the stationary solution when only one species is present. It is well-known that $w_{[\gamma,e]}$ exists if, and only if, $\lambda_1 < \gamma$, and $w_{[f,e]} \equiv 0$ if $\lambda_1 \geq \gamma$.
On the other hand, if $\lambda \leq \lambda_1$ or $\mu \leq \lambda_1$, then one of the two species (or both of them) will be driven to extinction. This extinction region was enlarged by López-Gómez and Sabina in [25] (Corollary 4.5) to a region in the (λ,μ)-plane delimited by the curves $\lambda = \lambda_0(\mu)$ and $\mu = \mu_0(\lambda)$. However, if λ and μ satisfy

$$\lambda > \varphi(\mu) \quad \text{and} \quad \mu > \psi(\lambda) \tag{2}$$

where $\varphi(\mu) = \lambda_1(b w_{[\mu,c]})$ and $\psi(\lambda) = \lambda_1(d w_{[\lambda,a]})$, then (1) is permanent (see Cantrell et al. [2], [4], [5] and López-Gómez [24]). We would like to point out that $\lambda = \lambda_1(b w_{[\mu,c]})$ and $\mu = \lambda_1(d w_{[\lambda,a]})$ define two curves in the (λ,μ)-plane whose behaviour is analyzed in detail in [2] and [24]. In Figure 1 we have summarized the autonomous case for particular values of the parameters. In this Figure we have denoted by $P := \{(\lambda,\mu) : \lambda,\, \mu \text{ satisfy } (2)\}$ and by $E := \{(\lambda,\mu) : \lambda < \lambda_0(\mu) \text{ or } \mu < \mu_0(\lambda)\}$.

In the non-autonomous case, previous work focuses on nonlinearities that are periodic in time, or that are bounded by periodic functions. In the first case, the spectral theory still works and similar results to the autonomous case can be obtained, see Hess [14] and Hess and Lazer [15]. The second case was studied

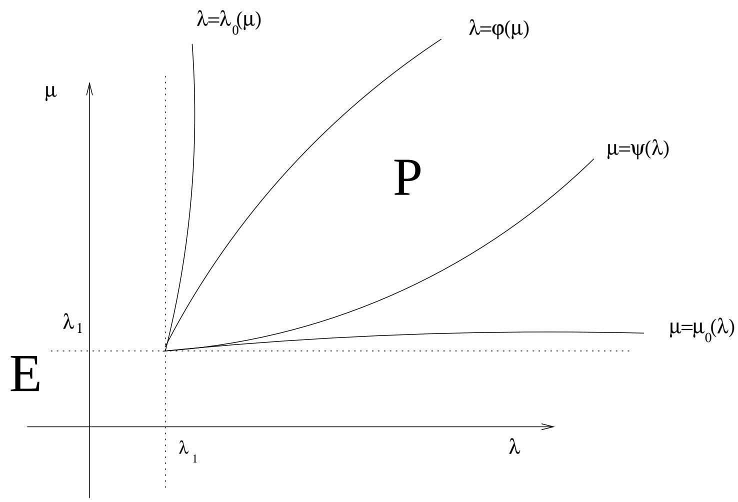

Fig. 1. Autonomous case. E: extinction region, P: permanent region.

by Cantrell and Cosner [3]. In [3] the authors assume that $0 < a_0 \leq a(t) \leq A$ for all $t \geq 0$, and using a comparison method, they show that if λ and μ satisfy

$$\lambda > \lambda_1 + \frac{\mu b}{a_0} \quad \text{and} \quad \mu > \lambda_1 + \frac{\lambda d}{c},$$

then (1) is permanent (Corollary 3.1 in [3]).

In this work, we do not assume that $a(t)$ is bounded below by a positive constant and in fact we are mainly interested in the case $a(t) \to 0$ as $t \to +\infty$. We prove in this case that there is no bounded absorbing set for (1), and so the system is not "permanent" in any conventional sense. In fact, we analyse the forward behaviour in time of (1) in detail and we show that one or both species are driven to extinction when

$$\lambda < \lambda_1 \quad \text{or} \quad \lambda > \varphi(\mu). \tag{3}$$

See Figure 2 where we have represented this case. We have denoted by $E = \{(\lambda, \mu) : \lambda,\, \mu \text{ satisfy } (3)\}$.

The idea of pullback convergence from the theory of random and non-autonomous attractors (cf. Crauel et al. [10], Kloeden and Schmalfuss [19], Schmalfuss [30]) allows us to ask (and answer) other questions about the behaviour of our model (1). In particular we define here a notion of *pullback permanence*: we say that (1) is *pullback permanent* if there exists a time-dependent family of (bounded) absorbing sets that are bounded away from zero in each component. This idea is not intended to replace the standard notion of permanence, but rather to complement it. This definition has an interesting biological interpretation: if we arrive at an island on which two species have already been

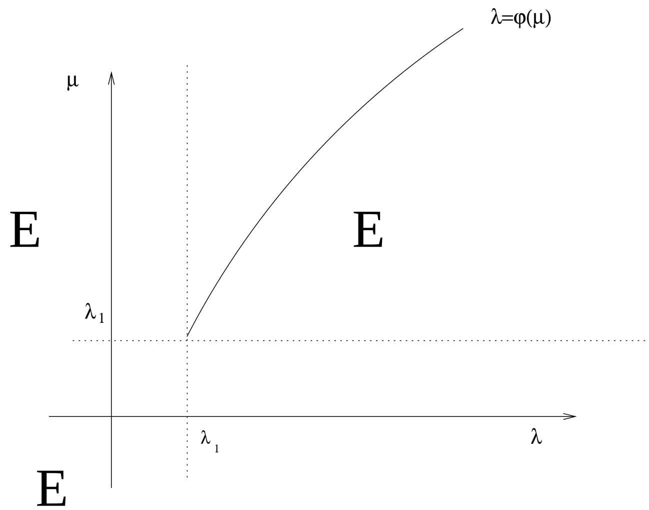

Fig. 2. Forward behaviour in time when $a(t) \to 0$ as $t \to \infty$.

competing (according to our model) for a long time then we can guarantee that neither species will have died out (and their numbers are bounded below in a uniform way, no matter how long this ecology has been running). This is new information, not available by considering the behaviour as $t \to +\infty$: indeed, one might expect from the inevitability of extinction as $t \to \infty$ that such behaviour would not occur.

We get here pullback extinction if $\lambda < \lambda_1$ or $\mu < \lambda_1$. Moreover, assuming that $a(t) \to a_0 > 0$ as $t \to -\infty$ and λ and μ satisfy

$$\lambda > \varphi(\mu) \quad \text{and} \quad \mu > \psi(\lambda, a_0), \tag{4}$$

where $\psi(\lambda, a_0) = \lambda_1(dw_{[\lambda,a_0]})$, then (1) is pullback permanent. We have summarized this in Figure 3, where $E = \{(\lambda, \mu) : \lambda \leq \lambda_1 \text{ or } \mu \leq \lambda_1\}$.

In fact we can give a bit more information about the structure of the pullback attracting states ("the non-autonomous attractor") by using the order-preserving property of (1) (we define an appropriate order in section 3, cf. [15], for example): a result due to Langa and Suárez [22] shows that (1) possesses two trajectories, maximal and minimal, that are globally stable from above and below respectively.

Finally, we should mention the use of skew product flows (Hale [11], Sell [29]) in studying non-autonomous problems, particularly in the periodic, quasi-periodic, or almost periodic case. The idea is to construct an autonomous semiflow $S(t)$ on the product space $H \times \mathcal{F}$, where H is the natural phase space where the dynamics take place (here the dynamics of u and v) and $\mathcal{F}$ is the *hull* (see [29]) of all the time dependent terms of the equation. Provided that $\mathcal{F}$ is compact in some appropriate topology the general theory of dissipative dynamical systems can be applied to study $S(t)$ on the space

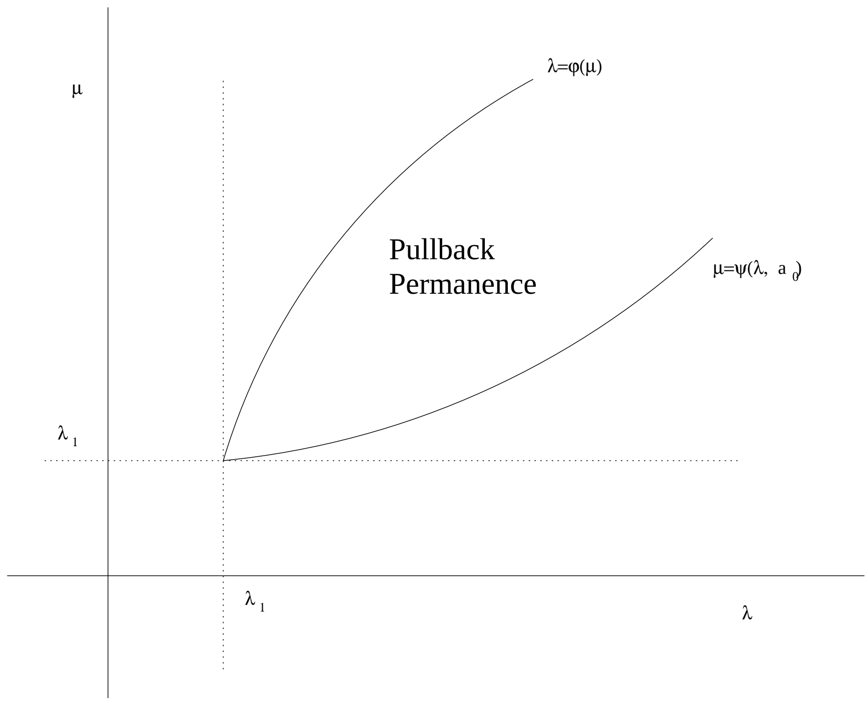

Fig. 3. Non-autonomous case. E: pullback extinction region.

$H \times \mathcal{F}$. However, with an entirely general non-autonomous term there is no clear choice of topology on $\mathcal{F}$ that will make it compact, a property crucial to this approach [3]. This is highlighted in the theory of attractors for non-autonomous equations developed by Chepyzhov & Vishik (see, for example, [6] and [7]): while their strongest results require almost periodicity, precisely in order to obtain a compact $\mathcal{F}$, they study general non-autonomous terms without appealing to skew product flows using the concepts of a "kernel" and "kernel sections", the latter corresponding exactly to the time slices $\mathcal{A}(t)$ of the non-autonomous attractor whose definition we recall below.

An outline of this paper is as follows: in Section 2 we introduce the concept of a process, give the definition of a non-autonomous attractor and state conditions that guarantee its existence. In Section 3 we study properties of order-preserving processes and in particular recall a result about their stability. In Section 4 we study in detail a non-autonomous logistic equation which governs the behaviour of one of the species in absence of the other: this section plays a crucial role throughout all that follows. In Section 5 we analyse both the forwards and pullback behaviour of system (1), and finish in section 6 with the existence of a non-autonomous attractor for (1) and conditions for

[3] Using uniform convergence on $\mathbb{R}$ requires almost periodicity. An interesting extension should be possible under the assumption that the non-autonomous terms enjoy a uniform modulus of continuity over $\mathbb{R}$, for then the topology of uniform convergence on compact subsets of $\mathbb{R}$ will make $\mathcal{F}$ compact, cf. Johnson & Kloeden [17], and the recent monograph by Chepyzhov & Vishik [8].

pullback permanence.

2 Non-autonomous attractors

In this section we introduce the definitions of a non-autonomous attractor and of pullback permanence.
Let (X, d) be a complete metric space (with metric d) and $\{S(t,s)\}_{t\geq s}$, $t, s \in \mathbb{R}$ be a family of mappings satisfying:

a) $S(t,s)S(s,\tau)u = S(t,\tau)u$, for all $\tau \leq s \leq t$, $u \in X$,
b) $S(t,\tau)u$ is continuous in t, τ and u.
c) $S(t,t)$ is the identity in X for all $t \in \mathbb{R}$.

Such a map is called a *process*. Usually $S(t,\tau)u$ will arise as the value of the solution of a non-autonomous equation at time t with "initial condition" u at time τ. As remarked in the introduction, for an autonomous equation the solutions only depend on $t-\tau$, and we can write $S(t,\tau) = S(t-\tau, 0)$.
Let $\mathcal{D}$ be a non-empty set of parameterized families of non-empty bounded sets $\{D(t)\}_{t\in\mathbb{R}}$. In particular, $D(t) \equiv B \in \mathcal{D}$, where $B \subset X$ is a bounded set. In what follows, we will consider a fixed *base of attraction* $\mathcal{D}$ and throughout our analysis the concepts of absorption and attraction will be referred to this fixed base.

For $A, B \subset X$ define the Hausdorff semidistances as,

$$\mathrm{dist}(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b) \qquad \mathrm{Dist}(A,B) = \inf_{a\in A} \inf_{b\in B} d(a,b).$$

Definition 1 *a) Given $t_0 \in \mathbb{R}$, we say that $K(t_0) \subset X$ is attracting at time t_0 if for every $\{D(t)\} \in \mathcal{D}$*

$$\lim_{\tau\to-\infty} \mathrm{dist}(S(t_0,\tau)D(\tau), K(t_0)) = 0.$$

A family $\{K(t)\}_{t\in\mathbb{R}}$ is attracting if $K(t_0)$ is attracting at time t_0, for all $t_0 \in \mathbb{R}$.

b) Given $t_0 \in \mathbb{R}$, we say that $B(t_0) \subset X$ is absorbing at time t_0 if for every $\{D(t)\} \in \mathcal{D}$ there exists $T = T(t, D) \in \mathbb{R}$ such that

$$S(t_0,\tau)D(\tau) \subset B(t_0), \text{ for all } \tau \leq T.$$

A family $\{B(t)\}_{t\in\mathbb{R}}$ is absorbing if $B(t_0)$ is absorbing at time t_0, for all $t_0 \in \mathbb{R}$.

Note that every absorbing set at time t_0 is attracting.
As discussed in the introduction, this notion takes the final time as fixed

and moves the initial time backwards towards $-\infty$. We are not evolving one trajectory backwards in time, but rather we consider the current state of the system (at the fixed time t_0) which would result from the same initial condition starting at earlier and earlier times. This is called *pullback attraction* in the literature (cf. [18], [19], [30]).

Definition 2 *Let $\{B(t)\}_{t\in\mathbb{R}}$ be a family of subsets of X. This family is said to be invariant with respect to the process S if*

$$S(t,\tau)B(\tau)=B(t),\ \textit{for all}\ \ (\tau,t)\in\mathbb{R}^2,\tau\le t.$$

Note that this property is a generalization of the classical property of an invariant set for a semigroup. However, in this case we have to define the invariance with respect to a family of sets depending on a parameter.

Definition 3 *The family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be the global non-autonomous (or pullback) attractor associated to the process S if it is invariant, attracts every $\{D\,(t)\}\in\mathcal{D}$ (for all $t_0\in\mathbb{R}$) and minimal in the sense that if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t)\subset C(t)$ for all $t\in\mathbb{R}$.*

The general result on the existence of non-autonomous attractors is a generalization of the abstract theory for autonomous dynamical systems (Temam [32], Hale [11]):

Theorem 4 *(Crauel et al. [10], Schmalfuss [30]) Assume that there exists a family of compact absorbing sets. Then, the family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ defined by*

$$\mathcal{A}(t)=\overline{\cup_{D\in\mathcal{D}}\Lambda(D,t)}$$

is the global non-autonomous attractor, where $\Lambda(D,t)$ is the omega-limit set at time t of $D\equiv\{D\,(t)\}\in\mathcal{D}$,

$$\Lambda(D,t_0)=\cap_{s\le t_0}\overline{\cup_{\tau\le s}S(t_0,\tau)D(\tau)}.$$

Using the pullback idea introduced above we can now give the following definition of "pullback permanence". As in [5] we suppose that $X=X_0\cup\partial X_0$, where X_0 is open, and X_0, ∂X_0 are invariant with respect to the process S. In our application, ∂X_0 will be the set of solutions with at least one component identically zero.

Definition 5 *We say that a system has the property of pullback permanence (or that it is permanent in the pullback sense) if there exists a time-dependent* *family of bounded sets* $U : \mathbb{R} \longmapsto X$*, satisfying*

a) $U(t)$ *absorbs every bounded set* $D \subset X$ *(cf. Definition 1).*
b) $\mathrm{Dist}(U(t), \partial X_0) > 0$ *for all* $t \in \mathbb{R}$*.*

Following Definition 3, we can define a global attractor $\mathcal{A}_+ \subset X_0$ that attracts every bounded set in X_0: its existence follows using Theorem 4.

3 Order-preserving non-autonomous differential equations

In this section we define what it means for a process to be order-preserving. For such a process we can determine some of the structure of the non-autonomous attractor and prove the existence of a minimal and maximal trajectory on the attractor with some particular stability properties.

Definition 6 *We say that the process* $\{S(t,s) : X \to X\}_{t \geq s}$ *is order-preserving if there exists an order relation '*$\preceq$*' in* X *such that, if* $w_1 \preceq w_2$*, then* $S(t,s)w_1 \preceq S(t,s)w_2$*, for all* $t \geq s$*.*

The next definition generelizes the concept of equilibria in Hess [14], (see also Arnold and Chueshov [1] in the stochastic case and Chueshov [9] in the non-autonomous case under stronger conditions).

Definition 7 *Let* S *be an order-preserving process. We call the continuous map* $w : \mathbb{R} \to X$ *a complete trajectory if, for all* $s \in \mathbb{R}$*, we have*

$$S(t,s)w(s) = w(t), \text{ for } t \geq s.$$

From $(\underline{w}, \overline{w})$ such that $\underline{w}(t) \preceq \overline{w}(t)$, for all $t \in \mathbb{R}$, we can define the "interval"

$$I_{\underline{w}}^{\overline{w}}(t) = \{w \in X : \ \underline{w}(t) \preceq w \preceq \overline{w}(t)\}.$$

The following result was proved in [22] and it gives sufficient conditions for the existence of upper and lower asymptotically stable complete trajectories, and provides some information about the structure of the non-autonomous attractor.

Theorem 8 *Let* S *be an order-preserving process and* $\mathcal{A}(t)$ *its associated pull-back attractor attracting time-dependent families of sets in a base of attraction* $\mathcal{D}$*. Let* $\underline{w}$*,* $\overline{w} \in \mathcal{D}$ *be such that* $\underline{w}(t) \preceq \overline{w}(t)$*, for all* $t \in \mathbb{R}$*, and assume that*

$$\mathcal{A}(t) \subset I_{\underline{w}}^{\overline{w}}(t), \quad \forall t \in \mathbb{R}.$$

Then there exist two trajectories $w_*(t)$, $w^*(t) \in \mathcal{A}(t)$ *such that*

i) $w_*(t) \preceq w \preceq w^*(t)$, $\forall t \in \mathbb{R}$ *and* $\forall w \in \mathcal{A}(t)$.

ii) w_* (w^*) *is minimal (maximal) in the sense that there is no complete trajectory in the interval* $I_{\underline{w}}^{w_*}(I_{w^*}^{\overline{w}})$.

iii) $w_*(t)$ *is globally asymptotically stable from below, that is, for all* $z \in \mathcal{D}$ *with* $\underline{w}(t) \preceq z(t) \preceq w_*(t)$, *for all* $t \in \mathbb{R}$, *we have*

$$\lim_{s\to-\infty} d(S(t,s)z(s), w_*(t)) = 0.$$

$w^*(t)$ *is globally asymptotically stable from above, that is, for all* $z \in \mathcal{D}$ *with* $w^*(t) \preceq z(t) \preceq \overline{w}(t)$, *for all* $t \in \mathbb{R}$, *we have*

$$\lim_{s\to-\infty} d(S(t,s)z(s), w^*(t)) = 0.$$

4 The non-autonomous logistic equation

In the absence of one species, the evolution of the other is given by the non-autonomous logistic equation

$$\begin{cases} w_t - \Delta w = q(x,t)w - b(t)w^2 & \text{in } \Omega \times (s, +\infty), \\ w = 0 & \text{on } \partial\Omega \times (s, +\infty), \\ w(x,s) = w_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where $q \in L^\infty(\Omega \times (s,\infty))$ and $0 < b(t) \le B$ for all $t \in \mathbb{R}$.
Firstly, we introduce some results which will be very useful. Given $f \in L^\infty(\Omega)$ we denote by $\lambda_1(f)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta w + f(x)w = \sigma w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

and by $\varphi_1(f)$ the unique positive eigenfunction such that $\|\varphi_1(f)\|_\infty = 1$. It is well known that $\lambda_1(f)$ is increasing and continuous in f, decreasing and continuous in Ω and that if $f(x) > 0$ in Ω then (see Theorem 6.4 in [23])

$$\lim_{\beta\to\infty} \lambda_1(\beta f) = \infty \tag{7}$$

We denote $\lambda_1 := \lambda_1(0)$. Finally, given $f \in L^\infty(\Omega)$ and $e \in \mathbb{R}$, $e > 0$, we consider the nonlinear elliptic equation

$$\begin{cases} -\Delta w = f(x)w - ew^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

The next result collects the main results on the existence and uniqueness of a positive solution for (8) and some important properties of this solution.

Lemma 9 *Problem* (8) *possesses a positive solution if, and only if,* $\lambda_1(-f) < 0$. *Furthermore, if such a solution exists then it is unique: we denote it by* $w_{[f,e]}$ *and set* $w_{[f,e]} \equiv 0$ *if* $\lambda_1(-f) \geq 0$. *In addition,*

a) $w_{[f,e]}$ *is bounded below:*

$$-\frac{\lambda_1(-f)}{e}\varphi_1(-f) \leq w_{[f,e]} \quad \text{in } \Omega, \text{ and} \tag{9}$$

b) The maps $f \in L^\infty(\Omega) \mapsto w_{[f,e]}$ *and* $e \in (0, \infty) \mapsto w_{[f,e]}$ *are continuous.*

PROOF. The existence and uniqueness of a positive solution are well-known, see for instance [14]. Observe that the pair

$$(\underline{w}, \overline{w}) = \left(-\frac{\lambda_1(-f)}{e}\varphi_1(-f), \frac{f_M}{e}\right)$$

is a sub-supersolution of (8), where $f_M := \operatorname{ess\,sup}_{x\in\Omega} f(x)$. Indeed, it is not hard to prove that $\underline{w}$ and $\overline{w}$ satisfy the following inequalities

$$-\Delta\underline{w} \leq f(x)\underline{w} - e\underline{w}^2, \qquad -\Delta\overline{w} \geq f(x)\overline{w} - e\overline{w}^2, \quad \text{in } \Omega,$$

and

$$\underline{w} = -\frac{\lambda_1(-f)}{e}\varphi_1(-f) \leq -\frac{\lambda_1(-f_M)}{e} = \frac{f_M - \lambda_1}{e} < \frac{f_M}{e} = \overline{w}.$$

Thus,

$$-\frac{\lambda_1(-f)}{e}\varphi_1(-f) \leq w_{[f,e]} \leq \frac{f_M}{e} \tag{10}$$

from which (9) follows. Now, by (10), the continuity of the maps $f \mapsto w_{[f,e]}$ and $e \mapsto w_{[f,e]}$ can be obtained. $\square$

The following result provides us with the existence and uniqueness of positive solution for (5), as well as its "forward" and "pullback" asymptotic behaviour. We consider the Banach space

$$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \quad \text{on } \partial\Omega\}$$

ordered by its positive cone $P := \{u \in C_0(\overline{\Omega}) : u \geq 0 \text{ in } \Omega\}$.

Proposition 10 *Given $w_0 \in P\backslash\{0\}$, there exists a unique positive solution of (5), denoted by $\theta_{[q,b]}(t,s;w_0)$, which is strictly positive for $t > s$. Moreover:*

a) $\theta_{[q,b]}(t,s;w_0)$ is increasing in q and decreasing in b.

Now, assume that $q(x,t) \equiv q(x)$. Then,

b) If $b(t) = b_0 > 0$, then $\|\theta_{[q,b_0]}(t,s;w_0) - w_{[q,b_0]}\|_\infty \to 0$ as $t \to \infty$ or $s \to -\infty$.
c) If $\lambda_1(-q) > 0$, then $\|\theta_{[q,b]}(t,s;w_0)\|_\infty \to 0$ as $t \to \infty$ or $s \to -\infty$.
d) If $\lambda_1(-q) < 0$ and $b(t) \to 0$ when $t \to \infty$, then $\|\theta_{[q,b]}(t,s;w_0)\|_\infty \to \infty$ as $t \to \infty$.
e) Given $t \in \mathbb{R}$ and $\lambda_1(-q) < 0$, there exist $V \in C(\overline{\Omega})$, $V > 0$, and $T(t,w_0)$ such that

$$V(x) \leq \theta_{[q,b]}(t,s;w_0) \leq r(t) \qquad \text{for any } s \leq T(t,w_0), \tag{11}$$

where

$$r(t) = \frac{\mathrm{e}^{\|q\|_\infty t}}{\frac{1}{2}\int_{-\infty}^{t} \mathrm{e}^{\|q\|_\infty \tau} b(\tau)\,\mathrm{d}\tau}.$$

PROOF. The existence and uniqueness follow in a standard way. The positivity of the solution for $t > s$ follows by the strong maximum principle for parabolic equations.
For part a), take $q_1(x,t) \leq q_2(x,t)$ for all $x \in \Omega, t \geq s$. Then, $\theta_{[q_1,b]}(t,s;w_0)$ is a subsolution of (5) with $q = q_2$, and so by the uniqueness of the solution it follows that

$$\theta_{[q_1,b]}(t,s;w_0) \leq \theta_{[q_2,b]}(t,s;w_0).$$

A similar reasoning shows the monotony with respect to b.
Part b) has been proved, for instance, in [4] Lemma 5.1 when $t \to \infty$. As we have indicated before in the autonomous case

$$\theta_{[q,b_0]}(t,s;w_0) = \theta_{[q,b_0]}(t-s,0;w_0),$$

and so the result follows when $s \to -\infty$.
For part c), since $\lambda_1(-q) > 0$ and by the continuity of the principal eigenvalue with respect to the domain, there exists a regular domain Ω_1 such that $\Omega \subset \overline{\Omega} \subset \Omega_1$ and

$$0 < \lambda_1^{\Omega_1}(-q) < \lambda_1(-q),$$

where $\lambda_1^{\Omega_1}(-q)$ stands for the principal eigenvalue of (6) in Ω_1 with $f = -q$. We denote by $\varphi_1^{\Omega_1}(-q)$ its associated positive eigenfunction and take $\overline{w} := K\mathrm{e}^{\gamma(t-s)}\varphi_1^{\Omega_1}(-q)$. Then, $\overline{w}$ is a supersolution of (5) provided that

$$K\varphi_1^{\Omega_1}(-q) \geq w_0 \quad \text{in } \Omega,$$
$$\gamma + \lambda_1^{\Omega_1}(-q) + Kb(t)\mathrm{e}^{\gamma(t-s)}\varphi_1^{\Omega_1}(-q) \geq 0 \quad \text{in } \Omega \times (s, \infty).$$

We can take K sufficiently large and $-\lambda_1^{\Omega_1}(-q) \leq \gamma < 0$, and so

$$\theta_{[q,b]}(t, s; w_0) \leq K\mathrm{e}^{\gamma(t-s)}\varphi_1^{\Omega_1}(-q)$$

whence the result is obtained.
We now prove d). Fix $M > 0$ and $\lambda_1(-q) < 0$. Taking

$$\varepsilon := -\frac{\lambda_1(-q)}{2M},$$

since $b(t) \to 0$ as $t \to \infty$, there exists $t_\varepsilon \in \mathbb{R}$ such that for any $t \geq t_\varepsilon$

$$b(t) \leq \varepsilon.$$

Observe that,

$$\theta_{[q,b]}(t, s; w_0) = \theta_{[q,b]}(t, t_\varepsilon; z_{\varepsilon,s}) \tag{12}$$

where

$$z_{\varepsilon,s} = \theta_{[q,b]}(t_\varepsilon, s; w_0).$$

Now, by part a) we have that

$$\theta_{[q,b]}(t, t_\varepsilon; z_{\varepsilon,s}) \geq \theta_{[q,\varepsilon]}(t, t_\varepsilon; z_{\varepsilon,s}) \quad \text{for } t \geq t_\varepsilon. \tag{13}$$

By part b), there exists $t_1 \in \mathbb{R}$ such that for $t \geq t_1$, we get

$$\theta_{[q,\varepsilon]}(t, t_\varepsilon; z_{\varepsilon,s}) \geq w_{[q,\varepsilon]} + \frac{\lambda_1(-q)}{2\varepsilon} \geq -\frac{\lambda_1(-q)}{\varepsilon}\varphi_1(-q) + \frac{\lambda_1(-q)}{2\varepsilon},$$

this last inequality thanks to (9). Therefore, by (12) and (13), for $t \geq t_1$ we get

$$\theta_{[q,b]}(t, s; w_0) \geq -\frac{\lambda_1(-q)}{\varepsilon}\varphi_1(-q) + \frac{\lambda_1(-q)}{2\varepsilon},$$

and so, since $\|\varphi_1(-q)\|_\infty = 1$, we obtain

$$\|\theta_{[q,b]}(t, s; w_0)\|_\infty \geq M.$$

This completes part d).
For part e), since $b(t) \leq B$ for all $t \in \mathbb{R}$, it follows that

$$\theta_{[q,B]}(t - s, 0; w_0) = \theta_{[q,B]}(t, s; w_0) \leq \theta_{[q,b]}(t, s; w_0)$$

and the existence of a positive function V follows by part b). On the other hand,

$$\overline{w} := y(t, s; \|w_0\|_\infty)$$

is a supersolution of (5), where $y(t, s; y_0)$ is the solution of

$$y' = \|q\|_\infty y - b(t)y^2, \qquad y(s) = y_0$$

which is given explicitly by

$$y(t, s; y_0) = \frac{\mathrm{e}^{\|q\|_\infty (t-s)}}{\frac{1}{y_0} + \int_s^t \mathrm{e}^{\|q\|_\infty (r-s)} b(r)\,\mathrm{d}r}.$$

Now, it suffices to let $s \to -\infty$. This completes the proof. $\square$

Proposition 10 provides us with a complete description of the long-time behaviour of the positive solution of (5). In the autonomous case, part b), $w_{[q,b_0]}$ is globally asymptotically stable, and so the species is driven to extinction when $\lambda_1(-q) \geq 0$ and (5) is permanent when $\lambda_1(-q) < 0$.
In the non-autonomous case, the species is driven to extinction in the "forward" and "pullback" senses when $\lambda_1(-q) > 0$. However, when $\lambda_1(-q) < 0$

there is a drastic change of behaviour: by part d), we cannot expect forward permanence, whereas in [22] it was proved that for $\lambda_1(-q) < 0$ equation (5) is permanent in the pullback sense.

5 Non-autonomous Lotka-Volterra competition model

Our first result in this section guarantees the existence and uniqueness of a positive solution of (1) and provides some helpful estimates.

Theorem 11 *Given $u_0, v_0 \in P\backslash\{0\}$, there exists a unique positive solution of (1), denoted by $(u(t,s;u_0,v_0)$,$v(t,s;u_0,v_0))$, which is strictly positive for $t > s$. Moreover,*

$$\theta_{[\lambda - b\theta_{[\mu,c]},a]} \leq u \leq \theta_{[\lambda,a]} \qquad \theta_{[\mu - d\theta_{[\lambda,a]},c]} \leq v \leq \theta_{[\mu,c]}. \tag{14}$$

PROOF. We take

$$(\underline{u},\overline{u}) := (\theta_{[\lambda - b\theta_{[\mu,c]},a]}, \theta_{[\lambda,a]}) \quad (\underline{v},\overline{v}) := (\theta_{[\mu - d\theta_{[\lambda,a]},c]}, \theta_{[\mu,c]}).$$

Firstly, by Proposition 10 a), it is clear that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$. Moreover, it is not hard to prove that this couple satisfies

$$\underline{u}_t - \Delta\underline{u} = \underline{u}(\lambda - a(t)\underline{u} - b\overline{v}), \quad \overline{u}_t - \Delta\overline{u} = \overline{u}(\lambda - a(t)\overline{u}) \geq \overline{u}(\lambda - a(t)\overline{u} - b\underline{v}),$$
$$\underline{v}_t - \Delta\underline{v} = \underline{v}(\mu - c\underline{v} - d\overline{u}), \quad \overline{v}_t - \Delta\overline{v} = \overline{v}(\mu - c\overline{v}) \geq \overline{v}(\mu - c\overline{v} - d\underline{u}).$$

Thus the existence of a positive solution of (1) follows from Theorem 8.3.2 in [31]. Uniqueness follows by a standard argument to complete the proof. $\square$

5.1 Asymptotic behaviour forward in time

The asymptotic behaviour of (1) depends on the values of λ and μ. The next result shows cases where the trivial solution and the semi-trivial one are globally asymptotically stable, and so at least one species is driven to extinction.

Proposition 12 *Suppose $\lambda < \lambda_1$.*

a) If $\mu \leq \lambda_1$, then $(u(t,s;u_0,v_0), v(t,s;u_0,v_0)) \to (0,0)$ as $t \to \infty$.
b) If $\mu > \lambda_1$, then $(u(t,s;u_0,v_0), v(t,s;u_0,v_0)) \to (0, w_{[\mu,c]})$ as $t \to \infty$.

PROOF. If $\lambda < \lambda_1$, then observe that $\lambda_1(-\lambda) = \lambda_1 - \lambda > 0$. Hence, from (14) and Proposition 10 c) we get that $u(t, s; u_0, v_0) \to 0$ as $t \to \infty$. Similarly, when $\mu \leq \lambda_1$ we get that $v(t, s; u_0, v_0) \to 0$ as $t \to \infty$.
Now, we assume $\mu > \lambda_1$. Let $\delta > 0$ be such that $\mu > \lambda_1 + d\delta$. For such δ there exists $t_0 \in \mathbb{R}$ such that

$$\|u(t, s; u_0, v_0)\|_\infty < \delta \qquad \text{for any } t \geq t_0.$$

On the other hand, using the definition of $\theta_{[q,b]}$ we obtain

$$u = \theta_{[\lambda - bv, a]} \quad \text{and} \quad v = \theta_{[\mu - du, c]}. \tag{15}$$

Then, by (14) and Proposition 10 a), we have

$$\theta_{[\mu - d\delta, c]} \leq \theta_{[\mu - du, c]} = v \leq \theta_{[\mu, c]}, \qquad \text{for } t \geq t_0.$$

It is sufficient to apply Proposition 10 b) and the continuity of the map $f \mapsto w_{[f,e]}$. $\square$

The following result shows that the system is not permament when λ and μ satisfy an easily verifiable condition. The system is not permanent because one species (u) increases indefinitely and drives the other to extinction.

We note here that although under the condition $a(t) \to 0$ the equation is "asymptotically autonomous" in the sense of Markus [26] (see also more recent works by Thieme [33], Mischaikow et al. [27]) the general results that are available for such systems are not sufficiently detailed to give us all the information we need: for example, it is known that if all the solutions of the limit equation are unbounded then so are the solutions of the non-autonomous equation [26], but we wish to show that while one species grows without bound the other is driven to extinction.

Proposition 13 *Suppose $a(t) \to 0$ as $t \to \infty$. If $\lambda > \lambda_1(bw_{[\mu,c]})$, then*

$$(u(t, s; u_0, v_0), v(t, s; u_0, v_0)) \to (\infty, 0) \qquad \text{as } t \to \infty.$$

Observe that $w_{[\mu,c]} = 0$ if $\mu \leq \lambda_1$, so $\lambda > \lambda_1(bw_{[\mu,c]})$ means $\lambda > \lambda_1$ when $\mu \leq \lambda_1$.

PROOF. Assume $\mu \leq \lambda_1$, then by Proposition 10 c) we have that $v \leq \theta_{[\mu,c]} \to 0$ as $t \to \infty$. Moreover, since $\lambda > \lambda_1$, we can obtain that

$$\lambda - b\theta_{[\mu,c]} > \lambda_1 \qquad \text{for } t \geq t_1.$$

Hence,

$$\lambda_1(-\lambda + b\theta_{[\mu,c]}) < \lambda_1(-\lambda_1) = 0,$$

and so, by Proposition 10 d)

$$\theta_{[\lambda - b\theta_{[\mu,c]},a]} \to \infty,$$

and the result follows by (14).
Now, suppose $\mu > \lambda_1$ and $\lambda > \lambda_1(bw_{[\mu,c]})$. We define

$$\varepsilon := \frac{\lambda - \lambda_1(bw_{[\mu,c]})}{2b}$$

Since $v \leq \theta_{[\mu,c]} \to w_{[\mu,c]}$ as $t \to \infty$, then there exists t_ε such that for $t \geq t_\varepsilon$

$$v \leq w_{[\mu,c]} + \varepsilon,$$

and so, by (15)

$$u = \theta_{[\lambda - bv,a]} \geq \theta_{[\lambda - b(w_{[\mu,c]}+\varepsilon),a]}, \quad \text{for } t \geq t_\varepsilon.$$

Since, $a(t) \to 0$ as $t \to \infty$, given $\delta \in (0,1]$ there exists t_δ such that for $t \geq t_\delta$ we have $a(t) \leq \delta$, and so,

$$u \geq \theta_{[\lambda - b(w_{[\mu,c]}+\varepsilon),a]} \geq \theta_{[\lambda - b(w_{[\mu,c]}+\varepsilon),\delta]}, \quad t \geq \max\{t_\varepsilon, t_\delta\}. \tag{16}$$

Now, observe that

$$\lambda_1(-\lambda + bw_{[\mu,c]} + b\varepsilon) = \lambda_1(bw_{[\mu,c]}) - \lambda + b\varepsilon = -\frac{\lambda - \lambda_1(bw_{[\mu,c]})}{2} < 0. \tag{17}$$

Taking account (16) and (17), a similar argument to the used in the proof of Proposition 10 d) shows that given a small positive $\sigma > 0$ there exists t_σ such that for $t \geq t_\sigma$, we have

$$u \geq \Phi := \frac{\lambda - \lambda_1(bw_{[\mu,c]})}{2\delta}\varphi_1(-\lambda + b(w_{[\mu,c]} + \varepsilon)) - \sigma. \tag{18}$$

Taking σ such that

$$0 < \sigma < \frac{\lambda - \lambda_1(bw_{[\mu,c]})}{4} \leq \frac{\lambda - \lambda_1(bw_{[\mu,c]})}{4\delta} \tag{19}$$

we get that

$$\|u\|_\infty \geq \|\Phi\|_\infty \geq \frac{\lambda - \lambda_1(bw_{[\mu,c]})}{4\delta}.$$

Hence, it is sufficient to take δ sufficiently small in order to show that u approaches infinity.
Finally, observe that by (18) we get

$$v = \theta_{[\mu - du, c]} \leq \theta_{[\mu - d\Phi, c]}, \quad t \geq t_\sigma,$$

and if we can take $\mu < \lambda_1(d\Phi)$, by Proposition 10 b) we obtain that v goes to 0. But, $\mu < \lambda_1(d\Phi)$ is equivalent to

$$\mu < \lambda_1 \left(\frac{\lambda - \lambda_1(bw_{[\mu,c]})}{2\delta} \varphi_1(-\lambda + b(w_{[\mu,c]} + \varepsilon)) \right) - \sigma,$$

which is true by (19) and (7) taking δ sufficiently small. This completes the proof. $\square$

5.2 Pullback asymptotic behaviour

The next two results show "pullback" extinction for some values of λ and μ. The first one is similar to Proposition 12 and so we omit the proof.

Proposition 14 *Suppose $\lambda < \lambda_1$.*

a) If $\mu \leq \lambda_1$, then $(u(t,s;u_0,v_0), v(t,s;u_0,v_0)) \to (0,0)$ as $s \to -\infty$.
b) If $\mu > \lambda_1$, then $(u(t,s;u_0,v_0), v(t,s;u_0,v_0)) \to (0, w_{[\mu,c]})$ as $s \to -\infty$.

Hereafter, we denote $\mathbf{A} : D(\mathbf{A}) \mapsto C_0(\overline{\Omega})$ the linear operator associated to the Laplacian.

Proposition 15 *Given $t \in \mathbb{R}$, $\lambda > \lambda_1$ and $\mu \leq \lambda_1$, then*

$$(u(t,s;u_0,v_0), v(t,s;u_0,v_0)) \to (\theta_{[\lambda,a]}(t,s;u_0), 0) \quad \text{as } s \to -\infty.$$

PROOF. Since $\mu \leq \lambda_1$, then $v \leq \theta_{[\mu,c]} \to 0$ as $s \to -\infty$. Now, given $\delta > 0$ there exists s_δ such that

$$v(t,s;u_0,v_0) \leq \delta \qquad \text{for } s \leq s_\delta.$$

Hence, by (15), we get

$$\theta_{[\lambda-b\delta,a]} \leq \theta_{[\lambda-bv,a]} = u \leq \theta_{[\lambda,a]}, \qquad \text{for } s \leq s_\delta,$$

and so,

$$\theta_{[\lambda-b\delta,a]} - \theta_{[\lambda,a]} \leq u - \theta_{[\lambda,a]} \leq 0, \qquad \text{for } s \leq s_\delta.$$

Thus, it suffices to prove that

$$w_\delta := \theta_{[\lambda-b\delta,a]} - \theta_{[\lambda,a]} \to 0, \qquad \text{as } \delta \to 0. \tag{20}$$

It is not hard to prove that w_δ satisfies

$$(w_\delta)_t - \Delta w_\delta = \lambda w_\delta - b\delta\theta_{[\lambda-b\delta,a]} - a(t) w_\delta(\theta_{[\lambda-b\delta,a]} + \theta_{[\lambda,a]}).$$

Now, if we denote by

$$g_\delta(r,s) = \lambda - a(r)(\theta_{[\lambda-b\delta,a]}(r,s;u_0) + \theta_{[\lambda,a]}(r,s;u_0))$$

and writing w_δ from the variation of constants formula, we obtain

$$w_\delta(t,s;u_0) = \int_s^t \mathrm{e}^{-\mathbf{A}(t-r)}(g_\delta(r,s)w_\delta(r,s;u_0) - b\delta\theta_{[\lambda-b\delta,a]}(r,s;u_0))\,\mathrm{d}r,$$

and so, since $\left\|\mathrm{e}^{-\mathbf{A}(t-r)}\right\|_{op} \leq 1$, we get

$$\|w_\delta(t,s;u_0)\|_\infty \leq \int_s^t \|g_\delta(r,s)\|_\infty \|w_\delta(r,s;u_0)\|_\infty \,\mathrm{d}r + b\delta \int_s^t \|\theta_{[\lambda-b\delta,a]}(r,s;u_0)\|_\infty \,\mathrm{d}r,$$

and by Gronwall's lemma we obtain

$$\|w_\delta(t,s;u_0)\|_\infty \leq b\delta \int_s^t \|\theta_{[\lambda-b\delta,a]}(r,s;u_0)\|_\infty \,\mathrm{d}r \cdot \mathrm{e}^{\int_s^t \|g_\delta(r,s)\|_\infty \,\mathrm{d}r}. \tag{21}$$

On the other hand, by Proposition 10 we have

$$\|\theta_{[\lambda-b\delta,a]}(t,s;u_0)\|_\infty \leq \|\theta_{[\lambda,a]}(t,s;u_0)\|_\infty \leq r(t) \qquad \text{for } s \leq T(t),$$

for some $T(t)$ and $r(t)$ independent of δ. Now, (20) follows by taking δ to zero in (21). $\square$

The next result shows that for a fixed final time t_0, the positive solution of (1) is bounded away by positive functions for s sufficiently small.

Proposition 16 *Fix $t_0 \in \mathbb{R}$. Assume that*

$$\inf_{s\in(-\infty,t_0]} a(s) = \alpha(t_0) > 0,$$

$$\lambda > \lambda_1(bw_{[\mu,c]}), \qquad \textit{and} \qquad \mu > \lambda_1(dw_{[\lambda,\alpha(t_0)]}).$$

Then, there exist $s_0 \le t_0$ and $e_i \in C_0(\overline{\Omega})$ positive functions (depending on t_0), such that for all $s \le s_0$:

$$u(t_0, s; u_0, v_0) \ge e_1 \quad \textit{and} \quad v(t_0, s; u_0, v_0) \ge e_2.$$

PROOF. Since $\alpha(t_0) \le a(t) \le A$ for all $t \le t_0$, we have

$$\theta_{[\lambda,A]}(t_0, s; u_0) \le \theta_{[\lambda,a]}(t_0, s; u_0) \le \theta_{[\lambda,\alpha(t_0)]}(t_0, s; u_0) \quad \text{for } s \le t_0.$$

Since $\lambda > \lambda_1(bw_{[\mu,c]})$, $\mu > \lambda_1(dw_{[\lambda,\alpha(t_0)]})$, we can choose $\varepsilon > 0$ sufficiently small such that

$$\lambda > \lambda_1(b(w_{[\mu,c]} + \varepsilon)), \quad \text{and} \quad \mu > \lambda_1(d(w_{[\lambda,\alpha(t_0)]} + \varepsilon)). \tag{22}$$

For such $\varepsilon > 0$, and by Proposition 10 b), we obtain

$$w_{[\lambda,A]} - \varepsilon \le \theta_{[\lambda,a]}(t_0, s; u_0) \le w_{[\lambda,\alpha(t_0)]} + \varepsilon \quad \text{for } s \le s_0,$$

for some s_0. Using again Proposition 10 a) and (14), we get

$$\theta_{[\mu - d(w_{[\lambda,\alpha(t_0)]}+\varepsilon),c]} \le v, \qquad \text{for } s \le s_0. \tag{23}$$

On the other hand, by Proposition 10 a)

$$\theta_{[\lambda - b\theta_{[\mu,c]},A]} \le \theta_{[\lambda - b\theta_{[\mu,c]},a]} \le u$$

and by part b),

$$w_{[\mu,c]} - \varepsilon \le \theta_{[\mu,c]}(t_0, s; u_0) \le w_{[\mu,c]} + \varepsilon \quad \text{for } s \le s_0,$$

and so,

$$\theta_{[\lambda - b(w_{[\mu,c]}+\varepsilon),A]} \le u. \tag{24}$$

Now, by Proposition 10 b), we have that as $s \to -\infty$,

$$\theta_{[\mu - d(w_{[\lambda,\alpha(t_0)]}+\varepsilon),c]} \to w_{[\mu - d(w_{[\lambda,\alpha(t_0)]}+\varepsilon),c]},$$
$$\theta_{[\lambda - b(w_{[\mu,c]}+\varepsilon),A]} \to w_{[\lambda - b(w_{[\mu,c]}+\varepsilon),A]}.$$

Proposition 10 b), (22), (23) and (24) complete the proof. $\square$

Assuming that $a(t)$ tends to a positive constant as $t \to -\infty$, we obtain a similar result to Proposition 16 but where the conditions on λ and μ do not depend on t.

Corollary 17 *Assume $a(t) \to a_0 > 0$ as $t \to -\infty$, for each $t \in \mathbb{R}$*

$$\inf_{s \in (-\infty,t]} a(s) = \alpha(t) > 0,$$

$$\lambda > \lambda_1(bw_{[\mu,c]}) \qquad \textit{and} \qquad \mu > \lambda_1(dw_{[\lambda,a_0]}).$$

Then, for all $t \in \mathbb{R}$, there exist $s_0(t) \le t$ and $f_i \in C_0(\overline{\Omega})$ positive functions (depending on t), such that for all $s \le s_0$ it holds:

$$u(t,s;u_0,v_0) \ge f_1 \quad \textit{and} \quad v(t,s;u_0,v_0) \ge f_2.$$

PROOF. Since $\mu > \lambda_1(dw_{[\lambda,a_0]})$ and from the continuity of the map $e \mapsto w_{[\lambda,e]}$, there exists $\varepsilon > 0$ such that $\mu > \lambda_1(dw_{[\lambda,a_0-\varepsilon]})$. On the other hand, since $a(t) \to a_0$ as $t \to -\infty$, there exists $T \in \mathbb{R}$ such that for all $t \le T$, $a_0 - \varepsilon \le \alpha(t) \le a(t) \le A$. Then, for any $t_0 \le T$ we have that

$$\mu > \lambda_1(dw_{[\lambda,\alpha(t_0)]}),$$

and so by Proposition 16, we get that there exist two positive functions e_i such that

$$u(t_0,s;u_0,v_0) \ge e_1 \quad \text{and} \quad v(t_0,s;u_0,v_0) \ge e_2.$$

Furthermore, for all $t \ge t_0$ we have

$$u(t,s;u_0,v_0) = u(t,t_0;u(t_0,s;u_0,v_0),v(t_0,s;u_0,v_0))$$

from which, by the strong maximum principle, we obtain the result. $\square$

6 Existence of a non-autonomous attractor and pullback permanence for the Lotka-Volterra competition model

We define $X := C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ and the following process in X: for $t, s \in \mathbb{R}$, $t \geq s$,

$$S(t,s) : X \mapsto X; \quad S(t,s)(u_0, v_0) = (u(t,s;u_0,v_0), v(t,s;u_0,v_0)),$$

where $(u(t,s;u_0,v_0), v(t,s;u_0,v_0))$ is the unique positive solution of (1) for $u_0, v_0 \in P$. Moreover, in X we define the following order: given $(u_1, v_1), (u_2, v_2) \in X$,

$$(u_1, v_1) \preceq (u_2, v_2) \qquad \text{if, and only if,} \qquad u_1 \leq u_2 \quad \text{and} \quad v_1 \geq v_2,$$

where "$\leq$" is the order defined by P in $C_0(\overline{\Omega})$. It is well-known, see [15], that $S(t,s)$ is an order-preserving process, that is, if $(u_1, v_1) \preceq (u_2, v_2)$, then $S(t,s)(u_1, v_1) \preceq S(t,s)(u_2, v_2)$. Moreover, we consider the norm $|(u,v)|_\infty = \|u\|_\infty + \|v\|_\infty$ in X.

In the next two sections we will prove the existence of a non-autonomous attractor for (1).

6.1 Absorbing set in X

Let $D \subset X$ be bounded, i.e., $\sup_{d \in D} |d|_\infty \leq M$, for $M > 0$, and $(u_0, v_0) \in D$. By (14) and Proposition 10 e), there exists $T(t, u_0, v_0) \in \mathbb{R}$ such that

$$\|u(t,s;u_0,v_0)\|_\infty \leq \left\|\theta_{[\lambda,a]}(t,s;u_0)\right\|_\infty \leq r_\lambda(t) \quad \text{for } s \leq T(t), \tag{25}$$

where

$$r_\lambda(t) = \frac{2\mathrm{e}^{\lambda t}}{\int_{-\infty}^t \mathrm{e}^{\lambda \tau} a(\tau)\, \mathrm{d}\tau}.$$

Similarly,

$$\|v(t,s;u_0,v_0)\|_\infty \leq r_\mu(t) \quad \text{for } s \leq T(t), \tag{26}$$

where

$$r_\mu(t) = \frac{2\mathrm{e}^{\mu t}}{c\int_{-\infty}^t \mathrm{e}^{\mu \tau}\, \mathrm{d}\tau} = \frac{2\mu}{c}.$$

Clearly, this means that the ball in X with radius $r_1(t) = r_\lambda(t) + r_\mu(t)$, $B_X(0, r_1(t))$, is absorbing for the process $S(t,s)$.

6.2 Absorbing set in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$

In order to obtain a family of absorbing sets in $C_0^1(\overline{\Omega})$ we need the following result from [28], see also Lemma 3.1 in [4]. Here, for a Banach space Y, Y^β will denote the usual fractional power spaces with norm $|\cdot|_\beta$. Recall that $\mathbf{A} : D(\mathbf{A}) \mapsto C_0(\overline{\Omega})$ is the linear operator associated to the Laplacian.

Lemma 18 *The operator* $\mathbf{A}$ *generates an analytic semigroup on* $Y = C_0^k(\overline{\Omega})$ *for* $k = 0, 1$. *Moreover*

$$Y^\beta \hookrightarrow C_0^{k+q}(\overline{\Omega}) \quad \textit{for } q = 0, 1 \textit{ and } 2\beta > q.$$

Given $D \subset X$ bounded, we define for $r \geq s$

$$h(r,s) = \lambda u(r,s;u_0,v_0) - a(r)u^2(r,s;u_0,v_0) - bu(r,s;u_0,v_0)v(r,s;u_0,v_0).$$

Then, writing u from the variation of constants formula, we obtain

$$u(t,s;u_0,v_0) = \mathrm{e}^{-\mathbf{A}(t-s)}u_0 + \int_s^t \mathrm{e}^{-\mathbf{A}(t-r)}h(r,s)\,\mathrm{d}r.$$

Hence, between $t-1$ and t, we get

$$u(t,s;u_0,v_0) = \mathrm{e}^{-\mathbf{A}}u(t-1,s;u_0,v_0) + \int_{t-1}^t \mathrm{e}^{-\mathbf{A}(t-r)}h(r,s)\,\mathrm{d}r.$$

Hence,

$$|u(t,s;u_0,v_0)|_\beta = \left\|\mathbf{A}^\beta u(t,s;u_0,v_0)\right\|_\infty \leq \left\|\mathbf{A}^\beta \mathrm{e}^{-\mathbf{A}}\right\|_{op} \|u(t-1,s;u_0,v_0)\|_\infty + \\ \sup_{r\in[t-1,t]} \|h(r,s)\|_\infty \int_{t-1}^t \left\|\mathbf{A}^\beta \mathrm{e}^{-\mathbf{A}(t-r)}\right\|_{op}\,\mathrm{d}r.$$

Now, using the estimate

$$\left\|\mathbf{A}^\beta \mathrm{e}^{-\mathbf{A}(t-r)}\right\|_{op} \leq C_\beta (t-r)^{-\beta}\mathrm{e}^{-\delta(t-r)}$$

for some constants $C_\beta, \delta > 0$ (cf. Henry [13]), and the estimates (25) and (26), we obtain the existence of $M(t)$ and $T_0(t)$ such that

$$|u(t, s; u_0, v_0)|_\beta \leq M(t) \quad \text{for all } s \leq T_0(t),$$

with $\beta < 1 - \varepsilon$, and any $\varepsilon \in (0, 1)$. Applying now Lemma 18 with $q = 1$ and $\beta > 1/2$, we obtain

$$\|u(t, s; u_0, v_0)\|_{C^1} \leq R_1(D, t) \quad \text{for all } s \leq T_0(t).$$

Similarly, it can be proven that

$$\|v(t, s; u_0, v_0)\|_{C^1} \leq R_2(D, t) \quad \text{for all } s \leq T_0(t),$$

for some $R_2(D, t)$, and so the ball in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, $B(0, R(t))$ is absorbing in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, for $R(t) = R_1(t) + R_2(t)$, where again we have used the norm $|(u, v)|_{C^1(\overline{\Omega})} = \|u\|_{C^1(\overline{\Omega})} + \|v\|_{C^1(\overline{\Omega})}$ in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$.

We can repeat the argument taking $Y = C_0^1(\overline{\Omega})$ and D a bounded set in $Y \times Y$. In this case, using Lemma 18 again, we obtain

$$\|u(t, s; u_0, v_0)\|_{C^2} \leq N(D, t) \quad \text{for all } s \leq T_1(t),$$

and hence, the existence of an absorbing set that is bounded in $C_0^2(\overline{\Omega}) \times C_0^2(\overline{\Omega})$, and so compact in X.

Analogously we can show the existence of the global attractor $\mathcal{A}_+$ attracting every bounded set in X_0.

6.3 On the structure of the pullback attractor and pullback permanence

In this section we apply the results of Section 3 to our model. We take

$$\underline{w}(t) = (0, r_\mu(t)) \qquad \text{and} \qquad \overline{w}(t) = (r_\lambda(t), 0).$$

Firstly, observe that $\underline{w}(t) \preceq \overline{w}(t)$. On the other hand, by (25) and (26) it follows that

$$\mathcal{A}(t) \subset I_{\underline{w}}^{\overline{w}}(t).$$

Finally, we define the base of attraction in our model as

$$\mathcal{D} := \{w : \mathbb{R} \mapsto X \text{ continuous, such that, } \lim_{s \to -\infty} \frac{\mathrm{e}^{\gamma s}}{\|w(s)\|_\infty} = 0\}$$

where $\gamma = \min\{\lambda, \mu\}$. Note, that given $w = (u, v) \in \mathcal{D}$,

$$\lim_{s \to -\infty} \operatorname{dist}(S(t,s)(u(s), v(s)), \mathcal{A}(t)) = 0. \tag{27}$$

Indeed, we have that for s small enough

$$\|u(t,s;u(s),v(s))\|_\infty \leq \|\theta_{[\lambda,a]}(t,s;u(s))\|_\infty \leq \frac{\mathrm{e}^{\lambda t}}{\frac{\mathrm{e}^{\lambda s}}{\|u(s)\|_\infty} + \int_s^t \mathrm{e}^{\lambda \tau} a(\tau)\, \mathrm{d}\tau} \leq r_\lambda(t).$$

Moreover, it is clear that $(\underline{w}, \overline{w}) \in \mathcal{D}$. So, applying Theorem 8, there exist complete trajectories w_* (minimal) and w^* (maximal) that are stable in the sense of Theorem 8.
In a similar way, for $\mathcal{A}_+$ we can also apply Theorem 8 for

$$\underline{w}(t) = (f_1(t), r_\mu(t)), \qquad \overline{w}(t) = (r_\lambda(t), f_2(t)),$$

so that, for strictly positive initial data, the non-autonomous attractor is bounded above and below by strictly positive bounds. Finally, we can conclude the pullback permanence of our model.

Theorem 19 *Assume that $a(t) \to a_0 > 0$ as $t \to -\infty$, for each $t \in \mathbb{R}$*

$$\inf_{s \in (-\infty, t]} a(s) = \alpha(t) > 0,$$

$$\lambda > \lambda_1(bw_{[\mu,c]}) \qquad \text{and} \qquad \mu > \lambda_1(dw_{[\lambda,a_0]}).$$

Then (1) *is permanent in the pullback sense.*

PROOF. We write $X = X_0 \cup \partial X_0$, where $X_0 = (intP)^2$ and $\partial X_0 = X \setminus X_0$. The permanence follows with

$$U(t) = \{w \in X : (f_1(t), r_\mu(t)) \preceq w \preceq (r_\lambda(t), f_2(t))\},$$

where f_1, f_2 are defined in Corollary 17 and r_λ and r_μ in (25) and (26), respectively. By Section 6.1, $U(t)$ is absorbing and by Corollary 17 $\operatorname{Dist}(U(t), \partial X_0) > 0$. This completes the proof. $\square$

7 Conclusions

We have considered a Lotka-Volterra system with a non-autonomous term that produces only a weak dissipativity effect. This effect is so weak that there are no bounded absorbing sets, and hence we cannot expect any kind of permanence as $t \to \infty$. In order to understand the dynamics of the system further we have introduced the concept of "pullback permanence": for our example we could show the existence of a time dependent family of sets, bounded above and below by positive functions, that absorbs every trajectory of the system "in the pullback sense". This gives a sense in which, even though one species will eventually die out, the system exhibits some kind of permanence: at any time t_0, no matter how long the system has been running, the species numbers are uniformly bounded below.

We note here that the region in the (λ, μ)-plane defined by $\lambda > \lambda_1(bw_{[\mu,c]})$ and $\mu > \lambda_1(dw_{[\lambda,a_0]})$ can be empty depending on the values of the parameters a_0, b, c and d (cf. Section 7 in [24]). Even in the autonomous case, $a(t) = a > 0$, results of permanence are not known when bc is large with respect to ad. In this case the region defined by $\lambda > \lambda_1(bw_{[\mu,c]})$ and $\mu > \lambda_1(dw_{[\lambda,a]})$ is empty, and it is known that if λ and μ belong to the region defined by $\lambda < \lambda_1(bw_{[\mu,c]})$ and $\mu < \lambda_1(dw_{[\lambda,a]})$, then there exists an unstable stationary positive solution of (1) (cf. Theorem 5.3 in [25]).

To understand the behaviour of this model in more detail would require an analysis of the local stability and instability of the complete trajectories that play a major role in the dynamics. There is some progress on this for the ODE version of (1) (see Langa et al. [21]), but in general the subject is still in its infancy: even one-dimensional non-autonomous examples show a much richer and more complex dynamics than their corresponding autonomous counterparts (cf. Langa et al. [20]).

As emphasized above, the notion of "pullback permanence" that we have defined is not intended as a candidate to replace the standard definition. Rather we believe that the results presented here offer strong evidence that the pullback procedure is a valuable tool with which we can further our understanding of non-autonomous systems.

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