Existence and nonexistence of unbounded forward attractor for a class of non-autonomous reaction diffusion equations*

J.A. Langa[1], J.C. Robinson[2], A. Rodríguez-Bernal[3], A. Suárez[1] & A. Vidal-López[3]

[1] Departamento de Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla,
Apdo. de Correos 1036,
41080–Sevilla, SPAIN

[2] Mathematics Institute,
University of Warwick,
Coventry CV4 7AL, UK

[3] Departamento de Matemática Aplicada,
Universidad Complutense de Madrid,
Madrid 28040 SPAIN

Abstract

The goal of this work is to study the forward dynamics of positive solutions for the non-autonomous logistic equation $u_t - \Delta u = \lambda u - b(t)u^p$, with $p > 1$, $b(t) > 0$, for all $t \in \mathbb{R}$, $\lim_{t\to\infty} b(t) = 0$. While the pullback asymptotic behaviour for this equation is now well understood, several different possibilities are realised in the forward asymptotic regime.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with smooth boundary $\partial\Omega$. Consider the following non-autonomous scalar logistic equation

$$\begin{cases} u_t - \Delta u = \lambda u - b(t)u^p \\ Bu = 0 \\ u(s) = u_s, \end{cases} \tag{1.1}$$

with $p > 1$, $\lambda \in \mathbb{R}$ and $b \in C^1(\mathbb{R})$, assuming that there exists a positive constant B_0 such that

$$0 < b(t) \leq B_0, \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

and the boundary operator

$$Bu = u, \quad \text{Dirichlet case}, \quad \text{or} \quad Bu = \frac{\partial u}{\partial n}, \quad \text{Neumann case},$$

*Partially supported by Ministerio de Educación y Ciencia (Spain) Projects BFM 2003-03810, MTM2005-01412 and BFM 2003-06446. JCR is a Royal Society University Research Fellow.

being n the outward normal vector-field to $\partial\Omega$.

From now on, suppose that

$$\liminf_{t\to-\infty} b(t) \geq b_0 > 0. \tag{1.3}$$

$$\lim_{t\to\infty} b(t) = 0, \tag{1.4}$$

$$\lambda > \lambda_1, \tag{1.5}$$

where λ_1 denotes the principal eigenvalue associated to $-\Delta$ under $Bu = 0$.

As will be recalled in the next section, under these assumptions (1.1) has a unique complete non-negative trajectory $u^*(t)$, that is, a solution of (1.1) defined for all $t \in \mathbb{R}$. This trajectory is bounded as $t \to -\infty$, unbounded as $t \to \infty$, bounded away from zero for all $t \in \mathbb{R}$, and describes the dynamics of positive solutions of (1.1) in a pullback sense: see Proposition 1 for a more detailed statement.

Our goal here is to investigate to what extent $u^*(x,t)$ still describes the forward asymptotic behavior of positive solutions of (1.1).

For this, we will first show that all positive solutions of (1.1) grow at the same rate as $u^*(t)$. Then we will scale the solutions appropriately so as to capture the behavior of their unbounded leading term and study the relative and absolute errors between a solution and $u^*(t)$.

We will prove that if $b(t)$ vanishes slowly at infinity, then the relative errors with respect to $u^*(t)$ tend to zero as $t \to \infty$. In this sense, $u^*(\cdot)$ remains a 'first order approximation' to the forwards attractor for positive solutions of (1.1) since it still captures the asymptotic behavior of all positive solutions.

Then we will show that there are some regimes for $b(t)$ and λ for which the absolute errors with respect to $u^*(t)$ tend to zero as $t \to \infty$: in this case, $u^*(t)$ is a forwards attractor for positive solutions of (1.1) in the conventional sense, even though it is unbounded.

However, we will also show that there are other regimes for $b(t)$ and λ, in which the absolute errors with respect to $u^*(t)$ become unbounded as $t \to \infty$. In this case $u^*(t)$ is no longer a forward attractor for positive solutions of (1.1) in any strong sense.

On the other hand, when $b(t)$ vanishes fast at infinity, we will show that all positive solutions of (1.1) differ strongly in their leading terms. In this way we will show in particular that there is no forward attractor of positive solutions of (1.1) in any sense, although the existence of a pullback attractor holds in all cases.

We note here that this behaviour is not particular to the PDE case. If one neglects the Laplacian, the equivalent ODE model

$$\dot{u} = \lambda u - b(t)u^p \qquad \text{with} \quad u(s) = u_s$$

with $\lambda > 0$ and $p > 1$ has exact solution

$$u(t,s;u_s)^{1-p} = e^{-(p-1)\lambda(t-s)}u_s^{1-p} + (p-1)\int_s^t e^{-(p-1)\lambda(t-r)}b(r)\,dr.$$

In this case the pullback attracting trajectory $u^*(\cdot)$ is given explicitly by

$$u^*(t)^{1-p} = (p-1)\int_{-\infty}^t e^{-(p-1)\lambda(t-r)}b(r)\,dr \tag{1.6}$$

and the analysis of the asymptotic behavior is therefore simplified. Also note that solutions of the ode above are space-homogeneous solutions of (1.1) in the case of Neumann boundary conditions.

However, we are able to obtain detailed information about the asymptotic behavior in the PDE case for non space-homogeneous initial data and also in the case of Dirichlet boundary conditions despite the lack of such an explicit solution.

2 Pullback and forward dynamics: preliminary results

Given a fixed regular domain $\Omega \subset \mathbb{R}^N$, let λ_1 and φ_1 stand for the principal eigenvalue and the positive eigenfunction associated to $-\Delta$ under the homogeneous condition $Bu = 0$, normalized such that $\max_{x\in\bar{\Omega}} \varphi_1(x) = 1$. Hence for Neumann boundary conditions we have $\lambda_1 = 0$ and $\varphi_1 = 1$, while $\lambda_1 > 0$ for Dirichlet boundary conditions.

Consider $X = C^1_0(\overline{\Omega})$ (in the case of Dirichlet boundary conditions) and $X = C^0(\overline{\Omega})$ (in the case of Neumann boundary conditions) with sup norm denoted by $\|\cdot\|$. Observe that the choice of space X in the Dirichlet case is not a severe restriction due to the regularization effect of parabolic problems. For (1.1), we can define an order on X in a natural way, namely

$$u_0 \le v_0 \quad \text{iff} \quad v_0(x) - u_0(x) \ge 0, \quad x \in \Omega.$$

On the other hand, we say u is *strictly positive* if

$$u(x) > 0 \quad x \in \Omega, \qquad \frac{\partial u}{\partial n} < 0, \; x \in \partial\Omega$$

in the case of Dirichlet boundary conditions, and more simply if $u(x) > 0$ for all $x \in \overline{\Omega}$ in the case of Neumann boundary conditions.

Under these conditions, we have existence and uniqueness of solutions:

Theorem 1 *Assume that* (1.2) *holds and that* $u_0 \ge 0$, $u_0 \ne 0$. *Then, there exists a unique solution* $u(t) = u(t, s; u_0) \in X$ *of* (1.1), *which is strictly positive for* $t > s$.

We can define the following order preserving flow in X, for $t, s \in \mathbb{R}$, $t \ge s$:

$$\begin{aligned} S(t,s) &: X \to X \\ S(t,s)u_s &= u(t,s;u_s), \end{aligned}$$

with $u(t, s; u_s)$ the unique solution of (1.1).

Definition 1
i) $v : \mathbb{R} \to X$ *is a complete trajectory of problem (1.1) if*

$$u(t, s; v(s)) = v(t) \text{ in } X, \text{ for all } t \ge s, \; t, s \in \mathbb{R},$$

with $u(t, s; v(s))$ *the unique solution of* (1.1) *with initial condition* $u(s) = v(s)$.
ii) A complete trajectory $v(t)$ *is non-degenerate at* $+\infty$ *(respectively at* $-\infty$*), if there exist* $t_0 \in \mathbb{R}$ *and* $\varphi_0 > 0$, *such that* $v(x, t) \ge \varphi_0(x)$, *for all* $t \ge t_0$ *(respectively for* $t \le t_0$*).*

Concerning the asymptotic behavior of solutions of (1.1) we have the following results from [4, Section 4.2], [5] and [8] ((i) and (ii) below) and [10] ((iii) and (iv) below). (Note that in [4] $p = 3$ but the arguments extend easily to the more general $p > 1$ considered here.) In the statement of the proposition,

$$\mathcal{V}_+ = \{u \in X : \ u \geq 0\}.$$

Proposition 1

i) Assume that (1.2) *holds, i.e. that* $0 < b(t) \leq B_0$ *for all* $t \in \mathbb{R}$*, then*

a) There exists a complete trajectory $u^*(\cdot) : \mathbb{R} \to \mathcal{V}_+$ *such that, for every* $u(\cdot) : \mathbb{R} \to \mathcal{V}_+$ *with* $u(s) \geq u^*(s)$ *for all* $s \in \mathbb{R}$*, we have that, for all* $t \in \mathbb{R}$*,*

$$\lim_{s \to -\infty} \|u(t, s; u(s)) - u^*(t)\| = 0 \quad (pullback\ attracting\ from\ above)$$

b) There exists a time dependent family $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ *of compact sets, known as the pullback attractor, such that*

b.i) $S(t, s)\mathcal{A}(s) = \mathcal{A}(t)$ *for all* $t \geq s$ *and*

b.ii) $\lim_{s \to -\infty} dist(S(t, s)D, \mathcal{A}(t)) = 0,$ *for all* $t \in \mathbb{R}$ *fixed and* $D \subset \mathcal{V}_+$ *bounded.*

b.iii) $\mathcal{A}(t) \subset \{u \in X : \ 0 \leq u \leq u^*(t)\}$*. In addition,* $u^*(t) \in \mathcal{A}(t)$*, for all* $t \in \mathbb{R}$*.*

ii) Under the assumptions above, if $\lambda < \lambda_1$ *then, for all* $t \in \mathbb{R}$*,*

$$\mathcal{A}(t) = u^*(t) = 0,$$

and for all $s \in \mathbb{R}$ *and* $D \subset \mathcal{V}_+ \setminus \{0\}$ *bounded*

$$\lim_{t \to \infty} u(t, s; u_s) = 0$$

uniformly for $u_s \in D$*.*

iii) If $\lambda > \lambda_1$ *and* (1.3) *holds, i.e.* $\liminf_{t \to -\infty} b(t) \geq b_0 > 0$*, then* u^* *is the unique complete trajectory non-degenerate and bounded at* $-\infty$ *and for all* $t \in \mathbb{R}$*, and for every* $D \subset \mathcal{V}_+$ *bounded, we have that, for all* $t \in \mathbb{R}$*,*

$$\lim_{s \to -\infty} \|u(t, s; u_s) - u^*(t)\| = 0$$

uniformly for $u_s \in D$*.*

iv) Finally, if in addition we have

$$\liminf_{t \to \infty} b(t) > 0$$

(in contrast to (1.4) *where this limit exists and is zero) then* u^* *is bounded in* X *and*

$$\lim_{t \to \infty} \|u(t, s; u_s) - u^*(t)\| = 0$$

uniformly for $u_s \in D$*.*

In the rest of the paper we will consider the case in which (1.2), (1.3), (1.4) and (1.5) apply: then parts (i–iii) of the above theorem hold, and $u^*(t)$ is the unique complete trajectory, bounded away from zero at $-\infty$ and pullback attracting for all non-negative solutions of (1.1). Moreover it is unbounded, since as proved in [4] for all $s \in \mathbb{R}$ and $u_s \geq 0$, $u_s \neq 0$,

$$\lim_{t\to\infty} \|u(t,s;u_s)\| = \infty.$$

Even more, $u^*(t)$ is non-degenerate at $+\infty$.

Our goal is to investigate to what extent $u^*(t)$ still describes the forward asymptotic behavior of positive solutions of (1.1).

3 Forward asymptotic behaviour: a case study

3.1 Growth rate of positive solutions

Since we are concerned with the case in which unbounded positive non-degenerate solutions exist, we first prove the following result which shows that all positive solutions of (1.1) grow at the same rate as $u^*(\cdot)$. This will be a corollary of the following Lemma concerning solutions of linear equations:

Lemma 1 *Consider the following linear equation with potential $q(x,t)$*

$$\begin{cases} u_t - \Delta u = q(x,t)u, \\ Bu = 0, \\ u(s) = u_s \in X, \end{cases} \tag{3.1}$$

where $q \in C^\alpha(\mathbb{R}, L^p(\Omega))$, with $p > N/2$ and $0 < \alpha \leq 1$. Let $T_q(t,s)$ be the evolution operator associated to (3.1). Let us suppose that there exists a trajectory $z(t) \geq 0$, non-degenerate at ∞. In addition assume that $z(t)$ is unbounded in X as $t \to \infty$.

Then, if $u_s > 0$ there exist $\alpha_s, \beta_s > 0$ such that, for all $t \geq s$ we have

$$\alpha_s z(t) \leq u(t,s;u_s) \leq \beta_s z(t),$$

Proof. Simply note that if $u_s > 0$ there exist $\alpha_s, \beta_s > 0$ such that,

$$\alpha_s z(s) \leq u_s \leq \beta_s z(s).$$

Thus, as (3.1) is order preserving, the result follows. ■

Corollary 1 *Let $u^*(x,t)$ be the unique complete trajectory of (1.1), non-degenerate at $\pm\infty$, which is moreover unbounded in X at ∞. Then, if $u_s > 0$ there exist $\alpha_s, \beta_s > 0$ such that, for all $t \geq s$ we have*

$$\alpha_s u^*(t) \leq u(t,s;u_s) \leq \beta_s u^*(t),$$

where $u(t,s;u_s)$ is the solution of (1.1).

Proof. We construct an appropriate $q(t,x)$ and apply Lemma 1. To this end, note that if $u_s \geq u^*(s)$, then $u(t,s;u_s) \geq u^*(t)$. Thus, as

$$\frac{\lambda u - b(t)u^p}{u}$$

is decreasing in u, we have that

$$u_t - \Delta u = \frac{\lambda u - b(t)u^p}{u} u \leq \frac{\lambda u^* - b(t)(u^*)^p}{u^*} u,$$

so that, for

$$q(x,t) = \frac{\lambda u^* - b(t)(u^*)^p}{u^*},$$

we have

$$u(t,s;u_s) \leq T_q(t,s)u_s.$$

But note that there exists $\beta_s \geq 1$ such that $u_s \leq \beta_s u^*(s)$, so that

$$u^*(t) \leq u(t,s;u_s) \leq u(t,s;\beta_s u^*(s)) \leq T_q(t,s)(\beta_s u^*(s)) = \beta_s T_q(t,s)u^*(s) = \beta_s u^*(t).$$

A similar argument, when $u_s \leq u^*(s)$ would give that there exists $\alpha_s \in (0,1)$ such that

$$\alpha_s u^*(t) \leq u(t,s;u_s) \leq u^*(t).$$

For an arbitrary initial data u_s note that there exist $0 < \alpha_s < 1$, $\beta_s > 1$ such that $\alpha_s u^*(s) \leq u_s \leq \beta_s u^*(s)$. The result follows. ■

Note that, as a consequence, all positive solutions of (1.1) grow at the same rate as $u^*(\cdot)$.

3.2 The rescaled equation

In order to analyze the forward asymptotic behavior of solutions of (1.1), we scale the solutions according to

$$z(x,t) = b(t)^{\frac{p'}{p}} u(x,t), \tag{3.2}$$

with $p' = \frac{p}{p-1}$ the conjugate value of p. Then, z satisfies the rescaled equation

$$\begin{cases} z_t - \Delta z = (\lambda + \dfrac{p'}{p}\dfrac{b'(t)}{b(t)})z - z^p \\ Bz = 0 \\ z(s) = z_s = b(s)^{\frac{p'}{p}} u_s. \end{cases} \tag{3.3}$$

where we have transferred the non-autonomous term in (1.1) to the linear part of the equation.

Remark 1 *Note that in the case $b(t) = e^{-\delta t}$, with $\delta > 0$, equation (3.3) becomes the autonomous equation*

$$\begin{cases} z_t - \Delta z = (\lambda - \frac{\delta p'}{p})z - z^p \\ Bz = 0. \end{cases} \tag{3.4}$$

We now study the properties of the linear equation associated with (3.3). These will be central to the remainder of our analysis of the equation's asymptotic behaviour.

Lemma 2 *In (3.3) the evolution operator $T_a(t,s)$ of the associated linear equation*

$$\begin{cases} z_t - \Delta z = a(t)z \\ Bz = 0 \\ z(s) = z_s \end{cases} \tag{3.5}$$

with $a(t) = (\lambda + \frac{p'}{p}\frac{b'(t)}{b(t)})$ satisfies
i) There exists $M > 0$ such that

$$\|T_a(t,s)\|_{\mathcal{L}(X)} \leq M\big(\frac{b(t)}{b(s)}\big)^{\frac{p'}{p}} e^{(\lambda-\lambda_1)(t-s)}, \qquad t > s.$$

ii) If $z_s \geq 0$ then

$$\|T_a(t,s)z_s\| \geq C_1(\Omega)\big(\int_\Omega z_s\varphi_1\big)\big(\frac{b(t)}{b(s)}\big)^{\frac{p'}{p}} e^{(\lambda-\lambda_1)(t-s)}, \qquad t > s.$$

and $C_1(\Omega) = \frac{1}{\|\varphi_1\|_{L^1(\Omega)}}$.
iii) Given $\varepsilon > 0$, for any z_s, define the projection $Pz_s = C_2(\Omega)\big(\int_\Omega z_s\varphi_1\big)\varphi_1$ with where $C_2(\Omega) = \frac{1}{\|\varphi_1\|^2_{L^2(\Omega)}}$ onto the linear space spanned by φ_1, and its complement $Q = I - P$. Then

$$PT_a(t,s)z_s = T_a(t,s)Pz_s = v(t,s) = C_2(\Omega)\big(\int_\Omega z_s\varphi_1\big)\big(\frac{b(t)}{b(s)}\big)^{\frac{p'}{p}} e^{(\lambda-\lambda_1)(t-s)}\varphi_1,$$

and, for every $\varepsilon > 0$,

$$\|QT_a(t,s)z_s\| = \|T_a(t,s)Qz_s\| \leq M_\varepsilon\big(\frac{b(t)}{b(s)}\big)^{\frac{p'}{p}} e^{-(\lambda_2-\lambda-\varepsilon)(t-s)}\|z_s\|, \qquad t > s$$

for some $M_\varepsilon > 0$, where λ_2 is the second eigenvalue of the Laplacian with boundary conditions given by B.

Proof. Note that $y(t) = z(t)e^{-\int_s^t a(r)\,dr} = y(t,s;z_s)$ satisfies

$$\begin{cases} y_t - \Delta y = 0 \\ By = 0 \\ y(s) = z_s \end{cases}$$

for which we have,

$$\|y(t)\| \leq Me^{-\lambda_1(t-s)}\|y(s)\|, \qquad t > s$$

and we get i).

If $z_s \geq 0$ then $y(t) \geq 0$ and

$$\|y(t)\| = \sup_{\phi\in L^1(\Omega)} \frac{\int_\Omega y(t)\phi}{\|\phi\|_{L^1(\Omega)}} \geq \frac{\int_\Omega y(t)\varphi_1}{\|\varphi_1\|_{L^1(\Omega)}}.$$

Multiplying the equation for $y(t)$ by φ_1 and integrating by parts, we get

$$\frac{d}{dt}\int_\Omega y(t)\varphi_1 + \lambda_1 \int_\Omega y(t)\varphi_1 = 0$$

and ii) follows.

Finally, note that $-\Delta$ with boundary conditions given by B is a sectorial operator $L^2(\Omega)$, [3], and from the results in [7] or [1], it is also sectorial in X. Also note that the spectrum of this operator is the same in both spaces. As λ_1 is a (simple and) isolated point of the spectrum, we can consider the associated projections and invariant subspaces as in [3], Theorem 1.5.2, page 30, given by

$$P = \frac{1}{2\pi i}\int_\gamma (A - \mu I)^{-1} d\mu, \quad Q = I - P \tag{3.6}$$

where γ denotes a small simple closed curve contained in $\rho(A)$ and surrounding λ_1. Since X is dense in $L^2(\Omega)$, then, for every $z \in X$, $(A-\mu I)^{-1}z$ gives the same element in either space setting. Then the spectral projections above in X coincide with the ones in $L^2(\Omega)$, which are given by P and Q in the statement. Therefore,

$$Py(t) = y(t, s; Pz_s) = C_2(\Omega)\Big(\int_\Omega z_s\varphi_1\Big)e^{-\lambda_1(t-s)}\varphi_1$$

and Theorem 1.5.3 in [3] implies now that for every $\varepsilon > 0$,

$$\|Qy(t)\| = \|y(t, s; Pz_s)\| \le M_\varepsilon e^{-(\lambda_2-\varepsilon)(t-s)}\|z_s\|, \qquad t > s$$

and iii) follows. ■

3.3 When $b(t)$ vanishes slowly.

The next result shows that if $b(t)$ vanishes slowly then relative errors with respect to $u^*(t)$ tend to zero as $t \to \infty$. In this sense, we can consider $u^*(\cdot)$ to be a 'first order approximation' to a forwards attractor for positive solutions of (1.1) since it still describes the leading order asymptotic behavior of all positive solutions.

Proposition 2 *Suppose that*

$$\lambda_1 < \lambda + \frac{p'}{p}\liminf_{t\to\infty}\frac{b'(t)}{b(t)} \le \lambda + \frac{p'}{p}\limsup_{t\to\infty}\frac{b'(t)}{b(t)} < \infty. \tag{3.7}$$

Then the relative errors satisfy

$$\lim_{t\to\infty}\frac{\|u(t, s; u_s) - u^*(t)\|}{\|u^*(t)\|} = 0. \tag{3.8}$$

Proof. Note that there exist $t_0 \in \mathbb{R}$ and $C_0 > 0$ such that, for all $t \ge t_0$

$$\lambda_1 < \lambda + \frac{p'}{p}\frac{b'(t)}{b(t)} \le C_0 < \infty. \tag{3.9}$$

Then, from [10] we know that there exists a unique complete trajectory $\varphi(\cdot)$, non-degenerate at $\pm\infty$, for (3.3) which is the pullback attractor in $int(\mathcal{V}_+)$. Hence, we must have

$$u^*(t) = b(t)^{\frac{-p'}{p}}\varphi(t). \tag{3.10}$$

Since, also from [10], $\varphi(t)$ is bounded above and below by positive functions and $\|\varphi(t) - z(t)\| \to 0$ as $t \to \infty$, the relative errors satisfy

$$\lim_{t\to\infty} \frac{\|u(t,s;u_s) - u^*(t)\|}{\|u^*(t)\|} = \lim_{t\to\infty} \frac{\|z(t,s;z_s) - \varphi(t)\|}{\|\varphi(t)\|} = 0.$$

■

Remark 2 *Note that (3.9) is equivalent to*

$$1 \le e^{-(\lambda_1-\lambda)(t-s)}\Big(\frac{b(t)}{b(s)}\Big)^{\frac{p'}{p}} \le e^{(C_0-\lambda_1)(t-s)},$$

see Lemma 2.

We now look at conditions on $b(t)$ and λ that imply different behaviors of the absolute error

$$w(t,s) = |u(t,s;u_s) - u^*(t)|.$$

Our goal is to show that there exist some regimes for $b(t)$ and λ for which the absolute errors tend to zero or, on the other hand, become unbounded, as $t \to \infty$. In the former case $u^*(t)$ is a forwards attractor for positive solutions of (1.1) in the conventional sense, while in the latter case it is not.

We start with the following

Lemma 3 *For every $u_s \le u^*(s)$ or $u_s \ge u^*(s)$ the absolute error satisfies an equation of the form*

$$\begin{cases} w_t - \Delta w = q(x,t)w, \\ Bw = 0 \\ w(s) = |u_s - u^*(s)| \end{cases} \tag{3.11}$$

where

$$\|q(t) - q_0(t)\| \to 0 \quad \textit{exponentially} \quad \textit{as} \quad t \to \infty$$

with

$$q_0(x,t) = \lambda - p\varphi^{p-1}(x,t) \ \textit{ and } \varphi \textit{ from } (3.10).$$

Proof. Suppose, for instance, that $u_s \ge u^*(s)$. Then, $w(t) = u(t,s;u_s) - u^*(t)$ satisfies

$$w_t - \Delta w = \lambda w - b(t)(u^p - (u^*)^p) = \lambda w - b(t)p\xi^{p-1}w$$

for some $u^* \le \xi \le u$, i.e. w is a solution of (3.11) with

$$q(x,t) = \lambda - pb(t)\xi^{p-1}(x,t), \tag{3.12}$$

and then using (3.2) and (3.10) we have

$$\lambda - p\varphi^{p-1}(x,t) \geq q(x,t) \geq \lambda - pz^{p-1}(x,t), \tag{3.13}$$

and since $\|\varphi(t) - z(t)\| \to 0$ exponentially, [9], as $t \to \infty$, we get the result.

The case $u_s \leq u^*(s)$ is analogous. ■

Remark 3 *For an arbitrary initial datum u_s, note that there exist $0 < \alpha_s < 1$, $\beta_s > 1$ such that $\alpha_s u^*(s) \leq u_s \leq \beta_s u^*(s)$. Hence,*

$$u(t,s;\alpha_s u^*(s)) - u^*(t) \leq u(t,s;u_s) - u^*(t) \leq u(t,s;\beta_s u^*(s)) - u^*(t)$$

and

$$u(t,s;\alpha_s u^*(s)) - u^*(t) \leq 0, \qquad u(t,s;\beta_s u^*(s)) - u^*(t) \geq 0$$

can both be treated using Lemma 3.

From the lemma above the forward asymptotic dynamics of the absolute error will be then given by the asymptotic behavior of the solutions of the linear equation

$$\begin{cases} w_t - \Delta w = q_0(x,t)w, \\ Bw = 0 \\ w(s) = |u_s - u^*(s)| \end{cases} \tag{3.14}$$

which we now analyze. Note that the results in [9] imply that if solutions of (3.14) decay or grow exponentially then the solutions of (3.11) behave in the same way. Now denote for $t_0 > 1$ large enough

$$k(t_0) = \inf_{t \geq t_0} \frac{b'(t)}{b(t)}, \qquad \text{and} \qquad K(t_0) = \sup_{t \geq t_0} \frac{b'(t)}{b(t)}.$$

and note that from (3.3)

$$z_k(x) \leq \varphi(x,t) \leq z_K(x), \quad t \geq t_0, \tag{3.15}$$

where z_k and z_K are the unique positive equilibria for

$$\begin{cases} -\Delta z = (\lambda + \frac{p'}{p} k(t_0))z - z^p \quad \text{in } \Omega, \\ Bz = 0 \quad \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta z = (\lambda + \frac{p'}{p} K(t_0))z - z^p \quad \text{in } \Omega, \\ Bz = 0 \quad \text{on } \partial\Omega, \end{cases}$$

respectively. Hence, in (3.14)

$$\lambda - pz_K^{p-1}(x) \leq q_0(x,t) \leq \lambda - pz_k^{p-1}(x), \quad t \geq t_0. \tag{3.16}$$

We will now distinguish the different boundary conditions.

3.3.1 Neumann boundary conditions

For the Neumann case, we have

Proposition 3 *Consider* (1.1) *under Neumann boundary conditions and suppose that* (3.7) *holds (with* $\lambda_1 = 0$*). Then,*

a) If $\lambda + \dfrac{p'}{p-1} \liminf_{t\to\infty} \dfrac{b'(t)}{b(t)} > 0$*, then for all* u_s *we have* $\lim_{t\to\infty} \|w(t)\| = 0$ *exponentially.*

Hence $u^*(t)$ *is an unbounded forwards attractor for positive solutions of* (1.1).

b) If $\lambda + \dfrac{p'}{p-1} \limsup_{t\to\infty} \dfrac{b'(t)}{b(t)} < 0$*, then for all* $u_s \le u^*(s)$ *or* $u_s \ge u^*(s)$*,* $u_s \ne u^*(s)$*, we have* $\lim_{t\to\infty} \|w(t)\| = \infty$ *exponentially.*

Hence $u^*(t)$ *is not a forwards attractor for positive solutions of* (1.1).

Proof. Note that as we have Neumann boundary conditions, $\lambda_1 = 0$ and then (3.9) reads

$$0 < \lambda + \frac{p'}{p}\frac{b'(t)}{b(t)} \le C_0 < \infty. \tag{3.17}$$

Also, we get explicit formulae for the above equilibria, since in this case they are constant and given by

$$z_k(x) = (\lambda + \frac{p'}{p}k(t_0))^{\frac{1}{p-1}}, \quad \text{and} \quad z_K(x) = (\lambda + \frac{p'}{p}K(t_0))^{\frac{1}{p-1}}$$

(cf. (1.6)). Thus, from the assumptions a) and b) we get, for t_0 large enough, that $\lambda + \dfrac{p'}{p-1}k(t_0) > 0$ and $\lambda + \dfrac{p'}{p-1}K(t_0) < 0$ respectively, and (3.16) gives

$$(1-p)(\lambda + \frac{p'}{p-1}K(t_0)) \le q_0(x,t) \le (1-p)(\lambda + \frac{p'}{p-1}k(t_0)).$$

In case a) the upper bound for $q_0(x,t)$ is negative and this implies that $w(t)$ decays exponentially to zero in (3.14).

In case b) the lower bound for $q_0(x,t)$ is positive and this implies that $w(t)$ grows exponentially in (3.14). ■

Remark 4 *Note that for the case* $b(t) = e^{-\delta t}$*, with* $\delta > 0$*, the above inequalities are optimal, in the sense that* $k(t_0) = \delta = K(t_0)$*, for all* $t_0 \in \mathbb{R}$*, and so,* $\varphi = z_k = z_K$ *and, as* $t \to \infty$*,*

$$\|w(t)\| \to \infty \quad \text{for} \quad \lambda \in (\frac{p'}{p}\delta, \frac{p'}{p-1}\delta), \quad w(t) \to 0 \quad \text{for} \quad \lambda > \frac{p'}{p-1}\delta$$

and both limits above are exponentially fast.

3.3.2 Dirichlet boundary conditions

Now for the case of Dirichlet boundary conditions, we get the following behavior of absolute errors:

Proposition 4 *Consider* (1.1) *under Dirichlet boundary conditions and suppose that* (3.7) *holds. Then,*

a) There exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, then for all u_s we have $\lim_{t\to\infty} \|w(t)\| = 0$ exponentially.

Hence $u^*(t)$ is an unbounded forwards attractor for positive solutions of (1.1).

b) If

$$\lambda + \frac{p'}{p-1} \limsup_{t\to\infty} \frac{b'(t)}{b(t)} < \frac{-\lambda_1}{p-1} \tag{3.18}$$

then for all $u_s \leq u^*(s)$ or $u_s \geq u^*(s)$, $u_s \neq u^*(s)$, we have $\lim_{t\to\infty} \|w(t)\| = \infty$ exponentially.

Hence $u^*(t)$ is not a forwards attractor for positive solutions of (1.1).

Proof. Observe that for t_0 large $\lambda < \dfrac{1}{p-1}(-\lambda_1 - p'K(t_0))$ and in the Dirichlet case

$$\varphi(x,t) \leq z_K(x) \leq (\lambda + \frac{p'}{p} K(t_0))^{\frac{1}{p-1}},$$

and so from (3.18)

$$q_0(x,t) = \lambda - p\varphi^{p-1}(x,t) \geq \lambda - p z_K^{p-1}(x) \geq \lambda(1-p) - p'K(t_0) > \lambda_1$$

and then b) follows.

On the other hand, from (3.16) $0 \leq w \leq v$ being v the unique solution of

$$\begin{cases} v_t - \Delta v = q_1(x)v, & \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{cases}$$

with $q_1(x) := \lambda - p(z_k(x))^{p-1}$.

Therefore, $v(x,t)$ decays zero if, and only if, $\lambda_1(-\Delta - q_1) > 0$, where $\lambda_1(-\Delta - q_1) > 0$ denotes the principal eigenvalue of the eigenvalue problem for the operator $-\Delta - q_1$ under homogeneous Dirichlet boundary conditions. Note that $\lambda_1(-\Delta - q_1) > 0$, can be written as $\lambda_1(-\Delta + p(z_k)^{p-1}) > \lambda$ or equivalently

$$\frac{\lambda_1(-\Delta + p(z_k)^{p-1})}{\lambda} > 1.$$

But since

$$\frac{\lambda_1(-\Delta + p(z_k)^{p-1})}{\lambda} = \lambda_1(-\frac{1}{\lambda}\Delta + p\frac{(z_k)^{p-1}}{\lambda}),$$

and from Theorem 2.1 in [2]

$$\frac{(z_k)^{p-1}}{\lambda} \to 1 \quad \text{uniformly in compacts of } \Omega,$$

it follows, using Theorem 5.1 in [6], that

$$\lim_{\lambda \uparrow \infty} \frac{\lambda_1(-\Delta + p(z_k)^{p-1})}{\lambda} = p > 1,$$

from where a) follows. ■

3.4 When $b(t)$ vanishes quickly.

Assume now that, instead of (3.7), we have

$$\lambda + \frac{p'}{p} \limsup_{t \to \infty} \frac{b'(t)}{b(t)} < \lambda_1. \tag{3.19}$$

Then there exists $t_0 \in \mathbb{R}$ such that, for all $t \geq t_0$

$$\lambda + \frac{p'}{p} \frac{b'(t)}{b(t)} < \lambda_1, \tag{3.20}$$

or equivalently,

$$\left(\frac{b(t)}{b(s)}\right)^{\frac{p'}{p}} e^{-(\lambda_1 - \lambda)(t-s)} \leq e^{-\delta(t-s)}, \quad t, s \geq t_0. \tag{3.21}$$

for some $\delta > 0$. Hence, from Lemma 2, solutions of the linearized equation (3.5) decay exponentially. Moreover, in this case in (3.3) we have $0 \leq z(t) \leq T_a(t,s)z(s)$ and, in particular, $\lim_{t \to \infty} \|z(t, s; z_s)\| = 0$, for all $z_s \geq 0$, so that zero is the forward attractor.

Note that in this situation for any two continuous curves in X, $u(t)$, $v(t)$ we have

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} = \frac{\|z(t) - y(t)\|}{\|y(t)\|}$$

where $z(t)$ and $y(t)$ denote the corresponding curves given by the scaling (3.2) ($z = b^{p/p'}u$, $y = b^{p/p'}v$).

Therefore our goal in this section is to show that for certain regimes for $b(t)$ and λ, if $u(t)$ above is a solution of (1.1) we can find a solution $y(t)$ of a linear equation, such that the above relative error goes to zero as $t \to \infty$. Hence, again up to first order approximation, the corresponding $v(t)$ describes the asymptotic behavior of $u(t)$.

We will show that such a construction is possible in such a way that for different solutions of (1.1), the corresponding curves $v(t)$, differ strongly at infinity. In this way we will show in particular that $u^*(t)$ is no longer the forward attractor of positive solutions of (1.1) in any sense. Note that the proof of the next result is inspired by the proof of Theorem 5.1.2 in [3].

Proposition 5 *Assume that* (3.19) *holds. Then there exists a continuous function*

$$K_0 : X \times \mathbb{R} \to \mathbb{R}$$

such that for any $u(t, s; u_s) \geq 0$ *solution of* (1.1)*, and for* t_0 *sufficiently large, the function*

$$v(x, t) = K_0(u(t_0, s; u_s), t_0)e^{(\lambda-\lambda_1)(t-t_0)}\varphi_1(x), \quad \textit{for} \quad t \geq t_0$$

satisfies, as $t \to \infty$*,*

$$\frac{\|u(t, s; u_s) - v(t)\|}{\|v(t)\|} \to 0.$$

Hence, up to first order approximation, $v(t)$ *describes the forward asymptotic behavior of the solution* $u(t, s; u_s)$*. Also,* t_0 *can be taken uniform for bounded sets of initial data* u_s *in* (1.1)*.*

Moreover for any two initial data u_s^1*,* u_s^2*, if for some large* t_0*,*

$$\int_\Omega u(t_0, s; u_s^1)\varphi_1 \neq \int_\Omega u(t_0, s; u_s^2)\varphi_1$$

then

$$K_0(u(t_0, s; u_s^1), t_0) \neq K_0(u(t_0, s; u_s^2), t_0)$$

and

$$0 < C_0 \leq \frac{\|v(t, u_s^1) - v(t, u_s^2)\|}{\|v(t, u_s^1)\|} \leq C_1,$$

hence, they have different leading asymptotic terms. In particular there is no forward attractor for (1.1)*.*

Proof. Observe that we will prove all results for the equation (3.3) and then transfer them to (1.1) by means of (3.2). Then, note that from (3.3), we have, for any $t \geq t_0 \geq s$,

$$z(t) = T_a(t, t_0)z(t_0) - \int_{t_0}^t T_a(t, r)z^p(r)\, dr.$$

Then, considering the projections introduced in Lemma 2, we take

$$z_0(t) = Pz(t), \qquad z_1(t) = Qz(t)$$

Then using that $0 \leq z(r) \leq T_a(r, t_0)z(t_0)$ and Lemma 2, we get

$$\|z_0(t)\| \leq M_0\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}} e^{(\lambda-\lambda_1)(t-t_0)}\|Pz(t_0)\| +$$

$$+M_0\|z(t_0)\|^p \int_{t_0}^t \Big(\frac{b(t)}{b(r)}\Big)^{\frac{p'}{p}} e^{(\lambda-\lambda_1)(t-r)} \Big(\frac{b(r)}{b(t_0)}\Big)^{p'} e^{p(\lambda-\lambda_1)(r-t_0)}\, dr,$$

where we have denoted $M_0 = \max\{M, M^p\}$. Note that the second term above can be estimated above by

$$M_0\|z(t_0)\|^p \Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}} e^{-(\lambda_1-\lambda)(t-t_0)} \int_{t_0}^t \frac{b(r)}{b(t_0)} e^{(p-1)(\lambda-\lambda_1)(r-t_0)}\, dr,$$

which, using (3.21), is bounded above by

$$M_0\|z(t_0)\|^p\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{(\lambda-\lambda_1)(t-t_0)}\int_{t_0}^{t}e^{-\delta(p-1)(r-t_0)}\,dr,$$

which gives a bound

$$M_0\|z(t_0)\|^p\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{(\lambda-\lambda_1)(t-t_0)}\frac{1-e^{-\delta(p-1)(t-t_0)}}{\delta(p-1)}. \tag{3.22}$$

In particular,

$$K(z(t_0),t_0)=\lim_{t\to\infty}\Big(\frac{b(t_0)}{b(t)}\Big)^{\frac{p'}{p}}e^{-(\lambda-\lambda_1)(t-t_0)}z_0(t)=$$

$$=Pz(t_0)+\int_{t_0}^{\infty}\Big(\frac{b(r)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{-(\lambda-\lambda_1)(r-t_0)}Pz^p(r)\,dr=K_0(z(t_0),t_0)\varphi_1$$

is well defined and from (3.22),

$$\|K(z(t_0),t_0)-Pz(t_0)\|\le M_1\|z(t_0)\|^p. \tag{3.23}$$

We define then

$$y(x,t)=K_0(z(t_0),t_0)\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{(\lambda-\lambda_1)(t-t_0)}\varphi_1(x).$$

Then, from (3.22) and (3.23),

$$\|y(t)-z_0(t)\|\le M_1\|z(t_0)\|^p\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{(\lambda-\lambda_1)(t-t_0)}\frac{e^{-\delta(p-1)(t-t_0)}}{\delta(p-1)}. \tag{3.24}$$

On the other hand, using Lemma 2, for every $\varepsilon>0$,

$$\|z_1(t)\|\le M_\varepsilon\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}e^{-(\lambda_2-\lambda-\varepsilon)(t-t_0)}\|z(t_0)\|+$$

$$M_0\|z(t_0)\|^p\int_{t_0}^{t}\Big(\frac{b(t)}{b(r)}\Big)^{\frac{p'}{p}}e^{-(\lambda_2-\lambda-\varepsilon)(t-r)}\Big(\frac{b(r)}{b(t_0)}\Big)^{p'}e^{p(\lambda-\lambda_1)(r-t_0)}\,dr.$$

Note that the integral term above can be written as

$$M_0\|z(t_0)\|^p\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}\int_{t_0}^{t}e^{-(\lambda_2-\lambda-\varepsilon)(t-r)}\Big(\frac{b(r)}{b(t_0)}\Big)e^{p(\lambda-\lambda_1)(r-t_0)}\,dr$$

and using (3.21), we get a bound above of the type

$$M_0\|z(t_0)\|^p\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}}\int_{t_0}^{t}e^{-(\lambda_2-\lambda-\varepsilon)(t-r)}e^{-\big(\lambda_1-\lambda+\delta(p-1)\big)(r-t_0)}\,dr.$$

Therefore, for $\delta,\varepsilon>0$ small enough, for every γ such that

$$\lambda_1-\lambda<\gamma<\lambda_1-\lambda+\delta(p-1)<\lambda_2-\lambda-\varepsilon$$

we get

$$\|z_1(t)\| \le M_1\Big(\frac{b(t)}{b(t_0)}\Big)^{\frac{p'}{p}} e^{-\gamma(t-t_0)}\Big(\|z(t_0)\| + \|z(t_0)\|^p\Big). \tag{3.25}$$

Now observe that from (3.23), since $z(t_0) \ge 0$ is not zero and converges to zero, if t_0 is sufficiently large, then $Pz(t_0) \neq 0$ and we get $K_0(z(t_0), t_0) \neq 0$. Also, note that t_0 can be taken uniform for bounded sets of initial data z_s in (3.3).

Hence, using (3.24) and (3.25),

$$\frac{\|z(t) - y(t)\|}{\|y(t)\|} \le \frac{\|y(t) - z_0(t)\|}{\|y(t)\|} + \frac{\|z_1(t)\|}{\|y(t)\|} \le$$

$$\le \frac{M_2\|z(t_0)\|^p e^{-\delta(p-1)(t-t_0)}}{|K_0(z(t_0), t_0)|\delta(p-1)} + \frac{M_2\Big(\|z(t_0)\| + \|z(t_0)\|^p\Big)}{|K_0(z(t_0), t_0)|} e^{-(\gamma-(\lambda_1-\lambda))(t-t_0)} \to 0$$

as $t \to \infty$.

Note that the function corresponding to $y(t)$ through (3.2) is $v(t)$ in the statement and we get the first part.

Moreover, for different initial data z_s^1, z_s^2, if for some large t_0,

$$\int_\Omega z(t_0, s; z_s^1)\varphi_1 \neq \int_\Omega z(t_0, s; z_s^2)\varphi_1$$

then

$$K_0(z(t_0, s; z_s^1), t_0) \neq K_0(z(t_0, s; z_s^2), t_0)$$

and clearly

$$0 < C_0 \le \frac{\|y(t, z_s^1) - y(t, z_s^2)\|}{\|y(t, z_s^1)\|} \le C_1.$$

■

Conclusion

We have performed a detailed study of the asymptotic behavior of a canonical non-autonomous reaction-diffusion problem. Despite the fact that the equation possesses a complete trajectory that is pullback attracting for all positive initial conditions, the attracting properties of this trajectory forwards in time depend sensitively one the rate of decay of the non-autonomous term.

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