

# Some eigenvalue problems with non-local boundary conditions and applications

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ABSTRACT. In this paper we study an elliptic eigenvalue problem with non-local boundary condition. We prove the existence of the principal eigenvalue and its main properties. As consequence, we show the existence and uniqueness of positive solution of a nonlinear problem arising from population dynamics.

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## 1 Introduction

In this paper we consider the following nonlinear equation with non-local boundary conditions

$$(1) \quad \begin{cases} -\Delta u = \lambda u - u^p & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary,  $\lambda \in \mathbb{R}$ ,  $p > 1$ ,  $K \in C(\overline{\Omega})$  is a nonnegative and nontrivial function satisfying some hypotheses that we will establish later and

$$\mathcal{B}u := \alpha_0 \partial_{\eta} u + \beta(x)u,$$

where  $\beta \in C(\partial\Omega)$ ,  $\eta$  is the outward unit vector and two cases are considered:  $\alpha_0 = 1$  and  $\beta$  could change sign (Robin case) or  $\alpha_0 = 0$  and  $\beta \equiv 1$  (Dirichlet case).

Parabolic nonlinear problems with non-local boundary conditions

$$(2) \quad \begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega \times (0, \infty), \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

have been extensively studied. Some physical phenomena can be formulated as (2), for instance problems arise from quasi-static thermoelasticity, where  $u(x, t)$  describes entropy per volume of the material, see [3]. Also, (2) can model the behaviour of a population inhabiting in  $\Omega$ , where the flux across the boundary  $\partial\Omega$  at any given point is proportional to the total population, in this case in a linear way. We refer to [2] for the case in which this flux is proportional to the density.

In [7] the sub-supersolution method is used to show some existence and comparison results for (2), see also [5] and [6] and references therein for the case  $f(x, u) = -c(x)u^p$ . However, the elliptic case associated to (2) has been only analyzed in some specific case. In [7], the author showed that the sub-supersolution method can be applied to the stationary problem associated to (2). Moreover, when  $f$  is decreasing in  $u$  and imposing the condition

$$(3) \quad K(x) \geq 0, \quad \int_{\Omega} K(x)dx < 1$$

the author proved the uniqueness of solution. Condition (3) was used to assure the validity of the maximum principle. This condition was relaxed in [8] allowing that  $\int_{\Omega} K(x)dx = 1$  and either  $\alpha_0 > 0$  or  $K(x) > 0$ .

In the first part of this paper, we analyze the validity of the maximum principle by means of the existence of a principal eigenvalue. For that, we study the eigenvalue problem

$$(4) \quad \begin{cases} -\Delta u + c(x)u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega. \end{cases}$$

Our first result can be summarized as follows:

**Theorem 1.1** *1. There exists the principal eigenvalue of (4), denoted by  $\lambda_1[-\Delta + c; \mathcal{B}; K]$ , that is, the only eigenvalue of (4) possessing a positive eigenfunction.*

*2.  $\lambda_1[-\Delta + c; \mathcal{B}; K]$  is increasing in  $c$  and decreasing in  $K$ .*

3. *The maximum principle for the linear equation*

$$(5) \quad \begin{cases} -\Delta u + c(x)u = f(x) & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx + g(x) & \text{on } \partial\Omega, \end{cases}$$

is satisfied if, and only if,  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$ . We say that (5) satisfies the maximum principle, if given  $f \in C(\overline{\Omega})$ ,  $g \in C(\partial\Omega)$ ,  $f \geq 0$ ,  $g \geq 0$  and some of them non-trivial, then  $u \geq 0$  in  $\Omega$  for any solution  $u$  of (5).

We mention that (4) was only analyzed in [9] when  $K(x) \equiv k$  is a positive constant and Dirichlet boundary condition. The author proved that if there exists a negative eigenvalue then  $k > 1/|\Omega|$  and if  $k < 1/|\Omega|$  then all the eigenvalues are positive. Our results improve this one, because we prove the existence of principal eigenvalue for general function  $K$ , not only constant, see Remark 2.7. On the other hand, in [7] it is proved that when  $c \equiv 0$ ,  $\beta \equiv 1$  and  $K$  verifies (3), then the maximum principle is satisfied. We prove that in this particular case  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$  and then the maximum principle is satisfied. Obviously our result generalizes and improves the above result, see again Remark 2.7 where a detailed comparison with the results of [7] and [8] is made.

With respect to the nonlinear problem (1), our main result is:

**Theorem 1.2** *Equation (1) possesses at least a positive solution if, and only if,  $\lambda > \lambda_1[-\Delta; \mathcal{B}; K]$ . In this case, (1) possesses a unique positive solution.*

We apply the sub-supersolution method to prove the existence and we use properties of the principal eigenvalue, to show that in fact this condition is also necessary to have positive solution and also to show the uniqueness of positive solution.

An outline of the paper is as follows: Section 2 is devoted to study in detail the eigenvalue problem (4). In Section 3, using the sub-supersolution method, we study (1).

## 2 Eigenvalue problems

In this section, we consider the eigenvalue problem

$$(6) \quad \begin{cases} -\Delta u + c(x)u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega. \end{cases}$$

We will show the existence of a principal eigenvalue of (6), that is, an eigenvalue with a positive eigenfunction. Before proving our main result, we introduce some notations.

Consider the space  $X = C(\overline{\Omega})$  and its respective cone  $P =: \{u \in X : u \geq 0 \text{ in } \Omega\}$ . Recall that

$$\text{int}(P) =: \{u \in P : u > \delta_0 \text{ in } \Omega \text{ for some } \delta_0 > 0\}.$$

We say that a function  $u$  is positive, we write  $u \geq 0$ , if  $u \in P$ ;  $u$  is strongly positive if  $u \in \text{int}(P)$ .

Given  $f \in X$ , we denote by

$$f_L := \min_{x \in \overline{\Omega}} f(x) \quad \text{and} \quad f_M := \max_{x \in \overline{\Omega}} f(x).$$

Finally, we denote by  $\lambda_1[-\Delta + c; \mathcal{B}]$  the principal eigenvalue of the problem

$$(7) \quad \begin{cases} -\Delta u + c(x)u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We are ready to state our first main result:

**Theorem 2.1** *There exists the principal eigenvalue of (6), denoted by  $\lambda_1[-\Delta + c; \mathcal{B}; K]$ . This eigenvalue is simple and possesses a unique eigenfunction, up to multiplicative constants, which can be taken positive. Moreover, the principal eigenfunction is strongly positive and  $\lambda_1[-\Delta + c; \mathcal{B}; K]$  is the only eigenvalue of (6) possessing a positive eigenfunction. Furthermore, any other eigenvalue  $\sigma$  of (6) satisfies*

$$(8) \quad \text{Re } \sigma > \lambda_1[-\Delta + c; \mathcal{B}; K].$$

The proof of this result consists of several steps. Let  $R > 0$  be an enough large constant such that  $c_R(x) := c(x) + R > 0$ . Then, the above equation (6) becomes

$$(9) \quad \begin{cases} -\Delta u + c_R(x)u = (\lambda + R)u & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega. \end{cases}$$

We are going to apply the Krein-Rutmann Theorem to this equation. So, we consider the following linear equation:

$$(10) \quad \begin{cases} -\Delta u + c_R(x)u = f(x) & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx & \text{on } \partial\Omega. \end{cases}$$

for  $f \in X$ .

**Proposition 2.2** Consider the operator  $T : X \mapsto X$  defined by  $T(f) := u$ , where  $u$  is the solution of (10). Then, there exists  $R_0 > 0$  such that for  $R \geq R_0$ ,  $T$  is well defined and it is compact. Moreover,

$$T(P \setminus \{0\}) \subset \text{int}(P),$$

that is,  $T$  is a strongly positive operator.

*Proof:* First, take  $R$  large enough such that  $\lambda_1[-\Delta + c_R; \mathcal{B}] > 0$ . The proof depends on the boundary condition. So, we distinguish two cases:

1.- Assume that  $\alpha_0 = 1$ . Consider the unique positive solution  $e_R \in C(\overline{\Omega})$  of

$$(11) \quad \begin{cases} -\Delta e_R + c_R(x)e_R = K(x) & \text{in } \Omega, \\ \mathcal{B}e_R = 0 & \text{on } \partial\Omega. \end{cases}$$

First, we will prove that for  $R$  large we have that

$$\int_{\partial\Omega} e_R dS < 1.$$

Consider  $\varphi_1 > 0$  a positive eigenfunction associated to  $\lambda_1[-\Delta; \mathcal{B}]$ , that is,

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega, \quad \mathcal{B}\varphi_1 = 0 \quad \text{on } \partial\Omega.$$

Then,  $M\varphi_1$  is supersolution of (11) if

$$M\varphi_1(\lambda_1 + c(x) + R) \geq K(x),$$

for which is sufficient that (for  $R$  large)

$$M = \frac{K_M}{(\lambda_1 + c_L + R)(\varphi_1)_L}.$$

Hence

$$e_R \leq \frac{K_M}{(\lambda_1 + c_L + R)(\varphi_1)_L} \varphi_1,$$

from where

$$\int_{\partial\Omega} e_R(x) dS \leq \frac{K_M}{(\lambda_1 + c_L + R)(\varphi_1)_L} \int_{\partial\Omega} \varphi_1 dS \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Now, multiplying (10) by  $e_R$  and integrating we get

$$\int_{\Omega} K(x) u dx \left( 1 - \int_{\partial\Omega} e_R dS \right) = \int_{\Omega} f(x) e_R(x) dx$$

and we conclude for some  $R > 0$  large enough that

$$\int_{\Omega} K(x)u dx = \frac{\int_{\Omega} f e_R dx}{\left(1 - \int_{\partial\Omega} e_R dS\right)}.$$

Hence, (10) is equivalent to

$$(12) \quad \begin{cases} -\Delta u + c_R(x)u = f(x) & \text{in } \Omega, \\ \mathcal{B}u = \frac{\int_{\Omega} f e_R dx}{\left(1 - \int_{\partial\Omega} e_R dS\right)} dx & \text{on } \partial\Omega. \end{cases}$$

Now, it is clear that (12) possesses a unique solution, and so  $T$  is well defined and compact. Moreover, if  $f \geq 0$ ,  $f \neq 0$ , the strong maximum principle implies that  $u$  is strongly positive.

2.- For the Dirichlet case,  $\alpha_0 = 0$  and  $\beta \equiv 1$ , the argument is completely different. For each  $M \in \mathbb{R}$  constant, denote by  $e_{R,M}$  the unique solution of

$$(13) \quad \begin{cases} -\Delta u + c_R(x)u = f(x) & \text{in } \Omega, \\ u = M & \text{on } \partial\Omega. \end{cases}$$

Define the map

$$h(M) := \int_{\Omega} K(x)e_{R,M} dx.$$

We claim that  $u$  is a solution of (10) if, and only if,  $h$  has a fixed point,  $M = h(M)$ . Indeed, assume that  $u$  is solution of (10), denote by

$$M_0 = \int_{\Omega} K(x)u dx.$$

Now, it is clear that  $M_0 = h(M_0)$ .

On the other hand, assume that there exists  $M_0 \in \mathbb{R}$  such that  $M_0 = h(M_0)$ . Then,  $u = e_{R,M_0}$  is solution of (10).

We study now the map  $h$ . Observe that  $h''(M) = 0$ , and then

$$h(M) = aM + b,$$

where

$$b = \int_{\Omega} K(x)e_{R,0} dx, \quad a = \int_{\Omega} K(x)e'_{R,0} dx$$

and  $e'_{R,0}$  is the unique solution of

$$(14) \quad \begin{cases} -\Delta u + c_R(x)u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

Now, we prove that there exists  $R_0 > 0$  such that for  $R \geq R_0$  we have that

$$a = \int_{\Omega} K(x)e'_{R,0} dx < 1,$$

concluding that  $h$  possesses a unique fixed point. Take  $\varphi \in C^2(\overline{\Omega})$ ,  $\varphi > 0$  and  $\varphi = 0$  on  $\partial\Omega$ . Then,  $e^{-C\varphi}$  is supersolution of (14) if

$$C^2|\nabla\varphi|^2 - C\Delta\varphi - (c(x) + R) \leq 0,$$

for which is enough that

$$C^2(|\nabla\varphi|^2)_M - C(\Delta\varphi)_L - (c_L + R) = 0.$$

Then, it is clear that for each  $R > 0$  large enough, there exists  $C(R)$  such that  $C(R) \rightarrow +\infty$  as  $R \rightarrow \infty$ . Then,

$$e'_{R,0} \leq e^{-C(R)\varphi},$$

and hence

$$a = \int_{\Omega} K(x)e'_{R,0} dx \leq \int_{\Omega} K(x)e^{-C(R)\varphi} dx < 1, \quad \text{for } R \text{ large.}$$

This implies that  $h(M) = M$  has a unique solution, denoted by  $M_0$ . Hence, (10) possesses at least a solution  $u$ . We show now that this solution is unique. Assume that (10) has two solutions  $u_1 \neq u_2$ . It is clear that if

$$\int_{\Omega} K(x)u_1 dx = \int_{\Omega} K(x)u_2 dx,$$

then  $u_1 = u_2$ . If  $\int_{\Omega} K(x)u_1 dx \neq \int_{\Omega} K(x)u_2 dx$ , then  $h$  has two fixed points. This shows that (10) possesses a unique solution.

Moreover, when  $f \geq 0$ ,  $f \neq 0$ , we get that  $a, b > 0$ , and then  $M_0 > 0$ ; that is, the unique solution is strictly positive. This proves that  $T$  is well-defined and strongly positive. Finally, by the elliptic regularity, it follows that  $T$  is compact.  $\square$

We are ready to prove the result.

*Proof of Theorem 2.1:* It is enough to apply the Krein-Rutmann Theorem, see for instance [1], and conclude the existence of a positive spectral radius of  $T$ . Then,

$$\lambda_1[-\Delta + c_R; \mathcal{B}; K] = \frac{1}{\text{spr}(T)},$$

and hence,

$$\lambda_1[-\Delta + c; \mathcal{B}; K] = \frac{1}{\text{spr}(T)} - R.$$

□

Now, we want to study the relation of the principal eigenvalue with the maximum principle and its properties. The first result compares the eigenvalues  $\lambda_1[-\Delta + c; \mathcal{B}]$  and  $\lambda_1[-\Delta + c; \mathcal{B}; K]$  and characterizes the sign of  $\lambda_1[-\Delta + c; \mathcal{B}; K]$ .

**Proposition 2.3** 1. *It holds that*

$$\lambda_1[-\Delta + c; \mathcal{B}; K] \leq \lambda_1[-\Delta + c; \mathcal{B}].$$

2. *Assume that  $\lambda_1[-\Delta + c; \mathcal{B}] \leq 0$ , then  $\lambda_1[-\Delta + c; \mathcal{B}; K] \leq 0$ .*

3. *Assume that  $\lambda_1[-\Delta + c; \mathcal{B}] > 0$ , then*

$$\text{sgn}(\lambda_1[-\Delta + c; \mathcal{B}; K]) = \text{sgn}\left(1 - \int_{\partial\Omega} e(x) dS\right),$$

*for the Robin case, and*

$$\text{sgn}(\lambda_1[-\Delta + c; \mathcal{B}; K]) = \text{sgn}\left(1 + \int_{\partial\Omega} \partial_\eta e(x) dS\right),$$

*for the Dirichlet case, where  $e$  is the unique positive solution of*

$$(15) \quad \begin{cases} -\Delta e + c(x)e = K(x) & \text{in } \Omega, \\ \mathcal{B}e = 0 & \text{on } \partial\Omega. \end{cases}$$

*Proof:* Denote by  $\varphi_1$  and  $\psi_1$  the positive eigenfunction associated to  $\lambda_1[-\Delta + c; \mathcal{B}; K]$  and  $\lambda_1[-\Delta + c; \mathcal{B}]$ , respectively. Multiplying the equation of  $\varphi_1$  by  $\psi_1$ , we can integrate by parts and obtain

$$\int_{\partial\Omega} (\partial_\eta \psi_1 \varphi_1 - \partial_\eta \varphi_1 \psi_1) dS = (\lambda_1[-\Delta + c; \mathcal{B}; K] - \lambda_1[-\Delta + c; \mathcal{B}]) \int_{\Omega} \psi_1 \varphi_1 dx.$$



For both boundary conditions, we obtain the first statement, and consequently also the second one. Observe that in the Dirichlet case we have used that  $\partial_\eta \psi_1 < 0$  on  $\partial\Omega$ .

Assume now that  $\lambda_1[-\Delta + c; \mathcal{B}] > 0$ , then there exists a unique positive solution of (15). Multiplying (15) by  $\varphi_1$  and integrating by parts, we obtain

$$\int_{\Omega} K(x)\varphi_1 dx + \int_{\partial\Omega} (\partial_\eta e\varphi_1 - \partial_\eta \varphi_1 e) dS = \lambda_1[-\Delta + c; \mathcal{B}; K] \int_{\Omega} e\varphi_1 dx.$$

When  $\alpha_0 = 1$ , we have

$$\int_{\Omega} K(x)\varphi_1 dx \left(1 - \int_{\partial\Omega} e dS\right) = \lambda_1[-\Delta + c; \mathcal{B}; K] \int_{\Omega} e\varphi_1 dx,$$

and for  $\alpha_0 = 0$

$$\int_{\Omega} K(x)\varphi_1 dx \left(1 + \int_{\partial\Omega} \partial_\eta e dS\right) = \lambda_1[-\Delta + c; \mathcal{B}; K] \int_{\Omega} e\varphi_1 dx,$$

that is, the third statement is satisfied. This completes the result.  $\square$

In the following result, we give a useful result to prove that the principal eigenvalue is positive. First, we need a definition:

**Definition 2.4** *Given  $f \in C(\overline{\Omega})$ ,  $g \in C(\partial\Omega)$ ,  $f \geq 0$ ,  $g \geq 0$  and some of them non-trivial. We say that*

$$(16) \quad \begin{cases} -\Delta u + c(x)u = f(x) & \text{in } \Omega, \\ \mathcal{B}u = \int_{\Omega} K(x)u(x)dx + g(x) & \text{on } \partial\Omega, \end{cases}$$

*satisfies the maximum principle, if for any solution  $u$  of (16) lies in  $P$ . If  $u \in \text{int}(P)$ , we say that (16) satisfies the strong maximum principle.*

**Proposition 2.5** *The following conditions are equivalent:*

1.  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$ ,
2. *There exists a strict positive supersolution, that is, a function  $\bar{u} > 0$  in  $\Omega$  such that*

$$(-\Delta + c(x))\bar{u} \geq 0 \quad \text{in } \Omega, \quad \mathcal{B}\bar{u} \geq \int_{\Omega} K(x)\bar{u} dx \quad \text{on } \partial\Omega,$$

*with some inequality strict.*

3. (16) satisfies the maximum principle.

4. (16) satisfies the strong maximum principle.

*Proof:* It is clear that if  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$ , then the positive eigenfunction  $\varphi_1$  is a strict positive supersolution.

Assume now the existence of a strict positive supersolution  $\bar{u}$ . Multiplying the equation (15) by  $\bar{u}$  we obtain

$$\int_{\Omega} (-\Delta \bar{u} + c(x)\bar{u})e dx + \int_{\partial\Omega} (\partial_{\eta} \bar{u}e - \partial_{\eta} e\bar{u})dS = \int_{\Omega} K(x)\bar{u}dx,$$

and then for the case  $\alpha_0 = 1$ ,

$$0 \leq \int_{\Omega} K(x)\bar{u}dx \left(1 - \int_{\partial\Omega} edS\right),$$

whence, using Proposition 2.3, we conclude that  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$ . A similar argument works for the case  $\alpha_0 = 0$ .

Suppose that  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$  and  $f \geq 0$ ,  $g \geq 0$ , some of them non-trivial and consider  $u$  the solution of (16). Since  $\lambda_1[-\Delta + c; \mathcal{B}; K] > 0$ , by Proposition 2.3 we get that  $\lambda_1[-\Delta + c; \mathcal{B}] > 0$ , and so there exists a unique positive solution  $e$  of (15). Multiplying the above inequality by  $e$ , we obtain

$$\int_{\Omega} fedx \leq \int_{\Omega} K(x)udx \left(1 - \int_{\partial\Omega} edS\right),$$

whence we obtain that  $\int_{\Omega} K(x)udx \geq 0$  and so  $u \geq 0$  in  $\Omega$ . Since, some inequality is strict, it follows that  $u$  is strongly positive, and so the strong maximum principle is satisfied.

Finally, it is clear that if the strong maximum principle is satisfied, then there exists a positive supersolution. This completes the proof. □

Finally, we get the following properties:

**Proposition 2.6** *The principal eigenvalue  $\lambda_1[-\Delta + c; \mathcal{B}; K]$  is increasing in  $c$  and decreasing in  $K$ .*

*Proof:* Assume that  $K_1 \geq K_2$ ,  $K_1 \neq K_2$  in  $\Omega$ , and denote  $\varphi_1$  the positive eigenfunction associated to  $\lambda_1[-\Delta + c; \mathcal{B}; K_1]$ . Then,

$$(-\Delta + c(x) - \lambda_1[-\Delta + c; \mathcal{B}; K_1])\varphi_1 = 0 \quad \text{in } \Omega,$$

and

$$\mathcal{B}\varphi_1 = \int_{\Omega} K_1(x)\varphi_1 dx > \int_{\Omega} K_2(x)\varphi_1 dx \quad \text{on } \partial\Omega,$$

hence,  $\varphi_1$  is a supersolution of the above problem. So, by Proposition 2.5 we obtain that

$$\lambda_1[-\Delta + c - \lambda_1[-\Delta + c; \mathcal{B}; K_1]; \mathcal{B}; K_2] > 0,$$

that is,

$$\lambda_1[-\Delta + c; \mathcal{B}; K_2] > \lambda_1[-\Delta + c; \mathcal{B}; K_1].$$

In a similar way, it can be proved that the principal eigenvalue is increasing in  $c$ . □

**Remark 2.7** *We show that our results improve those in [7], [8] and [9].*

1. *Assume as in [7] that  $c \equiv 0$  and  $\beta \equiv 1$ . If  $K$  verifies (3), then positive constants are supersolutions and then the maximum principle is satisfied, as it is proven in [7]. In fact, this condition is also necessary for the maximum principle holds. Indeed, observe that in this case, equation (15) is*

$$-\Delta e = K(x) \quad \text{in } \Omega, \quad \mathcal{B}e = 0 \quad \text{on } \partial\Omega,$$

*and then, when  $\alpha_0 = 1$  we have that*

$$\int_{\partial\Omega} e(x) dS = \int_{\Omega} K(x) dx,$$

*and for  $\alpha_0 = 0$*

$$-\int_{\partial\Omega} \partial_{\eta} e(x) dS = \int_{\Omega} K(x) dx.$$

*So, by Proposition 2.3,  $\text{sgn}(\lambda_1[-\Delta; \mathcal{B}; K]) = \text{sgn}(1 - \int_{\Omega} K(x) dx)$ . Hence, by Proposition 2.5, (3) is a necessary and sufficient condition to hold the maximum principle.*

2. *Assume that  $c \geq 0$ ,  $c \neq 0$ ,  $\beta \equiv 1$  and  $\int_{\Omega} K(x) = 1$  as in [8]. Again, positive constants are supersolutions, and so the maximum principle is satisfied.*

3. *Assume that  $K(x) = k$  as in [9]. So,*

$$\int_{\Omega} K(x) dx = k|\Omega|.$$

*Then, if  $k|\Omega| < 1$  and by (8) all the real eigenvalues are positive. On the other hand, if  $k|\Omega| > 1$  then  $\lambda_1[-\Delta; \mathcal{B}; K] < 0$ , implying Proposition 2.1 in [9].*

### 3 Nonlinear Problem

In this section we study the nonlinear problem (1) and prove Theorem 1.2.

*Proof of Theorem 1.2:* Assume that (1) has a unique positive solution  $u$ . Then, using Proposition 2.6

$$\lambda = \lambda_1[-\Delta + u^{p-1}; \mathcal{B}; K] > \lambda_1[-\Delta; \mathcal{B}; K].$$

Reciprocally, let us suppose that  $\lambda > \lambda_1[-\Delta; \mathcal{B}; K]$  and let  $\varphi_1$  be an eigenfunction related to  $\lambda_1[-\Delta; \mathcal{B}; K]$ . If we consider  $0 < \varepsilon < [(\lambda - \lambda_1[-\Delta; \mathcal{B}; K]) / (\varphi_1)_M]^{\frac{1}{p-1}}$ , it follows that

$$-\Delta(\varepsilon\varphi_1) = \lambda_1\varepsilon\varphi_1 \leq \lambda(\varepsilon\varphi_1) - (\varepsilon\varphi_1)^p.$$

So, by denoting  $\underline{u} := \varepsilon\varphi_1$ , we have that  $\underline{u}$  is a lower solution of (1). By the other hand, if we consider  $M > [(\lambda - \lambda_1[-\Delta; \mathcal{B}; K]) / (\varphi_1)_L]^{\frac{1}{p-1}}$ , it follows that

$$-\Delta(M\varphi_1) = \lambda_1M\varphi_1 \geq \lambda(M\varphi_1) - (M\varphi_1)^p.$$

Then, by denoting  $\bar{u} := M\varphi_1$ , we have that  $\bar{u}$  is an upper solution of (1). Hence, using for example Theorem 3.1 in [7], we conclude that (1) has a positive solution.

Now, we will prove that the solution of (1) is unique. If  $u_1$  and  $u_2$  are solutions of the equation (1) such that

$$(17) \quad \int_{\Omega} K(x)u_1 dx = \int_{\Omega} K(x)u_2 dx,$$

then we have that  $u_1 = u_2$ . Indeed, observe that  $u_1$  is a positive solution of the logistic equation

$$(18) \quad -\Delta w = \lambda w - w^p \quad \text{in } \Omega, \quad \mathcal{B}w = \int_{\Omega} K(x)u_2 dx.$$

But, this equation possesses a unique positive solution, see for instance [4], and so  $u_1 = u_2$ .

Then, we will prove uniqueness when (17) does not occur. Let us suppose that

$$\int_{\Omega} K(x)u_1 dx > \int_{\Omega} K(x)u_2 dx.$$

Observe that  $u_1$  is supersolution of (18), and so  $u_1 > u_2$ . Denoting  $w := u_1 - u_2 > 0$ , we get that

$$\begin{cases} -\Delta w = \lambda w - (u_1^p - u_2^p) & \text{in } \Omega, \\ \mathcal{B}w = \int_{\Omega} K(x)w(x) dx & \text{on } \partial\Omega, \end{cases}$$

and then

$$\begin{cases} -\Delta w + \frac{u_1^p - u_2^p}{u_1 - u_2} w = \lambda w & \text{in } \Omega, \\ \mathcal{B}w = \int_{\Omega} K(x)w(x)dx & \text{on } \partial\Omega. \end{cases}$$

But,

$$\frac{u_1^p - u_2^p}{u_1 - u_2} > u_1^{p-1}$$

and then

$$\lambda = \lambda_1[-\Delta + \frac{u_1^p - u_2^p}{u_1 - u_2}; \mathcal{B}; K] > \lambda_1[-\Delta + u_1^{p-1}; \mathcal{B}; K] = \lambda,$$

which is an absurd. □

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