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Discrete breathers collisions in nonlinear Schrödinger and Klein–Gordon lattices

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Collisions between moving localized modes (moving breathers) in non-integrable lattices present a rich outcome. In this paper, some features of the interaction of moving breathers in Discrete Nonlinear Schrödinger and Klein–Gordon lattices, together with some plausible explanations, are exposed.

Keywords: Discrete breathers; collisions; NLS lattices; Klein–Gordon lattices.

1. Introduction

Discrete breathers are localized modes that arise in nonlinear discrete lattices.[1] Under certain conditions, these localized entities can move through the lattice and are denoted as moving breathers. Most of the models where discrete breathers exist are non-integrable, and, contrary to continuum nonlinear localized excitations (solitons),[2] discrete breathers collisions exhibit a rich behaviour. One of the systems where discrete breathers have been extensively studied is the Discrete Nonlinear Shcrödinger (DNLS) chain,

$$i\dot{u}_n + f(|u_n|^2)u_n + (u_{n+1} + u_{n-1} - 2u_n) = 0. \tag{1}$$

Stationary solutions, oscillating with frequency ω_b, are given by $u_n(t) = \exp(i\omega_\mathrm{b} t)v_n$, and can be calculated using methods based on the anti-continuous limit.[3] This equation has two conserved quantities: the Hamiltonian and the power (or norm). The latter is given by $P = \sum_n |u_n|^2$. Breathers can be put into movement by adding a momentum q so that the initial condition has the form $u_n(0) = v_n \exp(iqn)$.

Another system where discrete breathers appear is the Klein–Gordon chain,

$$\ddot{u}_n + V'(u_n) - C(u_{n+1} + u_{n-1} - 2u_n) = 0 \tag{2}$$

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where C is a coupling constant and $V(u)$ is the substrate potential. The method for calculating stationary breathers is similar to the used in the DNLS equation. Moving breathers are also generated by adding a momentum, an analogously method to the originally used in Ref. 4. Both approaches consist in breaking the shift translational symmetry of the stationary breather.

The aim of this paper is to show some features of binary symmetric collisions between breathers in DNLS and Klein–Gordon lattices, and pose some plausible explanations to the observed outcome.

2. DNLS lattices

Symmetric collisions of discrete breathers in the DNLS equation with a cubic (or Kerr) nonlinearity, $f(|u|^2) = |u|^2$ in Eq. (1), was considered in.[5] The results of this paper show that, for small incoming velocities, breathers get trapped. For high velocities, the collision is quasi-elastic and breathers are refracted. This outcome is shown in Fig. 1. Both behaviours are separated by a critical value of the initial momentum, q_c. In that paper, inter-site and on-site collisions were considered. In the first case the value of q_c is an order of magnitude higher than the one of the second case.

Collisions in the DNLS equation with saturable nonlinearity, $f(|u|^2) = -\beta/(1 + |u|^2)$ in Eq. (1), were studied in Ref. 6 for $\beta = 2$ and for arbitrary values of β in Refs. 7,8. This equation allows the existence of moving breathers with any arbitrarily high power, contrary to the cubic DNLS equation, where the power of the moving breathers is limited. In addition, three regimes are observed when moving breathers collide in this model, separated by two critical values, $q_{c1} > q_{c2}$. The order of magnitude of these critical values is the same for inter-site and on-site collisions. For $q < q_{c1}$, breathers get trapped; for $q \in (q_{c1}, q_{c2})$, breathers are refracted; and, for $q > q_{c2}$, breathers are refracted, remaining a great part of the energy trapped. This regime, also called *breather creation* takes place only if the power is high enough. Figure 1(left) shows the above mentioned behaviours and Fig. 2(left) depicts the parameter ranges for which these regimes are observed.

3. Klein–Gordon lattices

Discrete breathers collisions in Klein–Gordon lattices have been considered in Ref. 9, where a Morse substrate potential, $V(u) = (\exp(-u) - 1)^2/2$, was

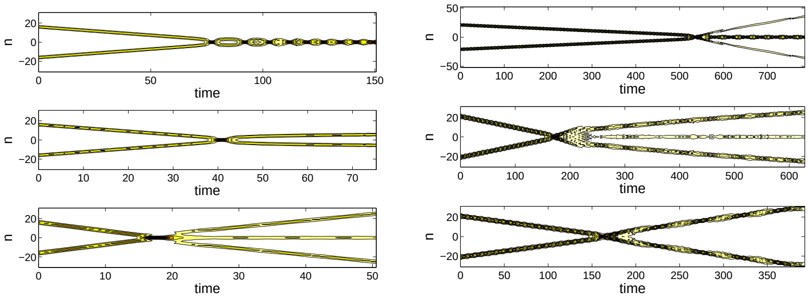

Fig. 1. Outcomes of discrete breathers collisions for saturable DNLS (left) and Klein–Gordon chains with $\omega_\mathrm{b} = 0.8$ (right).

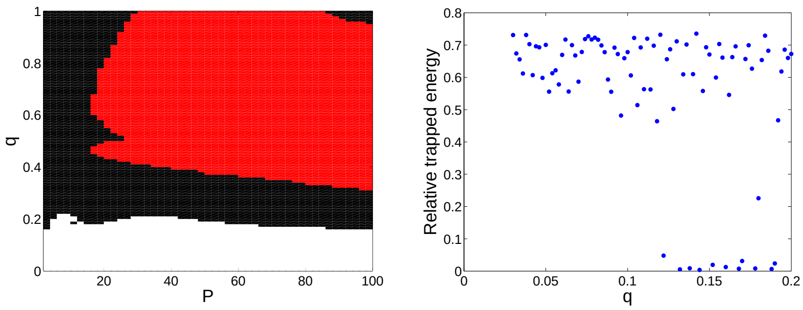

Fig. 2. (Left panel) Different regimes observed in inter-site collisions in the saturable DNLS equation with $\beta = 2$. Colours represent the following: white-trapping; black-refraction; and red-breather creation. (Right panel) Relative trapped energy after the collision in the Klein–Gordon lattice with $C = 0.32$ and $\omega_\mathrm{b} = 0.8$.

chosen[a]. Two breather frequencies ω_b were chosen. For $\omega_\mathrm{b} = 0.95$, which is close to the linear modes band, the outcome is similar to the observed in the cubic DNLS equation, except for the fact that q_c has the same order of magnitude for both inter-site and on-site collisions. For $\omega_\mathrm{b} = 0.8$, i.e. a frequency far from the linear modes band, the nonlinearity is high and a radically different outcome is observed. First of all, the collisions are strongly phase-dependent, depending the outcome even on the initial distance between the incoming breathers. For this reason, it has no sense to distinguish between inter-site and on-site collisions. For small incoming

[a]Breather collisions in FPU lattices (i.e. Klein–Gordon lattices without substrate potential) have been extensively studied in Ref. 10

velocities, partial trapping is observed. As Fig. 1(right) shows, part of the energy is trapped and another part is emitted as a pair of small-amplitude or high-amplitude moving breathers. For high velocities, the collision can be quasi-elastic although partial trapping is also observed. In consequence, there exists a critical value q_c below which no breather refraction (or equivalently, only partial trapping) is observed. Figure 2(right) shows the relative trapped energy after the collision. For quasi-elastic collisions, this value drops to a value close to zero.

4. Interpretation

In the section, we provide some plausible explanations for the symmetric collision scenario observed in the previous sections.

Some of the moving breathers features can be explained supposing that they behave as quasi-particles moving in a periodic potential, known as *Peierls-Nabarro potential*. In order to move a breather, a kinetic energy must be transferred to a stationary breather. The minimum value of this energy receives the name of Peierls–Nabarro barrier (PNB), and its value is the energy difference between an inter-site breather and an on-site breather, with the same power or action, for DNLS and Klein–Gordon breathers respectively.[11] Figure 3(left) shows the PNB as a function of the power for the cubic and saturable DNLS. It can be observed that in the former, the PNB grows monotonically with the power, and, in consequence, there are no breathers for high values of the power. On the contrary, for the saturable DNLS, the PNB is bounded, and it is possible to find moving breathers for high powers. This is also related to the existence of non-radiating breathers (i.e. free of PNB) for the saturable DNLS,[8] and the non-existence of those entities in the cubic DNLS.[13] Figure 3(right) shows the PNB as a function of the coupling constant C for Klein–Gordon lattices. It can be observed that the PNB is bounded, as in the saturable DNLS equation. The high value of the PNB in the cubic DNLS equation is the reason for the high difference of the value of q_c when inter-site and on-site collisions are considered in that framework.

The scenario observed in the cubic DNLS equation was explained through a variational approach in Ref. 5. Apart from this mathematical point of view, there is a rough physical explanation: when two breathers interact, they attract each other, creating a potential well. Thus, if their velocity is below the "escape" velocity of the potential well, they get trapped. If the velocity of the incoming breathers is above that critical value, they are refracted and no trapping is observed. The existence of the critical value q_c

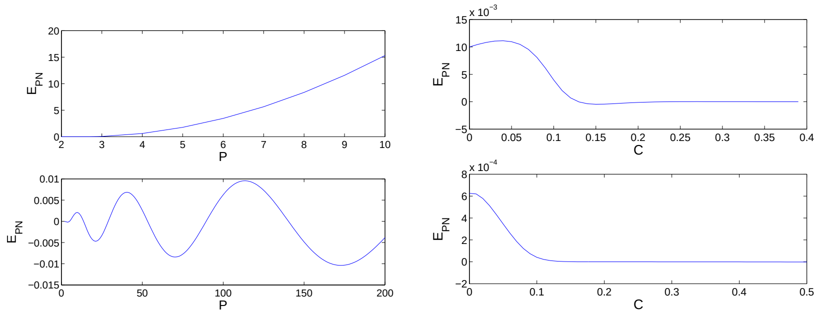

Fig. 3. (Left panel) Peierls–Nabarro barrier for saturable DNLS breathers with $\beta = 2$ (top) and cubic DNLS breathers (bottom). (Right panel) Peierls–Nabarro barrier for Klein–Gordon breathers with $C = 0.32$ and $\omega_{\rm b} = 0.8$ (top) and $\omega_{\rm b} = 0.95$ (bottom).

is explained by the increasing dependence of the incoming breathers velocity with q. This physical explanation is valid also for explaining the existence of q_{c1} in the saturable DNLS equation. In the Klein–Gordon equation for $\omega_{\rm b} = 0.95$, which is close to the linear modes band, breathers can be approximated by envelope discrete breathers of the cubic DNLS equation.[12] Thus, the scenario should be similar to the observed in the cubic DNLS equation. This is not found for $\omega_{\rm b} = 0.8$. Instead, no total trapping of the energy after the collision is observed, as two moving breathers escape from the potential well, additionally to the trapped one. Apart from this, there is a no clearly defined escape velocity. Contrary to the saturable DNLS breathers, in the Klein–Gordon case, the partial trapping can be observed for smaller velocities than those where the reflection takes place.

The existence of partial trapping in Klein–Gordon lattices can be explained by means of an energetic balance. Incoming breathers can be described as quasiparticles with energy E_i, whereas the outgoing breathers energy is E_o; the trapped energy is denoted as $U_{\rm trap}$. Then, neglecting the phonon radiation, it is fulfilled that $E_i = U_{\rm trap}/2 + E_o$. For a localized trapped breather to be formed, its energy must be similar to that of a static breather. For a given value of C, the static breather energy possesses a maximum $\tilde{E}$, which corresponds to the minimum value of the frequency (i.e. to the resonance of the second harmonic of the breather frequency with the phonon band — see Fig. 4). Thus, $U_{\rm trap} < \tilde{E}$, and, if $E_i > \tilde{E}/2$, there is an exceeding energy that is emitted as two outgoing moving breathers. This is the case of breathers with $\omega_{\rm b} = 0.8$. When $\omega_{\rm b} = 0.95$, $E_i < \tilde{E}/2$, which explains the non existence of outgoing breathers apart from the trapped one (see Fig. 4). This analysis cannot be done for the saturable DNLS as in this case the energy is not bounded.[14]

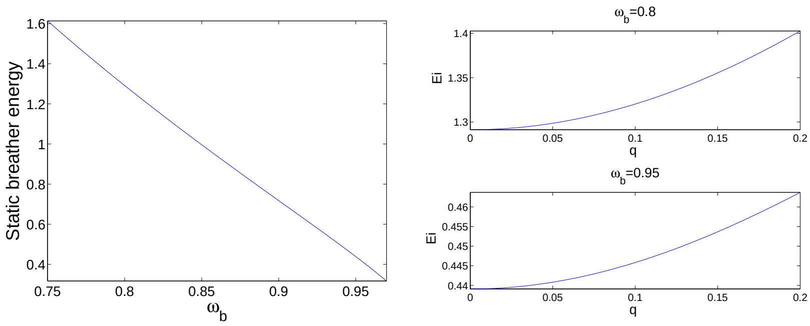

Fig. 4. (Left panel) Energy of static breathers in Klein–Gordon lattices with respect to $\omega_{\rm b}$ for $C = 0.32$. The maximum value of the energies is $\tilde{E} = 1.5798$ and corresponds to $\omega_{\rm b} = 0.7550$. (Right panel) Incoming moving breathers energies versus q for $\omega_{\rm b} = 0.8$ and $\omega_{\rm b} = 0.95$. Clearly, for $\omega_{\rm b} = 0.95$, $E_i < \tilde{E}/2\ \forall q$.

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