UNIVERSITY OF SEVILLE

Faculty of Mathematics

Department of Geometry and Topology

ABELIAN SUBALGEBRAS AND IDEALS OF MAXIMAL DIMENSION IN LIE ALGEBRAS

Dissertation presented by Manuel Ceballos González to obtain the Ph.D. degree in Mathematics from the University of Seville.

Signed:

Manuel Ceballos González
M.Sc. in Mathematics

Seen and approved by the advisors:

Juan Núñez Valdés
Associate Professor
Dept. Geometry and Topology
University of Seville

Ángel F. Tenorio Villalón
Assistant Professor
Dept. Economics, Quantitative Methods, and Economic History
Pablo de Olavide University

Seville, January 2012

ACKNOWLEDGEMENTS

It is really difficult to find the correct words to thank all the people that have been with me through this period of my life. First of all, I should start with my advisors. D. Juan Núñez and D. Ángel Francisco Tenorio are a very important part of this work. I believe that without their suggestions and support in every moment, this dissertation would not exist.

I would like to thank D. Juan Núñez for his suggestions and his continuous support to keep improving this work. He has also encouraged me to spend some months abroad during my Ph.D. in order to learn more. Moreover, he was the first person who introduced me to the mathematical research world.

I would like to thank D. Ángel Francisco Tenorio for his patience and the multiple corrections that he has done on this work. He introduced me to the study of abelian subalgebras and ideals and he has also given me a very useful point of view so that we could obtain more and new results.

There are other people that have helped me and I would like to dedicate a few words for them. First, Professor Dietrich Burde, from the University of Vienna introduced me to the general concepts of α and β invariants. Then, Professor David Towers from the University of Lancaster, who taught me a lot about supersolvable Lie algebras, maximal subalgebras and other related topics. Finally, I would like to thank Professor Francisco J. Castro from the University of Seville for his useful comments and ideas related to the relations between the coefficients in the general laws of filiform Lie algebras.

Of course, I also have to thank my family, relatives and friends since they have supported me in every moment. There seems to have been a very long process since I started studying Mathematics until this moment that I am writing these words trying to obtain the doctoral degree.

Contents

Introduction

Apart from the study of Lie Theory from a purely theoretical view-point, extensive research about this theory exists due to its many applications in Engineering, Physics and, above all, Applied Mathematics. However, some aspects of Lie algebras still remain to be studied. Indeed, the classification of nilpotent and solvable Lie algebras is still an open problem, although the classification of other types of Lie algebras (like semisimple and simple ones) was already obtained in 1890. In order to solve these and other problems, the need of studying additional properties of Lie algebras arises. For example, conditions on the lattice of subalgebras of a Lie algebra often lead to information about the Lie algebra itself. In fact, studying abelian Lie subalgebras and ideals of a finite-dimensional Lie algebra constitutes the main goal of this dissertation.

It is also convenient to indicate the motivations leading to obtain more information about Lie algebras in general. In a high percentage, the reasons lie in the possibility of using Lie algebras and their properties as tools in the study of several topics in Physics and Economics, for instance. Indeed, at present, Lie algebras and groups are widely used in Modern Physics. For example, a classic use of Lie Theory corresponds to the study of symmetries (see [55, 87]). Nowadays, symmetries are not limited to those geometrical versions of space-time; but there are other new symmetries associated with “internal” degrees of freedom of particles and fields. Regarding to possible economic and finance applications, we advise to consult [53] as starting point.

Another application of Lie groups and algebras corresponds to Einstein spaces and manifolds. To classify Einstein spaces according to their respective isometry groups in dimension 4, Petrov [85] used Bianchi’s works and applied the classification of low dimensional Lie algebras to General Relativity (see also [93]). Besides, a particular type of Einstein solvmanifold (which is called standard) is based on computing an abelian orthogonal complement [51] and, therefore, it would be interesting to obtain an algorithmic method which computes abelian subalgebras in a given non-abelian Lie algebra.

Let us consider a finite-dimensional Lie algebra, $\mathfrak{g}$. Let $\alpha(\mathfrak{g})$ denote the maximal dimension of an abelian subalgebra of $\mathfrak{g}$, and $\beta(\mathfrak{g})$ the maximal dimension of an abelian ideal of $\mathfrak{g}$. Both invariants are important for many subjects. First of all they are very useful in the study of Lie algebra contractions and degenerations. There is extensive literature on these topics, in particular for low-dimensional Lie algebras, as can be seen in [19, 48, 49, 77, 92] and the references given therein.

The first author dealing with the invariant $\alpha(\mathfrak{g})$ was Schur [91], who studied in 1905 the abelian subalgebras of maximal dimension contained in the Lie algebra of $n \times n$ square matrices. Schur proved that *the maximum number of linearly independent commuting $n \times n$ matrices over an algebraically closed field is* $\left[\frac{n^2}{4}\right] + 1$, which is the maximal dimension of abelian ideals of Borel subalgebras in the general linear Lie algebra $\mathfrak{gl}(n)$ (where $[x]$ denotes the integer part of a real number x). Let us note that this result was obtained only over algebraically closed fields such as the complex number field. Almost forty years later, in 1944, Jacobson [57] gave a simpler proof of Schur's results, extending them from algebraically closed fields to arbitrary fields. This fact allowed several authors to deal later with the study of the abelian subalgebras of maximal dimension of many different types of Lie algebras.

As it is well-known, Lie algebras can be distinguished in three different types: solvable ones, semisimple ones and the remaining algebras which do not belong to any of the two previous types. More concretely, Levi [69] and Malcev [72] proved (in 1905 and 1945, respectively) that every finite-dimensional Lie algebra can be expressed as a semi-direct sum of a semisimple subalgebra and its radical (i.e. a solvable ideal). Regarding this, it was also proved that every semisimple Lie algebra can be decomposed in a direct sum of simple Lie algebras. Hence, the classification of Lie algebras can be reduced to obtain the classification of both semisimple and solvable Lie algebras.

Additionally, Killing [59, 60, 61, 62] and Cartan [20] (among others) obtained the classification of simple Lie algebras at the end of the nineteenth century. Moreover, taking advantage of Killing and Cartan's classification, Malcev [72] computed the α invariant for semisimple Lie algebras in 1945 giving the value of this invariant for simple ones according to Table 1.

Since there are no abelian ideals in a simple Lie algebra $\mathfrak{s}$, we have $\beta(\mathfrak{s}) = 0$. Very recently, the study of abelian ideals in a Borel subalgebra $\mathfrak{b}$ of a simple complex Lie algebra $\mathfrak{s}$ has drawn considerable attention. In this case, $\alpha(\mathfrak{s}) = \beta(\mathfrak{b})$ holds, and this value can be computed purely in terms of certain root system invariants, see [97]. Konstant [66] also dealt with the topic of abelian ideals in a Borel subalgebra of a given Lie algebra $\mathfrak{g}$ by studying the proof of the following theorem, which has been

Table 1: The invariant α for simple Lie algebras

$\mathfrak{s}$	$\dim(\mathfrak{s})$	$\alpha(\mathfrak{s})$
A_n, $n \geq 1$	$n(n+2)$	$\lfloor(\frac{n+1}{2})^2\rfloor$
B_3	21	5
B_n, $n \geq 4$	$n(2n+1)$	$\frac{n(n-1)}{2}+1$
C_n, $n \geq 2$	$n(2n+1)$	$\frac{n(n+1)}{2}$
D_n, $n \geq 4$	$n(2n-1)$	$\frac{n(n-1)}{2}$
G_2	14	3
F_4	52	9
E_6	78	16
E_7	133	27
E_8	248	36

attributed to Peterson in [66]: *the number of abelian ideals in a fixed Borel subalgebra of* $\mathfrak{g}$ *is* 2^r, *where* $r = \mathrm{rank}(\mathfrak{g})$. Regarding this, some authors like Cellini, Papi and Orsina [30, 80] have recently obtained new properties of the maximal abelian ideals in a Borel subalgebra as well as a generalization of Peterson's theorem from abelian ideals to ad-nilpotent ones.

Furthermore, Kostant [66, 67] found a relation of these invariants α and β with discrete series representations of the corresponding Lie group, and with powers of the Euler product. In fact, there are much more results concerning the invariants α and β for simple Lie algebras and their Borel subalgebras.

There are also several results concerning the question of how large or small these maximal dimensions can be, in comparison with the dimension of the Lie algebra (see [73, 88, 94] for example). The results show, roughly speaking, that a Lie algebra of large dimension contains abelian subalgebras of large dimension. For example, the dimension of a nilpotent Lie algebra $\mathfrak{g}$ satisfying $\alpha(\mathfrak{g}) = \ell$ is bounded by $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+1)}{2}$ [88, 94]. There is a better bound in [74] for 2-step nilpotent Lie algebras with $\alpha(\mathfrak{g}) = l$ and $\ell \geq 8$: $dim(\mathfrak{g}) \leq [\frac{\ell^2+4}{8}] + \ell$. If $\mathfrak{g}$ is a complex solvable Lie algebra with $\alpha(\mathfrak{g}) = \ell$, then the bound $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+3)}{2}$ was proved in [73]. In general, $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+17)}{2}$ for any complex Lie algebra $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = \ell$, see [73].

The problem of classifying abelian subalgebras of Lie algebras also has a long history, starting from the Killing-Cartan classification of semisimple Lie algebras. However, the problem of determining such subalgebras is far from being solved, due to the lack of structural criteria for general types of Lie algebras.

Bearing in mind the results obtained by Schur [91], Jacobson [57] and Malcev [72], some authors have dealt with this topic in order to achieve new results for abelian subalgebras of an arbitrary Lie algebra. For example, let us recall that Suprunenko and Tyshkevich [96] dealt in 1968 with the problem of determining abelian subalgebras of maximal dimension of nilpotent type. Now, we are going to analyze and summarize some important papers about this subject.

First, Stewart [94] studied in 1970 some properties about nilpotent Lie algebras containing abelian ideals of maximal dimension. Some results in this paper are the following

Lemma.

1. A maximal abelian ideal of a nilpotent Lie algebra is self-centralizing.
2. The Fitting ideal (sum of all nilpotent ideals) of a solvable Lie algebra contains its centralizer.

Lemma. Let L be a finite-dimensional nilpotent Lie algebra and any maximal abelian ideal A of L. Suppose that $dim(A) \leq a$. Then L/A is isomorphic to a Lie algebra of $a \times a$ zero-triangular matrices.

Theorem. If L is nilpotent and the dimension of all its abelian ideals is at most n, then L has dimension $\leq \frac{n(n+1)}{2}$, nilpotency class $\leq 2n-1$ and derived length $\leq 2 + log_2(n)$.

After that, Bratzlavsky [14] studied in 1974 the law of some nilpotent Lie algebras of dimension n and class $n-1$. These algebras are known as filiform Lie algebras and were introduced by Vergne [107]. More concretely, Bratzlavsky obtained the canonical forms for the structure of filiform Lie algebras having an abelian derived Lie algebra. One of the results obtained in that article was the following

Theorem. For a filiform Lie algebra whose derived Lie algebra is abelian, there exists a basis $\{x_1, \ldots, x_n\}$ such that

$$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq n-1; \quad [x_i, x_j] = 0,\ \text{for } 3 \leq i < j \quad \text{and}$$

$$[x_2, x_i] = \sum \lambda_r x_{i+2+r},\ \text{for } 3 \leq i \leq n-2,\ \text{and } r \leq n-i-2.$$

Later, Kubo [68] in 1978 wondered whether there would exist some inclusions between the following families of Lie algebras: a) those containing finite-dimensional abelian ideals; b) those containing finite-dimensional nilpotent ideals; c) those satisfying the maximal condition for abelian, nilpotent and solvable ideals respectively; and finally, d) those satisfying the minimal condition for abelian, nilpotent and solvable ideals respectively.

Answering himself that question in his own paper [68], Kubo obtained the two following results by using several tools like tensorial extensions, adjoint transformations and central simple Lie algebras, for instance.

Theorem. The class of Lie algebras which contains finite-dimensional abelian ideals is not equal to the one which contains finite-dimensional nilpotent ideals.

Theorem. It is possible to find a Lie algebra verifying the maximal and minimal condition for abelian ideals which does not verify the same conditions for solvable ideals.

Three years later, in 1981, Zaicev [110] proved a result about the relative distribution of an abelian ideal and a positive polarization in an arbitrary Lie algebra, applying this result to find representations of Lie groups with an abelian normal subgroup. Additionally, Zaicev also developed in that reference a theory (introduced by Kirillov [63]) about the extension of orbits to solvable Lie groups. He used the notion of polarization to deal with arbitrary Lie algebras and Lie groups instead of solvable ones. Moreover, Zaicev considered abelian ideals instead of abelian subalgebras, which requires more restrictive conditions. Let us note that this notion of polarization has also an independent algebraic interest.

In his work, Zaicev used the following main tools: real Lie groups, their associated Lie algebras, dual spaces of Lie algebras, stationary groups, the total positive polarization of an element in the dual space of a given Lie algebra with respect to a stationary group and the regular intersection with a polarization.

However, some authors thought that introducing new models of Lie algebras was completely necessary, since the classification of Lie algebras was an unsolved problem. For example, Bowman and Towers [11] studied in 1996 those Lie algebras whose proper subalgebras are nilpotent-by-abelian but which themselves are not nilpotent-by-abelian. They analyzed the structure and existence of such algebras. Previously, other authors (like Elduque [39], Farnsteiner [41, 42], Gein [45, 46] and Varea [106], for instance) had studied simple semiabelian Lie algebras. Besides this, *almost nilpotent* Lie algebras (i.e. those containing a finite-dimensional nilpotent ideal) were studied and analyzed by Stitzinger [95], Gein and Kuznecov [47] and Towers [100, 102], whereas *almost supersolvable* Lie algebras were dealt with by

Towers [101] and Elduque and Varea [40].

At this respect, Bowman and Towers [11] obtained several results when studying almost nilpotent-by-abelian Lie algebras. Some of them are the following

Lemma. Let L be any Lie algebra. Then L is nilpotent-by-abelian if and only if its derived algebra $\mathcal{C}^2(L) = L^2$ is nilpotent.

Theorem. Let L be any Lie algebra over a field F of characteristic zero. Then the following statements are equivalent

- L is almost nilpotent-by-abelian.
- L is simple semiabelian or else $L = sl_2(F)$.

To obtain both results, the main tools used were Frattini subalgebras and ideals, algebraically closed fields and the following structures of Lie algebras: Heisenberg, solvable, nilpotent, semisimple and simple. Let us note that Frattini structures have relation with maximal subalgebras and maximal ideals. In their article, Bowman and Towers also considered the cases of an algebraically closed field of characteristic zero and one of characteristic $p > 0$. In this sense, some theorems were stated about the structure of certain solvable almost nilpotent-by-abelian algebras.

In addition, some articles also deal with important and useful properties of Lie subalgebras like decomposability. For example, Petravchuk [84] obtained in 1999 a Lie algebra L over an arbitrary field decomposed into the sum $L = A + B$ of an almost abelian subalgebra A and a subalgebra B finite-dimensional over its center. His main goal was to prove that this algebra was almost solvable (i.e. containing a solvable ideal of finite dimension) and that the sum of an abelian Lie algebra and an almost abelian one was an almost solvable Lie algebra.

To do this, he firstly wrote an historical introduction of the problem, recalling one of Ito's classic theorems ([31, 56], for instance) about the solvability of a product of two abelian groups. This result can also be translated into Lie algebras: *A Lie algebra decomposable into the sum of two of its abelian subalgebras is solvable.* Furthermore, Petravchuk himself recalled the following open problem: *it is unknown whether the product of two almost abelian groups is solvable.* In this sense, the main result in [84] pursued to answer this open problem. The statement is the following

Theorem. Let L be a Lie algebra over an arbitrary field that is decomposable into the sum $L = A + B$ of a subalgebra A finite-dimensional over its center and an almost abelian subalgebra B. Then, the algebra L is almost solvable.

As an immediate consequence, the following result holds: The sum of abelian and almost abelian Lie groups is almost solvable.

To prove the previous theorem, some local results were considered in the article. Some of them were necessary conditions for almost solvable Lie algebras and sufficient conditions for almost solvable Lie algebras or Lie algebras containing a solvable or almost solvable subalgebra. In addition, some statements about operations with brackets and sums were also proved.

Later, some papers appeared in about 2004 dealing with abelian ideals in Borel subalgebras. As a sake of example, Suter [97] started from a complex Lie algebra $\mathfrak{g}$ and a fixed Borel subalgebra $\mathfrak{b}$ of it, describing all abelian ideals of $\mathfrak{b}$ in a uniform way and independently of the classification of complex simple Lie algebras. Besides, as an application of this description, a formula was obtained for the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$. In this article, the maximal dimension among abelian subalgebras of $\mathfrak{g}$ was determined in terms of certain invariants such as the dual Coxeter number and the number of positive roots of some associated root subsystems of g. Other tools used in that article were fundamental alcoves (see [3, page 70]), symmetric groups, Dihedral and Weyl groups and the Hasse graph of Young's lattice (i.e. the lattice of integer partitions).

Taking into account the result commented in the previous paragraph, Suter answered and solved Panyushev and Röhrle's question [83] about a uniform explanation for the one-to-one correspondence between maximal abelian ideals in $\mathfrak{b}$ and long simple roots. This answer was given by emerging all positive long roots in a natural way (a very interesting overview about this question can be consulted in the very recent paper [70]). Finally, Suter also gave a generalization of the symmetry property of a certain subposet of Young's lattice.

Simultaneously, Cellini, Frajria and Papi [29] studied some properties of abelian subalgebras in the particular case of $\mathbb{Z}_2$-graded Lie algebras. More concretely, they considered a simple $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ and a fixed Borel subalgebra of $\mathfrak{g}_0$, $\mathfrak{b}_0$. The main goal of [29] was to describe and enumerate abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$, which is a problem previously posed by Panyushev [82]. Besides, some formulas were obtained in terms of combinatorial data associated to the $\mathbb{Z}_2$-graduation.

The main interest of this question lies in Kostant's theorem [65] relating abelian subalgebras to the maximal eigenvalue of the Casimir element. This theorem was later generalized to $\mathbb{Z}_2$ by Panyushev [81], who solved the problem posed in [82] for the very special case of the little adjoint module by identifying the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ with the abelian ideals of long roots of a Borel subalgebra of the

Langlands dual of $\mathfrak{g}_0$.

The approach given in [29] for describing abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ is based on a suitable combination of ideas given by Garland, Lepowsky [44] and Kostant [66, 67], using several types of Lie algebras such as affine Kac-Moody Lie algebras, Cartan algebras, graded algebras and Borel algebras.

Later, a recent new research was developed in 2008 by Romanovskii and Shestakov [89], who proved some properties of abelian Lie algebras. More concretely, they studied whether a wreath product of abelian Lie algebras is Noetherian with respect to the equations of the universal enveloping algebra. This was done with the objective of obtaining some improvements about algebraic geometry over Lie algebras, by proving that a wreath product of two finite-dimensional abelian Lie algebras over a field of characteristic zero is Noetherian with respect to the equations of the universal enveloping algebra. They previously recalled several earlier papers constructing examples of groups that are not equationally Noetherian in several cases, as well as similar examples of Lie algebras over any field.

By using different techniques (like the concepts and properties of equationally Noetherian Lie algebras, abelian normal subgroups, Noetherian Lie algebras, soluble free Lie algebras, algebraic subsets or coordinate algebras), the main result obtained was the following

Theorem. A wreath product of two finite-dimensional abelian Lie algebras over a field of characteristic 0 is Noetherian with respect to the equations of the universal enveloping algebra.

As an immediate consequence of this theorem, Romanovskii and Shestakov proved that an index 2 soluble free Lie algebra of finite rank is Noetherian with respect to the equations of a universal enveloping algebra.

In order to advance in this subject, it was very useful to deal with the dimension of abelian subalgebras of a given Lie algebra. In this sense, Milentyeva [73] studied this topic, obtaining some functions as bounds for the dimension of an abelian subalgebra in finite-dimensional associative algebras and Lie algebras. The same results were also obtained for the largest abelian subgroup of a Lie group.

Moreover, the growth of the functions previously developed to bound these dimensions was studied in [73] too. To do so, several different cases were considered depending on the field which the algebras are defined over. In this way, the functions were well-defined and finite for the case of complex and real number fields, having quadratic growth in other cases.

The families of associative and Lie algebras satisfying that the dimension of all its abelian subalgebras is at most n (denoted by condition $A(n)$) were also studied by Milentyeva in [73]. Besides, the set of the greatest integer h such that there exists a Lie algebra (respectively, an associative algebra) of dimension h and verifying the condition $A(n)$ was considered. Finally, the goal was to find the greatest k satisfying that there exists a Lie group of dimension k over an arbitrary field verifying that the dimension of all its abelian Lie subgroups is less than or equal to n.

Because of the great importance of Milentyeva's article, we think appropriate to describe the procedure. The general structure of the article was the following: first, an introduction showed the most important and essential concepts and definitions, as well as expounding the main result to be proved about inequalities for the bound functions. The next section was devoted to study quadratic upper bounds for the relation between the dimension of a given Lie algebra and the dimension of an abelian subalgebra of maximal dimension. This whole study was carried out for Lie algebras over both the complex and real number fields, as well as repeating this study for associative algebras. The third section computed quadratic lower bounds for Lie algebras over an arbitrary field. Finally, more details were shown about the bounds given for nilpotent algebras and groups.

One year later, Milentyeva [74] continued her work computing the functions which bounded the dimensions of finite-dimensional nilpotent both associative and Lie algebras of class 2 over an algebraically closed field in terms of the dimensions of their abelian subalgebras. Whereas her previous article only gave bounds for these functions, they were now completely determined and computed as a expression of a value n which bounds (or is exactly equal to) the α invariant of the associative or Lie algebra given for these functions.

In this way, the main theorem of [74] consisted of the mathematical expression of these functions expressed with respect to the value n. To obtain this result, Milentyeva had to prove a lemma assuring the existence of a vector subspace in a fixed and given vector space and which is simultaneously isotropic for all the components in any tuple (with a fixed dimension) of alternating bilinear forms. The proof of this lemma was based on the application of Zariski topology over projective spaces and the notions of projective and quasiprojective varieties in those spaces, as well as the notion of regular map from quasiprojective varieties to projective spaces (not necessarily the same containing such varieties). Other mathematical objects used in the proof are Grassmann varieties, Plücker coordinates of a vector subspace and Schubert cells in a Grassmann variety.

The last stage of Milentyeva's theorem was to reduce the associative case to the Lie case. In fact, she proved that the value of the functions was the same independently of using associative algebras or Lie ones. After this, she determined both upper and lower bounds of the function for the Lie case, by using the previously referred lemma and the results already obtained in [73]. The proof concluded when she obtained that upper and lower bounds were the same.

After having commented this brief historical view on Lie algebras in general and on abelian Lie subalgebras in particular, now we will deal with this dissertation. Its structure is the following: Chapter 1 recalls those more general concepts on Lie algebras which will be cited throughout this dissertation. Chapter 2 constitutes a theoretical study of abelian subalgebras and ideals contained in Lie algebras. First, we give some general properties and bounds. More concretely, we want to point out an interesting result for solvable Lie algebras: if $\mathfrak{g}$ is a solvable Lie algebra over an algebraically closed field of characteristic zero, then $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$. This means that, given an abelian subalgebra of maximal dimension m there exists also an abelian ideal of dimension m. For a given value of $\alpha(\mathfrak{g})$, the dimension of $\mathfrak{g}$ is bounded in terms of this value, as mentioned above. It is natural to ask what we can say on an n-dimensional Lie algebra $\mathfrak{g}$ involving the value of $\alpha(\mathfrak{g})$ is close to n. Indeed, if $\alpha(\mathfrak{g}) = n$, then $\mathfrak{g}$ is abelian and $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$. If $\alpha(\mathfrak{g}) = n-1$, then also $\beta(\mathfrak{g}) = n-1$. This means that $\mathfrak{g}$ has an abelian ideal of codimension 1 and is almost abelian. In particular, $\mathfrak{g}$ is 2-step solvable. In this case the structure of $\mathfrak{g}$, and even all its degenerations are quite well-understood, see [48]. These two easy cases suggest to consider Lie algebras $\mathfrak{g}$ satisfying $\alpha(\mathfrak{g}) = n - 2$. Here we can classify all such non-solvable Lie algebras. For the solvable case, we characterize these algebras and prove that every supersolvable Lie algebra with an abelian subalgebra of codimension 2 has also an abelian ideal with the same dimension. We also give a method to obtain an abelian ideal of codimension 2 from an abelian subalgebra of the same dimension in a nilpotent Lie algebra. Let us note that for many problems concerning the cohomology of nilpotent Lie algebras, the subclass of those having an abelian ideal of codimension 1 or 2 is very important, as can be seen in [2, 86] and the references given therein. In order to conclude the chapter, we prove that nilpotent Lie algebras with an abelian subalgebra of codimension 3 contain an abelian ideal with the same dimension, provided that the characteristic of the underlying field is not two. We also give several examples to clarify some results.

Chapter 3 is devoted to show several algorithmic methods and results about the abelian subalgebras of maximal dimension for the most important families of solvable Lie algebras: Lie algebra $\mathfrak{g}_n$, of $n \times n$ strictly upper-triangular matrices, Lie algebra $\mathfrak{h}_n$, given by $n \times n$ upper-triangular matrices, Heisenberg algebras and filiform Lie algebras. First, we show a method to compute an abelian subalgebra of maximal dimension for Lie algebras $\mathfrak{g}_n$, $\mathfrak{h}_n$ and Heisenberg algebras. This procedure is defined as follows: given a Lie algebra, $\mathfrak{g}$, we consider an arbitrary basis of an r-dimensional subalgebra with respect to the basis $\mathcal{B}_\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ in which its law is expressed. Then the vectors in $\mathcal{B}_\mathfrak{g}$ are divided in main vectors or non-main vectors according to a echelon matrix associated with $\mathcal{B}_\mathfrak{g}$. Finally it is studied whether the subalgebra is abelian or not. When only non-abelian subalgebras are obtained, we have proved that the α invariant of $\mathfrak{g}$ is less than r. The main reason why we have focused on the Lie algebras $\mathfrak{h}_n$ and $\mathfrak{g}_n$ is that every finite-dimensional solvable Lie algebra can be represented by a Lie subalgebra of some $\mathfrak{h}_n$ (see [43, Theorem 9.11] or [105, Theorem 3.7.3]) and every finite-dimensional nilpotent Lie algebra is isomorphic to a subalgebra of $\mathfrak{g}_n$ ([105, Proposition 3.6.6]). Another reason is that its applications to Physic are many and varied (e.g. [54, 87]). In order to conclude the chapter, we study the abelian subalgebras of maximal dimension for filiform Lie algebras. We prove that there exists a unique abelian ideal of maximal dimension for these algebras and we give several results concerning the description of the general law of a filiform Lie algebras by using several invariants. Finally, we give an algorithmic procedure which computes the law of an n-dimensional non-model filiform Lie algebra $\mathfrak{g}$ starting from the value of $\alpha(\mathfrak{g})$.

In Chapter 4, we show an algorithmic method to compute abelian subalgebras and ideals of any finite-dimensional Lie algebra, starting from the non-zero brackets in its law. To implement this algorithm we use the symbolic computation package Maple 12. Additionally, we give a brief computational study considering both the computing time and the memory used in the two main routines of the implementation. We have also studied the complexity and number of operations. Moreover, we have included in this chapter two different applications of the previous results. The first is related to the computation of the α and β invariants for low dimensional Lie algebras. More concretely, we compute these invariants for Lie algebras of dimension less than 5 in general, solvable Lie algebras of dimension less than 7 and nilpotent Lie algebras of dimension less than 8. The second application consists in the study of minimal faithful unitriangular matrix representations for filiform Lie algebras.

Spanish summary

En esta sección exponemos un resumen en español del contenido de este trabajo, en el que se estudian dos invariantes de las álgebras de Lie: las dimensiones maximales de sus subálgebras e ideales abelianos. Así, si $\mathfrak{g}$ es un álgebra de Lie de dimensión finita, denotamos por $\alpha(\mathfrak{g})$ y $\beta(\mathfrak{g})$ al máximo entre la dimensión de todas las subálgebras e ideales abelianos de $\mathfrak{g}$, respectivamente. Estos invariantes fueron introducidos por Schur en 1905 al estudiar las subálgebras abelianas maximales en el álgebra formada por matrices cuadradas. Se usan en el estudio de contracciones y degeneraciones de álgebras de Lie. Hay muchos artículos tratando estos temas, por ejemplo: [49, 19, 77, 92, 48]. Asimismo, algunos resultados comparan el valor de estos invariantes con la dimensión de la propia álgebra ([94, 88, 73]).

En el Capítulo 1 exponemos conceptos preliminares necesarios para la adecuada comprensión del resto del trabajo. A continuación, en el Capítulo 2, se analizan propiedades básicas y generales de los invariantes α y β, como son la aditividad y monotonía, y diversas cotas. También se exponen una serie de resultados comparando ambos invariantes. En concreto, se demuestra que si $\mathfrak{g}$ un álgebra de Lie de dimensión finita y se consideran su descomposición de Levi-Malcev $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, entonces $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \leq \alpha(\mathfrak{s} \oplus \mathfrak{r})$. También se prueba que si $\mathfrak{g}$ es un álgebra de Lie resoluble sobre un cuerpo algebraicamente cerrado de característica cero, entonces ambos invariantes coinciden. A continuación, se estudian dos casos particulares correspondientes a subálgebras abelianas de codimensión 1 y 2. Para el primer caso, se prueba que si $\mathfrak{g}$ un álgebra de Lie de dimensión n verificando $\alpha(\mathfrak{g}) = n-1$, entonces $\beta(\mathfrak{g}) = n-1$. Las álgebras de Lie verificando $\beta(\mathfrak{g}) = n-1$ son álgebras de Lie casi abelianas, que son álgebras de Lie resolubles 2-step. En el caso de codimensión 2, también podemos dar una caracterización: Si $\mathfrak{g}$ es un álgebra de Lie de dimensión n verificando $\alpha(\mathfrak{g}) = n-2$, entonces, o bien $\mathfrak{g}$ es isomorfa a $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$ o bien es resoluble. Además damos una caracterización en ambos casos. También se prueba que toda álgebra de Lie nilpotente con una subálgebra abeliana de codimensión 2, contiene un ideal abeliano de la misma dimensión. Notemos que los ideales abelianos de álgebras de Lie nilpotentes de codimensión 1 y 2 juegan un papel importante en problemas de cohomología de estas álgebras (ver [2, 86]). Por último, estudiamos el caso de codimensión 3 para las álgebras nilpotentes sobre un cuerpo de característica distinta de dos en el que se muestra un método para obtener un ideal abelian de codimensión 3 a partir de una subálgebra abeliana de la misma dimensión.

En el tercer capítulo, se analizan las subálgebras e ideales abelianos en varias familias de álgebras de Lie resolubles. En concreto, estudiamos las álgebras de Lie $\mathfrak{g}_n$, dadas por matrices cuadradas estrictamente triangulares superiores, las álgebras $\mathfrak{h}_n$, dadas por matrices cuadradas triangulares superiores y las álgebras de Heisenberg. La principal razón por la que hemos estudiado estas familias es que cada álgebra de Lie resoluble de dimensión finita se puede representar por una subálgebra de $\mathfrak{h}_n$ (ver [43, Theorem 9.11] o [105, Theorem 3.7.3]) y cada álgebra de Lie nilpotente de dimensión finita es isomorfa a una subálgebra de $\mathfrak{g}_n$ ([105, Proposition 3.6.6]). Otra razón es por sus múltiples aplicaciones en Física (por ejemplo [54, 87]). Para concluir este capítulo, estudiamos las álgebras de Lie filiformes. Se prueba que para estas álgebras existe un único ideal abeliano de máxima dimensión. También damos algunos resultados sobre la descripción de la ley general de estas álgebras a paritr de varios invariantes. Finalmente damos un procedimiento algorítmico para calcular la ley general de un álgebra de Lie filiforme a partir del valor del invariante alfa.

En el Capítulo 4, mostramos un método algorítmico para calcular subálgebras e ideales abelianos de un álgebra de Lie arbitraria a partir de su ley. Para implementar este procedimiento, hemos utilizado el programa de computación simbólica Maple 12. Además, damos un breve estudio computacional de las principales rutinas analizando el tiempo computacional y la memoria usada. Se adjunta también una tabla con la complejidad computacional y el número de operaciones para estas rutinas. A continuación, y para finalizar este capítulo, hemos incluido dos aplicaciones correspondientes al cálculo del invariante α para álgebras de Lie de pequeña dimensión y la representación matricial de álgebras de Lie filiformes. Más concretamente, estudiamos el valor de α para álgebras de Lie de dimensión menor que 5 de cualquier tipo, álgebras de Lie resolubles de dimensión menor que 7 y nilpotentes de dimensión menor que 8. Para las representaciones matriciales de álgebras de Lie filiformes, se consideran las álgebras $\mathfrak{g}_n$ y sus subálgebras abelianas calculadas en el capítulo anterior. Mostramos la representación general de las álgebras de Lie filiformes modelos, un método para el cálculo de la representación para las no modelos y, como ejemplo, varias tablas con la representación de álgebras de Lie filiformes de dimensión menor que 9.

Chapter 1

Preliminaries

This chapter is devoted to recall some preliminary concepts and results on Lie algebras. For a general overview, the interested reader can consult [105]. Throughout this dissertation, $\mathfrak{g}$ will denote a finite-dimensional Lie algebra over a field $\mathbb{K}$. The assumptions on $\mathbb{K}$ will be specified in each section or result.

1.1 Definitions and notations

Definition 1.1. *A* Lie algebra $\mathfrak{g}$ *over an arbitrary field* $\mathbb{K}$ *is a vector space over* $\mathbb{K}$ *endowed with a second inner law, named the* bracket product*, and verifying the following three properties*

1. *Bilinearity:* $[\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w], \quad [u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w],$ *for* $\alpha, \beta \in \mathbb{K}, \forall u, v \in \mathfrak{g}$.

2. $[u, u] = 0, \quad \forall u \in \mathfrak{g}$.

3. *Jacobi Identity:* $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0, \quad \forall u, v, w \in \mathfrak{g}$.

Definition 1.2. *The dimension of a Lie algebra is its dimension as a vector space. Let us consider a Lie algebra* $\mathfrak{g}$ *with a basis* $\{e_i\}_{i=1}^n$*. Such a basis can be characterized by the* structure constants *(or* Maurer-Cartan constants*), defined by* $[e_i, e_j] = \sum c_{i,j}^h e_h$ *for any* $1 \leq i < j \leq n$*. These constants determine the whole structure of the Lie algebra.*

Remark 1.1. *The second condition in Definition 1.1, together with the bilinearity of the bracket product, implies the skew-symmetry over a field of characteristic different from 2; i.e.* $[u, v] = -[v, u]$*, for all* $u, v \in \mathfrak{g}$.

Definition 1.3. *Given a Lie algebra* $\mathfrak{g}$*, a vector subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is a* subalgebra *if* $[u, v] \in \mathfrak{h}$*, for all* u*,* $v \in \mathfrak{h}$*. Moreover, the subalgebra* $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$ *if* $[h_1, g_1] \in \mathfrak{h}$*, for all* $h_1 \in \mathfrak{h}$ *and for all* $g_1 \in \mathfrak{g}$*.*

Definition 1.4. *Given a subalgebra* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$*, the* core *of* $\mathfrak{h}$*, denoted by* $\mathfrak{h}_{\mathfrak{g}}$*, is the largest ideal of* $\mathfrak{g}$ *contained in* $\mathfrak{h}$*.*

Definition 1.5. *Given a subalgebra* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$*, the* ideal closure *of* $\mathfrak{h}$*, denoted by* $\mathfrak{h}^{\mathfrak{g}}$*, is the smallest ideal of* $\mathfrak{g}$ *containing* $\mathfrak{h}$*.*

Definition 1.6. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra and* $\mathfrak{h} \subseteq \mathfrak{g}$ *a subalgebra. The* centralizer *of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *is defined as* $\mathcal{C}_{\mathfrak{g}}(\mathfrak{h}) = \{g \in \mathfrak{g} \mid [g, h] = 0, \ \forall h \in \mathfrak{h}\}$*.*

Definition 1.7. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra and* $\mathfrak{h} \subseteq \mathfrak{g}$ *a subalgebra. We define the* normalizer *of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *as* $\mathcal{N}_{\mathfrak{g}}(\mathfrak{h}) = \{g \in \mathfrak{g} \mid [g, h] \in \mathfrak{h}, \ \forall h \in \mathfrak{h}\}$*.*

Definition 1.8. *Given an element* $x \in \mathfrak{g}$*,* the adjoint endomorphism *or* adjoint action *is a Lie-algebra endomorphism* $ad_x : \mathfrak{g} \to \mathfrak{g}$ *defined as* $y \in \mathfrak{g} \mapsto ad_x(y) = [x, y]$*.*

Definition 1.9. *A* derivation *on a Lie algebra* $\mathfrak{g}$ *is a linear map* $D : \mathfrak{g} \to \mathfrak{g}$ *verifying*

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

The set of all derivations of $\mathfrak{g}$ *will be denoted by* $Der(\mathfrak{g})$*. Let us note that* ad_x *is a derivation for all* $x \in \mathfrak{g}$ *and* $ad : \mathfrak{g} \to Der(\mathfrak{g})$*.*

Definition 1.10. *The* Frattini subalgebra $F(\mathfrak{g})$ *of a Lie algebra* $\mathfrak{g}$ *is defined as the intersection of all maximal subalgebras of* $\mathfrak{g}$*. The* Frattini ideal $\phi(\mathfrak{g})$ *is the largest ideal contained in* $F(\mathfrak{g})$*.*

Definition 1.11. *The* linear adjoint group *of* G*,* AdG*, is the image of the Lie group or algebraic group* G *under the adjoint representation. The adjoint group* AdG *is contained in the group* $Aut(\mathfrak{g})$ *of automorphisms of the Lie algebra* $\mathfrak{g}$ *of* G *and its Lie algebra coincides with the adjoint algebra* $ad(\mathfrak{g})$ *of* $\mathfrak{g}$*.*

Theorem 1.1 (Borel fixed-point theorem in Borel [9])**.** *Let* G *be a connected, solvable algebraic group acting regularly on a non-empty, complete algebraic variety* V *over an algebraically closed field* $\mathbb{K}$*. Then* G *has a fixed point in* V*.*

There exist three different types of Lie algebras: solvable algebras, semisimple ones and those which do not belong to these two previous types, but can be expressed as a semidirect sum of two algebras of the previous types.

Definition 1.12. *A Lie algebra* $\mathfrak{g}$ *over a a field of characteristic zero is* semisimple *if* $\mathfrak{g}$ *is not abelian and does not contain any non-zero proper abelian ideal. In addition, the Lie algebra* $\mathfrak{g}$ *is said to be* simple *if it is not abelian and does not contain any non-zero proper ideal.*

Definition 1.13. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. The* upper central series *or* derived series *of* $\mathfrak{g}$ *is defined by*

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}],\ \mathcal{C}_3(\mathfrak{g}) = [\mathcal{C}_2(\mathfrak{g}),\mathcal{C}_2(\mathfrak{g})],\ \ldots,\ \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}),\mathcal{C}_{k-1}(\mathfrak{g})],\ \ldots$$

If there exists $m \in \mathbb{N}$ *such that* $\mathcal{C}_m(\mathfrak{g}) \equiv 0$*, the Lie algebra* $\mathfrak{g}$ *is called* solvable. *Moreover,* $\mathfrak{g}$ *is* $(m-1)$*-step solvable if there exists* $m \in \mathbb{N}$ *such that* $\mathcal{C}_m(\mathfrak{g}) \equiv \{0\}$ *and* $\mathcal{C}_{m-1}(\mathfrak{g}) \neq \{0\}$*.*

A special class of solvable Lie algebras is formed by abelian algebras.

Definition 1.14. *A Lie algebra* $\mathfrak{g}$ *is* abelian *if* $[v,w] = 0$*, for all* v*,* $w \in \mathfrak{g}$*.*

A very important and interesting abelian subalgebra (in fact, an abelian ideal) in a given Lie algebra $\mathfrak{g}$ is the *center* of the algebra.

Definition 1.15. *The* center $Z(\mathfrak{g})$ *of a Lie algebra* $\mathfrak{g}$*, is defined as follows* $Z(\mathfrak{g}) = \{u \in \mathfrak{g} \mid [u,v] = 0,\ \forall\ v \in \mathfrak{g}\}$*.*

Definition 1.16. *The* lower central series *or simply* central series *of a Lie algebra* $\mathfrak{g}$ *is defined by*

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}^2(\mathfrak{g}) = [\mathcal{C}^1(\mathfrak{g}),\mathfrak{g}], \mathcal{C}^3(\mathfrak{g}) = [\mathcal{C}^2(\mathfrak{g}),\mathfrak{g}],\ \ldots,\ \mathcal{C}^k(\mathfrak{g}) = [\mathcal{C}^{k-1}(\mathfrak{g}),\mathfrak{g}],\ \ldots$$

If there exists $m \in \mathbb{N}$ *such that* $\mathcal{C}^m(\mathfrak{g}) \equiv 0$*, the Lie algebra* $\mathfrak{g}$ *is called* nilpotent. *We will say that* $\mathfrak{g}$ *is* $(m-1)$*-step nilpotent if there exists* $m \in \mathbb{N}$ *such that* $\mathcal{C}^m(\mathfrak{g}) \equiv \{0\}$ *and* $\mathcal{C}^{m-1}(\mathfrak{g}) \neq \{0\}$*.*

Definition 1.17. *The derived Lie algebra* $\mathcal{D}(\mathfrak{g})$ *of a Lie algebra* $\mathfrak{g}$ *is given by* $\mathcal{D}(\mathfrak{g}) = \mathcal{C}_2(\mathfrak{g}) = \mathcal{C}^2(\mathfrak{g})$*.*

Remark 1.2. *Lie algebras with its derived algebra being abelian correspond to* 2*-step solvable ones and are usually called* metabelian.

Remark 1.3. *Let us note that every nilpotent Lie algebra is also solvable, because* $\mathcal{C}_i(\mathfrak{g}) \subseteq \mathcal{C}^i(\mathfrak{g})$*,* $\forall i$*.*

Related to the lower central series associated with a subalgebra of $\mathfrak{g}$, the following result holds

Proposition 1.1. *Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. Then $\mathcal{C}^k(\mathfrak{h}) \subseteq \mathcal{C}^k(\mathfrak{g})$, for all $k \in \mathbb{N}$.*

Definition 1.18. *The* radical *of a Lie algebra $\mathfrak{g}$ is defined as its maximal solvable ideal and the* nilradical *is its maximal nilpotent ideal.*

Definition 1.19. *A* Borel subalgebra *of a Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.*

Definition 1.20. *A* Cartan subalgebra *of $\mathfrak{g}$ is a nilpotent subalgebra which is self-normalizing. The rank of $\mathfrak{g}$ is given by the dimension of its Cartan subalgebras.*

Definition 1.21. *A Lie algebra $\mathfrak{g}$ is* supersolvable *if there is a chain $0 = I_0 \subset I_1 \subset \ldots \subset I_{n-1} \subset I_n = \mathfrak{g}$, where I_k is a k-dimensional ideal of $\mathfrak{g}$*

Remark 1.4. *It is well-known that every supersolvable Lie algebra is also solvable. Moreover, these classes coincide over an algebraically closed field of characteristic zero (Lie's Theorem [105]). There are, however, examples of solvable Lie algebras over algebraically closed field of non-zero characteristic which are not supersolvable, as can be seen in [5] or [58, page 53].*

Definition 1.22. *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be* triangulable *on $\mathfrak{g}$ if $ad_{\mathfrak{g}}\mathfrak{h} = \{ad_{\mathfrak{g}}x \mid x \in \mathfrak{h}\}$ is a Lie algebra of linear transformations of $\mathfrak{g}$ which is triangulable over the algebraic closure of $\mathbb{K}$.*

Proposition 1.2 (Theorem 2.2 in [109]). *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is triangulable on $\mathfrak{g}$ if and only if every element of $\mathcal{D}(\mathfrak{h})$ acts nilpotently on $\mathfrak{g}$.*

Definition 1.23. *Let us consider a vector space V and an endomorphism f over it. Then, V can be decomposed into the direct sum of two subspaces which are invariants under the action of f as $V = V_0 \oplus V_1$, where $f_{|V_0}$ is nilpotent and $f_{|V_1}$ is an isomorphism. The definition of these subspaces is as follows*

$$V_1 = \bigcap_{i=1}^{\infty} f^i(V) \quad \text{and} \quad V_0 = \{v \in V \mid \exists r \in \mathbb{N},\, f^r v = 0\}.$$

This is known as the Fitting decomposition.

Definition 1.24. *A nilpotent Lie algebra $\mathfrak{g}$ is said to be* filiform *if it verifies that*

$$dim(\mathcal{C}^2(\mathfrak{g})) = n-2; \;\; \ldots \;\; dim(\mathcal{C}^k(\mathfrak{g})) = n-k; \;\; \ldots \;\; dim(\mathcal{C}^n(\mathfrak{g})) = 0,$$

where $dim(\mathfrak{g}) = n$.

A basis $\{e_i\}_{i=1}^n$ *of* $\mathfrak{g}$ *is called* adapted *if*

$$[e_1, e_2] = 0, \quad [e_1, e_h] = e_{h-1} \ (3 \leq h \leq n),$$

$$[e_2, e_h] = 0 \ (3 \leq h \leq n), \quad [e_3, e_h] = 0 \ (4 \leq h \leq n).$$

Remark 1.5. *Note that the definition of filiformity assures that every filiform Lie algebra has an adapted basis, as it was proved in [107].*

Remark 1.6. *It is easy to deduce that, with respect to adapted bases, the following conditions hold.*

$$\mathcal{C}^2(\mathfrak{g}) \equiv \{e_i\}_{i=2}^{n-1},\ \mathcal{C}^3(\mathfrak{g}) \equiv \{e_i\}_{i=2}^{n-2}, \ldots, \mathcal{C}^{n-1}(\mathfrak{g}) \equiv \{e_2\},\ \mathcal{C}^n(\mathfrak{g}) \equiv \{0\}.$$

Proposition 1.3. *Let* $\mathfrak{g}$ *be a* n*-dimensional filiform Lie algebra with an adapted basis* $\{e_i\}_{i=1}^n$*, then it is verified that* $[\mathcal{C}^p(\mathfrak{g}), \mathcal{C}^q(\mathfrak{g})] \subset \mathcal{C}^{min\{p,q\}+1}(\mathfrak{g})$.

Proof. Let us note that if $p < q$ then $\mathcal{C}^q(\mathfrak{g}) \subset \mathcal{C}^p(\mathfrak{g})$. Therefore, $[\mathcal{C}^p(\mathfrak{g}), \mathcal{C}^q(\mathfrak{g})] \subset [\mathcal{C}^p(\mathfrak{g}), \mathcal{C}^p(\mathfrak{g})] \subset [\mathcal{C}^p(\mathfrak{g}), \mathfrak{g}] = \mathcal{C}^{p+1}(\mathfrak{g})$. □

Definition 1.25. *A filiform Lie algebra* $\mathfrak{g}$ *is said to be* model *if the only non-zero brackets between the elements of an adapted basis are:* $[e_1, e_h] = e_{h-1}$*, for* $h = 3, \ldots, n$.

In [34] (although by using a different notation, which was later improved in [35]), the following invariants were introduced

Definition 1.26. *Let* $\mathfrak{g}$ *be an* n*-dimensional complex filiform Lie algebra. Then, the following invariants of* $\mathfrak{g}$ *are defined*

$$z_1 = z_1(\mathfrak{g}) = max\{k \in \mathbb{N} \mid \ \mathcal{C}_{\mathfrak{g}}(\mathcal{C}^{n-k+2}(\mathfrak{g})) \supset \mathcal{C}^2(\mathfrak{g})\},$$

$$z_2 = z_2(\mathfrak{g}) = max\{k \in \mathbb{N} \,|\, \mathcal{C}^{n-k+1}(\mathfrak{g}) \textit{ is abelian}\}.$$

Remark 1.7. *The definition of* z_1 *means that* $\mathcal{C}^{n-z_1+2}(\mathfrak{g})$ *is the largest ideal of* $\mathfrak{g}$ *whose centralizer contains* $\mathcal{C}^2(\mathfrak{g})$*; i.e., the ideal whose centralizer is the ideal* $\overline{\mathfrak{g}}$*, generated by* $\{e_i\}_{i=2}^{n-1}$ *with respect to an adapted basis* $\{e_i\}_{i=1}^n$

In addition, $z_1(\mathfrak{g})$ *is an invariant for non-model filiform Lie algebras. In terms of an adapted basis, it is deduced in [34] that it can be written as follows*

$$z_1 = min\{k \in \mathbb{N} - \{1\} \mid \ [e_k, e_n] \neq 0\}.$$

and

$$[e_h, e_k] = 0, \text{ for } 1 < h < z_1 \text{ and } k > 1. \tag{1.1}$$

Remark 1.8. *By definition, z_2 means that z_2 is an invariant of complex non-model filiform Lie algebras and the ideal $\mathcal{C}^{n-z_2+1}(\mathfrak{g}) \equiv \{e_i\}_{i=2}^{z_2}$ is the largest abelian ideal in the lower central series.*

Moreover, there exists at least some bracket $[e_k, e_{k+1}] \neq 0$, for $k < n$, in every complex non-model filiform Lie algebra of dimension n. Consequently, an equivalent definition for z_2 is the following $z_2(\mathfrak{g}) = min\ \{k \in \mathbb{N} \mid [e_k, e_{k+1}] \neq 0\}$.

If the sets in the definition of $z_1(\mathfrak{g})$ and $z_2(\mathfrak{g})$ are empty, then $\mathfrak{g}$ is a model filiform Lie algebra and both invariants are not defined. Moreover, the smallest possible value for $z_2(\mathfrak{g})$ is 4, because of $[e_1, e_2] = [e_2, e_3] = [e_3, e_4] = 0$ for all adapted basis of $\mathfrak{g}$.

With respect to these invariants, the following results were proved in [36]

Lemma 1.1. *Under previous conditions, it is deduced that $[e_{z_1+k-1}, e_{z_2+1}] = \alpha_1 e_{k+1} + \alpha_2 e_k + \ldots + \alpha_{k-1} e_3 + \alpha_k e_2$, with $1 \leq k \leq z_2 - z_1 + 1$. Moreover, $\alpha_p \neq 0$, for some p such that $1 \leq p \leq z_2 - z_1 + 1$.*

Lemma 1.2. *Under previous conditions, it is deduced that $[e_{z_1}, e_{z_2+k}] = \alpha_1 e_{k+1} + \alpha_2^1 e_k + \ldots + \alpha_k^{k-1} e_2$, with $1 \leq k \leq n - z_2$. Moreover, $\alpha_q^{q-1} \neq 0$, for some q such that $1 \leq q \leq n - z_2$.*

Finally, it is also proved in [37] the following relation among the invariants z_1, z_2 and n, which will be used later

$$4 \leq z_1 \leq z_2 < n \leq 2z_2 - 2. \tag{1.2}$$

1.2 Families of solvable Lie algebras

In this section, we show some special families of solvable Lie algebras. More concretely, we are interested in the Lie algebra $\mathfrak{h}_n$, of $n \times n$ upper-triangular matrices; the Lie algebra $\mathfrak{g}_n$, of $n \times n$ strictly upper-triangular matrices; and the Heisenberg algebra $\mathfrak{H}_k$.

The main reason to deal with the Lie algebras $\mathfrak{h}_n$ and $\mathfrak{g}_n$ is that every finite-dimensional solvable Lie algebra is isomorphic to a subalgebra of $\mathfrak{h}_n$ [105, Proposition 3.7.3] and every finite-dimensional nilpotent Lie algebra is isomorphic to a subalgebra of $\mathfrak{g}_n$ [105, Proposition 3.6.6]).

Regarding Heisenberg algebras, they constitute a special subclass of nilpotent Lie algebras and are interesting for their applications to both the theory of nilpotent

Lie algebras itself and Theoretical Physics. With respect to the first subject, a well-known result [90] sets that a nilpotent Lie algebra $\mathfrak{g}$ is abelian if and only if $\mathfrak{g}$ does not contain a subalgebra isomorphic to the 3-dimensional Heisenberg algebra $\mathfrak{H}_1$. In Theoretical Physics, these algebras are also very interesting because of several reasons. The first of them is based on the origin itself of Heisenberg algebras. These algebras appeared at the beginnings of the 20th century when introducing Quantum Mechanics. In Classic Mechanics, the state of a particle in a given time t is determined by both its position vector $\mathbf{Q} \in \mathbb{R}^3$ and its momentum vector $\mathbf{P} \in \mathbb{R}^3$. Heisenberg [52] took the components of these two vectors, considering them as operators in a Hilbert space such that the following commutation relations were verified

$$[Q_i, Q_j] = 0, \ [P_i, P_j] = 0, \ [P_i, Q_j] = -\mathbf{i}\ \hbar\ \delta_{ij}, \ \forall i, j = 1, 2, 3,$$

where δ_{ij} is the named *Kronecker delta* and $\hbar$, called the *deformation parameter*, represents the choice of a measure unit (usually $\hbar = 1$).

1.2.1 Lie algebras $\mathfrak{h}_n$

Let us denote by $\mathfrak{h}_n$ the complex solvable Lie algebra of $n \times n$ upper-triangular matrices having the following structure

$$h_n(x_{r,s}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{nn} \end{pmatrix}.$$

The basis $\mathcal{B}_n$ of $\mathfrak{h}_n$ is formed by the vectors $X_{ij} = h_n(x_{r,s})$, where $1 \leq i \leq j \leq n$ and

$$x_{r,s} = \begin{cases} 1, & \text{if } (r,s) = (i,j); \\ 0, & \text{if } (r,s) \neq (i,j). \end{cases}$$

In this way, the dimension of $\mathfrak{h}_n$ is $\dim(\mathfrak{h}_n) = d_{\mathfrak{h}_n} = \frac{n(n+1)}{2}$. Let us note that the center $Z(\mathfrak{h}_n)$ of the Lie algebra $\mathfrak{h}_n$ is generated by the vector $\sum_{i=1}^{n} X_{i,i}$, coming from the main diagonal. This vector is the only one which commutes with all the vectors in $\mathfrak{h}_n$. Additionally, the non-zero brackets of $\mathfrak{h}_n$ with respect to the basis $\mathcal{B}_n$ are the following

$$\begin{array}{ll} [X_{i,j}, X_{j,k}] = X_{i,k}, & \forall i = 1 \ldots n-2, \forall j = i+1 \ldots n-1, \forall k = j+1 \ldots n; \\ [X_{i,i}, X_{i,j}] = X_{i,j}, & \forall j > i; \\ [X_{k,i}, X_{i,i}] = X_{k,i}, & \forall k < i. \end{array}$$

1.2.2 Lie algebras $\mathfrak{g}_n$

Let us denote by $\mathfrak{g}_n$ the complex nilpotent Lie algebra of $n \times n$ strictly upper triangular matrices, with $n \in \mathbb{N} \setminus \{1\}$. The expression of the vectors in $\mathfrak{g}_n$ is the following

$$g_n(x_{r,s}) = \begin{pmatrix} 0 & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & 0 & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The dimension of $\mathfrak{g}_n$ is $dim(\mathfrak{g}_n) = d_{\mathfrak{g}_n} = \frac{n(n-1)}{2}$ and, from now on, we will use the basis of $\mathfrak{g}_n$ given by the vectors

$$X_{i,j} = g_n(x_{r,s}), \qquad \text{with } 1 \leq i < j \leq n, \text{ and } x_{r,s} = \begin{cases} 1, & \text{if } (r,s) = (i,j); \\ 0, & \text{if } (r,s) \neq (i,j) \end{cases}$$

Let us note that the center of this algebra is given by $\langle X_{1,n} \rangle$. The law of $\mathfrak{g}_n$ with respect to this basis is expressed as follows

$$[X_{i,j}, X_{j,k}] = X_{i,k}, \quad 1 \leq i < j < k \leq n$$

1.2.3 Heisenberg algebras

Heisenberg Lie algebras constitutes a special class of nilpotent Lie algebras and the applications of these algebras are many and varied (e.g. [54, 87]).

Definition 1.27. *For a given $k \in \mathbb{N}$, the* Heisenberg algebra $\mathfrak{H}_k$ *is the* $(2k+1)$*-dimensional Lie algebra having the following law with respect to a certain basis* $\{x_1, \ldots x_k, y_1, \ldots, y_k, z\}$

$$[x_i, y_i] = z, \quad \forall i = 1, \ldots, k.$$

Let us note that the center of the Heisenberg algebra is given by $\langle z \rangle$.

1.3 Main and non-main vectors

Fixed and given a n-dimensional Lie algebra $\mathfrak{g}$, every r-dimensional (abelian) subalgebra (with $r \leq n$) is generated by a basis $\mathcal{B} = \{v_h\}_{h=1}^{r}$. Each vector $v_h \in \mathcal{B}$ is expressible as a linear combination $v_h = \sum_i a_i^h X_i$ of the vectors in the basis $\mathcal{B}_{\mathfrak{g}} = \{X_i\}_{i=1}^{n}$ of $\mathfrak{g}$. Therefore, the basis $\mathcal{B}$ can be translated into a matrix in which the h^{th} row corresponds to the coordinates of v_h with respect to the basis $\mathcal{B}_{\mathfrak{g}}$

$$\begin{pmatrix} a_{1,2}^1 & a_{1,3}^1 & \cdots & a_{1,n}^1 & a_{2,3}^1 & \cdots & a_{n-1,n}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1,2}^r & a_{1,3}^r & \cdots & a_{1,n}^r & a_{2,3}^r & \cdots & a_{n-1,n}^r \end{pmatrix} \tag{1.3}$$

This matrix is equivalent to the following echelon form, obtained by using elementary row and column transformations

$$\begin{pmatrix} b_{1,1} & 0 & \cdots & 0 & b_{1,r+1} & \cdots & b_{1,n} \\ 0 & b_{2,2} & \cdots & 0 & b_{2,r+1} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{r,r} & b_{r,r+1} & \cdots & b_{r,n} \end{pmatrix} \tag{1.4}$$

Let us note that the vector $X_{i,j} \in \mathcal{B}_{\mathfrak{g}}$ associated with each row in (1.4) can be different from the one in (1.3).

Therefore, given a subalgebra $\mathfrak{h}$ of $\mathfrak{h}_n$, we can suppose, without loss of generality, that every basis $\mathcal{B}$ of $\mathfrak{h}$ is expressible by a matrix similar to (1.4) and that each vector in $\mathcal{B}$ is a linear combination of two different types of vectors $X_{i,j}$: the ones corresponding to the pivots in (1.4) and the remaining ones. Each vector $X_{i,j}$ corresponding to a pivot position is called a *main vector* of $\mathcal{B}$, with respect to the basis $\mathcal{B}_n$, whereas the rest are called *non-main vectors*. These concepts and notations were introduced firstly in [8] in order to study the maximal dimension of abelian subalgebras.

Chapter 2

Theoretical study

In this chapter, we develop a theoretical study about abelian subalgebras and ideals of maximal dimension contained in a finite-dimensional Lie algebra. Hereafter, algebra direct and semidirect sums will be denoted by $\oplus$ and $\ltimes$, respectively; whereas vector space direct sums will be denoted by $\dot{+}$. The content of this chapter can be seen in the papers [18] and [28]. In the first place, we define the following invariants.

$$\alpha(\mathfrak{g}) = \max\{\,\dim(\mathfrak{a}) \,|\, \mathfrak{a} \text{ is an abelian subalgebra of } \mathfrak{g}\},$$

$$\beta(\mathfrak{g}) = \max\{\,\dim(\mathfrak{b}) \,|\, \mathfrak{b} \text{ is an abelian ideal of } \mathfrak{g}\}.$$

2.1 Some properties and bounds

First, let us note that an abelian subalgebra of maximal dimension is maximal abelian with respect to inclusion. However, a maximal abelian subalgebra may not be of maximal dimension. We show the following

Example 2.1. *Let $\mathfrak{f}_n$ be the model filiform nilpotent Lie algebra of dimension n. Let $\{e_i\}_{i=1}^n$ be an adapted basis, such that $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$. Then $\mathfrak{a} = \langle e_1, e_2 \rangle$ is a maximal abelian subalgebra of dimension 2, but $\alpha(\mathfrak{f}_n) = \beta(\mathfrak{f}_n) = n - 1$.*

Clearly, we have $\beta(\mathfrak{g}) \leq \alpha(\mathfrak{g})$. In general, both invariants are different. A complex semisimple Lie algebra $\mathfrak{s}$ has no abelian ideals, hence $\beta(\mathfrak{s}) = 0$. We already saw in Table 1 in Introduction, that this is not true for the invariant $\alpha(\mathfrak{s})$. As mentioned before, the following result holds, see [97]

Proposition 2.1. *Let $\mathfrak{s}$ be a complex simple Lie algebra and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{s}$. Then the maximal dimension of an abelian ideal in $\mathfrak{b}$ coincides with the* *maximal dimension of a commutative subalgebra of* $\mathfrak{s}$*, i.e.,* $\alpha(\mathfrak{s}) = \beta(\mathfrak{b})$*. Furthermore the number of abelian ideals in* $\mathfrak{b}$ *is* $2^{\mathrm{rank}(\mathfrak{s})}$*.*

This implies $\alpha(\mathfrak{b}) = \beta(\mathfrak{b})$, because we have $\alpha(\mathfrak{b}) \leq \alpha(\mathfrak{s}) = \beta(\mathfrak{b})$, since α is monotone.

Lemma 2.1. *The invariant* α *is monotone and additive: for a subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$*,* $\alpha(\mathfrak{h}) \leq \alpha(\mathfrak{g})$ *holds; and for two Lie algebras* $\mathfrak{a}$ *and* $\mathfrak{b}$*,* $\alpha(\mathfrak{a} \oplus \mathfrak{b}) = \alpha(\mathfrak{a}) + \alpha(\mathfrak{b})$*.*

The invariant β need not be monotone. For example, consider a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then $\beta(\mathfrak{h}) = 1 > 0 = \beta(\mathfrak{g})$.

If $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$ we put $\overline{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ for every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We have the following

Lemma 2.2. *It is verified that* $\alpha(\overline{\mathfrak{g}}) \geq \alpha(\mathfrak{g})$ *and* $\beta(\overline{\mathfrak{g}}) \geq \beta(\mathfrak{g})$*.*

Considering the Levi decomposition, we can set the following result.

Lemma 2.3. *Let* $\mathfrak{g}$ *be a complex Lie algebra with a Levi decomposition* $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$*. Then,* $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$*.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra in $\mathfrak{g}$ of maximal dimension and $\pi : \mathfrak{s} \ltimes \mathfrak{r} \longrightarrow \mathfrak{s}, (x, a) \longrightarrow (x, 0)$ be the projection. Restricting this Lie algebra homomorphism to $\mathfrak{a}$ yields $dim(\mathfrak{a}) = dim(ker(\pi_{\mathfrak{a}})) + dim(im(\pi_{\mathfrak{a}}))$. Since $im(\pi_{\mathfrak{a}})$ is the homomorphic image of a subalgebra of $\mathfrak{s} \ltimes \mathfrak{r}$, we can assume that $im(\pi_{\mathfrak{a}})$ is an abelian subalgebra of $\mathfrak{s}$. In particular we have $dim(im(\pi_{\mathfrak{a}})) \leq \alpha(\mathfrak{s})$. Furthermore we have $ker(\pi_{\mathfrak{a}}) = \mathfrak{a} \cap \mathfrak{r}$. Hence $ker(\pi_{\mathfrak{a}})$ is an abelian subalgebra of $\mathfrak{r}$ and we have $dim(ker(\pi_{\mathfrak{a}})) \leq \alpha(\mathfrak{r})$. Finally we obtain $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) = dim(\mathfrak{a}) = dim(ker(\pi_{\mathfrak{a}})) + dim(im(\pi_{\mathfrak{a}})) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$. □

We will also need the following lemma.

Lemma 2.4. *The center* $Z(\mathfrak{g})$ *of* $\mathfrak{g}$ *is contained in any abelian subalgebra of maximal dimension.*

Proof. We know that an abelian subalgebra $\mathfrak{a}$ of maximal dimension is self-centralizing, i.e., $\mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{a}] = 0\}$. Since $Z(\mathfrak{g}) \subset C_{\mathfrak{g}}(\mathfrak{a})$, the claim follows. □

The fact that $\alpha(\mathfrak{b}) = \beta(\mathfrak{b})$ for a Borel subalgebra $\mathfrak{b}$ of a complex simple Lie algebra can be generalized to all solvable Lie algebras.

Proposition 2.2. *Let* $\mathfrak{g}$ *be a solvable Lie algebra over an algebraically closed field* $\mathbb{K}$ *of characteristic zero. Then,* $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$*.*

Proof. The result follows easily from the proof of Theorem 4.1 of [38]. For the convenience of the reader we give the details. Let G be the adjoint algebraic group of $\mathfrak{g}$. This is the smallest algebraic subgroup of the automorphisms of $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$, such that its Lie algebra $\mathrm{Lie}(G)$ contains $\mathrm{ad}(\mathfrak{g})$. Then $\mathrm{Lie}(G)$ is the algebraic hull of $\mathrm{ad}(\mathfrak{g})$. Since $\mathrm{ad}(\mathfrak{g})$ is solvable, so is $\mathrm{Lie}(G)$. Therefore G is a connected solvable algebraic group. Let us suppose that $m = \alpha(\mathfrak{g})$. Consider the set $\mathcal{C}$ of all abelian subalgebras of $\mathfrak{g}$ of dimension m. This is, by assumtion, a non-empty set, which can be considered as a subset of the Grassmannian $Gr(\mathfrak{g}, m)$, which is an irreducible complete algebraic variety. Hence, $\mathcal{C}$ is a non-empty complete variety, and G operates morphically on it, mapping each commutative subalgebra $\mathfrak{h}$ on $g(\mathfrak{h})$, for $g \in G$. By Borel's fixed point theorem, G has a fixed point I in $\mathcal{C}$, i.e., a subalgebra I of $\mathfrak{g}$ with $g(I) = I$ for all $g \in G$. In particular we have $\mathrm{ad}(x)(I) = I$ for all $x \in \mathfrak{g}$. Hence I is an abelian ideal of dimension m of $\mathfrak{g}$. □

Borel's fixed point theorem relies on the closed orbit lemma. As a corollary one can also obtain the Lie-Kolchin theorem [64]. We note that the assumption on $\mathbb{K}$ is really necessary.

Example 2.2. *Let $\mathfrak{g}$ be the solvable Lie algebra of dimension 4 over $\mathbb{R}$ defined by*

$$[x_1, x_2] = x_2 - x_3, \quad [x_1, x_4] = 2x_4,$$
$$[x_1, x_3] = x_2 + x_3, \quad [x_2, x_3] = x_4.$$

We prove that, over $\mathbb{R}$, we have $\alpha(\mathfrak{g}) = 2$, but $\beta(\mathfrak{g}) = 1$. Let $\mathbb{K}$ be equal to $\mathbb{R}$ or $\mathbb{C}$. Obviously, $\langle x_3, x_4 \rangle$ is an abelian subalgebra of dimension 2 over $\mathbb{K}$. Assume that $\alpha(\mathfrak{g}) = 3$. Then $\mathfrak{g}$ is almost abelian, hence 2-step solvable. This is impossible, as $\mathfrak{g}$ is 3-step solvable. Hence $\alpha(\mathfrak{g}) = 2$ over $\mathbb{K}$.
Assume that I is a 2-dimensional abelian ideal over $\mathbb{K}$. It is easy to see that we can represent I as $\langle ax_2 + bx_3, x_4 \rangle$ with $a, b \in \mathbb{K}$. Obviously both x_2 and x_3 cannot belong to I. Hence $a \neq 0$ and $b \neq 0$. We have $ax_2 + bx_3 \in I$ and $[x_1, ax_2 + bx_3] = (a+b)x_2 - (a-b)x_3 \in I$. This implies $a^2 + b^2 = 0$. This is a contradiction over $\mathbb{R}$, so that $\beta(\mathfrak{g}) = 1$ in this case. Over $\mathbb{C}$ we may take $a = 1$ and $b = i$, and $I = \langle x_2 + ix_3, x_4 \rangle$ is a 2-dimensional abelian ideal.

In Section 2.4, we will see an example concerning the fact that the assumption on the characteristic of the field $\mathbb{K}$ is really necessary.

As a consequence, we have the following bounds for α invariant.

Lemma 2.5. *Let $\mathfrak{g}$ be a complex, non-abelian, nilpotent Lie algebra of dimension n. Then*

$$\frac{\sqrt{8n+1}-1}{2} \leq \alpha(\mathfrak{g}) \leq n-1$$

Proof. The estimate is well-known for $\beta(\mathfrak{g})$, see [38]. By Proposition 2.2, it follows for $\alpha(\mathfrak{g})$. □

Lemma 2.6. *Let $\mathfrak{g}$ be any solvable Lie algebra with nilradical N. Then $C_{\mathfrak{g}}(N) \subseteq N$.*

Proof. Suppose that $C_{\mathfrak{g}}(N) \not\subseteq N$. Then there is a non-trivial abelian ideal $I/(N \cap C_{\mathfrak{g}}(N))$ of $\mathfrak{g}/(N \cap C_{\mathfrak{g}}(N))$ inside $C_{\mathfrak{g}}(N)/(N \cap C_{\mathfrak{g}}(N))$. But now $\mathcal{C}^3(I) \subseteq [I, N] = 0$, so I is a nilpotent ideal of $\mathfrak{g}$. It follows that $I \subseteq N \cap C_{\mathfrak{g}}(N)$, which is a contradiction. □

Next, we have a bound of $\beta(\mathfrak{g})$ for certain metabelian Lie algebras.

Proposition 2.3. *Let $\mathfrak{g}$ be a metabelian Lie algebra of dimension n, and suppose that $\dim(\mathcal{D}(\mathfrak{g})) = k$. Then $\dim(\mathfrak{g}/C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g}))) \leq [k^2/4] + 1$. If, further, $\mathfrak{g}$ splits over $\mathcal{D}(\mathfrak{g})$, then $\beta(\mathfrak{g}) \geq n - [k^2/4] - 1$.*

Proof. Let $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\, \mathcal{D}(\mathfrak{g})$ be defined by $\mathrm{ad}\, x(y) = [y, x]$ for all $y \in \mathcal{D}(\mathfrak{g})$. Then ad is a homomorphism with kernel $C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g}))$. It follows that $\mathfrak{g}/C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g})) \cong D$ where D is an abelian subalgebra of $\mathrm{Der}\, \mathcal{D}(\mathfrak{g}) \cong gl(k, \mathbb{K})$. It follows from Schur's Theorem on commuting matrices (see [58]) that $\dim(\mathfrak{g}/C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g})) \leq [k^2/4] + 1$.

Now suppose that $\mathfrak{g} = \mathcal{D}(\mathfrak{g}) \oplus B$, where B is an abelian subalgebra of $\mathfrak{g}$. Then $C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g})) = \mathcal{D}(\mathfrak{g}) \oplus B \cap C_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g}))$ which is an abelian ideal of $\mathfrak{g}$. □

Now, we obtain bounds for supersolvable Lie algebras by following a development similar to [94, Lemma 2].

Lemma 2.7. *Let $\mathfrak{g}$ be a supersolvable Lie algebra and let I be a maximal abelian ideal of $\mathfrak{g}$. Then, $\mathcal{C}_{\mathfrak{g}}(I) = I$.*

Proof. We have that $\mathcal{C}_{\mathfrak{g}}(I)$ is an ideal of $\mathfrak{g}$. Suppose that $\mathcal{C}_{\mathfrak{g}}(I) \neq I$. Let J/I be a minimal ideal of $\mathfrak{g}/I$ with $J \subset \mathcal{C}_{\mathfrak{g}}(I)$. Then, for some $j \in J$, $J = I + \langle j \rangle$, which is an abelian ideal of $\mathfrak{g}$, contradicting the maximality of I. □

Proposition 2.4. *Let $\mathfrak{g}$ be a supersolvable Lie algebra and let I be any maximal abelian ideal of $\mathfrak{g}$. Suppose that dim$(I) \leq k$. Then, $\mathfrak{g}/I$ is isomorphic to a Lie algebra of $k \times k$ lower triangular matrices.*

Proof. Let $ad : \mathfrak{g} \to Der(I)$ be defined by $ad\,x(y) = [y, x]$ for all $x \in \mathfrak{g}$ and $y \in I$. Then, ad is a homomorphism with kernel $\mathcal{C}_{\mathfrak{g}}(I) = I$ by Lemma 2.7. Since $\mathfrak{g}$ is supersolvable there is a flag of ideals $0 = I_0 \subset I_1 \subset \ldots \subset I_k = I$ of $\mathfrak{g}$. Choose a basis $\{e_i\}_{i=1}^k$ for I with $e_i \in I_i$. With respect to this basis the action of $\mathfrak{g}$ on I is represented by $k \times k$ lower triangular matrices, since $[I_i, \mathfrak{g}] \subseteq I_i$ for each $0 \leq i \leq k$. □

Corollary 2.1. *Let $\mathfrak{g}$ be a supersolvable Lie algebra with a maximal abelian ideal I of dimension at most k. Then $dim(\mathfrak{g}) \leq \frac{k(k+3)}{2}$ and $\mathfrak{g}$ has derived length at most $k+1$.*

Proof. The Lie algebra of $k \times k$ lower triangular matrices has dimension $\frac{k(k+1)}{2}$ and derived length k. □

Corollary 2.2. *If $\mathfrak{g}$ is supersolvable of dimension n, then*

$$\beta(\mathfrak{g}) \geq \left\lfloor \frac{\sqrt{8n+9}-3}{2} \right\rfloor.$$

Corollary 2.3. *If $\mathfrak{g}$ is solvable, but non-abelian of dimension n over an algebraically closed field $\mathbb{K}$ of characteristic zero, then*

$$\left\lfloor \frac{\sqrt{8n+9}-3}{2} \right\rfloor \leq \alpha(\mathfrak{g}) \leq n-1.$$

Proof. Simply use Corollary 2.2 and Proposition 2.2. □

2.2 Abelian subalgebras of codimension 1

Let $\mathfrak{g}$ be a Lie algebra satisfying $\alpha(\mathfrak{g}) = n-1$. We will show that $\beta(\mathfrak{g}) = n-1$ without using Proposition 2.2. Our proof will be constructive. We do not only show the existence of an abelian ideal of dimension $n-1$, but really construct such an ideal from a given abelian subalgebra of dimension $n-1$. Note that Lie algebras $\mathfrak{g}$ with $\beta(\mathfrak{g}) = n-1$ are called *almost abelian*. As mentioned before, they are 2-step solvable, and their structure is well-known (see [48, Section 3] and [98]).

Proposition 2.5. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra satisfying $\alpha(\mathfrak{g}) = n-1$. Then we have $\beta(\mathfrak{g}) = n-1$, and $\mathfrak{g}$ is almost abelian.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra of dimension $n-1$. If $\mathcal{D}(\mathfrak{g}) \subseteq \mathfrak{a}$, then $\mathfrak{a}$ is also an abelian ideal, and we are done. Otherwise we choose a basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$ such that $\mathfrak{a} = \langle e_2, \ldots, e_n\rangle$. We have $[e_j, e_\ell] = 0$ for all $j, \ell \geq 2$. There exists $k \geq 2$ such that $[e_1, e_k]$ is not contained in $\mathfrak{a}$. We may assume that $k = 2$ by relabelling e_2 and e_k. For $j \geq 2$, let us consider

$$[e_1, e_j] = \alpha_{j,1}e_1 + \alpha_{j,2}e_2 + \cdots + \alpha_{j,n}e_n.$$

We have $\alpha_{2,1} \neq 0$. Rescaling e_1 we may assume that $\alpha_{2,1} = 1$. Using the Jacobi identity we have for all $j \geq 2$

$$0 = [e_1, [e_2, e_j]] = -[e_2, [e_j, e_1]] - [e_j, [e_1, e_2]] = -\alpha_{j,1}[e_1, e_2] + [e_1, e_j]$$

This implies $[e_1, e_j] = \alpha_{j,1}[e_1, e_2]$ and $[e_1, \alpha_{j,1}e_2 - e_j] = 0$ for all $j \geq 2$. Let us define $v_j = \alpha_{j,1}e_2 - e_j$. Note that every v_j lies in the center of $\mathfrak{g}$, and that the derived subalgebra $\mathcal{D}(\mathfrak{g})$ is 1-dimensional, generated by $[e_1, e_2]$. Now define

$$I := \langle [e_1, e_2], v_3, \ldots, v_n\rangle.$$

This is an abelian subalgebra of dimension $n-1$ which contains the derived subalgebra $\mathcal{D}(\mathfrak{g})$. Hence I is an abelian ideal of maximal dimension $n-1$, and we have $\beta(\mathfrak{g}) = n-1$. □

Remark 2.1. *Here, we give an alternative proof for the fact that $\mathfrak{g}$ is almost abelian.*

Let $\mathfrak{a}$ be an abelian subalgebra of dimension $n-1$. We choose a complement space $\langle x\rangle$ with $\mathfrak{a} \oplus \langle x\rangle = \mathfrak{g}$. If $[\mathfrak{a}, \langle x\rangle] \subset \mathfrak{a}$, then $\mathfrak{a}$ is an abelian ideal of codimension 1 and we are done. If $[\mathfrak{a}, \langle x\rangle] \nsubseteq \mathfrak{a}$, it is non-zero and contained in $\langle x\rangle$, hence it is verified that $[\mathfrak{a}, \langle x\rangle] = \langle x\rangle$. Then, every two linear independent vectors in the Lie algebra $\mathfrak{g}$ generate a 2-dimensional subalgebra. By applying Lemma 5.3 of [111], we can conclude that $\mathfrak{g}$ is almost abelian.

Proposition 2.6. *Let $\mathfrak{g}$ be a supersolvable Lie algebra and let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$. Suppose that $\mathfrak{b} = \mathfrak{a} + \langle e_1\rangle$ is a subalgebra of $\mathfrak{g}$, and that there is an $x \in \mathfrak{g}$ such that $[x, \mathfrak{b}] \subseteq \mathfrak{b}$, but $[x, \mathfrak{a}] \nsubseteq \mathfrak{a}$. Then, $\mathcal{D}(\mathfrak{b})$ is one dimensional and $Z(\mathfrak{b})$ has codimension at most one in $\mathfrak{a}$.*

Proof. Let $\{e_i\}_{i=2}^k$ be a basis for $\mathfrak{a}$ such that $e_1 = [x, e_2]$. Let $[x, e_j] = \sum_{i=1}^k \alpha_{j,i}e_i$ for $1 \leq j \leq k$. Then $[e_2, [x, e_j]] = \alpha_{j,1}[e_2, e_1]$, so, for $2 \leq j \leq k$,

$$0 = [x, [e_2, e_j]] = -[e_2, [e_j, x]] - [e_j, [x, e_2]] = \alpha_{j,1}[e_2, e_1] - [e_j, e_1].$$

Hence $[e_1, e_j] = \alpha_{j,1}[e_1, e_2]$. It follows that $\mathcal{D}(\mathfrak{b}) = \langle [e_1, e_2]\rangle$. Put $v_j = \alpha_{j,1}e_2 - e_j$ for $3 \leq j \leq k$. Then $\{v_i\}_{i=3}^k \in Z(\mathfrak{b}) \cap \mathfrak{a}$. □

The above result deals with the case where an abelian subalgebra of maximal dimension has codimension one in an ideal of $\mathfrak{g}$.

Corollary 2.4. *Let* $\mathfrak{g}$ *be a supersolvable Lie algebra and let* $\mathfrak{a}$ *be an abelian subalgebra of maximal dimension in* $\mathfrak{g}$. *If* $\mathfrak{a} \subset \mathfrak{b}$ *where* $\mathfrak{b}$ *is an ideal of* $\mathfrak{g}$ *and* $\mathfrak{a}$ *has codimension* 1 *in* $\mathfrak{b}$, *then* $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$.

Proof. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ then the result is clear, so suppose that it is not. With the same notation as Proposition 2.6 the hypotheses of that result are satisfied. Then $\{v_3, \ldots, v_k\} \subset Z(\mathfrak{b})$; in fact, the maximality of $\mathfrak{a}$ gives $Z(\mathfrak{b}) = \langle v_3, \ldots, v_k \rangle$. Let $\mathfrak{c}/\langle v_3, \ldots, v_k \rangle$ be a chief factor of $\mathfrak{g}$ (minimal ideal of $\mathfrak{g}/\langle v_3, \ldots, v_k \rangle$) with $\mathfrak{c} \subset \mathfrak{b}$. Then $\mathfrak{c}$ is an abelian ideal of $\mathfrak{g}$ with the same dimension as $\mathfrak{a}$. The result follows. □

Next we consider the situation where $\mathfrak{g}$ has a maximal subalgebra that is abelian: first when $\mathfrak{g}$ is any non-abelian Lie algebra and $\mathbb{K}$ is algebraically closed, and then when $\mathfrak{g}$ is solvable but $\mathbb{K}$ is arbitrary.

Proposition 2.7. *Let* $\mathfrak{g}$ *be a non-abelian Lie algebra of dimension* n *over an algebraically closed field* $\mathbb{K}$ *of any characteristic. Then,* $\mathfrak{g}$ *has a maximal subalgebra* $\mathfrak{a}$ *which is abelian if and only if* $\mathfrak{g}$ *has an abelian ideal of codimension one in* $\mathfrak{g}$. *So* $\alpha(\mathfrak{g}) = \beta(\mathfrak{g}) = n - 1$.

Proof. Suppose first that $\mathfrak{g}$ has a maximal subalgebra $\mathfrak{a}$ which is abelian. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ the proof is finished. So suppose that $\mathfrak{a}$ is self-idealising, in which case it is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{g}_1(\mathfrak{a})$ be the Fitting decomposition of $\mathfrak{g}$ relative to $\mathfrak{a}$. Then $\{(ad\, m)_{|\mathfrak{g}_1(\mathfrak{a})} : m \in \mathfrak{a}\}$ is a set of simultaneously triangulable linear mappings. So, there exists $0 \neq b \in \mathfrak{g}_1(\mathfrak{a})$ such that $[m, b] = \lambda(m) b$ for every $m \in \mathfrak{a}$, where $\lambda(m) \in \mathbb{K}$. Then we have that $\mathfrak{a} + \langle b \rangle$ is a subalgebra of $\mathfrak{g}$ strictly containing $\mathfrak{a}$, whence $\mathfrak{a} \dot{+} \langle b \rangle = \mathfrak{g}$ and $\mathcal{D}(\mathfrak{g}) = \langle b \rangle$. But now $dim(\mathfrak{g}/\mathcal{C}_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g}))) = 1$ and $\mathcal{C}_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g})) = \mathcal{D}(\mathfrak{g}) \dot{+} (\mathfrak{a} \cap \mathcal{C}_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g})))$ is an abelian ideal of codimension one in $\mathfrak{g}$. The converse is clear. □

The following is a generalisation of [103, Proposition 3.1]

Proposition 2.8. *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Then* $\mathfrak{g}$ *has a maximal subalgebra* $\mathfrak{a}$ *which is abelian if and only if either*

(i) $\mathfrak{g}$ *has an abelian ideal of codimension one in* $\mathfrak{g}$*; or*

(ii) $\mathcal{C}_3(\mathfrak{g}) = \phi(\mathfrak{g}) = Z(\mathfrak{g})$, $\mathcal{C}_2(\mathfrak{g})/\mathcal{C}_3(\mathfrak{g})$ *is a chief factor of* $\mathfrak{g}$, *and* $\mathfrak{g}$ *splits over* $\mathcal{C}_2(\mathfrak{g})$.

Proof. Suppose first that $\mathfrak{g}$ has a maximal subalgebra $\mathfrak{a}$ which is abelian. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ we are in case (i). So suppose that $\mathfrak{a}$ is self-idealising, in which case it is a Cartan subalgebra of $\mathfrak{g}$. There is a $k \geq 1$ such that $\mathcal{C}_k(\mathfrak{g}) \nsubseteq \mathfrak{a}$ but $\mathcal{C}_{k+1}(\mathfrak{g}) \subseteq \mathfrak{a}$. Then $\mathfrak{g} = \mathfrak{a} + \mathcal{C}_k(\mathfrak{g})$, which implies that $\mathcal{C}_2(\mathfrak{g}) \subseteq \mathcal{C}_k(\mathfrak{g})$. It follows that $\mathcal{C}_3(\mathfrak{g}) \subseteq \mathfrak{a}$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ denote by $\widetilde{\mathfrak{h}}$ its image under the canonical homomorphism onto $\mathfrak{g}/\mathcal{C}_3(\mathfrak{g})$. Then $\widetilde{\mathfrak{a}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ has a Fitting decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{a}} \dotplus \widetilde{\mathfrak{g}}_1$. Now $\widetilde{\mathfrak{g}}_1 \subseteq \mathcal{C}_2(\widetilde{\mathfrak{g}}) = \widetilde{\mathcal{C}_2(\mathfrak{g})}$, which is abelian, so $\widetilde{\mathfrak{g}}_1$ is an ideal of $\widetilde{\mathfrak{g}}$. Moreover, since $\widetilde{\mathfrak{a}}$ is a maximal subalgebra of $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{g}}_1$ is a minimal ideal of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathcal{C}_2(\mathfrak{g})} = \widetilde{\mathfrak{g}}_1$. It follows that $\mathcal{C}_2(\mathfrak{g})/\mathcal{C}_3(\mathfrak{g})$ is a chief factor of $\mathfrak{g}$. Clearly $\phi(\widetilde{\mathfrak{g}}) = 0$, whence $\phi(\mathfrak{g}) \subseteq \mathcal{C}_3(\mathfrak{g})$. Also $\mathfrak{g} = \mathfrak{a} + \mathcal{C}_2(\mathfrak{g})$, so letting $\mathfrak{b}$ be a subspace of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \dotplus (\mathfrak{a} \cap \mathcal{C}_2(\mathfrak{g}))$ we see that $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ splits over $\mathcal{C}_2(\mathfrak{g})$. Next, $[\mathfrak{a}, \mathcal{C}_3(\mathfrak{g})] \subseteq \mathcal{C}_2(\mathfrak{a}) = 0$, so $\mathfrak{a} \subseteq C_{\mathfrak{g}}(\mathcal{C}_3(\mathfrak{g}))$. Since $\mathfrak{a}$ is a self-idealising maximal subalgebra of $\mathfrak{g}$ and $C_{\mathfrak{g}}(\mathcal{C}_3(\mathfrak{g}))$ is an ideal of $\mathfrak{g}$, we have $C_{\mathfrak{g}}(\mathcal{C}_3(\mathfrak{g})) = \mathfrak{g}$, whence $\mathcal{C}_3(\mathfrak{g}) = Z(\mathfrak{g})$. Finally, this means that $\mathcal{C}_2(\mathfrak{g})$ is nilpotent, giving $\mathcal{C}_3(\mathfrak{g}) = \phi(\mathcal{C}_2(\mathfrak{g})) \subseteq \phi(\mathfrak{g})$ by [99, Lemma 4.1 and Section 5], whence $\phi(\mathfrak{g}) = \mathcal{C}_3(\mathfrak{g})$.

Consider now the converse. If (i) holds the converse is clear. So suppose that (ii) holds. Then $\mathfrak{g} = \mathfrak{b} \dotplus \mathcal{C}_2(\mathfrak{g})$, where $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}$. Put $\mathfrak{a} = \mathfrak{b} + \mathcal{C}_3(\mathfrak{g})$, so $\mathfrak{a}$ is clearly abelian. Let $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathfrak{g}$. Then $\widetilde{\mathfrak{a}} \subseteq \widetilde{\mathfrak{c}} \subseteq \widetilde{\mathfrak{g}}$. But $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{a}} \dotplus \widetilde{\mathcal{C}_2(\mathfrak{g})}$ and $\widetilde{\mathcal{C}_2(\mathfrak{g})}$ is a minimal abelian ideal of $\widetilde{\mathfrak{g}}$. So $\widetilde{\mathfrak{a}} \neq \widetilde{\mathfrak{c}}$ implies that $\widetilde{\mathfrak{c}} = \widetilde{\mathfrak{g}}$. It follows that $\mathfrak{a}$ is a maximal subalgebra of $\mathfrak{g}$. □

2.3 Abelian subalgebras of codimension 2

Let $\mathfrak{g}$ be a complex Lie algebra of dimension n satisfying $\alpha(\mathfrak{g}) = n - 2$. We will show that $\mathfrak{g}$ must be solvable except for the cases $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$, for $\ell \geq 0$. We use the convention that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is included in this family, for $\ell = 0$.

Proposition 2.9. *Let $\mathfrak{g}$ be an n-dimensional complex Lie algebra with $\alpha(\mathfrak{g}) = n-2$, then either $\mathfrak{g}$ is isomorphic to one of the Lie algebras $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$, or $\mathfrak{g}$ is a solvable Lie algebra.*

Proof. Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition, where $\mathfrak{r}$ denotes the solvable radical of $\mathfrak{g}$. For a semisimple Levi subalgebra $\mathfrak{s}$ we have

$$\alpha(\mathfrak{s}) \leq \dim(\mathfrak{s}) - 2,$$

where equality holds if and only if $\mathfrak{s}$ is $\mathfrak{sl}_2(\mathbb{C})$. This follows from Table 1 and Lemma 2.1. By Lemma 2.3, we have that $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$. Assume that $\mathfrak{s} \neq 0$. Then it follows that

$$\alpha(\mathfrak{g}) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r}) \leq \dim(\mathfrak{s}) - 2 + \dim(\mathfrak{r}) = n - 2.$$

Since we must have an equality, it follows that $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and $\alpha(\mathfrak{r}) = \dim(\mathfrak{r})$. Therefore $\mathfrak{r}$ is abelian and $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C}) \ltimes_\phi \mathbb{C}^\ell$ with a homomorphism $\phi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathbb{C}^\ell)$. This Lie algebra contains an abelian subalgebra of codimension 2 if and only if ϕ is trivial. Indeed, the Lie bracket is given by $[(x,a),(y,b)] = ([x,y], \phi(x)b - \phi(y)a)$, for $x, y \in \mathfrak{sl}_2(\mathbb{C})$ and $a, b \in \mathbb{C}^\ell$. Since there is an abelian subalgebra of codimension 2, there must be a non-zero element $(x,0)$ commuting with all elements $(0,b)$, i.e., $(0,0) = [(x,0),(0,b)] = (0, \phi(x)b)$ for all $b \in \mathbb{C}^\ell$. It follows that $\ker(\phi)$ is non-trivial. Since $\mathfrak{sl}_2(\mathbb{C})$ is simple, $\phi = 0$. In the other remaining case we have $\mathfrak{s} = 0$. In that case, $\mathfrak{g}$ is solvable. □

It is easy to classify such Lie algebras in low dimensions.

Proposition 2.10. *Let $\mathfrak{g}$ be a complex Lie algebra of dimension n and $\alpha(\mathfrak{g}) = n-2$.*

(1) For $n = 3$, it follows $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$.

(2) For $n = 4$, $\mathfrak{g}$ is isomorphic to one of the following Lie algebras

Table 2.1: Lie algebras of dimension 4 whose α invariant is 2.

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_1 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1,e_2] = e_2,\ [e_3,e_4] = e_4$
$\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1,e_2] = e_2,\ [e_1,e_3] = -e_3,\ [e_2,e_3] = e_1$
$\mathfrak{g}_3$	$[e_1,e_2] = e_2,\ [e_1,e_3] = e_3,\ [e_1,e_4] = 2e_4,\ [e_2,e_3] = e_4$
$\mathfrak{g}_4(\alpha),\ \alpha \in \mathbb{C}$	$[e_1,e_2] = e_2,\ [e_1,e_3] = e_2 + \alpha e_3,\ [e_1,e_4] = (\alpha+1)e_4,\ [e_2,e_3] = e_4$

Proof. The proof is straightforward, using a classification of low-dimensional Lie algebras (see [19] for example). Note that $\mathfrak{g}_4(\alpha) \simeq \mathfrak{g}_4(\beta)$ if and only if $\alpha\beta = 1$ or $\alpha = \beta$. □

Moreover, we can characterise solvable Lie algebras $\mathfrak{g}$ whose biggest abelian subalgebras have codimension two in $\mathfrak{g}$.

Theorem 2.1. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension n with $\alpha(\mathfrak{g}) = n-2$, and let $\mathfrak{a}$ be an abelian subalgebra of dimension $n-2$. Then one of the following occurs*

(i) $\beta(\mathfrak{g}) = n-2$;

(ii) $\mathfrak{g} = \mathcal{C}_2(\mathfrak{g}) \dot{+} \mathfrak{b}$, *where $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}$, $\mathcal{C}_2(\mathfrak{g})$ is the three-dimensional Heisenberg algebra, $\mathcal{C}_3(\mathfrak{g}) = \phi(\mathfrak{g}) = Z(\mathfrak{g})$ and $\mathcal{C}_2(\mathfrak{g})/Z(\mathfrak{g})$ is a two-dimensional chief factor of $\mathfrak{g}$ (in which case $\beta(\mathfrak{g}) \leq n-3$);*

(iii) *$\mathfrak{a}$ has codimension one in the nilradical, N, of $\mathfrak{g}$, which itself has codimension one in $\mathfrak{g}$. Moreover, $\mathcal{C}_2(N)$ is one dimensional, $Z(N)$ is an abelian ideal of maximal dimension and $\beta(\mathfrak{g}) = n-3$.*

Proof. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ of dimension $n-2$ and suppose that (i) does not hold. Suppose first that $\mathfrak{a}$ is a maximal subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ is as in Proposition 2.8(ii) and $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{g}_1$ be the Fitting decomposition of $\mathfrak{g}$ relative to $\mathfrak{a}$. Then $\mathfrak{g}_1 \subseteq \mathcal{C}_2(\mathfrak{g})$ and $\dim \mathfrak{g}_1 = 2$. Let $\mathfrak{g}_1 = \langle x, y \rangle$. If $[x,y] = 0$ then $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{g}_1$ is abelian, so $\mathcal{C}_2(\mathfrak{g}) \subseteq \mathfrak{g}_1 \subseteq \mathcal{C}_2(\mathfrak{g})$. This yields that $\mathfrak{g}$ is metabelian and $\mathcal{C}_2(\mathfrak{g})$ is a two dimensional minimal ideal over which $\mathfrak{g}$ splits. It follows from Proposition 2.3 that $\beta(\mathfrak{g}) \geq n-2$, a contradiction.

If $[x,y] \neq 0$, then $\mathcal{C}_2(\mathfrak{g}) = \langle [x,y] \rangle + \mathfrak{g}_1$ and $\langle [x,y] \rangle \subseteq \mathcal{C}_3(\mathfrak{g}) = Z(\mathfrak{g})$, so $\langle [x,y] \rangle = Z(\mathfrak{g})$ and we have case (ii). Let $\mathfrak{c}$ be a maximal abelian ideal of $\mathfrak{g}$. Then $Z(\mathfrak{g}) \subseteq \mathfrak{c}$ (Lemma 2.4) and $\mathcal{C}_2(\mathfrak{g}) \not\subseteq \mathfrak{c}$. It follows that $\mathfrak{c} \cap \mathcal{C}_2(\mathfrak{g}) = Z(\mathfrak{g})$. If $\dim \mathfrak{c} = n-2$, then $\dim(\mathcal{C}_2(\mathfrak{g}) + \mathfrak{c}) = \dim \mathcal{C}_2(\mathfrak{g}) + \dim \mathfrak{c} - \dim(\mathfrak{c} \cap \mathcal{C}_2(\mathfrak{g})) = 3 + n - 2 - 1 = n$, so $\mathfrak{g} = \mathcal{C}_2(\mathfrak{g}) + \mathfrak{c}$. But then $\mathcal{C}_2(\mathfrak{g}) = Z(\mathfrak{g})$, a contradiction. Hence, $\beta(\mathfrak{g}) \leq n-3$.

So suppose that $\mathfrak{a}$ is not a maximal subalgebra of $\mathfrak{g}$. Then $\mathfrak{a} \subset \mathfrak{d} \subset \mathfrak{g}$, where $\dim \mathfrak{d} = n-1$. Moreover, there is such a subalgebra $\mathfrak{a}$ of $\mathfrak{d}$ which is an ideal of $\mathfrak{d}$, by Proposition 2.5. Suppose first that $\mathfrak{a}$ does not act nilpotently on $\mathfrak{g}$. Then the Fitting decomposition of $\mathfrak{g}$ relative to $\mathfrak{a}$ is $\mathfrak{g} = \mathfrak{d} \dot{+} \mathfrak{g}_1$, and $\mathfrak{g}_1$ is a one-dimensional ideal of $\mathfrak{g}$. Put $\mathfrak{b} = \mathfrak{a} \dot{+} \mathfrak{g}_1$, which is an ideal of $\mathfrak{g}$. Then $C_{\mathfrak{b}}(\mathfrak{g}_1)$ has codimension one in $\mathfrak{b}$ and so is an abelian ideal of codimension two in $\mathfrak{g}$. It follows that $\beta(\mathfrak{g}) = n-2$, a contradiction.

Finally, suppose that $\mathfrak{a}$ is an ideal of $\mathfrak{d}$ and that $\mathfrak{a}$ acts nilpotently on $\mathfrak{g}$. Then there is a $k \geq 0$ such that $\mathfrak{g}(\operatorname{ad} \mathfrak{a})^k \not\subseteq \mathfrak{d}$ but $\mathfrak{g}(\operatorname{ad} \mathfrak{a})^{k+1} \subseteq \mathfrak{d}$. Let $x \in \mathfrak{g}(\operatorname{ad} \mathfrak{a})^k \setminus \mathfrak{d}$, so $\mathfrak{g} = \mathfrak{d} \dot{+} \langle x \rangle$. Suppose first that $\mathfrak{d}$ is not an ideal of $\mathfrak{g}$. Then the core of $\mathfrak{d}$, $\mathfrak{d}_{\mathfrak{g}}$ has codimension one in $\mathfrak{d}$, by [1, Theorem 3.1]. If $\mathfrak{a} = \mathfrak{d}_{\mathfrak{g}}$ then we have case (i), so suppose that $\mathfrak{a} \neq \mathfrak{d}_{\mathfrak{g}}$ and $\mathfrak{d} = \mathfrak{a} + \mathfrak{d}_{\mathfrak{g}}$. Then $[\mathfrak{a}, x] \subseteq \mathfrak{d}$ which implies that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{d}$ and $[\mathfrak{g}, \mathfrak{d}] = [\mathfrak{g}, \mathfrak{d}_{\mathfrak{g}}] + [\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{d}$; that is, $\mathfrak{d}$ is an ideal of $\mathfrak{g}$.

Let N be the nilradical of $\mathfrak{g}$. If $N \subseteq \mathfrak{a}$ then $\mathfrak{a} \subseteq C_{\mathfrak{g}}(N) \subseteq N$, so $N = \mathfrak{a}$ and we have case (i) again. If $\mathfrak{a} \subset N$ then $N = \mathfrak{g}$ or we can assume that $N = \mathfrak{d}$. If $\mathfrak{a} \not\subseteq N$ and $N \not\subseteq \mathfrak{a}$ then either $\mathfrak{a} + N = \mathfrak{g}$, in which case $\mathfrak{g}$ is nilpotent, or we can assume that $\mathfrak{a} + N = \mathfrak{d}$, in which case $\mathfrak{d}$ is a nilpotent ideal of $\mathfrak{g}$ and so $\mathfrak{d} = N$. If $\mathfrak{g}$ is nilpotent, then we have case (i), as it will be seen in Proposition 2.11.

So suppose that $\mathfrak{g} = N + \langle e_1 \rangle$, $N = \mathfrak{a} + \langle e_2 \rangle$ and $\mathfrak{a} = \langle e_3, \ldots, e_n \rangle$. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ then we have case (i) again, so suppose further that $[e_1, e_3] = e_2$ and $[e_1, e_j] = \sum_{i=2}^n \alpha_{j,i} e_i$ for $j \geq 2$. Following the proof of Proposition 2.6, we get $[e_2, e_j] = \alpha_{j,2}[e_2, e_3]$ for $j \geq 3$ and $\mathcal{C}_2(N) = \langle [e_2, e_3] \rangle$. Let $[e_2, e_3] = \sum_{i=2}^n \lambda_i e_i$. Then $[[e_2, e_3], e_3] = \lambda_2 [e_2, e_3]$, and nilpotency of N implies that $\lambda_2 = 0$. It follows that $[e_2, e_3] \in \mathfrak{a}$. Similarly, $[e_2, [e_2, e_3]] = \mu [e_2, e_3]$ (where $\mu = \lambda_3 + \sum_{j=4}^n \lambda_j \alpha_{j,2}$), whence $\mu = 0$. We thus have that $\mathcal{C}^3(N) = 0$. Put $v_j = \alpha_{j,2} e_3 - e_j$ for $j \geq 3$. Then $\langle v_3, \ldots, v_n \rangle \subseteq Z(N)$, which is an abelian ideal of $\mathfrak{g}$. This is case (iii). □

Corollary 2.5. *Let $\mathfrak{g}$ be a supersolvable Lie algebra with $\alpha(\mathfrak{g}) = n - 2$. Then $\beta(\mathfrak{g}) = n - 2$.*

Proof. We use the same notation as in Theorem 2.1 and show that cases (ii) and (iii) cannot occur. Clearly case (ii) cannot occur, since in that case $\mathcal{C}_2(\mathfrak{g})/Z(\mathfrak{g})$ is a two-dimensional minimal ideal of $\mathfrak{g}/Z(\mathfrak{g})$. So suppose that case (iii) occurs. Then $\dim Z(N) = n - 3$. Since $\mathfrak{g}$ is supersolvable, there is an ideal $\mathfrak{b} \subset N$ of $\mathfrak{g}$ with $\dim(\mathfrak{b}/Z(N)) = 1$. But clearly $\mathfrak{b}$ is abelian, contradicting the maximality of $Z(N)$. □

Note that algebras of the type described in Theorem 2.1 (ii) and (iii) do exist over the real field, as the following examples show.

Example 2.3. *Let $\mathfrak{g}$ be the four-dimensional Lie algebra over $\mathbb{R}$ with basis $\{e_1, e_2, e_3, e_4\}$ and non-zero products*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_4.$$

Then this algebra is as described in Theorem 2.1(ii). For $\mathcal{C}_2(\mathfrak{g}) = \langle e_2, e_3, e_4 \rangle$ is the three-dimensional Heisenberg algebra, $\mathfrak{g} = \mathcal{C}_2(\mathfrak{g}) \dot{+} \langle e_1 \rangle$, $\mathcal{C}_3(\mathfrak{g}) = \langle e_4 \rangle = Z(\mathfrak{g}) = \phi(\mathfrak{g})$, and $\mathcal{C}_2(\mathfrak{g})/Z(\mathfrak{g})$ is a two-dimensional chief factor of $\mathfrak{g}$. This algebra has $\alpha(\mathfrak{g}) = 2$ and $\beta(\mathfrak{g}) = 1$. We could take $\mathfrak{a} = \langle e_1, e_4 \rangle$, for example, but $\langle e_4 \rangle$ is the unique maximal abelian ideal of $\mathfrak{g}$.

Example 2.4. *Let $\mathfrak{g}$ be the four-dimensional Lie algebra over $\mathbb{R}$ with basis $\{e_1, e_2, e_3, e_4\}$ and non-zero products*

$$[e_1, e_2] = e_2 - e_3, \quad [e_1, e_4] = 2e_4, \quad [e_1, e_3] = e_2 + e_3, \quad [e_2, e_3] = e_4.$$

Then this algebra is as described in Theorem 2.1(iii). We can take $\mathfrak{a} = \langle e_3, e_4 \rangle$, $N = \langle e_2 \rangle + \mathfrak{a}$, *so* $\mathcal{C}_2(N) = \langle e_4 \rangle$, $Z(N) = \langle e_4 \rangle$. *We have* $\alpha(\mathfrak{g}) = 2$ *and* $\beta(\mathfrak{g}) = 1$.

Now, we study the case of nilpotent Lie algebras containing an abelian subalgebra of codimension 2. In a nilpotent Lie algebra $\mathfrak{g}$ any subalgebra of codimension 1 is automatically an ideal. Hence given an abelian subalgebra of maximal dimension $n-1$ we obtain an abelian ideal of dimension $n-1$. In particular, $\alpha(\mathfrak{g}) = n-1$ for a nilpotent Lie algebra implies $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$, and we can explicitly provide such ideals. We are able to extend this result to the case $\alpha(\mathfrak{g}) = n-2$. Given an abelian subalgebra of dimension $n-2$ we can construct an abelian ideal of dimension $n-2$. This is non-trivial, since the abelian subalgebra of maximal dimension $n-2$ need not be an ideal in general. Of course, over an algebraically closed field of characteristic zero, the existence of such an ideal already follows from Proposition 2.2, as does the equality $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$. However, the existence proof is not constructive. Our proof will be constructive and elementary, which might be more appropriate to our special situation.

Proposition 2.11. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of dimension* n *satisfying* $\alpha(\mathfrak{g}) = n-2$. *Then there exists an algorithm to construct an abelian ideal of dimension* $n-2$ *from an abelian subalgebra of dimension* $n-2$. *In particular we have* $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$.

Proof. Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$ of maximal dimension $n-2$. Choose a basis $(e_3, \dots, e_n)$ for $\mathfrak{a}$. The normalizer of $\mathfrak{a}$, $N_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{a}] \subseteq \mathfrak{a}\}$, is a subalgebra strictly containing $\mathfrak{a}$. We may assume that $N_{\mathfrak{g}}(\mathfrak{a})$ has dimension $n-1$, because otherwise $N_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g}$, implying that $\mathfrak{a}$ is already an abelian ideal of maximal dimension $n-2$.
We may extend the basis of $\mathfrak{a}$ to a basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$, such that $N_{\mathfrak{g}}(\mathfrak{a}) = \langle e_2, \dots, e_n \rangle$. Since $N_{\mathfrak{g}}(\mathfrak{a})$ has codimension 1, it is an ideal in $\mathfrak{g}$. In particular we have $[e_1, N_{\mathfrak{g}}(\mathfrak{a})] \subseteq N_{\mathfrak{g}}(\mathfrak{a})$.
On the other hand, $[e_1, \mathfrak{a}]$ is not contained in $\mathfrak{a}$, since e_1 is not in $N_{\mathfrak{g}}(\mathfrak{a})$. Hence there exists a vector e_k such that $[e_1, e_k]$ is not in $\mathfrak{a}$. By relabelling e_3 and e_k we may assume that $k = 3$. Hence writing $[e_1, e_j] = \alpha_{j,2}e_2 + \cdots + \alpha_{j,n}e_n$ for $j \geq 2$, we may assume that $\alpha_{32} = 1$, i.e., $[e_1, e_3] = e_2 + \alpha_{3,3}e_3 + \cdots + \alpha_{3,n}e_n$.

In order to continue with our line of argument, we have to prove the following lemma. The result says

Lemma 2.8. *The following statements hold*

(1) $[e_2, e_j] = \alpha_{j,2}[e_2, e_3]$, *for all* $j \geq 3$.

(2) The element $[e_2, e_3]$ *is non-zero and contained in the center of* $\mathfrak{g}$.

(3) The normalizer $N_{\mathfrak{g}}(\mathfrak{a})$ *is two-step nilpotent.*

(4) $[N_{\mathfrak{g}}(\mathfrak{a}), v_j] = 0$*, for all* $j \geq 3$*, where* $v_j = \alpha_{j,2}e_3 - e_j$*.*

Proof. The first statement follows from the Jacobi identity. We have, for all $j \geq 3$,

$$0 = [e_1, [e_3, e_j]] = -[e_3, [e_j, e_1]] - [e_j, [e_1, e_3]] = -\alpha_{j,2}[e_2, e_3] + [e_2, e_j].$$

Concerning (2), assume first that $[e_2, e_3] = 0$. Then the subalgebra given by $\langle e_2, e_3, v_4, \ldots, v_n \rangle$ would be an abelian subalgebra of dimension $n-1$, with the v_j defined as in (4). This is a contradiction to $\alpha(\mathfrak{g}) = n-2$. Hence $[e_2, e_3]$ is non-zero. Since $e_2 \in N_{\mathfrak{g}}(\mathfrak{a})$, we have that $[e_2, e_3] \in \mathfrak{a}$. We write, $[e_2, e_3] = \beta_{3,3}e_3 + \cdots + \beta_{3,n}e_n$. We have $[e_3, [e_2, e_3]] = 0$ and $[e_2, [e_2, e_3]] = (\beta_{3,3}\alpha_{3,2} + \ldots + \beta_{3,n}\alpha_{n,2})[e_2, e_3]$. Since $\operatorname{ad}(e_2)$ is nilpotent, it follows $[e_2, [e_2, e_3]] = 0$. In the same way, $[e_1, [e_2, e_3]] = [e_2, [e_1, e_3]] - [e_3, [e_1, e_2]] = \lambda[e_2, e_3]$, so that $[e_1, [e_2, e_3]] = 0$, because $\operatorname{ad}(e_1)$ is nilpotent. Finally, $[e_j, [e_2, e_3]] = 0$ for all $j \geq 3$, since $[e_2, e_3] \in \mathfrak{a}$. It follows that $[e_2, e_3]$ lies in the center of $\mathfrak{g}$.
To show (3), note that $[N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})]$ is generated by $[e_2, e_3]$, so that $[N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})] \subseteq Z(\mathfrak{g})$. This proves (3). The statement (4) follows from (1). □

Now we continue with the proof of Proposition 2.11. Let us consider $\mathfrak{a}_1 = \langle v_4, \ldots v_n \rangle$. This is an abelian subalgebra $\mathfrak{a}_1 \subseteq \mathfrak{a} \subseteq \mathfrak{g}$ of dimension $n-3$. There exists an integer $\ell \geq 1$ satisfying

$$\operatorname{ad}(e_1)^{\ell-1}(e_2) \notin \mathfrak{a}_1,$$
$$\operatorname{ad}(e_1)^{\ell}(e_2) \in \mathfrak{a}_1,$$

because $\operatorname{ad}(e_1)$ is nilpotent. We define

$$I := \langle \operatorname{ad}(e_1)^{\ell-1}(e_2), v_4, \ldots, v_n \rangle$$

We will show that I is an abelian ideal of maximal dimension $n-2$. First of all, I is a subalgebra of dimension $n-2$. It is also abelian: because $N_{\mathfrak{g}}(\mathfrak{a})$ is an ideal, $\operatorname{ad}(e_1)^k(e_2) \in N_{\mathfrak{g}}(\mathfrak{a})$ for all $k \geq 0$. Then

$$[\operatorname{ad}(e_1)^k(e_2), v_j] = [\lambda_2 e_2 + \cdots \lambda_n e_n, \alpha_{j,2}e_3 - e_j] = \lambda_2\alpha_{j,2}[e_2, e_3] - \lambda_2[e_2, e_j] = 0.$$

It remains to show that I is an ideal, i.e., that $\operatorname{ad}(e_i)(I) \subseteq I$, for all $i \geq 1$. We have

$$[e_1, \operatorname{ad}(e_1)^{\ell-1}(e_2)] = \operatorname{ad}(e_1)^{\ell}(e_2) \in \mathfrak{a}_1 \subseteq I,$$
$$[e_k, \operatorname{ad}(e_1)^{\ell-1}(e_2)] \in [N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})] \subseteq Z(\mathfrak{g}) \subseteq I,$$

for all $k \geq 2$. Here we have used Lemma 2.4 to conclude that $Z(\mathfrak{g}) \subseteq I$. Also, $[e_k, v_j] = 0 \in I$, for all $k \geq 2$ and $j \geq 4$. We only must show that $[e_1, v_j] \in I$, for all $j \geq 4$. Effectively, we have

$$[e_2, [e_1, v_j]] = [e_1, [e_2, v_j]] + [v_j, [e_1, e_2]] = 0.$$

This implies that $[e_1, v_j]$ commutes with all elements from I. If it were not in I, then $\langle [e_1, v_j], I \rangle$ would be an abelian subalgebra of dimension $n - 1$, which is impossible. It follows that $[e_1, v_j] \in I$. □

Remark 2.2. *In connection with the Toral Rank Conjecture (TCR), introduced in [50] and asserting that any finite-dimensional, complex nilpotent Lie algebra should satisfy*

$$\dim H^*(\mathfrak{g}, \mathbb{C}) \geq 2^{\dim Z(\mathfrak{g})},$$

there are interesting examples of nilpotent Lie algebras $\mathfrak{g}$ with $\beta(\mathfrak{g}) = n - 2$ and of dimension $n \geq 10$, as can be seen in [86]. These algebras also have the property that all its derivations are singular.

2.4 Abelian subalgebras of codimension 3

In this section, we prove that nilpotent Lie algebras with an abelian subalgebra of codimension 3 contain an abelian ideal with the same dimension, provided that the characteristic of the underlying field is not two. We also give an example to show that the restriction on the field is necessary.

Theorem 2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n over a field of characteristic different from two with $\alpha(\mathfrak{g}) = n - 3$. Then, $\beta(\mathfrak{g}) = n - 3$.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$ with $\dim \mathfrak{a} = n - 3$, let $\mathfrak{c}$ be a maximal subalgebra containing $\mathfrak{a}$ and suppose that $\mathfrak{a}$ is not an ideal of $\mathfrak{g}$. Then $\mathfrak{c}$ is an ideal of $\mathfrak{g}$, and $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{c}$ of codimension 2 in $\mathfrak{c}$. By Proposition 2.11, we can assume that $\mathfrak{a}$ is an ideal of $\mathfrak{c}$. Let $\{e_i\}_{i=4}^n$ be an arbitrary basis for $\mathfrak{a}$ and $\mathfrak{g} = \langle e_1 \rangle + \mathfrak{c}$. We may suppose that $e_3 = [e_1, e_4] \notin \mathfrak{a}$; set $\mathfrak{b} = \langle e_3 \rangle + \mathfrak{a}$. If $[e_1, \mathfrak{b}] \subseteq \mathfrak{b}$, then $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, and the result follows from Corollary 2.4. Hence there exists k such that $e_2 = [e_1, e_k] \notin \mathfrak{b}$, where $3 \leq k \leq n$ and $k \neq 4$. Clearly, $\mathfrak{c} = \langle e_2 \rangle + \mathfrak{b}$. Let $[e_1, e_j] = \sum_{i=2}^n \alpha_{j,i} e_i$, for $2 \leq j \leq n$. Then,

$$[e_3, e_j] = [[e_1, e_4], e_j] = -[[e_4, e_j], e_1] - [[e_j, e_1], e_4] \quad (2.1)$$

$$= \alpha_{j,2}[e_2, e_4] + \alpha_{j,3}[e_3, e_4] \text{ for } j \geq 4. \quad (2.2)$$

Put $u_j = e_j - \alpha_{j,3}e_4$, for $j \geq 5$. Then $[e_3, u_j] = \alpha_{j,2}[e_2, e_4]$, for $j \geq 5$. Therefore, we can choose the elements $\{e_i\}_{i=1}^5$ in our basis so that

$$[e_3, e_j] = \alpha_{j,2}[e_2, e_4] \text{ and } \alpha_{j,3} = 0, \ \forall j \geq 5. \tag{2.3}$$

Case 1: Suppose that $\alpha_{j,2} = 0$, for all $j \geq 4$. So $[e_3, e_j] = 0$, for $j \geq 5$, $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{b}$ and we can put $e_2 = [e_1, e_3]$. We have

$$[e_2, e_j] = [[e_1, e_3], e_j] = -[[e_3, e_j], e_1] - [[e_j, e_1], e_3] \tag{2.4}$$

$$= \alpha_{j4}[e_4, e_3] \text{ for } j \geq 5. \tag{2.5}$$

If $\alpha_{j,4} = 0$, for all $j \geq 5$, then $\dim Z(\mathfrak{c}) \geq n - 4$, and if we choose $I/Z(\mathfrak{c})$ to be a chief factor of $\mathfrak{g}$ with $I \subset \mathfrak{c}$, I is an abelian ideal of $\mathfrak{g}$ with $\dim I \geq n - 3$.

So suppose that $\alpha_{5,4} \neq 0$. Put $v_j = \alpha_{5,4}e_j - \alpha_{j,4}e_5$, for $j \geq 6$. Then $[e_2, v_j] = 0$, for $j \geq 6$ and $\dim Z(\mathfrak{c}) \geq n - 5$. So, we can choose the terms $\{e_i\}_{i=6}^n$ in the initial basis such that they belong to $Z(\mathfrak{c})$. Let us note that $\mathcal{C}_2(\mathfrak{c})$ is spanned by $[e_2, e_3]$, $[e_2, e_4]$ and $[e_3, e_4]$. Now

$$[e_1, [e_2, e_5]] = -[e_2, [e_5, e_1]] - [e_5, [e_1, e_2]] = \alpha_{5,4}[e_2, e_4] + \alpha_{5,5}[e_2, e_5] + \alpha_{2,2}[e_2, e_5],$$

and

$$[e_1, [e_4, e_3]] = -[e_4, [e_3, e_1]] - [e_3, [e_1, e_4]] = [e_4, e_2].$$

It follows from (2.5) that

$$2\alpha_{5,4}[e_2, e_4] = (\alpha_{5,5} + \alpha_{2,2})\alpha_{5,4}[e_3, e_4],$$

so $\dim \mathcal{C}_2(\mathfrak{c}) \leq 2$. Let $I/Z(\mathfrak{c})$ be a chief factor of $\mathfrak{g}$ with $I \subset \mathfrak{c}$. Then I is an abelian ideal of $\mathfrak{g}$ of dimension at least $n - 4$. Suppose that $\dim I = n - 4$ and that this is a maximal abelian ideal of $\mathfrak{g}$. Then $C_{\mathfrak{g}}(I) = I$ by [94, Lemma 1]. Put $I = Z(\mathfrak{c}) + \langle b \rangle$. Now $[e_3, b]$, $[e_4, b]$, $[e_5, b] \in \mathcal{C}_2(\mathfrak{c})$. Since $\dim \mathcal{C}_2(\mathfrak{c}) \leq 2$ these elements are linearly dependent. Hence we have $\beta_3[e_3, b] + \beta_4[e_4, b] + \beta_5[e_5, b] = 0$ for some $\beta_3, \beta_4, \beta_5 \in \mathbb{K}$, not all zero. It follows that $\beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5 \in C_{\mathfrak{g}}(I) = I$, whence $I = \langle \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5 \rangle + Z(\mathfrak{c})$. Now $e_5 \in C_{\mathfrak{g}}(I) = I$, so $I = Z(\mathfrak{c}) + \langle e_5 \rangle$. But then $e_4 \in C_{\mathfrak{g}}(I) \setminus I$, which is a contradiction. It follows that there is an abelian ideal of dimension $n - 3$.

Case 2: Suppose that $\alpha_{5,2} \neq 0$, so $[e_1, e_5] \notin \mathfrak{b}$. Put $e_2 = [e_1, e_5]$, so that $\alpha_{5,2} = 1$, $\alpha_{5,j} = 0$, for $j \neq 2$; put also $v_j = e_j - \alpha_{j,2}e_5$, for $j \geq 6$. Then $[e_3, v_j] = 0$, for $j \geq 6$. Using (2.3), we also have

$$[e_1, v_j] = [e_1, e_j] - \alpha_{j,2}[e_1, e_5] \in \mathfrak{a}, \text{ for } j \geq 6.$$

Thus, $[e_2, v_j] = [[e_1, e_5], v_j] = -[[e_5, v_j], e_1] - [[v_j, e_1], e_5] = 0$ for $j \geq 6$. So again $\dim Z(\mathfrak{c}) \geq n-5$, we can choose $\{e_i\}_{i=6}^n$ in our original basis to belong to $Z(\mathfrak{c})$, and $\mathcal{C}_2(\mathfrak{c})$ is spanned by $[e_2, e_3]$, $[e_2, e_4]$, $[e_2, e_5]$ and $[e_3, e_4]$. Now

$$[e_1, [e_2, e_4]] = -[e_2, [e_4, e_1]] - [e_4, [e_1, e_2]] = [e_2, e_3] + \alpha_{2,2}[e_2, e_4] + \alpha_{2,3}[e_3, e_4],$$

and

$$[e_1, [e_3, e_5]] = -[e_3, [e_5, e_1]] - [e_5, [e_1, e_3]] = [e_3, e_2] + \alpha_{3,2}[e_2, e_5] + \alpha_{3,3}[e_3, e_5].$$

Since $[e_3, e_5] = [e_2, e_4]$, this yields

$$2[e_2, e_3] = (\alpha_{3,3} - \alpha_{2,2})[e_2, e_4] - \alpha_{2,3}[e_3, e_4] + \alpha_{3,2}[e_2, e_5], \tag{2.6}$$

so $\mathcal{C}_2(\mathfrak{c})$ is spanned by $[e_2, e_4]$, $[e_2, e_5]$ and $[e_3, e_4]$.

We have $\mathcal{C}_2(\mathfrak{c}) \subseteq \mathfrak{a}$, since $\dim \mathfrak{c}/\mathfrak{a} = 2$. Suppose first that $\mathcal{C}_2(\mathfrak{c}) \not\subseteq Z(\mathfrak{c})$. Then choose $I \subseteq \mathcal{C}_2(\mathfrak{c}) + Z(\mathfrak{c})$ such that $I/Z(\mathfrak{c})$ is a chief factor of $\mathfrak{g}$. Then $I \subset \mathfrak{a}$ and $\mathfrak{a} \subseteq C_{\mathfrak{g}}(I) \setminus I$. It follows that $\mathfrak{g}$ has an abelian ideal of dimension $n-3$.

So consider now the case where $\mathcal{C}^3(\mathfrak{c}) = 0$. We have $\dim \mathfrak{g}/(\mathcal{C}_2(\mathfrak{g}) + Z(\mathfrak{c})) \leq 3$ since $e_2, e_3 \in \mathcal{C}_2(\mathfrak{g})$. Suppose first that $\dim \mathfrak{g}/(\mathcal{C}_2(\mathfrak{g}) + Z(\mathfrak{c})) = 3$, so that $\mathcal{C}_2(\mathfrak{g}) + Z(\mathfrak{c}) = \langle e_2, e_3 \rangle + Z(\mathfrak{c}) = J$. This is an ideal of $\mathfrak{g}$, so $\alpha_{2,4} = \alpha_{2,5} = \alpha_{3,4} = \alpha_{3,5} = 0$. Now

$$[e_1, [e_2, e_3]] = -[e_2, [e_3, e_1]] - [e_3, [e_1, e_2]] = \alpha_{3,3}[e_2, e_3] + \alpha_{2,2}[e_2, e_3] = (\alpha_{3,3} + \alpha_{2,2})[e_2, e_3].$$

Since $\mathfrak{g}$ is nilpotent we must have $\alpha_{3,3} = -\alpha_{2,2}$. Now

$$\dim([\mathfrak{g}, J] + Z(\mathfrak{c}))/Z(\mathfrak{c}) \leq 1,$$

so $[e_1, e_3] + Z(\mathfrak{c}) = \lambda[e_1, e_2] + Z(\mathfrak{c})$ for some $\lambda \in \mathbb{K}$. It follows that $\alpha_{3,2} = \lambda\alpha_{2,2}$, $-\alpha_{2,2} = \alpha_{3,3} = \lambda\alpha_{2,3}$ and $I = \langle e_3 - \lambda e_2 \rangle + Z(\mathfrak{c})$ is an abelian ideal of $\mathfrak{g}$. Suppose first that $\lambda \neq 0$. Then

$$\begin{aligned}
&[e_3 - \lambda e_2, \lambda e_2 + e_3 - \lambda\alpha_{2,3}e_4 - \alpha_{3,2}e_5] \\
&= -2\lambda[e_2, e_3] + (\lambda^2\alpha_{2,3} - \alpha_{3,2})[e_2, e_4] - \lambda\alpha_{2,3}[e_3, e_4] + \lambda\alpha_{3,2}[e_2, e_5] \\
&= \lambda(-2[e_2, e_3] + (\alpha_{3,3} - \alpha_{2,2})[e_2, e_4] - \alpha_{2,3}[e_3, e_4] + \alpha_{3,2}[e_2, e_5]) \\
&= 0,
\end{aligned}$$

using (2.6). It follows that $C_{\mathfrak{g}}(I) \neq I$, so I is not a maximal abelian ideal of $\mathfrak{g}$ and the result holds.

If $\lambda = 0$, then $\alpha_{3,2} = \alpha_{3,3} = \alpha_{2,2} = 0$ and (2.6) becomes $2[e_2, e_3] = -\alpha_{2,3}[e_3, e_4]$. But this implies that $[2e_2 - \alpha_{2,3}e_4, e_3] = 0$, and $C_{\mathfrak{g}}(I) \neq I$ again.

So suppose now that $\dim \mathfrak{g}/(\mathcal{C}_2(\mathfrak{g}) + Z(\mathfrak{c})) = 2$. Then there is an $c_1 \in \mathfrak{c}$ such that $\mathfrak{g} = \langle e_1, c_1, c_2, c_3, c_4 \rangle + Z(\mathfrak{c})$, where $c_2 = [e_1, c_1]$, $c_3 = [e_1, c_2]$, $c_4 = [e_1, c_3]$, $[e_1, c_4] \in Z(\mathfrak{c})$. Now

$$\begin{aligned}
[e_1, [c_1, c_2]] &= -[c_1, [c_2, e_1]] - [c_2, [e_1, c_1]] = [c_1, c_3],\\
[e_1, [c_1, c_3]] &= -[c_1, [c_3, e_1]] - [c_3, [e_1, c_1]] = [c_1, c_4] + [c_2, c_3],\\
[e_1, [c_1, c_4]] &= -[c_1, [c_4, e_1]] - [c_4, [e_1, c_1]] = [c_2, c_4],\\
[e_1, [c_2, c_4]] &= -[c_2, [c_4, e_1]] - [c_4, [e_1, c_2]] = [c_3, c_4],\\
[e_1, [c_2, c_3]] &= -[c_2, [c_3, e_1]] - [c_3, [e_1, c_2]] = [c_2, c_4].
\end{aligned}$$

Since $\dim \mathcal{C}_2(\mathfrak{c}) \leq 3$ we have $0 = [e_1, [e_1, [e_1, [c_1, c_2]]]] = 2[c_2, c_4]$. Clearly $I = \langle c_4 \rangle + Z(\mathfrak{c})$ is an abelian ideal and $c_2 \in C_{\mathfrak{g}}(I) \setminus I$, which completes the proof. □

The restriction on the characteristic in the above result is necessary, as the following example shows.

Example 2.5. *Let $\mathfrak{g}$ be the nine-dimensional Lie algebra over any field $\mathbb{K}$ of characteristic two, with basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ and non-zero products*

$$[e_1, e_2] = e_6, \quad [e_1, e_3] = e_2, \quad [e_1, e_4] = e_3, \quad [e_1, e_5] = e_4, \quad [e_1, e_8] = e_7,$$

$$[e_1, e_9] = e_8, \quad [e_2, e_3] = e_7, \quad [e_2, e_4] = e_8, \quad [e_2, e_5] = e_9, \quad [e_3, e_4] = e_9.$$

This is a nilpotent Lie algebra whose abelian subalgebras of maximal dimension are

$$\langle e_3 + \lambda e_4, e_5, e_6, e_7, e_8, e_9 \rangle \quad \text{and} \quad \langle \lambda e_3 + e_4, e_5, e_6, e_7, e_8, e_9 \rangle \quad (\lambda \in \mathbb{K}),$$

so $\alpha(\mathfrak{g}) = 6$. However, none of these are ideals of $\mathfrak{g}$. In fact the abelian ideal of maximal dimension is

$$\langle e_2, e_6, e_7, e_8, e_9 \rangle,$$

so $\beta(\mathfrak{g}) = 5$.

Chapter 3

Abelian ideals and subalgebras in some solvable Lie algebras

In this chapter, we show several results and algorithmic methods to compute abelian ideals and subalgebras of some special families of solvable Lie algebras over the field $\mathbb{K} = \mathbb{C}$. More concretely, we deal with Lie algebras $\mathfrak{g}_n$, given by $n \times n$ strictly upper-triangular matrices; Lie algebras $\mathfrak{h}_n$, given by $n \times n$ upper-triangular matrices, Heisenberg algebras and filiform Lie algebras. The content of this chapter can be seen in the papers [23] and [26].

The main interest to deal with the Lie algebras $\mathfrak{h}_n$ and $\mathfrak{g}_n$ lies in the fact that every finite-dimensional solvable Lie algebra can be represented by a Lie subalgebra of some $\mathfrak{h}_n$ (see [43, Theorem 9.11] or [105, Theorem 3.7.3]) and every finite-dimensional nilpotent Lie algebra is isomorphic to a subalgebra of $\mathfrak{g}_n$ ([105, Proposition 3.6.6]). Another reason is that their applications to Physic are many and varied as can be seen in [54, 87].

Heisenberg algebras constitute a special subclass of nilpotent Lie algebras and are also very interesting for several reasons. For example, their applications to both the theory of nilpotent Lie algebras itself and Theoretical Physics. With respect to the first subject, a well-known result sets that a nilpotent Lie algebra $\mathfrak{g}$ is abelian if and only if $\mathfrak{g}$ does not contain a subalgebra isomorphic to the 3-dimensional Heisenberg algebra $\mathfrak{H}_1$ [90]. Heisenberg algebras appeared at the beginnings of the 20th century when introducing Quantum Mechanics. In Classic Mechanics, the state of a particle in a given time t is determined by both its position vector $\mathbf{Q} \in \mathbb{R}^3$ and its momentum vector $\mathbf{P} \in \mathbb{R}^3$. Heisenberg [52] took the components of these two vectors, considering them as operators in a Hilbert space such that the following commutation relations were verified

$$[Q_i, Q_j] = 0,\ [P_i, P_j] = 0,\ [P_i, Q_j] = -\mathbf{i}\ \hbar\ \delta_{ij},\ \forall i, j = 1, 2, 3,$$

where δ_{ij} is the named *Kronecker delta* and $\hbar$, called the *deformation parameter*, represents the choice of a measure unit (usually $\hbar = 1$).

Filiform Lie algebras constitute a very special subclass of nilpotent Lie algebras. In fact, they are the most structured Lie algebras in the nilpotent class and were introduced by Vergne [107] in 1966. The classification of solvable and nilpotent Lie algebras is still an open problem. Therefore, it seems convenient to reduce this problem by dealing with subclasses of nilpotent Lie algebras. Hence, studying and classifying filiform Lie algebras is a first step towards the classification of solvable Lie algebras in general.

The method to compute an abelian subalgebra of maximal dimension of a Lie algebra $\mathfrak{g}$ is based on the technique introduced in [6, 8] for subalgebras and consisting in considering an arbitrary basis of an r-dimensional subalgebra with respect to the basis $\mathcal{B}_\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ in which its law is expressed. Then the vectors in $\mathcal{B}_\mathfrak{g}$ are divided in main vectors or non-main vectors according to a echelon matrix associated with $\mathcal{B}_\mathfrak{g}$. Finally it is studied whether the subalgebra is abelian or not and if it is an ideal.

3.1 Lie algebra $\mathfrak{g}_n$

Now, we show an algorithmic procedure which allows us to obtain abelian ideals of the Lie algebra $\mathfrak{g}_n$. A first version of this algorithm for subalgebras was given in [6].

Case 1: n is even and $n \geq 4$ (i.e., $n = 2\,k$, with $k \in \mathbb{N} \setminus \{1\}$).

Step 1: $(2k)^{\text{th}}$ column. Let us consider the $2k-1$ vectors corresponding to this column. So, we obtain the abelian ideal $\langle X_{1,2k}, \ldots, X_{2k-1,2k}\rangle$.

Step 2: $(2k-1)^{\text{th}}$ column. Now, we add the $2k-2$ vectors corresponding to this column and the unique vector coming from the $(2k-1)^{\text{th}}$ row has to be removed. In this way, we obtain the abelian ideal $\langle X_{1,2k}, \ldots, X_{2k-2,2k}, X_{1,2k-1}, \ldots, X_{2k-2,2k-1}\rangle$.

Step $2k-i+1$**:** i^{th} column, with $2k > i > k+1$. After adding the $i-1$ vectors corresponding to the i^{th} column, the $2k-i$ vectors corresponding to i^{th} row are removed. In this way, the dimension of the obtained abelian ideal increases $2i-2k-1$, while $i > k+1/2$. So, k is the last step in which the number of vectors generating the obtained abelian ideal is greater than the one obtained in the previous step.

Step k**:** $(k+1)^{\text{th}}$ column. The k vectors corresponding to this column are added, whereas the $2k-(k+1) = k-1$ ones corresponding to the $(k+1)^{\text{th}}$ row are removed. So, a k^2-dimensional abelian ideal is obtained, being its basis formed by

$$\begin{matrix} X_{1,k+1} & \dots & X_{1,2k} \\ X_{2,k+1} & \dots & X_{2,2k} \\ \vdots & \ddots & \vdots \\ X_{k,k+1} & \dots & X_{k,2k} \end{matrix}$$

Case 2: n is odd and $n \geq 4$ (i.e., $n = 2k+1$, with $k \in \mathbb{N} \setminus \{1\}$).

Analogously to the Case 1, we can settle the following procedure to obtain an abelian ideal with dimension as large as possible.

Step 1: $(2k+1)^{\text{th}}$ column. Firstly, we consider the $2k$ vectors corresponding to this column. So we obtain the abelian ideal $\langle X_{1,2k+1}, \dots X_{2k,2k+1} \rangle$.

Step 2**:** $2k^{\text{th}}$ column. Now the $2k-1$ vectors corresponding to this column are added, removing the vector coming from the $2k^{\text{th}}$ row. In this way, the abelian ideal $\langle X_{1,2k+1}, \dots X_{2k-1,2k+1}, X_{1,2k}, \dots X_{2k-1,2k} \rangle$ is obtained.

Step $2k-i+2$**:** i^{th} column, with $2k+1 > i > k+2$. There exist $i-1$ vectors corresponding to the i^{th} column, which are added to the generators of the abelian ideal obtained in the previous step. Then the $2k+1-i$ vectors corresponding to the i^{th} row are removed, obtaining an abelian ideal whose dimension increases $2i-2k-2$ with respect to the previous step. The dimension increases only when $i > k+1$. Hence, Step k is the last step and the procedure will be stopped after it.

Step k**:** $(k+2)^{\text{th}}$ column. Now the $k+1$ vectors corresponding to the $(k+2)^{\text{th}}$ column are added, whereas the $k-1$ vectors corresponding to the $(k+2)^{\text{th}}$ row are removed, which allows us to obtain a (k^2+k)-dimensional abelian ideal with the basis

$$\begin{matrix} X_{1,k+1} & \dots & X_{1,2k+1} \\ X_{2,k+1} & \dots & X_{2,2k+1} \\ \vdots & \ddots & \vdots \\ X_{k,k+1} & \dots & X_{k,2k+1} \end{matrix}$$

By using a similar procedure we can compute this other (k^2+k)-dimensional abelian ideal

$$\begin{array}{ccc} X_{1,k+2} & \dots & X_{1,2k+1} \\ X_{2,k+2} & \dots & X_{2,2k+1} \\ \vdots & \ddots & \vdots \\ X_{k+1,k+2} & \dots & X_{k+1,2k+1} \end{array}$$

According to the results obtained, an abelian ideal of $\mathfrak{g}_n$ has been computed for all $n \in \mathbb{N}$. Indeed, the dimension of such a ideal is

$$\mathcal{A}_n = \begin{cases} n-1, & \text{if } n < 4; \\ k^2, & \text{if } n = 2k,\ n \geq 4; \\ k^2+k, & \text{if } n = 2\,k+1,\ n \geq 4. \end{cases}$$

Therefore, $\alpha(\mathfrak{g}_n) = \beta(\mathfrak{g}_n)$ is lower bounded by the value $\mathcal{A}_n$. Now we prove that the value of the invariants α and β for $\mathfrak{g}_n$ is $\mathcal{A}_n$, improving the bound given by Jacobson [57] and Schur [91].

Let us remember that $d_{\mathfrak{g}_n}$ denotes the dimension of $\mathfrak{g}_n$. Now, we give a sketch for the proof of the following fact: it is not possible to obtain an abelian subalgebra of the Lie algebra $\mathfrak{g}_n$ with dimension greater than $\mathcal{A}_n$. A more detailed information about this proof can be consulted in [8]. For it, three preliminary results are needed

Proposition 3.1 (Theorem 3.1 in [6]). *Let us consider $n \in \mathbb{N}$, with $n \geq 4$. Then the abelian Lie algebra of dimension $d_{\mathfrak{g}_n} - 1$ is not a subalgebra of $\mathfrak{g}_n$.* □

Lemma 3.1 (Lemma 4.1 in [8]). *Let us consider $n \in \mathbb{N}$, with $n \geq 3$. Then, $d_{\mathfrak{g}_n} - d_{\mathfrak{g}_{n-1}} = n-1$.* □

Lemma 3.2 (Lemma 4.2 in [8]). *Let us consider $n \in \mathbb{N}$, with $n \geq 3$. Then, $\mathcal{A}_n - \mathcal{A}_{n-1} = \left[\frac{n}{2}\right]$, where $\left[\frac{n}{2}\right]$ denotes the integral part of $\frac{n}{2}$.* □

Combining the previous results, a proof can be provided for

Theorem 3.1 (Theorem 4.1 in [8]). *Let us consider $n \in \mathbb{N}$, with $n \geq 2$. The α invariant of the Lie algebra $\mathfrak{g}_n$ is given by*

$$\alpha(\mathfrak{g}_n) = \mathcal{A}_n = \begin{cases} k^2, & \text{if } n = 2k, \quad \text{with } k \in \mathbb{N}, \\ k^2+k, & \text{if } n = 2k+1, \text{ with } k \in \mathbb{N}. \end{cases}$$

Proof. We use an iterative procedure based on studying the impossibility of obtaining a $(d_{\mathfrak{g}_n} - r)$-dimensional abelian Lie subalgebra in $\mathfrak{g}_n$ for $0 < r \leq d_{\mathfrak{g}_n} - \mathcal{A}_n - 1$. Cases $\mathfrak{g}_2$ and $\mathfrak{g}_3$ are trivial. Moreover, Proposition 3.1 implies this result for $\mathfrak{g}_4$. So we only need to prove it for $n \geq 5$.

Given an abelian subalgebra of $\mathfrak{g}_n$, our proof is based on the number of main vectors coming from the n^{th} column in an arbitrary basis $\mathcal{B}$ of that subalgebra.

$(d_{\mathfrak{g}_n} - 2)$-dimensional abelian subalgebra: We have to consider three different cases

Case 1: The two non-main vectors correspond to the n^{th} column. Each vector $X_{i,j} \in \mathfrak{g}_{n-1}$ is main, but $\mathfrak{g}_{n-1}$ is not abelian.

Case 2: A unique non-main vector corresponds to the n^{th} column. A $(d_{\mathfrak{g}_{n-1}} - 1)$-dimensional abelian subalgebra has to be contained in $\mathfrak{g}_{n-1}$, but it is not possible according to Proposition 3.1.

Case 3: No non-main vector corresponds to the n^{th} column. $[X_{i_0,n-1}, X_{n-1,n}]$ is non-zero where $X_{i_0,n-1}$ is a main vector, with i_0 as greater as possible.

$(d_{\mathfrak{g}_n} - 3)$-dimensional abelian subalgebra: There are three non-main vectors in $\mathcal{B}$ and the four following cases have to be considered

Case 1: The three non-main vectors correspond to the n^{th} column. This is in contradiction with the law of $\mathfrak{g}_{n-1}$.

Case 2: Two non-main vectors correspond to the n^{th} column. Due to Proposition 3.1, we cannot find a $(d_{\mathfrak{g}_{n-1}} - 1)$-dimensional abelian subalgebra in $\mathfrak{g}_{n-1}$.

Case 3: A unique non-main vector corresponds to the n^{th} column. Only one vector coming form the n^{th} column of $\mathfrak{g}_n$ is non-main. So we need to find a $(d_{\mathfrak{g}_{n-1}} - 2)$-dimensional abelian subalgebra in $\mathfrak{g}_{n-1}$.

- If $X_{n-1,n}$ is main, $[X_{i_0,n-1}, X_{n-1,n}]$ is non-zero where $X_{i_0,n-1}$ is a main vector with i_0 as greater as possible.
- If $X_{n-1,n}$ is non-main, $X_{i,n}$ is main for $1 \leq i \leq n-2$. If $n = 5$, we choose the main vectors $X_{2,5}$ and $X_{1,2}$ to obtain a non-zero bracket. If $n > 5$, $[X_{j_0,n-2}, X_{n-2,n}]$ is non-zero with j_0 as greater as possible such that $X_{j_0,n-2}$ is a main vector.

Case 4: No non-main vector corresponds to the n^{th} column. If $n > 5$, there exists i_0 as greater as possible such that $X_{i_0,n-1}$ is main and the bracket $[X_{i_0,n-1}, X_{n-1,n}]$ is non-zero. If $n = 5$, we consider the main vectors $X_{3,5}$ and $X_{1,3}$ to get a non-zero bracket.

$(d_{\mathfrak{g}_n} - r)$-dimensional abelian subalgebra, where $4 \le r \le d_{\mathfrak{g}_n} - \mathcal{A}_n - 1$: We have to prove the non-existence of $(d_{\mathfrak{g}_n} - r)$-dimensional abelian subalgebras in $\mathfrak{g}_n$, with $4 \le r \le d_{\mathfrak{g}_n} - \mathcal{A}_n - 1$. Our proof is based on an iterative procedure supposing that, for $1 \le s \le r-1$, no abelian subalgebra of dimension $d_{\mathfrak{g}_n} - s$ can be contained in $\mathfrak{g}_n$. Remember that $s \le d_{\mathfrak{g}_n} - \mathcal{A}_n$. We consider five cases

Case 1: r non-main vectors correspond to the n^{th} column. Then the algebra $\mathfrak{g}_{n-1}$ would be abelian and this is not possible.

Case 2: $r-1$ non-main vectors correspond to the n^{th} column. There exists one non-main vector in $\mathfrak{g}_{n-1}$ and a $(d_{\mathfrak{g}_{n-1}} - 1)$-dimensional abelian subalgebra is contained in $\mathfrak{g}_{n-1}$. It is not possible either due to Proposition 3.1.

Case 3: $r-k$ non-main vectors correspond to the n^{th} column (with $2 \le k \le \min\{r-1, d_{\mathfrak{g}_{n-1}} - \mathcal{A}_{n-1} - 1\}$). There are k non-main vectors in $\mathfrak{g}_{n-1}$ and a $(d_{\mathfrak{g}_{n-1}} - k)$-dimensional abelian subalgebra is looked for in $\mathfrak{g}_{n-1}$. This stands in contradiction with our iterative assumption.

Case 4: $r-k$ non-main vectors correspond to the n^{th} column (with $d_{\mathfrak{g}_{n-1}} - \mathcal{A}_{n-1} \le k \le r-1$). There exist k non-main vectors in $\mathfrak{g}_{n-1}$ and $n-1-r+k$ main ones coming from the n^{th} column, satisfying $1+\left[\frac{n}{2}\right] \le n-1-r+k \le n-1$. Thus, there are, at least, $1+\left[\frac{n}{2}\right]$ main vectors with the form $X_{i,n}$ and, at most, $n-\left[\frac{n}{2}\right]-2$ non-main ones. So, more than the half of the vectors coming from the n^{th} column are main. Now, we choose $\mathbf{w} \in \mathcal{B}$ such that its main vector X_{j_0,h_0}, with $1 \le j_0 < h_0 < n$, belongs to $\mathfrak{g}_{n-1}$ and the vector $X_{h_0,n}$ is the main vector of a basis vector $\mathbf{v} \in \mathcal{B}$. So the bracket $[\mathbf{v}, \mathbf{w}]$ is non-zero. To assert it, j_0 has to be chosen as greater as possible.

Case 5: No non-main vector corresponds to the n^{th} column. There exists a main vector $X_{i_0,j_0} \in \mathfrak{g}_{n-1}$ with i_0 as greater as possible and $[X_{i_0,j_0}, X_{j_0,n}] \neq 0$. □

3.2 Lie algebra $\mathfrak{h}_n$

We explain an algorithmic method to obtain abelian ideals in an arbitrary Lie algebra $\mathfrak{h}_n$, with $n \ge 4$. The content of this section can be seen in [21]. Depending on the parity of n, two possible cases have to be considered

Case 1: n is even and $n \geq 4$ (i.e., $n = 2\,k$, with $k \in \mathbb{N} \setminus \{1\}$).

We consider the vectors in the basis of $\mathfrak{h}_n$, corresponding to the columns in the matrix expression of $\mathfrak{h}_n$. When the vectors corresponding to the i^{th} column are chosen, all the vectors corresponding to the i^{th} row have to be removed. In this way, all the non-zero brackets are avoided and it is an ideal.

Step 1: $(2k)^{\text{th}}$ column. We add the $2k$ vectors corresponding to the $(2k)^{\text{th}}$ column. The vector coming from the $(2k)^{\text{th}}$ row has to be removed, obtaining the abelian subalgebra $\langle X_{1,2k}, \ldots, X_{2k-1,2k}\rangle$.

Step $2k - i + 1$**:** i^{th} column, with $2k > i > k + 1$. After adding the i vectors corresponding to the i^{th} column, the $2k - (i - 1)$ vectors corresponding to i^{th} row are removed. In this way, we obtain an abelian ideal whose dimension increases $2i - 2k - 1$, while $i > k + 1/2$. So k is the last step in which the dimension increases with this adding-removing procedure.

Step k**:** $(k + 1)^{\text{th}}$ column. The $k + 1$ vectors corresponding to the $(k + 1)^{\text{th}}$ column are added, whereas the k ones corresponding to the $(k+1)^{\text{th}}$ row are removed.

Step $k+1$**:** Adding the vector $\sum_{i=1}^{n} X_{i,i}$ to the basis computed in Step k, we obtain a $(k^2 + 1)$-dimensional abelian ideal whose basis is shown next

$$\begin{matrix} X_{1,k+1} & \ldots & X_{1,2k} \\ X_{2,k+1} & \ldots & X_{2,2k} \\ \vdots & \ddots & \vdots \\ X_{k,k+1} & \ldots & X_{k,2k} \end{matrix} \qquad \text{and} \qquad \sum_{i=1}^{n} X_{i,i}$$

Case 2: n is odd and $n \geq 4$ (i.e., $n = 2k + 1$, with $k \in \mathbb{N} \setminus \{1\}$).

By arguing analogously to the Case 1, we can settle the following procedure to obtain an abelian ideal with dimension as large as possible.

Step 1: $(2k + 1)^{\text{th}}$ column. After adding the $2k + 1$ vectors corresponding to this column, the unique vector coming from the $(2k+1)^{\text{th}}$ row is removed, obtaining the abelian ideal $\langle X_{1,2k+1}, \ldots, X_{2k,2k+1}\rangle$.

Step $2k - i + 2$**:** i^{th} column, with $2k+1 > i > k+2$. The i vectors corresponding to the i^{th} column are added and the $2k-(i-1)$ vectors corresponding to the i^{th} row are removed, obtaining an abelian ideal whose dimension increases $2i - 2k - 2$. This dimension increases while $i > k+1$. Hence, Step k is the last step for the adding-removing procedure.

Step k**:** $(k+2)^{\text{th}}$ column. Now the $k+2$ vectors corresponding to this column are added and, then, the $2k+1-(k+2-1) = k$ vectors corresponding to the $(k+2)^{\text{th}}$ row are removed, which allows us to obtain a (k^2+k)-dimensional abelian ideal.

Step $k+1$**:** Adding the vector $\sum_{i=1}^{n} X_{i,i}$ to the basis computed in Step k, we obtain the (k^2+k+1)-dimensional abelian ideal determined by the basis

$$\begin{matrix} X_{1,k+2} & \cdots & X_{1,2k+1} \\ X_{2,k+2} & \cdots & X_{2,2k+1} \\ \vdots & \ddots & \vdots \\ X_{k+1,k+2} & \cdots & X_{k+1,2k+1} \end{matrix} \qquad \text{and} \qquad \sum_{i=1}^{n} X_{i,i}$$

By using a similar procedure we can compute this other (k^2+k+1)-dimensional abelian ideal

$$\begin{matrix} X_{1,k+1} & \cdots & X_{1,2k+1} \\ X_{2,k+1} & \cdots & X_{2,2k+1} \\ \vdots & \ddots & \vdots \\ X_{k,k+1} & \cdots & X_{k,2k+1} \end{matrix} \qquad \text{and} \qquad \sum_{i=1}^{n} X_{i,i}$$

According to the results obtained in this section, an abelian ideal of $\mathfrak{h}_n$ has been computed for all $n \in \mathbb{N}$. Indeed, the dimension of such a subalgebra is

$$B_n = \begin{cases} n, & \text{if } n < 4; \\ k^2+1, & \text{if } n = 2k,\ n \geq 4; \\ k^2+k+1, & \text{if } n = 2\,k+1,\ n \geq 4. \end{cases}$$

Therefore, we can affirm that $\alpha(\mathfrak{h}_n) = \beta(\mathfrak{h}_n)$ is lower bounded by the value B_n, which is equal to the upper bound of $\alpha(\mathfrak{h}_n)$ given by Jacobson [57] and Schur [91]. So, for the Lie algebras $\mathfrak{h}_n$, α and β invariants are equal to B_n.

3.3 Heisenberg algebras

This subsection is devoted to compute the α invariant for complex Heisenberg algebras. To do so, lower and upper bounds of $\alpha(\mathfrak{H}_k)$ are computed for $k \in \mathbb{N}$ using the proof introduced in [78]. According to the law of $\mathfrak{H}_k$ and the basis $\mathcal{B}_k = \{X_i\}_{i=1}^{2k+1}$ expressed in preliminaries, $\mathfrak{a}_k = \langle\{X_{2i+1} | i = 0, \dots, k\}\rangle$ is a $(k+1)$-dimensional abelian ideal of $\mathfrak{H}_k$. So we can affirm that $k+1 \leq \alpha(\mathfrak{H}_k)$ for $k \in \mathbb{N}$. In this way, our main goal comes down to proving the non-existence of abelian subalgebras in $\mathfrak{H}_k$ with dimension greater than $k+1$. The following technical lemma has to be used to settle the previous statement, whose proof is based in solving a system with a symbolic computation package

Lemma 3.3. *The system of equations with complex unknowns a_i and b_i has not any complex solutions*

$$\begin{array}{llll} E_1: & a_1 \cdot b_2 - a_2 \cdot b_1 = -1; & E_6: & a_2 \cdot b_4 - a_4 \cdot b_2 = 0; \\ E_2: & a_1 \cdot b_3 - a_3 \cdot b_1 = 0; & E_7: & a_2 \cdot b_5 - a_5 \cdot b_2 = 0; \\ E_3: & a_1 \cdot b_4 - a_4 \cdot b_1 = 0; & E_8: & a_3 \cdot b_4 - a_4 \cdot b_3 = -1; \\ E_4: & a_1 \cdot b_5 - a_5 \cdot b_1 = 0; & E_9: & a_3 \cdot b_5 - a_5 \cdot b_3 = 0; \\ E_5: & a_2 \cdot b_3 - a_3 \cdot b_2 = 0; & E_{10}: & a_4 \cdot b_5 - a_5 \cdot b_4 = 0. \end{array} \tag{3.1}$$

□

By using the previous lemma, the following theorem can be proved, settling lower and upper bounds for $\alpha(\mathfrak{H}_k) = \beta(\mathfrak{H}_k)$.

Theorem 3.2 (Theorem 3.1 in [78])**.** *For $k \in \mathbb{N}$ and $\gamma \in \mathbb{N} \cup \{0\}$, $\alpha(\mathfrak{H}_k) = \beta(\mathfrak{H}_k)$ is bounded by $k+1 \leq \alpha(\mathfrak{H}_k) \leq 2k - \gamma$, for all $k \geq \gamma + 1$.*

Proof. The first inequality is trivial due to the existence of the subalgebra $\mathfrak{a}_k$ for $k \in \mathbb{N}$. To obtain the upper bound for $\alpha(\mathfrak{H}_k)$, we only need to prove the non-existence of abelian subalgebras $\mathfrak{b}$ of $\mathfrak{H}_k$ whose dimension is $2k$–γ+1. Since k is lower bounded by $\gamma + 1$, the dimension of $\mathfrak{H}_k$ can be lower bounded by

$$\dim(\mathfrak{H}_k) = 2k + 1 \geq 2\gamma + 3. \tag{3.2}$$

Any $(2k - \gamma + 1)$-dimensional subalgebra of $\mathfrak{H}_k$ has $(2k - \gamma + 1)$ main vectors and the number of the non-main ones is $\dim(\mathfrak{H}_k) - (2k - \gamma + 1) = \gamma$. By combining this expression with (3.2), the number of main vectors is greater than the number of non-main ones; that is $2k - \gamma + 1 \geq \gamma + 3$.

The basis $\mathcal{B}_k$ of $\mathfrak{H}_k$ is structured in pairs of basic vectors (X_i, X_{i+1}) such that i is an even natural number less than or equal to $2k$. For each of these pairs, its corresponding bracket is equal to the vector $X_1 \in \mathcal{B}_k$. Let us call *non-zero pair* to each of these pairs. Therefore, $\mathfrak{H}_k$ is formed by k non-zero pairs and the vector resulting from all these brackets.

There are, at most, $\left\lfloor \frac{\gamma}{2} \right\rfloor$ non-zero pairs which are formed by non-main vectors; whereas the number of non-zero pairs which only contain main vectors is at least one. Starting from this minimum, there is one non-zero pair formed by main vectors more for each new non-zero pair formed by non-main vectors.

To distinguish between main vectors and non-main vectors for the basis $\mathcal{B}$ of the subalgebra studied, the vectors in $\mathcal{B}_k$ are arranged in two sets, which are reordered as follows

$$\begin{array}{l} S_{NMV}=\{X_i\in\mathfrak{H}_k|i=1,\dots,2k+1 \wedge X_i \text{ is a non-main vector}\}=\{X_{i_h}|h=1,\dots,\gamma\} \\ S_{MV}=\{X_i\in\mathfrak{H}_k|i=1,\dots,2k+1 \wedge X_i \text{ is a main vector}\}=\{X_{i_h}|h=\gamma+1,\dots,2k+1\} \end{array} \tag{3.3}$$

So the proof of $\alpha(\mathfrak{H}_k)\leq 2k-\gamma$ starts by considering each possible case for non-zero pairs formed by non-main vectors and proving that the subalgebra cannot be abelian. Hence, if $\mathfrak{b}$ is an arbitrary $(2k-\gamma+1)$-dimensional subalgebra of $\mathfrak{H}_k$, the vectors in an arbitrary basis $\mathcal{B}=\{\omega_{i_r}\}_{r=\gamma+1}^{2k+1}$ of $\mathfrak{b}$ can be expressed as $\omega_{i_r}=X_{i_r}+\sum_{h=1}^{\gamma}\lambda_{i_h}^{r}\cdot X_{i_h}$, $\forall r\in\{\gamma+1,\dots,2k+1\}$. Now three steps are considered

No non-zero pair in S_{NMV}: The bracket between two vectors in $\mathcal{B}$ is given by

$$[\omega_{i_{r1}},\omega_{i_{r2}}]=[X_{i_{r1}},X_{i_{r2}}]+\sum_{h=1}^{\gamma}\lambda_{i_h}^{r2}\cdot[X_{i_{r1}},X_{i_h}]+\sum_{h=1}^{\gamma}\lambda_{i_h}^{r1}\cdot[X_{i_h},X_{i_{r2}}]. \tag{3.4}$$

We can suppose that the unique non-zero pair of main vectors is $(X_{i_{r1}},X_{i_{r2}})$. The bracket $[X_{i_{r1}},X_{i_{r2}}]$ is equal to X_1 and the rest of the brackets in (3.4) are zero because they do not correspond to non-zero pairs. Hence, the subalgebra $\mathfrak{b}$ cannot be abelian.

One non-zero pair in S_{NMV}: We can suppose that there exists only the non-zero pair formed by X_{i_1} and X_{i_2}. The bracket between two vectors in $\mathcal{B}$ is expressed as

$$\begin{aligned} [\omega_{i_{r1}},\omega_{i_{r2}}]= & \ [X_{i_{r1}},X_{i_{r2}}]+\sum_{h=1}^{\gamma}\lambda_{i_h}^{r2}\cdot[X_{i_{r1}},X_{i_h}]+\sum_{h=1}^{\gamma}\lambda_{i_h}^{r1}\cdot[X_{i_h},X_{i_{r2}}] \\ & +(\lambda_{i_1}^{r1}\cdot\lambda_{i_2}^{r2}-\lambda_{i_1}^{r2}\cdot\lambda_{i_2}^{r1})\cdot[X_{i_1},X_{i_2}]. \end{aligned} \tag{3.5}$$

We can suppose that the non-zero pairs of vectors in S_{MV} are $(X_{i_{\gamma+1}},X_{i_{\gamma+2}})$ and $(X_{i_{\gamma+3}},X_{i_{\gamma+4}})$ because there is one non-zero pair formed by vectors in S_{NMV}. Besides, there exists another vector $X_{i_{\gamma+5}}\in S_{MV}$ which does not form a non-zero pair with any vectors in S_{NMV}, because one of the following conditions is true

a) $X_1\in S_{MV}$: Hence, we can suppose $X_{i_{\gamma+5}}=X_1$, which generates the center of $\mathfrak{H}_k$.

b) $X_1 \in S_{NMV}$: Since there are, at least, $\gamma+3$ main vectors, one main vector does not correspond to the non-main vector nor to the non-zero pairs in S_{MV}.

So $\mathfrak{b}$ is abelian if and only if every bracket $[\omega_{i_{r1}}, \omega_{i_{r2}}]$ is zero. The brackets between two vectors in $\{X_{i_{\gamma+1}}, X_{i_{\gamma+2}}, X_{i_{\gamma+3}}, X_{i_{\gamma+4}}, X_{i_{\gamma+5}}\}$ leads to the system of equations without solutions shown in Lemma 3.3. Therefore, $\mathfrak{b}$ cannot be abelian.

General case: m non-zero pairs in S_{NMV}: There are m non-zero pairs in S_{NMV} and $m+1$ in S_{MV}. Hence we only have to consider the cases

a) A vector in a non-zero pair in S_{NMV} has zero coefficients in each vector ω_{i_r}. In this case, the problem is reduced to only $m-1$ non-zero pairs in S_{NMV} (i.e. those which do not contain the vector with zero coefficients).

b) A vector in a non-zero pair in S_{NMV} has non-zero coefficient in some vector ω_{i_r}. Let us suppose that such a vector is X_{i_1}. We apply the corresponding basis change on the basis $\mathcal{B}$ to place X_{i_1} as a main vector and X_{i_r} as a non-main vector. After this change, there are only $m-1$ non-zero pairs in S_{NMV} and, besides, there are m non-zero pairs in S_{MV}.

In this way, this case is reduced to $m-1$ non-zero pairs in S_{NMV}. By repeating $m-2$ times more this reduction, the resulting case is the following: there is one non-zero pair in S_{NMV}. As this case has already been solved, the proof is finished. □

Theorem 3.3 (Theorem 3.2 in [78]). *Given $k \in \mathbb{N}$, $\alpha(\mathfrak{H}_k) = k+1$. Moreover, a $(k+1)$-dimensional abelian ideal is $\mathfrak{a}_k = \langle \{X_{2i+1}\}_{i=0}^{k} \rangle$.*

Proof. According to the previous Theorem, $\alpha(\mathfrak{H}_k)$ is bounded by $k+1 \leq \alpha(\mathfrak{H}_k) \leq 2k-\gamma$, for $\gamma \leq k-1$. Replacing γ with $k-1$, the inequalities $k+1 \leq \alpha(\mathfrak{H}_k) \leq k+1$ are obtained. □

3.4 Filiform Lie algebras

Firstly, we show the following example of a characteristically nilpotent Lie algebra, that is, in fact, a filiform Lie algebra of dimension 7 with α invariant equals 5.

Example 3.1. *The Lie algebra of dimension $n = 7$ defined by $[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$ and $[x_2, x_3] = x_6 + x_7$, $[x_2, x_4] = x_7$ is characteristically nilpotent, i.e., all of its derivations are nilpotent. Furthermore it satisfies $\alpha(\mathfrak{g}) = n - 2 = 5$.*

We can find such examples in all dimensions $n \geq 7$. This suggests that nilpotent Lie algebras $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = n-2$ are not easy to understand. For the case of filiform nilpotent, we can say something more on $\alpha(\mathfrak{g})$.

Proposition 3.2. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra. Then, $\mathcal{C}^{n-z_2+1}(\mathfrak{g})$ is the unique abelian ideal of maximal dimension. Consequently, $\alpha(\mathfrak{g}) = \beta(\mathfrak{g}) = z_2 - 1$.*

Proof. Let us consider an adapted basis $\{e_h\}_{h=1}^n$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is filiform, its lower central series is given by

$$\mathcal{C}^1(\mathfrak{g}) = \langle e_1, \ldots, e_n \rangle,\ \mathcal{C}^2(\mathfrak{g}) = \langle e_2, \ldots, e_{n-1} \rangle,\ \ldots, \mathcal{C}^k(\mathfrak{g}) = \langle e_2, \ldots, e_{n-k+1} \rangle.$$

According to the definition of the invariant z_2, the ideal $\mathcal{C}^{n-z_2+1}(\mathfrak{g})$ is abelian but $\mathcal{C}^{n-z_2}(\mathfrak{g})$ is not. Let $\mathfrak{I}$ be an abelian ideal of $\mathfrak{g}$. We are going to prove that $\mathfrak{I} \subset \mathcal{C}^{n-z_2+1}(\mathfrak{g})$. To do so, we suppose that $\mathfrak{I} \nsubseteq \mathcal{C}^{n-z_2+1}(\mathfrak{g})$. In this way, let x be an element of $\mathfrak{I}$ such that $x \notin \mathcal{C}^{n-z_2+1}(\mathfrak{g})$. This implies that $x = \sum_{h=1}^n \alpha_h e_h$ such that $\exists h \in \{z_2 + 1, \ldots, n\}$ with $\alpha_h \neq 0$.

Additionally, if $\alpha_p \neq 0$, for some $p \in \mathbb{N} \cap [z_2+1, n]$, then $\mathcal{C}^{n-p+1}(\mathfrak{g}) = \langle e_2, \ldots, e_p \rangle \subseteq \mathfrak{I}$. This fact is because $\mathfrak{I}$ is an ideal and hence $ad^q(e_1)(x) = \alpha_{q+2}e_2 + \ldots + \alpha_p e_{p-q} + \ldots + \alpha_n e_{n-q} \in \mathfrak{I}$, for each $q \in \mathbb{N} \cap [0, n-2]$. In consequence, since $p \geq z_2+1$, we have the following chain of inclusion relations: $\mathcal{C}^{n-z_2}(\mathfrak{g}) \subset \mathcal{C}^{n-p+1}(\mathfrak{g}) \subset \mathfrak{I}$. This implies that $\mathfrak{I}$ is non-abelian, which comes into contradiction with our initial hypothesis. □

Proposition 3.3. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra. Then, the law of $\mathfrak{g}$ is given by the following brackets*

$[e_1, e_h] = e_{h-1}$, *for* $3 \leq h \leq n$;
$[e_{z_1+i}, e_{z_2+1}] = \alpha_1 e_{i+2} + \alpha_2 e_{i+1} + \ldots + \alpha_{i+1} e_2,\ 0 \leq i \leq z_2 - z_1$;
$[e_{z_1}, e_{z_2+j}] = \alpha_1 e_{j+1} + \alpha_2^1 e_j \ldots + \alpha_j^{j-1} e_2,\ 2 \leq j \leq n - z_2$;
$[e_{z_1+k}, e_{z_2+l}] = \sum_{h=2}^k P_h([e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}])e_{h+1} + \sum_{h=k+1}^{k+l} P_{h-k+1}(\sum_{m=0}^k \binom{k}{m} [e_{z_1+m}, e_{z_2+l-m}])e_{h+1} + \alpha_{k+l}^{k+l-1} e_2$,
where $2 \leq l \leq n - z_2$, $0 < k < z_2 - z_1 + l$ *and* P_r $(2 \leq r \leq n)$ *is the function*

$$P_r : \mathfrak{g} \to \mathbb{C} :\ u \mapsto P_r(u) := \text{coordinate of } u \text{ with respect to the basis vector } e_r.$$

Remark 3.1. *Let us note that the complex coefficients α_i and α_j^{j-1} are the ones indicated in Lemmas 1.1 and 1.2, respectively. We will say that $\mathfrak{g}$ is a filiform Lie algebra associated with the triple (z_1, z_2, n).*

Proof of Proposition 3.3. The brackets $[e_{z_1+i}, e_{z_2+1}]$ and $[e_{z_1}, e_{z_2+j}]$ are obtained from the Lemmas 1 and 2, respectively. We only need to prove the following condition for $2 \leq l \leq n - z_2$ and $0 < k < z_2 - z_1 + l$: $[e_{z_1+k}, e_{z_2+l}] = \sum_{h=2}^{k} P_h([e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}])e_{h+1} + \sum_{h=k+1}^{k+l} P_{h-k+1}(\sum_{m=0}^{k} \binom{k}{m}[e_{z_1+m}, e_{z_2+l-m}])e_{h+1} + \alpha_{k+l}^{k+l-1} e_2$.
In fact, the Jacobi identity $J(e_1, e_{z_1+k}, e_{z_2+l}) = 0$ involves

$$[e_1, [e_{z_1+k}, e_{z_2+l}]] = [[e_1, e_{z_1+k}], e_{z_2+l}] + [[e_{z_2+l}, e_1], e_{z_1+k}] =$$

$$= [e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}],$$

so $[e_{z_1+k}, e_{z_2+l}] = \sum_{h=2}^{k+l} P_h([e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}])e_{h+1} + \alpha_{k+l}^{k+l-1} e_2$.

Now, by using this relation, we get

$P_h([e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}]) = P_{h-1}([e_{z_1+k-2}, e_{z_2+l}] + 2[e_{z_1+k-1}, e_{z_2+l-1}] + [e_{z_1+k}, e_{z_2+l-2}]) = \ldots = P_{h-k+1}(\sum_{m=0}^{k}[e_{z_1+m}, e_{z_2+l-m}])e_{h+1}$ if $h - k + 1 \geq 2$.

Therefore, we obtain this family of brackets

$[e_{z_1+k}, e_{z_2+l}] = \sum_{h=2}^{k} P_h([e_{z_1+k-1}, e_{z_2+l}] + [e_{z_1+k}, e_{z_2+l-1}])e_{h+1} + \sum_{h=k+1}^{k+l} P_{h-k+1}(\sum_{m=0}^{k} \binom{k}{m}[e_{z_1+m}, e_{z_2+l-m}])e_{h+1}\alpha_{k+l}^{k+l-1} e_2$,

where $2 \leq l \leq n - z_2$, $0 < k < z_2 - z_1 + l$. □

Next, by using the definition of z_2 and Equation (1.2), we obtain the following two results

Corollary 3.1. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra with $\alpha(\mathfrak{g}) = k \in \mathbb{N}$. Then, $z_2 = k + 1$ and the following relation holds: $3 \leq z_1 - 1 \leq k < n - 1 \leq 2k - 1$.*

Proof. It is trivial starting from Proposition 2.2 and Equation (1.2). □

Corollary 3.2. *Let $\mathfrak{g}$ and $\{e_h\}_{h=1}^{n}$ be an n-dimensional non-model filiform Lie algebra and an adapted basis of $\mathfrak{g}$, respectively. If $\mathfrak{h}$ is the subalgebra $\langle e_2, \ldots, e_n\rangle$ of $\mathfrak{g}$, then the derived subalgebra $\mathcal{D}(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}]$ satisfies that $\mathcal{D}(\mathfrak{h}) = \mathcal{C}^2(\mathfrak{h}) \subset \langle e_2, e_3, \ldots, e_{2n-(z_1+z_2)}\rangle$.*

Proof. It is sufficient to consider the bracket $[e_{n-1}, e_n]$ in the general law given in Proposition 3.3. □

Let us note that Corollary 3.1 improves the bound for $\alpha(\mathfrak{g})$ in a complex non-abelian nilpotent Lie algebra $\mathfrak{g}$ given in Lemma 2.5: $\frac{\sqrt{8n+1}-1}{2} \leq \alpha(\mathfrak{g}) \leq n - 1$.

Next, we show several results concerning the value of $\beta(\mathfrak{g})$ and the coefficients given in the general law of Proposition 3.3. The first two results correspond to the cases $\beta(\mathfrak{g}) = n - 1$ and $n - 2$. For those cases, we can characterize the law of a general filiform Lie algebra.

Proposition 3.4. *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra verifying $\beta(\mathfrak{g}) = n - 1$, then $\mathfrak{g}$ is isomorphic to the model filiform Lie algebra.*

Proof. This is trivial, since $\langle e_2, \ldots, e_n \rangle$ is an abelian ideal. □

Proposition 3.5. *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra verifying $\beta(\mathfrak{g}) = n - 2$, then the law of $\mathfrak{g}$ is expressed as in Proposition 3.3 with no restrictions over the coefficients α_i and α_j^{j-1}.*

Proof. Since the derived Lie algebra $\mathcal{D}(\mathfrak{g})$ is abelian, all the Jacobi identities are trivially null. □

We can even obtain a more general result about the non-existence of restrictions over the coefficients α_i and α_j^{j-1} in Proposition 3.3 in virtue of a relation between the invariants z_1 and z_2.

Proposition 3.6. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra associated with the triple (z_1, z_2, n). If $z_1 \geq n - \frac{z_2}{2}$, then the law of $\mathfrak{g}$ is expressed as in Proposition 3.3 and there are no restrictions over the coefficients α_i and α_j^{j-1}.*

Proof. If $z_1 \geq n - \frac{z_2}{2}$, then $\mathcal{D}(\mathfrak{h}) \subset \langle e_2, \ldots, e_{z_1} \rangle$ in virtue of Corollary 3.2 and every Jacobi identity is satisfied using Equation (1.1). □

Now, we show this proposition where we have studied the first restriction over the coefficients in the law of $\mathfrak{g}$.

Proposition 3.7. *Let $\mathfrak{g}$ be a n-dimensional non-model filiform Lie algebra associated with the triple (z_1, z_2, n). If all the Jacobi identities are not identically null, then the first restriction for the coefficients in the law of $\mathfrak{g}$ is $\alpha_1 = 0$.*

Proof. In order to prove this result, we are going to consider the Jacobi identity $J(e_{z_2}, e_{z_2+1}, e_n) = 0$, which is given by

$$[[e_{z_2}, z_{z_2+1}], e_n] + [[e_{z_2+1}, e_n], e_{z_2}] + [[e_n, e_{z_2}], e_{z_2+1}] = 0.$$

According to Proposition 1.3, $[e_{z_2+1}, e_n] \in \langle e_2, \ldots, e_{z_2} \rangle$, which is an abelian ideal due to the definition of z_2. Therefore, the bracket $[[e_{z_2+1}, e_n], e_{z_2}]$ is zero. Now, we compute the expression of the bracket

$$\begin{aligned}
[[e_{z_2}, e_{z_2+1}], e_n] &= [\alpha_1 e_{z_2-z_1+2} + \alpha_2 e_{z_2-z_1+1} + \ldots + \alpha_{z_2-z_1} e_3 + \alpha_{z_2-z_1+1} e_2, e_n] \\
&= [\alpha_1 e_{z_2-z_1+2} + \alpha_2 e_{z_2-z_1+1} + \ldots + \alpha_{z_2-2z_1+3} e_{z_1}, e_n] \\
&= \alpha_{z_2-2z_1+3}[e_{z_1}, e_n] + \ldots + \alpha_2 [e_{z_2-z_1+1}, e_n] + \alpha_1 [e_{z_2-z_1+2}, e_n] \\
&= \alpha_{z_2-2z_1+3}(\alpha_1 e_{n-z_2+1} + \alpha_2^1 e_{n-z_2} + \ldots + \alpha_{n-z_2}^{n-z_2-1} e_2) + \ldots \\
&\quad + \alpha_1 (P_{n-2z_1+2}([e_{z_2-z_1+1}, e_n] + [e_{z_2-z_1+2}, e_{n-1}]) e_{n-2z_1+3} \\
&\quad + P_{n-2z_1+1}([e_{z_2-z_1+1}, e_n] + [e_{z_2-z_1+2}, e_{n-1}]) e_{n-2z_1+2} + \ldots \\
&\quad + \alpha_{n-2z_1+2}^{n-2z_1+1} e_2).
\end{aligned}$$

Now, we compute the expression of the bracket

$$\begin{aligned}
[[e_{z_2}, e_n], e_{z_2+1}] &= [\textstyle\sum_{h=2}^{n-z_1} P_h([e_{z_2-1}, e_n] + [e_{z_2}, e_{n-1}]) e_{h+1} + \alpha_{n-z_1}^{n-z_1-1} e_2, e_{z_2+1}] \\
&= [\textstyle\sum_{h=z_1-1}^{n-z_1} P_h([e_{z_2-1}, e_n] + [e_{z_2}, e_{n-1}]) e_{h+1}, e_{z_2+1}] \\
&= P_{z_1-1}(v)[e_{z_1}, e_{z_2+1}] + P_{z_1}(v)[e_{z_1+1}, e_{z_2+1}] + \ldots \\
&\quad + P_{n-z_1}(v)[e_{n-z_1+1}, e_{z_2+1}] = P_{z_1-1}(v)\alpha_1 e_2 + P_{z_1}(v)(\alpha_1 e_3 + \alpha_2 e_2) \\
&\quad + \ldots + P_{n-z_1}(v)(\alpha_1 e_{n-2z_1+3} + \ldots + \alpha_{n-2z_1+2} e_2) \\
&= \alpha_1 P_{n-z_1}(v) e_{n-2z_1+3} + (\alpha_1 P_{n-z_1-1} + \alpha_2 P_{n-z_1})(v) e_{n-2z_1+2} + \ldots \\
&\quad + (\alpha_1 P_{z_1-1} + \alpha_2 P_{z_1} + \ldots + \alpha_{n-2z_1+2} P_{n-z_1})(v) e_2,
\end{aligned}$$

where $v = [e_{z_2-1}, e_n] + [e_{z_2}, e_{n-1}]$. Therefore, the Jacobi identity $J(e_{z_2}, e_{z_2+1}, e_n) = 0$ is given by

$$\begin{aligned}
&\alpha_{z_2-2z_1+3}(\alpha_1 e_{n-z_2+1} + \alpha_2^1 e_{n-z_2} + \ldots + \alpha_{n-z_2}^{n-z_2-1} e_2) + \ldots \\
&+\alpha_2 (P_{n-2z_1+1}([e_{z_2-z_1}, e_n] + [e_{z_2-z_1+1}, e_{n-1}]) e_{n-2z_1+2} + \ldots + \alpha_{n-2z_1+1}^{n-2z_1} e_2) \\
&+\alpha_1 (P_{n-2z_1+2}([e_{z_2-z_1+1}, e_n] + [e_{z_2-z_1+2}, e_{n-1}]) e_{n-2z_1+3} \\
&+P_{n-2z_1+1}([e_{z_2-z_1+1}, e_n] + [e_{z_2-z_1+2}, e_{n-1}]) e_{n-2z_1+2} + \ldots + \alpha_{n-2z_1+2}^{n-2z_1+1} e_2) \\
&-\alpha_1 P_{n-z_1}(v) e_{n-2z_1+3} - (\alpha_1 P_{n-z_1-1} + \alpha_2 P_{n-z_1})(v) e_{n-2z_1+2} - \ldots \\
&-(\alpha_1 P_{z_1-1} + \alpha_2 P_{z_1} + \ldots + \alpha_{n-2z_1+2} P_{n-z_1})(v) e_2 = 0
\end{aligned}$$

Next, we compute the coefficient of the vector e_{n-2z_1+3}. To do so, we consider the following expressions

$$P_{n-2z_1+2}([e_{z_2-z_1+1}, e_n] + [e_{z_2-z_1+2}, e_{n-1}]) = P_{n-z_2+1}\left(\sum_{k=0}^{z_2-2z_1+2} \binom{z_2-2z_1+2}{k} [e_{z_1+k}, e_{n-k}]\right);$$

$$P_{n-z_1}(v) = P_{n-z_1}([e_{z_2-1}, e_n] + [e_{z_2}, e_{n-1}]) = P_{n-z_2+1}\left(\sum_{k=0}^{z_2-z_1} \binom{z_2-z_1}{k} [e_{z_1+k}, e_{n-k}]\right).$$

Therefore, the coefficient of e_{n-2z_1+3} is

$$\alpha_1 \cdot \left(\sum_{k=0}^{z_2-2z_1+2} \binom{z_2-2z_1+2}{k} P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) - \sum_{k=0}^{z_2-z_1} \binom{z_2-z_1}{k} P_{n-z_2+1}([e_{z_1+k}, e_{n-k}])\right).$$

Since $\binom{z_2-2z_1+2}{0} = \binom{z_2-z_1}{0} = 1$, the coefficient of e_{n-2z_1+3} is

$$\alpha_1 \cdot \left(\sum_{k=1}^{z_2-2z_1+2} \binom{z_2-2z_1+2}{k} P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) - \sum_{k=1}^{z_2-z_1} \binom{z_2-z_1}{k} P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) \right).$$

Let us note that $z_2 - z_1 > z_2 - 2z_1 + 2$, since $z_1 > 2$. Consequently, the coefficient of e_{n-2z_1+3} is

$$\alpha_1 \cdot \left(\sum_{k=1}^{z_2-2z_1+2} \left(\binom{z_2-2z_1+2}{k} - \binom{z_2-z_1}{k} \right) P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) - \sum_{k=z_2-2z_1+3}^{z_2-z_1} \binom{z_2-z_1}{k} P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) \right).$$

Now, we consider the family of functions $\{\Phi_h : \mathbb{Z} \to \mathbb{N} \cup \{0\}\}_{h \in \mathbb{N} \cup \{-1,0\}}$, defined as

$$\Phi_{-1}(n) = \begin{cases} 1, & \text{if } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \Phi_h(n) = \begin{cases} \sum_{i=1}^{n} \Phi_{h-1}(i), & \text{if } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases} \quad \forall h \in \mathbb{N} \cup \{0\}.$$

With this notation, we obtain

$$P_{n-z_2+1}([e_{z_1+k}, e_{n-k}]) = \left(\sum_{l=0}^{k} \binom{k}{l} \Phi_{k-1-l}(n - z_2 - 2k + l) \right) \alpha_1.$$

and the coefficient of e_{n-2z_1+3} turns out to be

$$(\alpha_1)^2 \cdot \left(\sum_{k=1}^{z_2-2z_1+2} \left(\binom{z_2-2z_1+2}{k} - \binom{z_2-z_1}{k} \right) \sum_{l=0}^{k} \binom{k}{l} \cdot \Phi_{k-1-l}(n - z_2 - 2k + l) - \sum_{k=z_2-2z_1+3}^{z_2-z_1} \binom{z_2-z_1}{k} \sum_{l=0}^{k} \binom{k}{l} \Phi_{k-1-l}(n - z_2 - 2k + l) \right).$$

If $n - z_2 - 2k + l \leq 0$ for $1 \leq k \leq z_2 - z_1$ and $0 \leq l \leq k$, then the Jacobi identity $J(e_{z_2}, e_{z_2+1}, e_n) = 0$ is identically null. Otherwise, the previous expression is zero if and only if $\alpha_1 = 0$. □

Next, we have this result for the case $\beta(\mathfrak{g}) = n - 3$.

Proposition 3.8. *Let $\mathfrak{g}$ be a n-dimensional filiform Lie algebra verifying $\beta(\mathfrak{g}) = n - 3$. Then the triples $([\frac{n-(2k+1)}{2}] + 2, n - 2, n)$, where $n \geq 2k + 5$, correspond to a filiform Lie algebra with unique restrictions given by $\{\alpha_i = 0\}_{i=1}^{k}$.*

Proof. In this case, we have to impose the Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$ over the general law given in Proposition 3.3 with $(z_1, z_2, n) = ([\frac{n-(2k+1)}{2}] + 2, n-2, n)$. This identity is given by $[[e_{n-2}, e_{n-1}], e_n] + [[e_n, e_{n-2}], e_{n-1}] = 0$. Next, we compute both brackets, starting with

$$\begin{aligned}
[[e_{n-2}, e_{n-1}], e_n] =& [\alpha_1 e_{n-z_1} + \alpha_2 e_{n-z_1-1} + \ldots + \alpha_{n-2-z_1} e_3 + \alpha_{n-z_1-1} e_2, e_n] \\
=& [\alpha_1 e_{n-z_1} + \alpha_2 e_{n-z_1-1} + \ldots + \alpha_{n-2z_1+1} e_{z_1}, e_n] \\
=& \alpha_1 [e_{n-z_1}, e_n] + \alpha_2 [e_{n-z_1-1}, e_n] + \ldots + \alpha_{2k-2} [e_{z_1}, e_n] \\
=& \alpha_1 ((2k-2)\alpha_1 e_{2k} + ((2k-3)\alpha_2 + \alpha_2^1) e_{2k-1} \\
&+((2k-4)\alpha_3 + \alpha_3^2) e_{2k-2} + \ldots + (\alpha_{2k-2} + \alpha_{2k-2}^{2k-3}) e_3 + \alpha_{2k-1}^{2k-2} e_2) \\
&+\alpha_2((2k-3)\alpha_1 e_{2k-1} + ((2k-4)\alpha_2 + \alpha_2^1) e_{2k-2} \\
&+((2k-5)\alpha_3 + \alpha_3^2) e_{2k-3} + \ldots + (\alpha_{2k-3} + \alpha_{2k-3}^{2k-4}) e_3 + \alpha_{2k-2}^{2k-3} e_2) \\
&+ \ldots + \alpha_{2k-2}(\alpha_1 e_3 + \alpha_2^1 e_2).
\end{aligned}$$

and continuing with the second taking into account that $[[e_n, e_{n-2}], e_{n-1}] = -[[e_{n-2}, e_n], e_{n-1}]$

$$\begin{aligned}
[[e_{n-2}, e_n], e_{n-1}] =& [(n-z_1-1)\alpha_1 e_{n-z_1+1} + ((n-z_1-2)\alpha_2 + \alpha_2^1) e_{n-z_1} + \ldots \\
&+(\alpha_{n-z_1-1} + \alpha_{n-z_1-1}^{n-z_1-2}) e_3 + \alpha_{n-z_1}^{n-z_1-1} e_2, e_{n-1}] \\
=& (n-z_1-1)\alpha_1 [e_{n-z_1+1}, e_{n-1}] + ((n-z_1-2)\alpha_2 + \alpha_2^1)[e_{n-z_1}, e_{n-1}] \\
&+ \ldots + (\alpha_{n-z_1-1} + \alpha_{n-z_1-1}^{n-z_1-2})[e_3, e_{n-1}] + \alpha_{n-z_1}^{n-z_1-1}[e_2, e_{n-1}] \\
=& (n-z_1-1)\alpha_1 [e_{n-z_1+1}, e_{n-1}] + ((n-z_1-2)\alpha_2 + \alpha_2^1)[e_{n-z_1}, e_{n-1}] \\
&+ \ldots + ((z_1-2)\alpha_{2k-1} + \alpha_{2k-1}^{2k-2})[e_{z_1}, e_{n-1}] \\
=& \quad (n-z_1-1)\alpha_1(\alpha_1 e_{2k} + \alpha_2 e_{2k-1} + \ldots + \alpha_{2k-1} e_2) \\
&+((n-z_1-2)\alpha_2 + \alpha_2^1)(\alpha_1 e_{2k-1} + \alpha_2 e_{2k-2} \\
&+ \ldots + \alpha_{2k-2} e_2) + \ldots + ((z_1-2)\alpha_{2k-1} + \alpha_{2k-1}^{2k-2})\alpha_1 e_2.
\end{aligned}$$

Combining both brackets, the Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$ is

$$\begin{aligned}
&\alpha_1((2k-2)\alpha_1 e_{2k} + ((2k-3)\alpha_2 + \alpha_2^1) e_{2k-1} + ((2k-4)\alpha_3 + \alpha_3^2) e_{2k-2} + \ldots \\
&+(\alpha_{2k-2} + \alpha_{2k-2}^{2k-3}) e_3 + \alpha_{2k-1}^{2k-2} e_2) + \alpha_2((2k-3)\alpha_1 e_{2k-1} + ((2k-4)\alpha_2 + \alpha_2^1) e_{2k-2} \\
&+((2k-5)\alpha_3 + \alpha_3^2) e_{2k-3} + \ldots + (\alpha_{2k-3} + \alpha_{2k-3}^{2k-4}) e_3 + \alpha_{2k-2}^{2k-3} e_2) + \ldots \\
&+(\alpha_{2k-3} + \alpha_{2k-3}^{2k-4}) e_3 + \alpha_{2k-2}^{2k-3} e_2) + \ldots + \alpha_{2k-2}(\alpha_1 e_3 + \alpha_2^1 e_2) \\
&-(n-z_1-1)\alpha_1(\alpha_1 e_{2k} + \alpha_2 e_{2k-1} + \ldots + \alpha_{2k-1} e_2) \\
&-((n-z_1-2)\alpha_2 + \alpha_2^1)(\alpha_1 e_{2k-1} + \alpha_2 e_{2k-2} + \ldots + \alpha_{2k-2} e_2) \\
&-((n-z_1-3)\alpha_3 + \alpha_3^2)(\alpha_1 e_{2k-2} + \alpha_2 e_{2k-3} + \ldots + \alpha_{2k-3} e_2) - \ldots \\
&-((z_1-2)\alpha_{2k-1} + \alpha_{2k-1}^{2k-2})(\alpha_1 e_2) = 0.
\end{aligned}$$

Note that, according to the previous computations, the coefficient of the vector e_{2k} is $(\alpha_1)^2 \cdot (2k - n + z_1 - 1)$. Fixing $0 < h \leq 2k-2$, we compute the coefficient of the vector e_{2k-h} as follows

$$\alpha_{h+1}(2k-(h+2))\alpha_1+\sum_{i=1}^{h}\alpha_i((2k-(h+2))\alpha_{h+2-i}+\alpha_{h+2-i}^{h+1-i})-(n-z_1-1)\alpha_1\alpha_{h+1}$$
$$-\sum_{i=2}^{h+1}((n-z_1-i)\alpha_i+\alpha_i^{i-1})\alpha_{h+2-i}=(2k-(h+2))\alpha_1\alpha_{h+1}+\sum_{i=1}^{h}\alpha_i\alpha_{h+2-i}(2k-(h+2))$$
$$-(n-z_1-1)\alpha_1\alpha_{h+1}-\sum_{i=2}^{h+1}\alpha_i\alpha_{h+2-i}(n-z_1-i)=(2k-n-h+1)\alpha_1\alpha_{h+1}$$
$$+\sum_{i=2}^{h}\alpha_i\alpha_{h+2-i}(2k-n-h-2+z_1+i).$$

In this way, we have to consider the following equations for $0<h\leq 2k-2$,

$$\begin{cases}(\alpha_1)^2(2k-n+z_1-1)=0,\\ (2k-n-h+1)\alpha_1\alpha_{h+1}+\displaystyle\sum_{i=2}^{h}\alpha_i\alpha_{h+2-i}(2k-n-h-2+z_1+i)=0.\end{cases}$$

From the first equation, we obtain $\alpha_1=0$, since $n\geq 2k+5$. Consequently, the second equation turns into

$$\sum_{i=2}^{h}\alpha_i\alpha_{h+2-i}(2k-n-h-2+z_1+i)=0,\qquad \forall\ 2<h\leq 2k-2.$$

This involves the condition $\alpha_l=0$ for $h=2l-2$, where $l=2,3,\dots,k$, leading to the conclusion that the Jacobi identity $J(e_{n-2},e_{n-1},e_n)=0$ implies $\{\alpha_i=0\}_{i=1}^{k}$. □

Moreover, we can set the following result for the coefficients in a filiform Lie algebra associated with a general triple (z_1,z_2,n)

Proposition 3.9. *Let $\mathfrak{g}$ be a n-dimensional non-model filiform Lie algebra associated with the triple (z_1,z_2,n). All the Jacobi identities in $\mathfrak{g}$ correspond to a system of equations whose solutions are a combination of the following expressions*

$$(\beta\alpha_p+\gamma\alpha_p^{p-1})(\lambda\alpha_q+\mu\alpha_q^{q-1})=0,\ \text{where}\,\beta,\gamma,\lambda,\mu\in\mathbb{C}\ \text{and}\ p,q\in\{2,\dots,n-z_2\}$$

Therefore, we always obtain linear relations between the coefficients in the law of $\mathfrak{g}$.

Proof. Let $\mathfrak{g}$ be a filiform Lie algebra associated with (z_1,z_2,n). We suppose that there are restrictions over the coefficients $\{\alpha_1,\alpha_2,\dots,\alpha_{z_2-z_1+1},\alpha_2^1,\dots,\alpha_{n-z_2}^{n-z_2-1}\}$. By using Proposition 3.7, we can take $\alpha_1=0$. The general law of $\mathfrak{g}$ is given by

$$[e_1, e_h] = e_{h-1},\ \forall 3 \le h \le n$$

$$[e_{z_1+i}, e_{z_2+1}] = \alpha_2 e_{i+1} + \ldots + \alpha_{i+1} e_2,\ 1 < i \le z_2 - z_1. \tag{3.6}$$

$$[e_{z_1}, e_{z_2+j}] = \alpha_2^1 e_j \ldots + \alpha_j^{j-1} e_2,\ 2 \le j \le n - z_2. \tag{3.7}$$

$$[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{p+k} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}])e_{h+1} + \alpha_{k+p}^{k+p-1} e_2,$$

$$2 \le k \le n - z_2, 0 < p < z_2 - z_1 + k. \tag{3.8}$$

The family of brackets (3.6), (3.7) and (3.8) will be called brackets of type 1, 2 and 3, respectively. We need to impose the Jacobi identities $J(e_a, e_b, e_c) = 0$, where $z_1 \le a < b < c \le n$. According to the law of $\mathfrak{g}$, we have to study the identities involving the brackets of type 1, 2 and 3. Therefore, we study the following Jacobi identities

$$J(e_{z_1+i}, e_{z_1+i'}, e_{z_2+1}) = 0,\ J(e_{z_1}, e_{z_1+i}, e_{z_2+1}) = 0,\ J(e_{z_1}, e_{z_1+i}, e_{z_2+j}) = 0,$$

$$J(e_{z_1}, e_{z_2+1}, e_{z_2+j}) = 0,\ J(e_{z_1+i}, e_{z_2+1}, e_{z_2+j}) = 0,\ J(e_{z_1+i}, e_{z_2+j}, e_{z_2+k}) = 0.$$

- For $J(e_{z_1+i}, e_{z_1+i'}, e_{z_2+1}) = 0$, all the brackets involved are of type 1. This identity is given by

  $$[[e_{z_1+i}, e_{z_1+i'}], e_{z_2+1}] + [[e_{z_1+i'}, e_{z_2+1}], e_{z_1+i}] + [[e_{z_2+1}, e_{z_1+i}], e_{z_1+i'}].$$

  The bracket $[e_{z_1+i}, e_{z_1+i'}] = 0$ since $\langle e_2, \ldots, e_{z_2} \rangle$ is an abelian ideal. Therefore, $J(e_{z_1+i}, e_{z_1+i'}, e_{z_2+1}) = 0$ is expressed as follows

  $$[\alpha_2 e_{i'+1} + \ldots + \alpha_{i'+1} e_2, e_{z_1+i}] - [\alpha_2 e_{i+1} + \ldots + \alpha_{i+1} e_2, e_{z_1+i'}].$$

  and both brackets are zero since $i' + 1, i + 1 \le z_2 - z_1 + 1 \le z_2 - 3 < z_2 + 1$. Consequently, this Jacobi identity is identically null.

- The Jacobi identity $J(e_{z_1}, e_{z_1+i}, e_{z_2+1}) = 0$, with $i \le z_2 - z_1$, is given by

  $$[[e_{z_1}, e_{z_1+i}], e_{z_2+1}] + [[e_{z_1+i}, e_{z_2+1}], e_{z_1}] + [[e_{z_2+1}, e_{z_1}], e_{z_1+i}],$$

  where $[e_{z_1}, e_{z_2+1}] = [e_{z_1}, e_{z_1+i}] = 0$ since $\alpha_1 = 0$ and $\langle e_2, \ldots, e_{z_2} \rangle$ is an abelian ideal. Therefore, the identity is given by $[[e_{z_1+i}, e_{z_2+1}], e_{z_1}] = [\alpha_2 e_{i+1} + \ldots +$

$\alpha_{i+1}e_2, e_{z_1}]$. Let us note that if $i+1 \leq z_2$, then this bracket is null. We consider $i+1 > z_2$. In that case, we have

$$\begin{aligned}
[[e_{z_1+i}, e_{z_2+1}], e_{z_1}] = & -[e_{z_1}, \alpha_2 e_{i+1} + \ldots + \alpha_{i+2-z_2} e_{z_2+1}] = -\alpha_{i+2-z_2}[e_{z_1}, e_{z_2+1}] \\
& -\alpha_{i+1-z_2}[e_{z_1}, e_{z_2+2}] - \ldots - \alpha_2[e_{z_1}, e_{i+1}] \\
= & -\alpha_{i+1-z_2}\alpha_2^1 e_2 - \alpha_{i-z_2}(\alpha_2^1 e_3 + \alpha_3^2 e_2) - \ldots \\
& -\alpha_2(\alpha_2^1 e_{i+2-z_2} + \ldots + \alpha_{i+2-z_2}^{i+1-z_2} e_2) \\
= & -\alpha_2\alpha_2^1 e_{i+2-z_2} - (\alpha_2\alpha_3^2 + \alpha_3\alpha_2^1)e_{i+1-z_2} - \ldots \\
& -(\alpha_2\alpha_{i-z_2}^{i-z_2-1} + \ldots + \alpha_{i+1-z_2}\alpha_2^1)e_2 = 0.
\end{aligned}$$

obtaining the following system of equations

$$\begin{cases}
\alpha_2\alpha_2^1 = 0, \\
\alpha_2\alpha_3^2 + \alpha_3\alpha_2^1 = 0, \\
\alpha_2\alpha_4^3 + \alpha_3\alpha_3^2 + \alpha_4\alpha_2^1 = 0, \\
\vdots \qquad \vdots \\
\alpha_2\alpha_{i-z_2}^{i-z_2-1} + \ldots + \alpha_{i+1-z_2}\alpha_2^1 = 0.
\end{cases}$$

which only returns solutions of type $\alpha_p = 0$ or $\alpha_q^{q-1} = 0$.

- For $J(e_{z_1}, e_{z_1+i}, e_{z_2+j}) = 0$, we have the following expression

$$[[e_{z_1}, e_{z_1+i}], e_{z_2+j}] + [[e_{z_1+i}, e_{z_2+j}], e_{z_1}] + [[e_{z_2+j}, e_{z_1}], e_{z_1+i}].$$

Let us note that $[e_{z_1}, e_{z_1+i}] = 0$ since $\langle e_2, \ldots, e_n\rangle$ is an abelian ideal. Moreover, $[e_{z_1}, e_{z_2+j}] = \alpha_2^1 e_j + \ldots + \alpha_j^{j-1} e_2$ with $2 \leq j \leq n - z_2$. From $j \leq n - z_2 \leq z_2 - 2 < z_2$ and the definition of z_2, we can affirm that $[[e_{z_1}, e_{z_2+j}], e_{z_1+i}] = 0$. Hence, the Jacobi identity $J(e_{z_1}, e_{z_1+i}, e_{z_2+j}) = 0$ is given by $[[e_{z_1+i}, e_{z_2+j}], e_{z_1}] = 0$. Developing the first term, we obtain

$$\begin{aligned}
[[e_{z_1+i}, e_{z_2+j}], e_{z_1}] = & [\textstyle\sum_{h=2}^{i+j} P_h([e_{z_1+i-1}, e_{z_2+j}] + [e_{z_1+i}, e_{z_2+j-1}])e_{h+1} + \alpha_{i+j}^{i+j-1} e_2, e_{z_1}] \\
= & [\textstyle\sum_{h=z_2+1}^{i+j} P_h(u)e_{h+1}, e_{z_1}] = -P_{z_2+1}(u)[e_{z_1}, e_{z_2+2}] - \ldots \\
- & P_{i+j}(u)[e_{z_1}, e_{i+j+1}] = -P_{z_2+1}(u)(\alpha_2^1 e_2) - \ldots \\
- & P_{i+j}(u)(\alpha_2^1 e_{i+j+1-z_2} + \ldots + \alpha_{i+j+1-z_2}^{i+j-z_2} e_2) \\
= & -\alpha_2^1 P_{i+j}(u)e_{i+j+1-z_2} - (\alpha_3^2 P_{i+j} + \alpha_2^1 P_{i+j-1})(u)e_{i+j-z_2} - \ldots \\
- & (\alpha_{i+j+1-z_2}^{i+j-z_2} P_{i+j} + \ldots + \alpha_2^1 P_{z_2+1})(u)e_2 = 0,
\end{aligned}$$

where $u = [e_{z_1+i-1}, e_{z_2+j}] + [e_{z_1+i}, e_{z_2+j-1}]$. Consequently, we have to solve the system of equations

$$\begin{cases}
\alpha_2^1 P_{i+j}(u) = 0, \\
\alpha_3^2 P_{i+j}(u) + \alpha_2^1 P_{i+j-1}(u) = 0, \\
\vdots \qquad \vdots \\
\alpha_{i+j+1-z_2}^{i+j-z_2} P_{i+j}(u) + \ldots + \alpha_2^1 P_{z_2+1}(u) = 0.
\end{cases}$$

Now, we compute the expression of $P_{i+j}(u), P_{i+j-1}(u), \ldots, P_{z_2+1}(u)$ by using the reasoning of the proof of Proposition 3.7.

$$\begin{aligned}
P_{i+j}(u) &= \alpha_1 \sum_{k=0}^{i} \sum_{l=0}^{k} \binom{i}{k}\binom{k}{l} \Phi_{k-1-l}(z_2+j-2k+l);\\
P_{i+j-1}(u) &= \alpha_2^1 \sum_{k=0}^{i} \sum_{l=0}^{k} \binom{i}{k}\binom{k}{l} \Phi_{k-1-l}(z_2+j-2k+l);\\
\vdots & \qquad\qquad \vdots\\
P_{z_2+1}(u) &= \alpha_{i+j-z_2}^{i+j-z_2-1} \sum_{k=0}^{i} \sum_{l=0}^{k} \binom{i}{k}\binom{k}{l} \Phi_{k-1-l}(z_2+j-2k+l).
\end{aligned}$$

When solving this system of equations, we obtain solutions of type $\alpha_q^{q-1} = 0$.

- For the Jacobi identities $J(e_{z_1}, e_{z_2+1}, e_{z_2+j}) = 0, J(e_{z_1+i}, e_{z_2+1}, e_{z_2+j}) = 0$ and $J(e_{z_1+i}, e_{z_2+j}, e_{z_2+k}) = 0$, only brackets of type 3 are involved. We will study this type of brackets.

  We define the degree function deg : $N \times N \to \mathbb{N}$, where $N = \{1, \ldots, n\}$, as $deg(a, b) = a + b$. Notice that, according to Proposition 3.3, $deg([a, b]) = a + b$ implies $[e_a, e_b] \in \langle e_2, \ldots, e_{a+b-z_1-z_2}\rangle$. The brackets of type 3 are given by

$$[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{p+k} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}])e_{h+1} + \alpha_{k+p}^{k+p-1} e_2$$

  Let us note that $deg(z_1+p-1, z_2+k) = z_1+z_2+p+k-1 = deg(z_1+p, z_2+k-1)$. In order to compute the term $\sum_{h=2}^{p+k} P_h([e_{z_1+p-1}, e_{z_2+k}]+[e_{z_1+p}, e_{z_2+k-1}])e_{h+1}$ in a bracket of type 3, we have to decompose it by using the following brackets of type 1 and 2

$$\begin{gathered}
[e_{z_1+l}, e_{z_2+1}] = \alpha_2 e_{l+1} + \ldots + \alpha_{l+1} e_2,\ 1 \le l \le z_2 - z_1;\\
[e_{z_1}, e_{z_2+l+1}] = \alpha_2^1 e_{l+1} + \ldots + \alpha_{l+1}^l e_2,\ 2 \le l+1 \le n - z_2,
\end{gathered}$$

  and their index pairs have the same degree. In fact, we obtain that

$$P_h([e_{z_1+l}, e_{z_2+1}]+[e_{z_1}, e_{z_2+l+1}]) = (\alpha_{l-h+3}+\alpha_{l-h+3}^{l-h+2}), \quad \text{for } \begin{cases} 1 \le l \le z_2-z_1;\\ 2 \le h \le l+1.\end{cases}$$

$$P_h([e_{z_1+l}, e_{z_2+1}]+[e_{z_1}, e_{z_2+l+1}]) = \alpha_{l-h+3}^{l-h+2}, \quad \text{for } \begin{cases} z_2-z_1 < l \le n-z_2-1;\\ 2 \le h \le l+1.\end{cases}$$

  From the remaining Jacobi identities, we obtain this type of equations

$$(\beta\alpha_p+\gamma\alpha_p^{p-1})(\lambda\alpha_q+\mu\alpha_q^{q-1}) = 0,\ \beta, \gamma, \lambda, \mu \in \mathbb{C},\ \text{and } p, q \in \{2, \ldots, n-z_2\}. \quad \square$$

Now, we show an algorithmic procedure which computes the law of an n-dimensional non-model filiform Lie algebra $\mathfrak{g}$ starting from the value of $\alpha(\mathfrak{g})$. More concretely, we give a step-by-step explanation of this algorithm for obtaining the law of $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = k \in \mathbb{N}$.

Input

1. The dimension n of a non-model filiform Lie algebra $\mathfrak{g}$.

2. The value k of the invariant $\alpha(\mathfrak{g})$.

Output

1. A list with the triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras associated with them.

2. The law of each family of filiform Lie algebras for each triple.

Method

1. First, computing the value of the invariant z_2 by using Proposition 3.2. According to this value and Expression (1.2), several possibilities appear for the invariant z_1.

2. For each value of z_1, computing all the possible non-zero brackets of $\mathfrak{g}$ given by Proposition 3.3.

3. Then, ruling out those values of z_1 not satisfying the Jacobi identities $J(e_h, e_k, e_l) = 0$, for $z_1 \leq h < k < l \leq n$.

4. Next, obtaining a list with all the triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras having such invariants.

5. By using Proposition 3.3 again, computing the law of each Lie algebra associated with a triple given in the previous step.

Now, we show the implementation of the previous algorithm. To do so, we have used the symbolic computation package MAPLE 12. Note that before running any sentence or loading a package, we must restart all the variables and delete all the computations with the command `restart`. After that, we load the library `DifferentialGeometry, LieAlgebras` to activate commands related to Lie algebras such as `BracketOfSubspaces`.

First, we show the implementation of a routine which computes the law of a filiform Lie algebra from the triple (z_1, z_2, n) by using Proposition 3.3. This routine, named `law`, receives the triple (z_1, z_2, n) as input and returns the filiform Lie algebra associated with this triple. For the implementation, we define a list as a local variable `L`. This list saves the indexes and the value of the structure constants corresponding with the non-zero brackets of the filiform Lie algebra. First, `L` saves the brackets given by filiformity (i.e. $[e_1, e_h] = e_{h-1}, \forall\, 3 \leq h \leq n$) and the bracket $[e_{z_1}, e_{z_2+1}] = \alpha_1 e_2$. Then, a loop is programmed for including the indexes of the rest of non-zero brackets in `L`.

```
> law:=proc(z_1,z_2,n)
> local L; L:=[seq([[1,h,h-1],1],h=3..n),
> [[z_1,z_2+1,2],a[1]]];
> for i from 1 to n do for j from 1 to n do c[i,i][j]:=0; od; od;
> for i from 1 to z_2-z_1 do for l from 1 to i+1 do
>    L:=[op(L),[[z_1+i,z_2+1,i+3-l],a[l]]];
>    c[z_1+i,z_2+1][i+3-l]:=a[l];
> od; od;
> for i from 2 to n-z_2 do
>    L:=[op(L),[[z_1,z_2+i,i+1],a[1]]];
>    c[z_1,z_2+i][i+1]:=a[1];
>    for j from 2 to i do
>       L:=[op(L),[[z_1,z_2+i,i+2-j],a[j][j-1]]];
>       c[z_1,z_2+i][i+2-j]:=a[j][j-1];
> od; od;
> for i from 2 to n-z_2 do for j from 1 to z_2-z_1+i-1 do
>    L:=[op(L),[[z_1+j,z_2+i,2],
>    a[i+j][i+j-1]]];
>    c[z_1+j,z_2+i][2]:=a[i+j][i+j-1];
>    for h from 2 to i+j do
>       L:=[op(L),[[z_1+j,z_2+i,h+1],
>       c[z_1+j-1,z_2+i][h]+c[z_1+j,z_2+i-1][h]]];
>       c[z_1+j,z_2+i][h+1]:=c[z_1+j-1,z_2+i][h]+
>       c[z_1+j,z_2+i-1][h];
> od;od;od;
> return _DG([["LieAlgebra",Alg1,[n]],L]);
> end proc:
```

Next, we implement the routine `coefficients`, which receives as input the triple (z_1, z_2, n) and returns a list with all the structure constants α_i and α_j^{j-1} involved in the law of the associated filiform Lie algebra.

```
> coefficients:=proc(z_1,z_2,n)
```

```
> local M; M:={}; for i from 1 to
> z_2-z_1+1 do M:={op(M),a[i]}; od;
> for j from 2 to 2*n-z_1-z_2-1 do
> M:={op(M),a[j][j-1]}; od;
> return M; end proc:

> DGsetup(law(z_1,z_2,n));
```

From here on, we can operate over the Lie algebra associated with the output given by `law`. This algebra is denoted by `Alg1`. Now, we change the format of the subindexes of the basis vectors with the command `assign`.

```
> assign([seq(e||i=e[i],i=1..n)]);
```

After that, we have to define the variable `Eq` from the output of the routine `law`. Now, we execute this sentence

```
Alg1 > L := LieAlgebraData(Eq,[seq(e[i],i=1..n)], Alg2);
> DGsetup(L);
```

From here on, we work over the Lie algebra `Alg2`. With the command `LieAlgebraData` we can evaluate the following sentence

```
Alg2 > TF, EQ, SOLN, AlgList :=Query(coefficients(z_1,z_2,n),"Jacobi");
```

This sentence provides us the conditions given by the Jacobi identities in terms of equations for the filiform Lie algebra. Additionally, `AlgList` shows us a list with all the non-zero brackets. Finally, for each term given by the previous output, we have to write the following commands

```
Alg2 > DGsetup(AlgList[i]);
Alg2_i > f:=proc() if BracketOfSubspaces([ez_1],[en])=[] then
Alg2_i > return "It is not a filiform Lie algebra"; end if; if
Alg2_i > BracketOfSubspaces([ez_2],[ez_2+1])=[] then return
Alg2_i > "It is not a filiform Lie algebra"; end if; else return
Alg2_i > AlgList[i],SOLN[i]; fi; end proc:
Alg2_i > f()
```

With these sentences, we program a routine which studies if the Lie algebra given in `AlgList[i]` is compatible with the definition of the invariants z_1 and z_2. In the affirmative case, the output is the set of conditions for the structure constants given by the routine `coefficients` and the non-zero brackets of the Lie algebra. In case of incompatibility, the output is the message ”It is not a filiform Lie algebra".

Using this algorithm and its implementation, we have computed the triples (z_1, z_2, n) associated with non-model filiform Lie algebras up to dimension 14. This triples are shown in Tables 3.1–3.3, structured according to the maximal dimension for their abelian ideals and including the restrictions for the coefficients α_i and α_j^{j-1}, if they exist.

Next, we show a computational study of the algorithm. To do so, we have used an Intel Core 2 Duo T 5600 with a 1.83 GHz processor and 2.00 GB of RAM. Table 3.4 shows some computational data about both the computing time and the memory usage. These data were obtained considering the family of filiform Lie algebras associated with the triple $(n-3, n-3, n)$.

Additionally, we show some brief statistics about the relation between the computing time and the memory usage by the implementation. In this sense, Fig. 3.1 shows the behavior of both the computing time and the memory usage according to the dimension n. In fact, the computing time increases more quickly than the memory usage. As we can observe in Fig. 3.2, the increase of the computing time corresponds to a positive exponential model, whereas the memory usage does not follow such a model. Finally, we have also studied the quotients between memory usage and computing time, obtaining the frequency diagram shown in Fig. 3.3. In this case, the behavior can be also considered exponential, although this time is negative.

Table 3.1: Triples for non-model filiform Lie algebras of dimension less than 12.

Dim	$\beta(\mathfrak{g})$	**Triples**	**Restrictions**
5	3	(4,4,5)	None
6	3	(4,4,6)	None
	4	(4,5,6)	None
		(5,5,6)	None
7	4	(4,5,7)	$\{\alpha_1 = 0\}$
		(5,5,7)	None
	5	(k,6,7) $k = 4..6$	None
8	4	(4,5,8)	$\{\alpha_1, \alpha_3^2 = 0, \alpha_2 = -\alpha_2^1\}$
		(5,5,8)	$\nexists$ algebra
	5	(4,6,8)	$\{\alpha_1 = 0\}$
		(5,6,8)	None
		(6,6,8)	None
	6	(k,7,8) $k = 4..7$	None
9	5	(4,6,9)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3^2 = (\frac{1}{2} \pm \frac{\sqrt{33}}{6})\alpha_3\}$
		(5,6,9)	$\{\alpha_1 = 0, \alpha_2^1 = -\frac{2}{3}\alpha_2\}, \{\alpha_1 = 0, \alpha_2^1 = \alpha_2\}$
		(6,6,9)	$\nexists$ algebra
	6	(4,7,9)	$\{\alpha_1, \alpha_2 = 0\}$
		(5,7,9)	$\{\alpha_1 = 0\}$
		(6,7,9)	None
		(7,7,9)	None
	7	(k,8,9) $k = 4..8$	None
10	5	(k,6,10) $k = 4...6$	$\nexists$ algebra
	6	(4,7,10)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3^2 = -\frac{1}{3}\alpha_3, \alpha_4^3 = -\frac{17}{10}\alpha_4\}$
		(5,7,10)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}, \{\alpha_1, \alpha_2 = 0, \alpha_3^2 = -7\alpha_3\},$ $\{\alpha_1 = 0, \alpha_2^1 = -\alpha_2, \alpha_3^2 = -\frac{5}{8}\alpha_3\}$
		(6,7,10)	$\nexists$ algebra
		(7,7,10)	None
	7	(4,8,10)	$\{\alpha_1, \alpha_2 = 0\}$
		(5,8,10)	$\{\alpha_1 = 0\}$
		(k,8,10) $k = 6..8$	None
	8	(k,9,10) $k = 4..9$	None
11	6	(k,7,11) $k = 4...7$	$\nexists$ algebra
	7	(4,8,11)	$\nexists$ algebra
		(5,8,11)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3^2 = \frac{1 \pm \sqrt{21}}{4}\alpha_3\}$
		(6,8,11)	$\{\alpha_1 = 0, \alpha_2^1 = -\frac{5}{4}\alpha_2\}$
		(7,8,11)	None
		(8,8,11)	None
	8	(4,9,11)	$\{\alpha_1, \alpha_2, \alpha_3 = 0\}$
		(5,9,11)	$\{\alpha_1, \alpha_2 = 0\}$
		(6,9,11)	$\{\alpha_1 = 0\}$
		(k,9,11) $k = 7..9$	None
	9	(k,10,11) $k = 4..10$	None

Table 3.2: Triples for non-model filiform Lie algebras of dimension 12 and 13.

Dim	$\beta(\mathfrak{g})$	**Triples**	**Restrictions**
12	6	(k,7,12)	$\nexists$ algebra
		$k = 4...7$	
	7	(4,8,12)	$\nexists$ algebra
		(5,8,12)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_2^1, \alpha_3^2, \alpha_5^4 = 0, \alpha_4 = \frac{-7\pm2\sqrt{21}}{5}\alpha_4^3\}$
		(6,8,12)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3^2 = \frac{11\pm\sqrt{321}}{20}\alpha_3\}$
		(7,8,12)	$\{\alpha_1 = 0, \alpha_3^2 = 0, \alpha_2^1 = -\frac{5}{9}\alpha_2\}$
		(8,8,12)	None
	8	(4,9,12)	$\nexists$ algebra
		(5,9,12)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3 = -\frac{4}{3}\alpha_3^2, \alpha_4 = -\frac{23}{47}\alpha_4^3\}$
		(6,9,12)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}, \{\alpha_1, \alpha_2^1 = 0, \alpha_3^2 = 14\alpha_3\}$
			$\{\alpha_1 = 0, \alpha_2^1 = -\frac{3}{2}\alpha_2, \alpha_3^2 = -\frac{28}{25}\alpha_3\}$
		(7,9,12)	$\{\alpha_1 = 0, \alpha_2^1 = \frac{14}{5}\alpha_2\}$
		(8,9,12)	None
		(9,9,12)	None
	9	(4,10,12)	$\{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$
		(5,10,12)	$\{\alpha_1 = \alpha_2 = 0\}$
		(6,10,12)	$\{\alpha_1 = 0\}$
		(k,10,12)	None
		$k = 7...11$	
	10	(k,11,12)	None
		$k = 4...11$	
13	7	(k,8,13)	$\nexists$ algebra
		$k \in [4, 8]$	
	8	(4,9,13)	$\nexists$ algebra
		(5,9,13)	$\{\alpha_i, \alpha_{i+1}^i = 0\}$, for $1 \leq i \leq 4$
		(6,9,13)	$\nexists$ algebra
		(7,9,13)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_2^1, \alpha_3^2 = 0, \alpha_4 = -2\alpha_4^3\}$
			$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}$
			$\{\alpha_1, \alpha_4^3 = 0, \alpha_2 = \frac{-5\pm3\sqrt{65}}{28}\alpha_2^1, \alpha_3^2 = -\frac{\alpha_3}{5}\frac{(297\pm3\sqrt{65})}{(-5\pm3\sqrt{65})}\}$
		(8,9,13)	$\{\alpha_1, \alpha_2^1 = 0\}, \{\alpha_1 = 0, \alpha_2 = -\frac{9}{5}\}$
		(9,9,13)	None
	9	(4,10,13)	$\nexists$ algebra
		(5,10,13)	$\nexists$ algebra
		(6,10,13)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3 = \pm\frac{\sqrt{5}}{3}\alpha_3^2\}$
		(7,10,13)	$\{\alpha_1 = 0, \alpha_2 = -\frac{5}{9}\alpha_2^1\}, \{\alpha_1 = 0, \alpha_2 = \alpha_2^1\}$
		(8,10,13)	$\{\alpha_1 = 0\}$
		(9,10,13)	None
		(10,10,13)	None
	10	(4,11,13)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0\}$
		(5,11,13)	$\{\alpha_1, \alpha_2, \alpha_3 = 0\}$
		(6,11,13)	$\{\alpha_1, \alpha_2 = 0\}$
		(7,11,13)	$\{\alpha_1 = 0\}$
		(k,11,13)	None
		$k \in [8, 11]$	
	11	(k,12,13)	None
		$k \in [4, 12]$	

Table 3.3: Triples for non-model filiform Lie algebras of dimension 14.

Dim	$\beta(\mathfrak{g})$	**Triples**	**Restrictions**
14	7	(k,8,14)	$\nexists$ algebra
		$k \in [4,8]$	
	8	(k,9,14)	$\nexists$ algebra
		$k \in [4,9]$	
	9	(4,10,14)	$\nexists$ algebra
		(5,10,14)	$\nexists$ algebra
		(6,10,14)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_2^1, \alpha_3^2 = 0, \alpha_4 = \frac{-17 \pm \sqrt{327}i}{28}\alpha_4^3,$
			$\alpha_5 = -\frac{1}{2}(9 + 5\frac{-17 \pm \sqrt{327}i}{28})\alpha_5^4\}$
		(7,10,14)	$\{\alpha_1, \alpha_2, \alpha_2^1, \alpha_4^3 = 0, \alpha_3 = -\frac{3}{2}\alpha_3^2\}$
			$\{\alpha_1, \alpha_2, \alpha_2^1, \alpha_4^3 = 0, \alpha_3 = \frac{5}{7}\alpha_3^2\}, \{\alpha_1, \alpha_2, \alpha_3, \alpha_2^1, \alpha_3^2 = 0\}$
		(8,10,14)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}, \{\alpha_1, \alpha_2 = 0, \alpha_3^2 = -\frac{5}{14}\alpha_3\}$
			$\{\alpha_1 = 0, \alpha_2 = \alpha_2^1, \alpha_3^2 = -\frac{5}{14}\alpha_3\}$
		(9,10,14)	$\{\alpha_1 = 0\}$
		(10,10,14)	None
	10	(4,11,14)	$\nexists$ algebra
		(5,11,14)	$\nexists$ algebra
		(6,11,14)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3 = -\frac{5}{6}\alpha_3^2, \alpha_4 = -\frac{46}{109}\alpha_4^3\}$
		(7,11,14)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}, \{\alpha_1, \alpha_2^1 = 0, \alpha_3 = \frac{4}{23}\alpha_3^2\},$
			$\{\alpha_1 = 0, \alpha_2 = -\frac{1}{2}\alpha_2^1, \alpha_3 = -\frac{18}{29}\alpha_3^2\}$
		(8,11,14)	$\{\alpha_1 = 0\}, \{\alpha_2^1 = \frac{23}{9}\alpha_2\}$
		(9,11,14)	None
		(10,11,14)	None
		(11,11,14)	None
	11	(4,12,14)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0\}$
		(5,12,14)	$\{\alpha_1, \alpha_2, \alpha_3 = 0\}$
		(6,12,14)	$\{\alpha_1, \alpha_2 = 0\}$
		(7,12,14)	$\{\alpha_1 = 0\}$
		(k,12,14)	None
		$k \in [8,12]$	
	12	(k,13,14)	None
		$k \in [4,13]$	

Table 3.4: Computing time and memory used.

Input	**Computing time**	**Memory used**
$n = 10$	0.51 s	5.31 MB
$n = 15$	0.69 s	5.37 MB
$n = 20$	0.92 s	5.49 MB
$n = 25$	1.07 s	5.53 MB
$n = 30$	1.62 s	5.56 MB
$n = 35$	1.95 s	5.59 MB
$n = 40$	2.46 s	5.62 MB
$n = 45$	3.03 s	5.65 MB
$n = 50$	3.59 s	5.68 MB

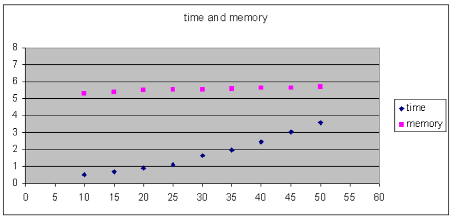


Figure 3.1: Comparative graph between C.T. and M.U. with respect to the dimension.

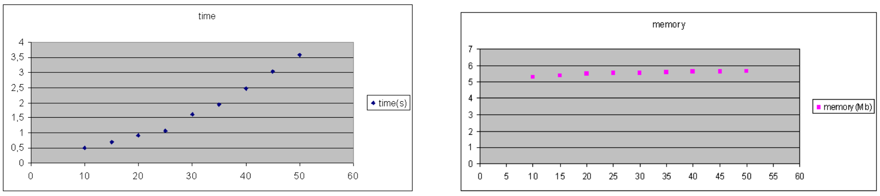


Figure 3.2: Graphs for C.T. and M.U. with respect to the dimension.

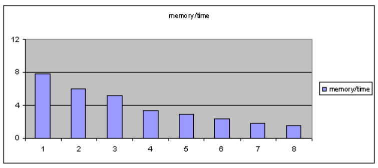


Figure 3.3: Graph for quotients C.T./M.U. with respect to the dimension.

Chapter 4

Algorithm and applications

In this chapter, we show an algorithmic method to compute α and β invariants, as well as the set of all abelian subalgebras and ideals of an arbitrary Lie algebra over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. After that, we develop two different applications. The first one corresponds to the computation of α and β invariants for Lie algebras of low dimension. To do so, we use the main and non-main vectors and check all the computations with the algorithm. Finally, we study and determine the minimal faithful unitriangular matrix representation of non-model filiform Lie algebras and filiform Lie algebras up to dimension 8 by applying both invariants and the algorithm to obtain abelian ideals and subalgebras. These representations are computed over the family of Lie algebras $\mathfrak{g}_n$ formed by strictly upper triangular matrices, using their abelian subalgebras to make the computations and obtain their representations. The content of this chapter can be seen in the papers [22, 24, 25, 27].

4.1 Algorithmic method

Let us consider an n-dimensional Lie algebra $\mathfrak{g}$ with basis $\mathcal{B}_n = \{Z_i\}_{i=1}^n$. If n is low, we can easily compute its abelian subalgebras and ideals because the number of non-zero brackets with respect to $\mathcal{B}_n$ is quite greater in proportion with the dimension of $\mathfrak{g}$. To solve this computational problem, we have implemented an algorithmic method which computes a basis of each non-trivial abelian subalgebra of $\mathfrak{g}$. In this algorithm, we will use the main and non-main vectors to express any given basis of the subalgebra in order to determine the existence of non-zero brackets. The vectors in this basis will be expressed as a linear combination of the vectors in $\mathcal{B}_n$.

To implement the algorithm, we have used the symbolic computation package MAPLE 12. We start loading the libraries `linalg` and `ListTools` to activate commands like `Flatten` and others related to Linear Algebra, since Lie algebras are vector spaces endowed with a second inner structure: the Lie bracket. Besides, the library `combinat` has to be also loaded to apply commands related to Combinatorial Algebra. Finally, we have also loaded the library `Maplets[Elements]` in order to display a message so that the user introduces the required input in the first subroutine, which is devoted to define the law of the Lie algebra considered.

Now, we show the different steps constituting the algorithm and its corresponding implementation. The structure of the algorithm is based on two main routines calling several other subroutines with different functions. Let us note that all the routines are written in the same worksheet in order to run it after introducing the data asked for the dialog window built with the library `Maplets[Elements]`.

1. Implementing a subroutine which computes the Lie bracket between two arbitrary basis vectors in $\mathcal{B}_n$. This subroutine depends on the law of $\mathfrak{g}$.

   The first subroutine, named `law`, receives two natural numbers as inputs. These numbers represent the subindexes of two basis vectors in $\mathcal{B}_n$. The subroutine returns the result of the bracket between these two vectors. Besides, conditional sentences are included to determine non-zero brackets (which are introduced in the subroutine) and the skew-symmetry property. Since the user has to complete the implementation of this subroutine with the non-zero brackets of $\mathfrak{g}$, we have also added a sentence at the beginning of the implementation, reminding this fact. Note that before running any other sentence, we must restart all the variables and delete all the computations saved for another law used before. Additionally, we must update the value of variable `dim`, which saves the dimension of the algebra to be studied.

```
> restart:
> maplet:=Maplet(AlertDialog("Don't forget to introduce non-zero brackets
 of the algebra and its dimension in subroutine law",
 'onapprove'=Shutdown("Continue"),'oncancel'=Shutdown("Aborted"))):
> Maplets[Display](maplet):
> assign(dim,...):
> law:=proc(i,j)
>   if i=j then return 0; end if;
>   if i>j then return -law(j,i); end if;
>   if (i,j)=... then return ...; end if;
>   if ....
>   else return 0; end if;
> end proc;
```

The ellipsis in command `assign` corresponds to write the dimension of the algebra $\mathfrak{g}$ to be studied. The following two suspension points are associated with the computation of $[Z_i, Z_j]$: First, the value of the subindexes (i, j) and second, the result of $[Z_i, Z_j]$ with respect to $\mathcal{B}_n$. The last ellipsis denotes the rest of non-zero brackets. For each non-zero bracket, a new sentence `if` has to be included in the cluster.

2. Programming a subroutine to compute the bracket between two vectors expressed as a linear combination of vectors from the basis $\mathcal{B}_n$ of $\mathfrak{g}$.

   Then, we implement a subroutine, `bracket`, which computes the bracket between two arbitrary vectors of $\mathfrak{g}$. These vectors are expressed as linear combination of the vectors in $\mathcal{B}_n$. The subroutine `law` is called in the implementation.

```
> bracket:=proc(u,v,n)
>   local exp; exp:=0;
>   for i from 1 to n do
>      for j from 1 to n do
>         exp:=exp + coeff(u,Z[i])*coeff(v,Z[j])*law(i,j);
>      end do;
>   end do;
>   return exp;
> end proc:
```

3. For each k-dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, computing the bracket between two arbitrary vectors in the basis of $\mathfrak{h}$. Those vectors are linear combinations of a main vector (whose coefficient is equal to 1) and the $n-k$ non-main ones. These expressions depend on the dimension of $\mathfrak{h}$.

   After introducing the law of $\mathfrak{g}$, we have to compute the brackets in an arbitrary subalgebra $\mathfrak{h}$. To do so, we implement the subroutine `eq`, which requires four inputs: the dimension `n` of $\mathfrak{g}$; the subindexes `i` and `l`, indicating the main vectors in the bracket to be computed; and a list `M` with the subindexes of the non-main vectors in $\mathfrak{h}$. To do so, three local variables `eqt`, `L` and `P` are defined. For computing the brackets between the vectors in $\mathcal{B}_n$, the subroutine `eq` calls the subroutine `bracket`, which is necessary to obtain each bracket in the law of $\mathfrak{h}$. Whereas the variable `eqt` saves the expression of the bracket belonging to the law of $\mathfrak{h}$, the list `P` takes the elements of `M` two by two and finally, `L` is a list containing all the coefficients in the expression of `eqt` with respect to $\mathcal{B}_n$. Precisely, the list `L` is the first term of the output of the subroutine `eq`. The second is a list with the subindexes `i` and `l` corresponding to `L`. Let us note that the subindexes of the main vectors have to be saved together with the coefficients in order to use them in a later subroutine.

Each vector in the subalgebra $\mathfrak{h}$ can be expressed as a linear combination of one main vector and the $n-k$ non-main ones according to expression (1.4), where each row represents the coefficients of one vector in the basis of $\mathfrak{h}$. Obviously, we can assume that the coefficient of each main vector is equal to 1, because the row of (1.4) corresponding to that main vector can be divided by its coefficient. To implement the subroutine `eq`, the coefficients of the non-main vectors are denoted by `b[i,k]`.

```
> eq:=proc(n,i,l,M::list)
>    local eqt,L,P; L:=[];
>    if nops(M)=1 then P:=[[M[1],M[1]]] else P:=choose (M,2); end if;
>    eqt:=law(i,l);
>    for k from 1 to nops(M) do
>      eqt:=eqt + b[l,M[k]]*law(i,M[k]) + b[i,M[k]]*law(M[k],l);
>    end do;
>    for j from 1 to nops(P) do
>      eqt:=eqt+(b[i,P[j][1]]*b[l,P[j][2]]-b[i,P[j][2]]*b[l,P[j][1]])*
           law(P[j][1],P[j][2]);
>    end do;
>    for m from 1 to n do
>      L:=[op(L),coeff(eqt,Z[m])];
>    end do;
>    return L,[i,l];
> end proc;
```

Let us note that it is also possible to program the subroutine `eq` by using the subroutine `bracket`. However, we will consider the previous implementation for the computational study due to the fact that if we consider an implementation of `eq` which calls the subroutine `bracket`, both the computing time and the used memory will increase.

```
> eq:=proc(n,i,l,M::list)
>   local eqt,L,u,v;
>   L:=[]; eqt:=0; u:=Z[i]; v:=Z[l];
>   for k from 1 to nops(M) do
>     u:=u+b[i,M[k]]*Z[M[k]]; v:=v+b[l,M[k]]*Z[M[k]];
>   end do;
>   eqt:=bracket(u,v,n);
>   for m from 1 to n do
>     L:=[op(L),coeff(eqt,Z[m])];
```

```
>   end do;
>   return L,[i,l];
> end proc:
```

4. Solving a system whose equations are obtained by imposing the abelian law to the brackects computed in the previous step for the subalgebra $\mathfrak{h}$.

   Next, we implement the subroutine `sys`, which receives two inputs: The dimension `n` of $\mathfrak{g}$ and a list `M` with the subindexes of the non-main vectors in the basis of $\mathfrak{h}$. This subroutine solves the system of equations generated by the subroutine `eq`. Four local variables `L`, `P`, `R` and `S` have been defined for its implementation: `L` is a list with the subindexes of the main vectors; the list `R` contains the expressions computed by the subroutine `eq`; `P` is defined as in the previous subroutine; and, finally, `S` is a set where the equations of the system are saved

```
> sys:=proc(n,M::list)
>   local L,P,R,S; L:=[]; R:=[]; S:={};
>   for x from 1 to n do
>     if member(x,convert(M,set))=false then L:=[op(L),x]; end if;
>   end do;
>   if nops(L)=1 then P:=[[L[1],L[1]]] else P:=choose (L,2); end if;
>   for j from 1 to nops(P) do
>     r[j]:=[eq(n,P[j][1],P[j][2],M)];
>   end do;
>   R:=[seq(r[i][1],i=1..nops(P))];
>   for y from 1 to nops(R) do
>     for k from 1 to n do
>       S:={op(S),R[y][k]=0};
>     end do;
>   end do;
>   return {solve(S)};
> end proc;
```

5. Programming a subroutine which determines the existence of abelian subalgebras in a fixed dimension.

   This subroutine, called `absub`, is implemented by introducing two natural numbers `n` and `k`, namely: `n` is the dimension of $\mathfrak{g}$ and `k` is less than `n`. This subroutine determines the existence of abelian subalgebras with dimension `k`. Two local variables are used by the subroutine: `L` and `S`. The first variable, `L`, is a list whose elements are lists with the subindexes of the `n`−`k` non-main vectors. The variable S is a set with the solutions given by the subroutine sys. In this way, absub returns a message indicating the non-existence of k-dimensional abelian subalgebras or, if there exist k-dimensional abelian subalgebras, returns the set S. Since the coefficient of each main vector is 1, the system given by the subroutine sys has not solutions when S vanishes. When the system has some solution, the family of computed vectors is linearly independent and forms a basis of the subalgebra. Let us note that, the solutions in S will be determined according to the the field $\mathbb{K}$ that we are considering. For example, if the solutions in S contain complex coefficients, there will be no real solutions for the system solved by sys and there do not exist any abelian subalgebras of dimension k for the case $\mathbb{K} = \mathbb{R}$. For this field, it would be necessary to include a conditional sentence for determining if such complex coefficients appear.

```
> absub:=proc(n,k)
>   local L,S; L:=choose(n,n-k); S:={ };
>   for i from 1 to nops(L) do
>     if sys(n,L[i])={{}} then S:=S else
>       for j from 1 to nops(sys(n,L[i])) do
>         S:={op(S),{convert(L[i],set),sys(n,L[i])[j]}};
>       end do;
>     end if;
>   end do;
>   if S={} then return "There is no abelian subalgebra"; end if;
>   if S={{}} then return "There is no abelian subalgebra" else return S;
>   end if;
> end proc;
```

6. Computing $\alpha(\mathfrak{g})$ by ruling out dimensions for abelian subalgebras.

   Next, we implement the subroutine alpha, which receives the dimension n of $\mathfrak{g}$ as its unique input and returns the α invariant of $\mathfrak{g}$. The subroutine starts studying if $\alpha(\mathfrak{g}) = n$ by using the subroutine absub. Then, a loop is programmed to stop when absub does not find abelian subalgebras.

```
> alpha:=proc(n)
>  if type(absub(n,n-1),set)=true then return n-1; end if;
>   for i from 2 to n-1 do
>     if absub(n,i)="There is no abelian subalgebra" then return i-1;
>    end if;
>   end do;
> end proc;
```

7. Computing the basis of an abelian subalgebra of maximal dimension, that is, a subalgebra with dimension $\alpha(\mathfrak{g})$.

   The following subroutine, named `asmd`, receives as input the dimension `n` of $\mathfrak{g}$ and returns the basis of an abelian subalgebra of maximal dimension. To do so, this subroutine calls the subroutines `alpha` and `absub`.

```
> asmd:=proc(n)
>   local u,L,R,S,B,k;
>   k:=alpha(n);S:={};L:={}; u:=absub(n,k);
>    if k=1 then return {seq({Z[i]},i=1..n)}; end if;
>    if type(u[1][1],set(integer))=true
>     then R:=u[1][1]; S:=u[1][2] else
>      R:=u[1][2]; S:=u[1][1];
>    end if;
>   for x from 1 to n do
>    if member(x,R)=false then L:={op(L),x};
>    end if;
>   end do;
>  for i from 1 to nops(L) do
>   b[i]:=Z[L[i]]; end do;
>    for i from 1 to nops(L) do
>     for j from 1 to nops(R) do
>      b[i]:=b[i]+a[L[i],R[j]]*Z[R[j]];
>     end do;
>    end do;
>   B:={seq(b[i],i=1..nops(L))};
>   return  eval(B,S);
> end proc:
```

8. Computing the basis of an abelian subalgebra for a fixed set of non-main vectors and some restrictions given by the previous subroutines.

   Now, we implement the subroutine `basabsub`, which receives three inputs: the dimension `n` of $\mathfrak{g}$ and two sets, `S` and `T`, with the subindexes of the non-main vectors in the basis of $\mathfrak{h}$. We will use this subroutine with the solution given by `sys`. We have defined four local variables `R`, `B`, `M` and `N` for its implementation. First, we introduce a conditional sentence `if` for the sets `M` and `N` in the cluster to find out whether `S` or `T` is the set of non-main vectors. This is due to the fact that MAPLE 12 sometimes returns the solutions in different order. `R` is a set with the subindexes of the main vectors and, in the set `B`, we compute the basis for the abelian subalgebra. In this way, `B` is the output of this subroutine.

```
> basabsub:=proc(n,S::set,T::set)
>   local R,B,M,N; R:={};B:={};
>   if type(S,set(integer))=true then M:=S; N:=T else M:=T; N:=S; end if;
>   for x from 1 to n do
>     if member(x,M)=false then R:={op(R),x}; end if;
>   end do;
>   for i from 1 to nops(R) do
>     a[i]:=Z[R[i]];
>   end do;
>   for i from 1 to nops(R) do
>     for j from 1 to nops(M) do
>       a[i]:=a[i] + b[R[i],M[j]]*Z[M[j]];
>     end do;
>   end do;
>   B:={seq(a[i],i=1..nops(R))};
>   return eval(B,N);
> end proc:
```

9. Programming a subroutine which computes a list with all the abelian subalgebras of $\mathfrak{g}$ with certain dimension k.

   The following subroutine, named `listabsub`, requires two inputs: the dimension `n` of $\mathfrak{g}$ and a natural number k, less than n and which corresponds with the dimension of the abelian subalgebra. To implement it, we consider two local variables `S` and `L`. This subroutine calls the subroutine `basabsub` to compute a basis for each k-dimensional abelian subalgebra. Whereas this value is saved in the local variable `S`, `L` is a set with the bases of each abelian subalgebra of $\mathfrak{g}$ with dimension k. Precisely, the list `L` is the output of the subroutine `listabsub`.

```
> listabsub:=proc(n,k)
>   local S,L; S:=absub(n,k);L:={};
>   if k=1 then return {seq({Z[i]},i=1..n)}; end if;
>   if S="There is no abelian subalgebra" then return {}; end if;
>   for i from 1 to nops(S) do
>     L:={op(L),basabsub(n,S[i][1],S[i][2])};
>   end do;
>   return L;
> end proc:
```

Let us note that it is also possible to give an equivalent implementation for the subroutine `asmd` by using the subroutine `listabsub`.

```
> asmd:=proc(n)
>  local k;
>   k:=alpha(n);
>  return listabsub(n,k);
> end proc:
```

10. Implementing the routine to compute a list with the bases of all the non-trivial abelian subalgebras of $\mathfrak{g}$ by using the previous subroutines.

    Now, we implement the routine `allabsub`, which receives the dimension `n` of $\mathfrak{g}$ as its unique input. The routine `allabsub` returns a set with the bases of all the abelian subalgebras of $\mathfrak{g}$ with dimension less than or equal to $\alpha(\mathfrak{g})$. In this way, the routine starts computing $\alpha(\mathfrak{g})$ and then, the output is defined by using the previous subroutine `listabsub`.

```
> allabsub:=proc(n)
>   local B,k; k:=alpha(n);B:={};
>   for i from 1 to k-1 do
>     B:={op(B),listabsub(n,i)};
>   end do;
>   return B;
> end proc:
```

11. Programming a subroutine which determines if there is an abelian ideal associated with a given abelian subalgebra.

    Next, we explain the subroutine `abideal`, which requires two inputs: a set `S` with the basis of an abelian subalgebra and the dimension `n` of $\mathfrak{g}$. The subroutine determines the existence of an abelian ideal from the basis `S` of an abelian subalgebra, obtained with the subroutine `listabsub` for a fixed dimension. To do so, we impose that `S` has to be the basis of an abelian ideal. Then, we solve the system: if there is no solution, the output of this subroutine is the message “It is not an abelian ideal" and if there is a solution, it returns the basis of an abelian ideal.

```
> abideal:=proc(S,n)
>   local w, R, L, Q, M; w:=0; R:=[]; L:=[]; Q:={}; M:={}; N:={};
>   for i from 1 to nops(S) do
>     w:=w + a[i]*S[i];
>   end do;
>   for i from 1 to nops(S) do
>     for j from 1 to n do
```

```
>       if bracket(S[i],Z[j],n)<>0 then
>         L:=[op(L),bracket(Z[j],S[i],n)]; else L:=L; end if;
>     end do;
>   end do;
>   for i from 1 to nops(L) do r[i]:=0;
>     for j from 1 to nops(S) do
>       r[i]:=r[i]+c[i,j]*S[j];
>     end do;
>   end do;
>   R:=[seq(r[i],i=1..nops(L))];
>   M:={seq(L[k]-R[k], k=1..nops(L))};
>   for i from 1 to nops(M) do
>     Q:={op(Q),seq(coeff(M[i],Z[j])=0,j=1..n)};
>   end do;
>   if {solve(Q)}={} then return "It is not an abelian ideal" else
>     return eval(S,solve(Q));
>   end if;
> end proc:
```

12. Computing $\beta(\mathfrak{g})$ from $\alpha(\mathfrak{g})$ and the previous subroutine.

    The subroutine `beta` receives the dimension `n` of $\mathfrak{g}$ as its unique input and returns the β invariant of $\mathfrak{g}$. Let us note that this value can be zero (semisimple Lie algebras). The subroutine starts computing the value of α. Then, a loop is programmed by using the previous subroutine and `listabsub`

```
> beta:=proc(n)
>   local r; r:=alpha(n);
>   for k from 0 to r-1 do
>     for i from 1 to nops(listabsub(n,r-k)) do
>       if abideal(listabsub(n,r-k)[i],n)<>"There is no abelian ideal"
>         then return r-k;
>       end if;
>     end do;
>   end do;
>   return 0;
> end proc:
```

13. Implementing a subroutine which determines the set of abelian ideals of maximal dimension, that is, abelian ideals with dimension $\beta(\mathfrak{g})$.

    Next, in this subroutine, named `aimd`, we compute the basis of an abelian ideal of maximal dimension; that is, an abelian ideal with dimension $\beta(\mathfrak{g})$. To do so, the routine `aimd` calls the subroutines `beta`, `listabsub` and `abideal`. First, we compute the set of all abelian subalgebras of dimension $\beta(\mathfrak{g})$ and then we apply the subroutine `abideal` to obtain abelian ideals.

```
> aimd:=proc(n)
>   local k,S,T; k:=beta(n);S:=listabsub(n,k);T:={};
>   for i from 1 to nops(S) do
>     T:={op(T),abideal(S[i],n)};
>   end do;
>   return T;
> end proc:
```

14. Programming the routine to compute a list with the bases of all the non-trivial abelian ideals of $\mathfrak{g}$ by using the previous subroutines.

    The routine `allabideal` receives the dimension `n` of $\mathfrak{g}$ as its unique input. This routine returns a set with the basis of all the abelian ideals of $\mathfrak{g}$ with dimension less than or equal to $\beta(\mathfrak{g})$. The output of this routine is defined by using the subroutines `listabsub` and `abideal`.

```
> allabideal:=proc(n)
>   local B, k; k:=beta(n); B:={};
>   if k=0 then return {}; else
>     for i from 1 to k do
>       for j from 1 to nops(listabsub(n,i)) do
>         if abideal(listabsub(n,i)[j],n)<>"There is no abelian ideal"
>           then B:={op(B),abideal(listabsub(n,i)[j],n)};
>         end if;
>       end do;
>     end do;
>   end if;
>   return B;
> end proc:
```

Now, we show an example using the 4-dimensional Lie algebra with non-zero brackets $[Z_1, Z_2] = Z_3$, $[Z_1, Z_3] = Z_4$. First, we have to complete the implementation of the subroutine `law` as follows

```
> restart:
> maplet:=Maplet(AlertDialog("Don't forget to introduce non-zero brackets
 of the algebra and its dimension in subroutine law",
 'onapprove'=Shutdown("Continue"), 'oncancel'=Shutdown("Aborted"))):
> Maplets[Display](maplet):
> assign(dim,4):
```

```
> law:=proc(i,j)
>   if i=j then return 0;end if;
>   if i>j then return -law(j,i);end if;
>   if (i,j)=(1,2) then return Z[3];end if;
>   if (i,j)=(1,3) then return Z[4]
>     else return 0;
>   end if;
> end proc:
```

After that, we must run all the routines. Once done this, we can compute the α and β invariants as well as the set of abelian subalgebras and ideals of $\mathfrak{g}$.

```
> alpha(dim);
                                       3
> listabsub(dim,alpha(dim));
          {{Z[2],Z[3],Z[4]}}
> allabsub(dim);
      {{{Z[2],Z[3],Z[4]}},{{Z[1]},{Z[2]},{Z[3]},{Z[4]}},
 {Z[4],Z[1]+b[1,2]*Z[2]+b[1,3]*Z[3]},{Z[4],Z[2]+b[2,1]*Z[1]+b[2,3]*Z[3]},
 {Z[4],Z[3]+b[3,1]*Z[1]+b[3,2]*Z[2]},{Z[2]+b[2,3]*Z[3],Z[4]+b[4,3]*Z[3]},
 {Z[2]+b[2,4]*Z[4],Z[3]+b[3,4]*Z[4]},{Z[3]+b[3,2]*Z[2],Z[4]+b[4,2]*Z[2]}}
> beta(dim);
                                       3
> allabideal(dim);
          {{Z[4]},{Z[3],Z[4]},{Z[2],Z[3],Z[4]}}
```

Now, we develop a computational study of the previous algorithm, which has been implemented with MAPLE 12, in an Intel Core 2 Duo T 5600 with a 1.83 GHz processor and 2.00 GB of RAM. Tables 4.1 and 4.2 show some computational data about both the computing time and the memory used to return the outputs according to the value of the dimension n of the algebra.

This computational study was done considering a particular family of Lie algebras: Lie algebra $\mathfrak{s}_n$ with basis $\{e_i\}_{i=1}^n$ and law $[e_i, e_n] = e_i$, for $i < n$. This family has been chosen because these algebras constitute a special subclass of non-nilpotent solvable Lie algebras, which allows us to check empirically the computational data given for both the computing time and the memory usage.

In Table 4.1, the set of all non-trivial abelian subalgebras has been computed for the algebras in this family up to dimension $n = 13$ inclusive. Starting from $n = 8$, the computing time is about three times greater when the dimension n is increased in one unit. In Table 4.2, the set of all non-trivial abelian ideals has been computed for the same family of Lie algebras up to dimension $n = 10$ inclusive.

Table 4.1: Computing time and memory usage for `allabsub`.

Input	**Computing time**	**Memory usage**
$n = 2$	0 s	0 MB
$n = 3$	0 s	0 MB
$n = 4$	0.11 s	3.13 MB
$n = 5$	0.15 s	5.06 MB
$n = 6$	0.43 s	5.38 MB
$n = 7$	1.05 s	5.56 MB
$n = 8$	2.67 s	6.06 MB
$n = 9$	6.98 s	7.06 MB
$n = 10$	20.27 s	8.25 MB
$n = 11$	61.17 s	11.50 MB
$n = 12$	187.89 s	13.87 MB
$n = 13$	804.73 s	51.93 MB

Table 4.2: Computing time and memory usage for `allabideal`.

Input	**Computing time**	**Memory usage**
$n = 2$	0 s	0 MB
$n = 3$	0.08 s	3.31 MB
$n = 4$	0.50 s	5.75 MB
$n = 5$	1.98 s	5.88 MB
$n = 6$	8.03 s	6.50 MB
$n = 7$	35.97 s	6.94 MB
$n = 8$	169.54 s	7.56 MB
$n = 9$	779.37 s	8.19 MB
$n = 10$	4118.78 s	9.31 MB

Next we show a brief statistics about the relation between the computing time and the memory usage of the main routines `allabsub` and `allabideal` for the Lie algebras $\mathfrak{s}_n$. These statistics are summarized in Figures 4.1–4.3.

Figure 4.1 and 4.2 show the behavior of the computing time (C.T.) and memory usage (M.U.) for both routines according to the dimension n of $\mathfrak{s}_n$. We can observe that the computing time increases more quickly than the memory usage in both cases. Moreover, whereas the increase of the computing time corresponds to a positive exponential model, the memory usage does not follow such a model.

We have also studied the quotients between memory usage and computing time. The resulting data can be observed in the frequency diagram of Figure 4.3. In this case, the behavior can be also considered exponential, although this time is negative.

Figure 4.1: Graphs for the C.T. and with respect to the dimension.

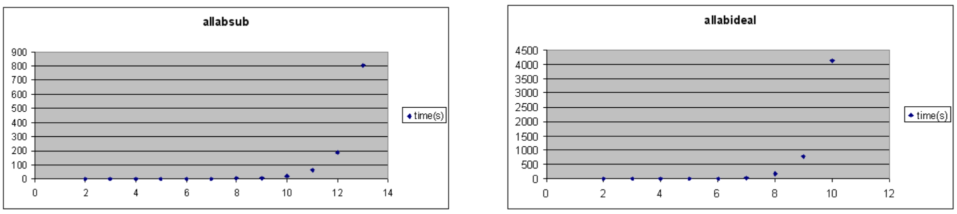


Figure 4.2: Graphs for the M.U. with respect to the dimension.

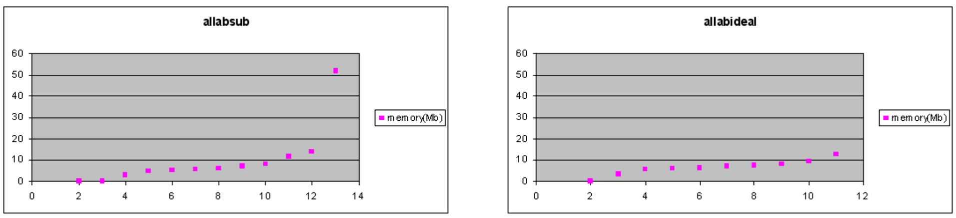
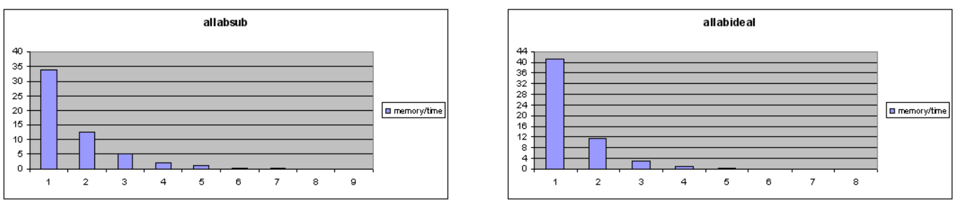


Figure 4.3: Graphs for quotients M.U./C.T. with respect to the dimension.



Next, we compute the complexity of the algorithm. To do so, we consider the number of operations carried out in the worst case. We use the big O notation to express the complexity. To recall the big O notation, the reader can consult [108]: given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) = O(g(x))$ if and only if there exist $M \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ such that $|f(x)| < M \cdot g(x)$, for all $x > x_0$.

We denote by $N_i(n)$ the order of the operations when considering the step i. This function depends on the dimension n of the Lie algebra. Table 4.3 shows the number of computations and the complexity of each step, as well as indicating the name of the routine corresponding to each step. In fact, we determine that the complexity of the algorithm has a polynomial order, where the two last routines are the most computationally expensive.

Table 4.3: Complexity and number of operations.

Step	Routine	Complexity	Operations
1	`law`	$O(n^2)$	$N_1(n) = O\left(\frac{n(n-1)}{2}\right)$
2	`bracket`	$O(n^4)$	$N_2(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} N_1(n)$
3	`eq`	$O(n^4)$	$N_3(n) = \sum_{j=1}^{\frac{n(n-1)}{2}} N_1(n)$
4	`sys`	$O(n^6)$	$N_4(n) = \sum_{i=1}^{\frac{n(n-1)}{2}} N_3(n)$
5	`absub`	$O(n^{10})$	$N_5(n) = \sum_{i=1}^{\frac{n(n-1)}{2}} \sum_{j=1}^{n^2} (N_4(n))$
6	`alpha`	$O(n^{11})$	$N_6(n) = \sum_{i=1}^{n} N_5(n)$
7	`basabsub`	$O(n^2)$	$N_7(n) = O(n^2) + \sum_{i=1}^{n} O(n) + \sum_{i=1}^{n}\sum_{j=1}^{n} O(1)$
8	`listabsub`	$O(n^{10})$	$N_8(n) = N_5(n) + \sum_{i=1}^{n} N_7(n)$
9	`allabsub`	$O(n^{11})$	$N_9(n) = N_6(n) + \sum_{i=1}^{n} N_8(n)$
10	`abideal`	$O(n^6)$	$N_{10}(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} N_2(n)$
11	`beta`	$O(n^{12})$	$N_{11}(n) = N_6(n) + \sum_{k=0}^{n-2}\sum_{i=1}^{n} (N_8(n) + N_{10}(n))$
12	`allabideal`	$O(n^{12})$	$N_{12}(n) = N_{11}(n) + \sum_{i=1}^{n}\sum_{j=1}^{n} (N_8(n) + N_{10}(n))$

4.2 Computation of α and β invariants for low-dimensional Lie algebras

In this section, α and β invariants are computed for Lie algebras of dimension less than 5, solvable Lie algebras of dimension less than 7 and nilpotent Lie algebras of dimension less than 8. To do so, we consider

- the classification of complex Lie algebras of dimension less than 5 given in [19].
- de Graaf's classification of non-decomposable complex solvable Lie algebras of dimension less than 5 [32].
- Mubarakzyanov's classification of 5-dimensional non-decomposable real solvable Lie algebras [75].
- Turkowski and Mubarakzyanov's classifications of 6-dimensional non-decomposable real solvable non-nilpotent Lie algebras [104, 76].
- de Graaf and Seeley's classifications of 6-dimensional non-decomposable complex nilpotent Lie algebras [33, 92].
- Magnin's classification of non-decomposable complex nilpotent Lie algebras up to dimension 7 [71].

According to Proposition 2.2, α and β invariants will be the same for all the cases with the exception of Mubarakzyanov and Turkowski's classifications of 5 and 6-dimensional non-decomposable real solvable Lie algebras. In fact, we will deal with the cases where α and β are different in Remarks 4.1 and 4.2.

4.2.1 Lie algebras of dimension less than 5

Now, we compute the α invariant for Lie algebras up to dimension 4.

Proposition 4.1. *Let $\mathfrak{g}$ be a Lie algebra of dimension less than 5. Then, the possible values of $\alpha(\mathfrak{g})$ are given in Table 4.4.*

Proof. Since α invariant of an abelian Lie algebra is exactly its dimension, then $\alpha(\mathbb{C}^n) = n$, for $n \in \mathbb{N}$. Additionally, in virtue of Lemma 2.1, α invariant is additive and, hence, its value for Lie algebras $\mathfrak{g}_{3,3}$, $\mathfrak{g}_{4,2}$, $\mathfrak{g}_{4,3}$, $\mathfrak{g}_{4,4}$, $\mathfrak{g}_{4,5}$, $\mathfrak{g}_{4,6}$ and $\mathfrak{g}_{4,7}$ can be computed directly. Moreover, α invariant of $\mathfrak{g}_{3,6} \equiv \mathfrak{sl}_2$ was already computed by

Malcev (see Table 1). Consequently, we only have to compute the value of α for the Lie algebras $\mathfrak{g}_{2,2}$, $\mathfrak{g}_{3,i}$, for $i = 2, 4, 5$ and $\mathfrak{g}_{4,j}$, for $j = 8, 9, \ldots, 16$.

The Lie algebra $\mathfrak{g}_{2,2}$ is generated by the vectors $\{e_1, e_2\}$ and there is a unique non-zero bracket: $[e_1, e_2] = e_1$. Hence, $\mathfrak{g}_{2,2}$ is non-abelian and $\alpha(\mathfrak{g}_{2,2}) < 2$. Since 1-dimensional Lie algebras are abelian, both $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are abelian subalgebras of $\mathfrak{g}_{2,2}$ and $\alpha(\mathfrak{g}_{2,2}) = 1$.

Fixed and given $i \in \{2, 4, 5\}$, let us prove that $\alpha(\mathfrak{g}_{3,i}) = 2$. We have to find a 2-dimensional abelian subalgebra in $\mathfrak{g}_{3,i}$, in addition to determining the non-existence of 3-dimensional abelian subalgebras. First, $\alpha(\mathfrak{g}_{3,i}) \leq 2$ because the Lie algebra $\mathfrak{g}_{3,i}$ is non-abelian. Obviously, the subalgebra $\langle e_2, e_3 \rangle$ of $\mathfrak{g}_{3,i}$ is abelian for $i \in \{2, 4, 5\}$. So, $\alpha(\mathfrak{g}_{3,i}) = 2$, for $i \in \{2, 4, 5\}$.

Now, we prove that $\alpha(\mathfrak{g}_{4,j}) = 3$, for $j = 8, \ldots, 13$. To do so, it is sufficient to bear in mind that these algebras are non-abelian and that $\langle e_2, e_3, e_4 \rangle$ is an abelian subalgebra.

Finally, we prove that $\alpha(\mathfrak{g}_{4,k}) = 2$, for $k = 14, 15, 16$. We have to find a 2-dimensional abelian subalgebra in $\mathfrak{g}_{4,k}$, as well as determining the non-existence of 3-dimensional abelian subalgebras. Since the Lie algebra $\mathfrak{g}_{4,k}$ is non-abelian, we can set that $\alpha(\mathfrak{g}_{4,k}) \leq 3$. First, $\langle e_2, e_4 \rangle$ and $\langle e_3, e_4 \rangle$ are abelian subalgebras of $\mathfrak{g}_{4,k}$. Now, we define the subalgebras

$$\mathfrak{a}_l = \langle \{e_i + \lambda_i e_l \mid 1 \leq i \leq 4 \wedge i \neq l\} \rangle,$$

where e_l is the non-main vector.

- For $l = 1, 4$, $[e_2 + \lambda_2 e_l, e_3 + \lambda_3 e_l] = e_4 + u$ is non-zero, because $u \in \langle e_2, e_3 \rangle$.
- For $l = 2, 3$, $[e_1 + \lambda_1 e_l, e_4 + \lambda_4 e_l] = \mu e_4 + v$ with $\mu \neq 0$ and $v \in \langle e_2, e_3 \rangle$. Consequently, this bracket is non-zero and $\alpha(\mathfrak{g}_{4,k}) = 2$, for $k = 14, 15, 16$. □

4.2.2 Solvable non-nilpotent Lie algebras of dimension 5 and 6

Now, α invariant is computed for non-decomposable real solvable non-nilpotent Lie algebras of dimension 5 and 6. In Tables 4.8-4.10, we denote by $\mathfrak{g}_{6,i}$, for $1 \leq i \leq 40$, the Lie algebras from Turkowski's classification [104] and for $41 \leq i \leq 143$ we have the remaining algebras from Mubarakzyanov's classification [76].

Proposition 4.2. *Let $\mathfrak{g}$ be a 5-dimensional non-decomposable real solvable non-nilpotent Lie algebra. Then, the possible values of $\alpha(\mathfrak{g})$ are given in Tables 4.5–4.6.*

Proof. The 4-dimensional subalgebra of $\mathfrak{g}_{5,j}$ generated by $\langle e_1, e_2, e_3, e_4\rangle$ is abelian for $j \in \{7, \ldots, 18\}$. Since $\mathfrak{g}_{5,j}$ is not abelian, $\alpha(\mathfrak{g}_{5,j}) = 4$ and the previous subalgebra is an abelian subalgebra of maximal dimension. Moreover, $\mathcal{D}(\mathfrak{g}_{5,j})$ is a 4-dimensional abelian ideal for $j \in \{7, \ldots, 18\}$. Consequently, these algebras are 2-step solvable.

The subalgebras $\langle e_1, e_3, e_4\rangle$ and $\langle e_1, e_2, e_3\rangle$ are abelian subalgebras of $\mathfrak{g}_{5,j}$ for $j \in \{19, \ldots, 29\}$ and for $j \in \{30, \ldots, 35, 38, 39\}$, respectively. So, we can set that $\alpha(\mathfrak{g}_{5,j}) \geq 3$, for $j \in \{19, \ldots 35, 38, 39\}$.

Consequently, it is sufficient to prove the non-existence of 4-dimensional abelian subalgebras of $\mathfrak{g}_{5,j}$. Once more, the reasoning is analogous for all these algebras. So we only study explicitly the algebra $\mathfrak{g}_{5,22}$, whose law, with respect to a certain basis $\{e_i\}_{i=1}^5$, is $[e_2, e_3] = e_1, [e_2, e_5] = e_3, [e_4, e_5] = e_4$.

By applying the reasoning and notation used in Proposition 4.1, assume the existence of a 4-dimensional abelian subalgebra and find a non-zero bracket in its law.

- For $k \in \{1, 4, 5\}$: $[e_2 + \lambda_2 e_k, e_3 + \lambda_3 e_k] = e_1 + v$ is non-zero, because $v \in \langle \{e_i\}_{i=2}^5\rangle$.
- For $k \in \{2, 3\}$:$[e_4 + \lambda_4 e_k, e_5 + \lambda_5 e_k] = e_4 + w$ is non-zero, because $w \in \langle e_1, e_2, e_3, e_5\rangle$.

So, there do not exist 4-dimensional abelian subalgebras of $\mathfrak{g}_{5,22}$ and $\alpha(\mathfrak{g}_{5,22}) = 3$.

The subalgebra $\langle e_1, e_2\rangle$ is abelian for $\mathfrak{g}_{5,36}$ and $\mathfrak{g}_{5,37}$. Hence, both $\alpha(\mathfrak{g}_{5,36})$ and $\alpha(\mathfrak{g}_{5,37}) \geq 2$ and we only have to prove the non-existence of abelian subalgebras of dimension 3. Both cases are analogous, so we will prove it for $\mathfrak{g}_{5,37}$.

Let us suppose the existence of a 3-dimensional abelian subalgebra. We can express 3-dimensional subalgebras as follows

$$\mathfrak{a}_{j,k} = \langle \{e_i + \lambda_i e_j + \mu_i e_k \mid 1 \leq i \leq 5 \wedge j, k \neq i\}\rangle,$$

where e_j and e_k are the two non-main vectors. Proving that $\mathfrak{a}_{j,k}$ is non-abelian is equivalent to finding a non-zero bracket in its law.

- For $(j, k) \in \{(1, 2), (1, 4)\}$: The bracket $[e_3 + \lambda_3 e_j + \mu_3 e_k, e_5 + \lambda_5 e_j + \mu_5 e_k] = e_2 + v$ is non-zero, because $v \in \langle e_1, e_3, e_4, e_5\rangle$.
- For $(j, k) \in \{(1, 3), (1, 5)\}$: The bracket $[e_2 + \lambda_2 e_j + \mu_2 e_k, e_4 + \lambda_4 e_j + \mu_4 e_k] = e_2 + v$ is non-zero, because $v \in \langle e_1, e_3, e_4, e_5\rangle$.
- For $(j, k) \in \{(2, 5), (3, 5)\}$: The bracket $[e_1 + \lambda_1 e_j + \mu_1 e_k, e_4 + \lambda_4 e_j + \mu_4 e_k] = 2e_1 + w$ is non-zero, because $w \in \langle e_i\rangle_{i=2}^5$.

- For $(j,k)=(4,5)$: The bracket $[e_2+\lambda_2 e_j+\mu_2 e_k, e_3+\lambda_3 e_j+\mu_3 e_k]=e_1+w$ is non-zero, because $w\in\langle e_i\rangle_{i=2}^5$.

- For $(j,k)=(2,3)$: If $\mathfrak{a}_{2,3}$ is abelian, the bracket

$$[e_1+\lambda_1 e_2+\mu_1 e_3, e_4+\lambda_4 e_2+\mu_4 e_3]=(2+\lambda_1\mu_4-\mu_1\lambda_4)e_1+\lambda_1 e_2+\mu_1 e_3$$

would be zero, obtaining the system $\{2+\lambda_1\mu_4-\mu_1\lambda_4=0,\ \lambda_1=\mu_1=0\}$, which has no solution.

- For $(j,k)=(2,4)$: Consider the brackets

$$[e_3+\lambda_3 e_2+\mu_3 e_4, e_5+\lambda_5 e_2+\mu_5 e_4]=-\lambda_5 e_1+(1+\lambda_3\mu_5-\mu_3\lambda_5)e_2+(\mu_5-\lambda_3)e_3,$$

$$[e_1+\lambda_1 e_2+\mu_1 e_4, e_5+\lambda_5 e_2+\mu_5 e_4]=2\mu_5 e_1-\lambda_1 e_3+(\lambda_1\mu_5-\mu_1\lambda_5)e_2.$$

If $\mathfrak{a}_{2,4}$ is abelian, these brackets would be zero and the system $\{1+\lambda_3\mu_5-\mu_3\lambda_5=0,\ \mu_5=\lambda_1=\lambda_5=0,\ \mu_5-\lambda_3=0, \lambda_1\mu_5-\mu_1\lambda_5=0\}$ without solutions would be obtained.

- For $(j,k)=(3,4)$: Consider the brackets

$$[e_2+\lambda_2 e_3+\mu_2 e_4, e_5+\lambda_5 e_3+\mu_5 e_4]=\lambda_5 e_1+(\lambda_2\mu_5-\mu_2\lambda_5-1)e_3+(\mu_5+\lambda_2)e_2,$$

$$[e_1+\lambda_1 e_3+\mu_1 e_4, e_2+\lambda_2 e_3+\mu_2 e_4]=-\mu_1 e_2+(2\mu_2-\lambda_1)e_1+(\lambda_1\mu_2-\mu_1\lambda_2)e_3,$$

$$[e_1+\lambda_1 e_3+\mu_1 e_4, e_5+\lambda_5 e_3+\mu_5 e_4]=2\mu_5 e_1+\lambda_1 e_2+(\lambda_1\mu_5-\mu_1\lambda_5)e_3.$$

If $\mathfrak{a}_{3,4}$ is abelian, these brackets would be zero, obtaining the following system without solutions $\{\lambda_2\mu_5-\mu_2\lambda_5-1=0, \mu_1=\mu_5=\lambda_1=\lambda_5=0,\ \mu_5+\lambda_2=0,\ 2\mu_2-\lambda_1=0, \lambda_1\mu_2-\mu_1\lambda_2=0, \lambda_1\mu_5-\mu_1\lambda_5=0\}$.

Hence, there are no 3-dimensional abelian subalgebras in $\mathfrak{g}_{5,37}$ and $\alpha(\mathfrak{g}_{5,37})=2$. □

Remark 4.1. *Regarding Lie algebras $\mathfrak{g}_{5,25}, \mathfrak{g}_{5,26}$ and $\mathfrak{g}_{5,37}$, we have an analogous situation to the one shown in Example 2.2. This is due to the fact that over $\mathbb{R}$, $\alpha(\mathfrak{g}_{5,25})=\alpha(\mathfrak{g}_{5,26})=3$, but $\beta(\mathfrak{g}_{5,25})=\beta(\mathfrak{g}_{5,26})=2$ since the abelian ideals of dimension 3 for both algebras are $\langle e_1,e_4,e_3\pm ie_2\rangle$ and $\langle e_1,e_4,e_2\pm ie_3\rangle$. For Lie algebra $\mathfrak{g}_{5,37}$ we have $\alpha(\mathfrak{g}_{5,37})=2$, but $\beta(\mathfrak{g}_{5,37})=1$ since the 2-dimensional abelian ideals of $\mathfrak{g}_{5,37}$ are $\langle e_1,e_2\pm ie_3\rangle$ and $\langle e_1,e_3\pm ie_2\rangle$.*

Proposition 4.3. *Let $\mathfrak{g}$ be a 6-dimensional non-decomposable real solvable non-nilpotent Lie algebra. Then, the possible values of $\alpha(\mathfrak{g})$ are given in Tables 4.8–4.14.*

Proof. In order to prove this result, we can use an analogous reasoning to the one used in the proof of Proposition 4.2 bearing in mind that

- $\langle e_3, e_4, e_5, e_6\rangle$ is an abelian subalgebra of $\mathfrak{g}_{6,j}$, for $j = 1, \dots, 27$.
- $\langle e_3, e_4, e_5\rangle$ and $\langle e_3, e_4, e_6\rangle$ are abelian subalgebras of $\mathfrak{g}_{6,28}$ and $\mathfrak{g}_{6,j}$, for $j = 29, \dots 40$, respectively.
- $\langle e_1, e_2, e_3, e_4, e_5\rangle$ is an abelian subalgebra of $\mathfrak{g}_{6,j}$, for $j = 41, \dots, 52$.
- $\langle e_1, e_2, e_4, e_5\rangle$ is an abelian subalgebra of $\mathfrak{g}_{6,j}$, for $j = 53, \dots, 78$.
- $\langle e_1, e_2, e_3, e_4\rangle$ is an abelian subalgebra of $\mathfrak{g}_{6,j}$, for $j = 79, \dots, 115$.
- $\langle e_1, e_2, e_3\rangle$ is an abelian subalgebra of $\mathfrak{g}_{6,j}$, for $j = 116, \dots, 139$.

□

Remark 4.2. *Table 4.7 shows the cases of* 6*-dimensional non-decomposable real solvable Lie algebras where* α *and* β *invariants are different, due to the fact that the abelian ideals of dimension* α *are defined over* $\mathbb{C}$.

4.2.3 Nilpotent Lie algebras of dimension less than 8

The invariant $\alpha(\mathfrak{g})$ for a complex nilpotent Lie algebra $\mathfrak{g}$ has been obtained up to dimension 6 in connection with degenerations [19, 92]. We want to give a list here, thereby correcting a few typos in [92]. In dimension 7 there is no list for $\alpha(\mathfrak{g})$, as far as we know. We use the classification of complex nilpotent Lie algebras up to dimension 7 by Magnin [71], and for dimension 6 also by de Graaf [33] and Seeley [92]. The result for the indecomposable algebras in dimension $n \leq 5$ is as follows

Proposition 4.4. *Let* $\mathfrak{g}$ *be a complex nilpotent Lie algebra of dimension less than* 6*. Then, the possible values of* $\alpha(\mathfrak{g})$ *are given in Table 4.15.*

Proof. The α invariant of Lie algebras $\mathfrak{n}_3$ and $\mathfrak{n}_4$ was computed and pointed out in Table 4.4. First, we prove that $\alpha(\mathfrak{g}_{5,2}) = \alpha(\mathfrak{g}_{5,5}) = 4$. Let us note that in this case, $\langle e_2, e_3, e_4, e_5\rangle$ is an abelian subalgebra of $\mathfrak{g}_{5,2}$ and $\mathfrak{g}_{5,5}$, that are non-abelian, so $\alpha(\mathfrak{g}_{5,2}) = \alpha(\mathfrak{g}_{5,5}) = 4$. Now, we compute a 3-dimensional abelian subalgebra of $\mathfrak{g}_{5,i}$, $\forall i \in \{1, 3, 4, 6\}$, and we have to prove that it is not possible to obtain an abelian subalgebra of them with dimension greater than 3. The algebra $\langle e_3, e_4, e_5\rangle$ is an abelian subalgebra of $\mathfrak{g}_{5,i}$, for $i \in \{1, 3, 4, 6\}$. Consequently, it is sufficient to prove that it is not possible to obtain a 4-dimensional abelian subalgebra of $\mathfrak{g}_{5,i}$, for

$i \in \{1,3,4,6\}$. For reasons of length, we only study explicitly one example of these algebras. The other cases are analogous.

Let us consider Lie algebra $\mathfrak{g}_{5,3}$ which law with respect to the basis $\{e_i\}_{i=1}^5$ is given by: $[e_2,e_4]=e_3, [e_2,e_5]=e_1, [e_4,e_5]=e_2$. Let us suppose that there exists a 4-dimensional abelian subalgebra in $\mathfrak{g}_{5,3}$. This subalgebra can be expressed as

$$\mathfrak{a}_k = \langle\{e_i+\lambda_i e_k | 1\le i\le 5 \wedge i\neq k\}\rangle,$$

where e_k is the non-main vector. To prove that $\mathfrak{a}_k$ is non-abelian it is necessary and sufficient to find a non-zero bracket in its law.

- For $k\in\{1,3,5\}$, the following bracket is non-zero: $[e_2+\lambda_2 e_k, e_4+\lambda_4 e_k]=e_3+v$, because the vector v belongs to $\langle e_1,e_2,e_4,e_5\rangle$.
- For $k=2$, a non-zero bracket is given by $[e_4+\lambda_4 e_2, e_5+\lambda_5 e_2] = e_2+v$, because the vector v belongs to $\langle e_1,e_3,e_4,e_5\rangle$.
- For $k=4$, the following bracket is non-zero: $[e_2+\lambda_2 e_4, e_5+\lambda_5 e_4]=e_1+v$, since v belongs to $\langle\{e_i\}_{i=2}^{i=5}\rangle$.

Consequently, there do not exist 4-dimensional abelian subalgebras in $\mathfrak{g}_{5,3}$ and $\alpha(\mathfrak{g}_{5,3})=3$.

□

Proposition 4.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension 6. Then, the possible values of $\alpha(\mathfrak{g})$ are given in Table 4.16.*

Proof. Clearly $\alpha(\mathbb{C}^6)=6$. Since α invariant is additive in virtue of Lemma 2.1, its value for Lie algebras $\mathfrak{g}_{5,i}\oplus\mathbb{C}$, for $i=1,2,\dots,6$; $\mathfrak{n}_3\oplus\mathfrak{n}_3$; $\mathfrak{n}_3\oplus\mathbb{C}^3$ and $\mathfrak{n}_4\oplus\mathbb{C}^2$ can be computed directly. Consequently, we only have to compute the value of α for Lie algebras $\mathfrak{g}_{6,i}$, for $i\in\{1,\dots,20\}$. These algebras are shown in Table 4.17.

Let us note that $\langle e_1,e_3,e_4,e_5,e_6\rangle$ and $\langle e_2,e_3,e_4,e_5,e_6\rangle$ are 5-dimensional abelian subalgebras of $\mathfrak{g}_{6,6}$ and $\mathfrak{g}_{6,16}$, respectively. Since these algebras are non-abelian, we can conclude that $\alpha(\mathfrak{g}_{6,6})=\alpha(\mathfrak{g}_{6,16})=5$.

Moreover, $\langle e_3,e_4,e_5,e_6\rangle$, $\langle e_2,e_4,e_5,e_6\rangle$ and $\langle e_1,e_4,e_5,e_6\rangle$ are abelian subalgebras of $\mathfrak{g}_{6,i}$, for $i\in\{1,3,4,5,7,8,11,12,14,15,17,19\}$; $\mathfrak{g}_{6,j}$, for $j\in\{2,10,13\}$ and $\mathfrak{g}_{6,9}$, respectively. So, α invariant of these algebras is, at least, 4. In this way, we have to see if there exists some abelian subalgebra of them with dimension 5. We study one example of the Lie algebras considered and the other are analogous.

We consider the algebra $\mathfrak{g}_{6,19}=\langle e_1,e_2,e_3,e_4,e_5,e_6\rangle$ with law $[e_1,e_2]=e_3, [e_1,e_3]=e_4, [e_1,e_4]=e_5, [e_1,e_5]=e_6, [e_2,e_3]=e_5, [e_2,e_4]=e_6$. A 5-dimensional abelian subalgebra is expressed by $\mathfrak{a}_k$ as in Proposition 4.4.

- For $k \in \{3, \ldots, 6\}$, the bracket $[e_1 + \lambda_1 e_k, e_2 + \lambda_2 e_k] = e_3 + v$, is non-zero since v belongs to $\langle \{e_i\}_{i \neq 3} \rangle$.
- For $k = 2$, a non-zero bracket is $[e_1 + \lambda_1 Z_2, e_3 + \lambda_3 Z_2] = e_4 + \lambda_3 e_3 + \lambda_1 e_5$.
- For $k = 1$, this bracket is non-zero: $[e_2 + \lambda_2 e_1, e_3 + \lambda_3 e_1] = e_5 - \lambda_3 e_3 + \lambda_2 e_4$.

There is no 5-dimensional abelian subalgebra in $\mathfrak{g}_{6,19}$, so $\alpha(\mathfrak{g}_{6,19}) = 4$. □

The Hasse diagram for degenerations of nilpotent Lie algebras in dimension 6 is shown in Figure 4.4, removing the typos in [92]. If $\mathfrak{g} \to_{\deg} \mathfrak{h}$, then $\alpha(\mathfrak{g}) \leq \alpha(\mathfrak{h})$.

In dimension 7, we use Magnin's classification of non-decomposable, complex nilpotent Lie algebras in [71] (see Table 4.18–4.21) to compute the α invariant of each isomorphism class. Note that $4 \leq \alpha(\mathfrak{g}) \leq 6$ in this case according to Lemma 2.5. The result is as follows

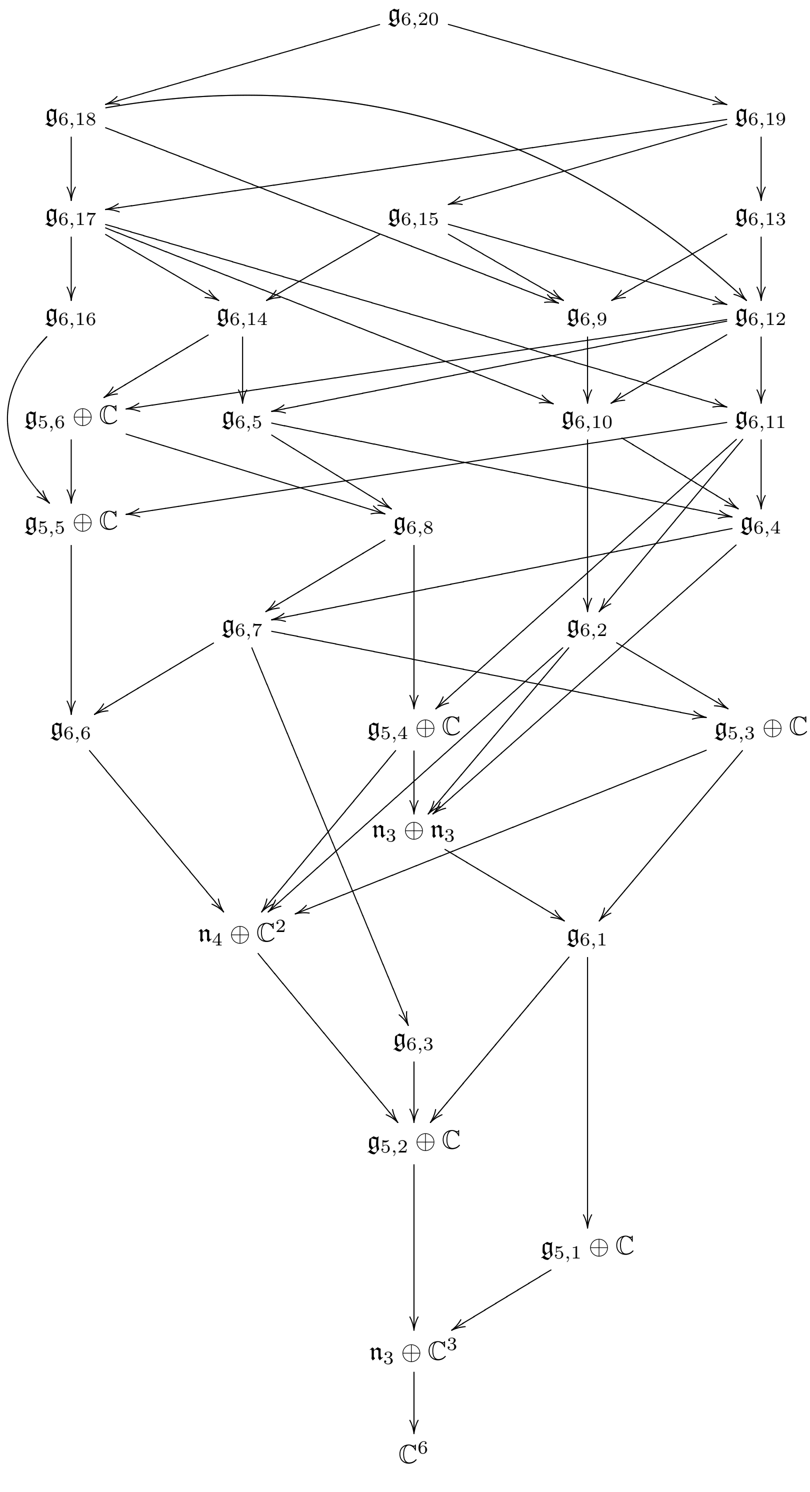
Proposition 4.6. *Let $\mathfrak{g}$ be a non-decomposable nilpotent Lie algebra of dimension 7. Then, the possible values of $\alpha(\mathfrak{g})$ are the following*

$\alpha(\mathfrak{g}) = 4 :$ $\mathfrak{g} =$ $\mathfrak{g}_{7,0.1}$, $\mathfrak{g}_{7,0.4(\mathfrak{a})}$, $\mathfrak{g}_{7,0.5}$, $\mathfrak{g}_{7,0.6}$, $\mathfrak{g}_{7,0.7}$, $\mathfrak{g}_{7,0.8}$, $\mathfrak{g}_{7,1.02}$, $\mathfrak{g}_{7,1.03}$, $\mathfrak{g}_{7,1.1(i_{\mathfrak{a}}),\mathfrak{a}\neq 1}$, $\mathfrak{g}_{7,1.1(ii)}$, $\mathfrak{g}_{7,1.1(iii)}$, $\mathfrak{g}_{7,1.1(iv)}$, $\mathfrak{g}_{7,1.1(v)}$, $\mathfrak{g}_{7,1.1(vi)}$, $\mathfrak{g}_{7,1.2(i_{\mathfrak{a}}),\mathfrak{a}\neq 1}$, $\mathfrak{g}_{7,1.2(ii)}$, $\mathfrak{g}_{7,1.2(iii)}$, $\mathfrak{g}_{7,1.2(iv)}$, $\mathfrak{g}_{7,1.3(i_{\mathfrak{a}}),\mathfrak{a}\neq 0}$, $\mathfrak{g}_{7,1.3(ii)}$, $\mathfrak{g}_{7,1.3(iii)}$, $\mathfrak{g}_{7,1.3(iv)}$, $\mathfrak{g}_{7,1.3(v)}$, $\mathfrak{g}_{7,1.5}$, $\mathfrak{g}_{7,1.8}$, $\mathfrak{g}_{7,1.11}$, $\mathfrak{g}_{7,1.14}$, $\mathfrak{g}_{7,1.17}$, $\mathfrak{g}_{7,1.19}$, $\mathfrak{g}_{7,1.20}$, $\mathfrak{g}_{7,1.21}$, $\mathfrak{g}_{7,2.1(i_{\mathfrak{a}}),\mathfrak{a}\neq 0,1}$, $\mathfrak{g}_{7,2.1(ii)}$, $\mathfrak{g}_{7,2.1(iii)}$, $\mathfrak{g}_{7,2.1(iv)}$, $\mathfrak{g}_{7,2.1(v)}$, $\mathfrak{g}_{7,2.2}$, $\mathfrak{g}_{7,2.4}$, $\mathfrak{g}_{7,2.5}$, $\mathfrak{g}_{7,2.6}$, $\mathfrak{g}_{7,2.10}$, $\mathfrak{g}_{7,2.12}$, $\mathfrak{g}_{7,2.13}$, $\mathfrak{g}_{7,2.17}$, $\mathfrak{g}_{7,2.23}$, $\mathfrak{g}_{7,2.26}$, $\mathfrak{g}_{7,2.28}$, $\mathfrak{g}_{7,2.29}$, $\mathfrak{g}_{7,2.30}$, $\mathfrak{g}_{7,2.34}$, $\mathfrak{g}_{7,2.35}$, $\mathfrak{g}_{7,2.37}$, $\mathfrak{g}_{7,3.1(i_{\mathfrak{a}}),\mathfrak{a}\neq 0,1}$, $\mathfrak{g}_{7,3.1(iii)}$, $\mathfrak{g}_{7,3.13}$, $\mathfrak{g}_{7,3.18}$, $\mathfrak{g}_{7,3.22}$, $\mathfrak{g}_{7,4.4}$.

$\alpha(\mathfrak{g}) = 5 :$ $\mathfrak{g} =$ $\mathfrak{g}_{7,0.2}$, $\mathfrak{g}_{7,0.3}$, $\mathfrak{g}_{7,1.01(i)}$, $\mathfrak{g}_{7,1.01(ii)}$, $\mathfrak{g}_{7,1.1(i_{\mathfrak{a}}),\mathfrak{a}=1}$, $\mathfrak{g}_{7,1.2(i_{\mathfrak{a}}),\mathfrak{a}=1}$, $\mathfrak{g}_{7,1.3(i_{\mathfrak{a}}),\mathfrak{a}=0}$, $\mathfrak{g}_{7,1.4}$, $\mathfrak{g}_{7,1.6}$, $\mathfrak{g}_{7,1.7}$, $\mathfrak{g}_{7,1.9}$, $\mathfrak{g}_{7,1.10}$, $\mathfrak{g}_{7,1.12}$, $\mathfrak{g}_{7,1.13}$, $\mathfrak{g}_{7,1.15}$, $\mathfrak{g}_{7,1.16}$, $\mathfrak{g}_{7,1.18}$, $\mathfrak{g}_{7,2.1(i_{\mathfrak{a}}),\mathfrak{a}=0,1}$, $\mathfrak{g}_{7,2.7}$, $\mathfrak{g}_{7,2.8}$, $\mathfrak{g}_{7,2.9}$, $\mathfrak{g}_{7,2.11}$, $\mathfrak{g}_{7,2.14}$, $\mathfrak{g}_{7,2.15}$, $\mathfrak{g}_{7,2.16}$, $\mathfrak{g}_{7,2.18}$, $\mathfrak{g}_{7,2.19}$, $\mathfrak{g}_{7,2.20}$, $\mathfrak{g}_{7,2.21}$, $\mathfrak{g}_{7,2.22}$, $\mathfrak{g}_{7,2.24}$, $\mathfrak{g}_{7,2.25}$, $\mathfrak{g}_{7,2.27}$, $\mathfrak{g}_{7,2.31}$, $\mathfrak{g}_{7,2.32}$, $\mathfrak{g}_{7,2.33}$, $\mathfrak{g}_{7,2.36}$, $\mathfrak{g}_{7,2.38}$, $\mathfrak{g}_{7,2.39}$, $\mathfrak{g}_{7,2.40}$, $\mathfrak{g}_{7,2.41}$, $\mathfrak{g}_{7,2.42}$, $\mathfrak{g}_{7,2.43}$, $\mathfrak{g}_{7,2.44}$, $\mathfrak{g}_{7,2.45}$, $\mathfrak{g}_{7,3.1(i_{\mathfrak{a}}),\mathfrak{a}=0,1}$, $\mathfrak{g}_{7,3.3}$, $\mathfrak{g}_{7,3.4}$, $\mathfrak{g}_{7,3.5}$, $\mathfrak{g}_{7,3.6}$, $\mathfrak{g}_{7,3.7}$, $\mathfrak{g}_{7,3.8}$, $\mathfrak{g}_{7,3.9}$, $\mathfrak{g}_{7,3.10}$, $\mathfrak{g}_{7,3.11}$, $\mathfrak{g}_{7,3.12}$, $\mathfrak{g}_{7,3.14}$, $\mathfrak{g}_{7,3.15}$, $\mathfrak{g}_{7,3.16}$, $\mathfrak{g}_{7,3.17}$, $\mathfrak{g}_{7,3.19}$, $\mathfrak{g}_{7,3.21}$, $\mathfrak{g}_{7,3.23}$ $\mathfrak{g}_{7,3.24}$, $\mathfrak{g}_{7,4.1}$, $\mathfrak{g}_{7,4.3}$.

$\alpha(\mathfrak{g}) = 6 :$ $\mathfrak{g} =$ $\mathfrak{g}_{7,2.3}$, $\mathfrak{g}_{7,3.2}$, $\mathfrak{g}_{7,3.20}$, $\mathfrak{g}_{7,4.2}$.

Figure 4.4: Hasse diagram for degenerations of nilpotent Lie algebras in dimension 6.



4.2.4 Tables

Table 4.4: Lie algebras of dimension less than 5

$\mathfrak{g}$	$\dim(\mathfrak{g})$	Lie brackets	$\alpha(\mathfrak{g})$
$\mathfrak{g}_{1,1} = \mathbb{C}^1$	1		1
$\mathfrak{g}_{2,1} = \mathbb{C}^2$	2		2
$\mathfrak{g}_{2,2} = \mathfrak{r}_2$	2	$[e_1, e_2] = e_1$	1
$\mathfrak{g}_{3,1} = \mathbb{C}^3$	3		3
$\mathfrak{g}_{3,2} = \mathfrak{n}_3$	3	$[e_1, e_2] = e_3$	2
$\mathfrak{g}_{3,3} = \mathfrak{r}_2 \oplus \mathbb{C}^1$	3	$[e_1, e_2] = e_1$	2
$\mathfrak{g}_{3,4} = \mathfrak{r}_3$	3	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$	2
$\mathfrak{g}_{3,5} = \mathfrak{r}_{3,\lambda}$	3	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3,\ \lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1$	2
$\mathfrak{g}_{3,6} = \mathfrak{sl}_2$	3	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$	1
$\mathfrak{g}_{4,1} = \mathbb{C}^4$	4		4
$\mathfrak{g}_{4,2} = \mathfrak{n}_3 \oplus \mathbb{C}^1$	4	$[e_1, e_2] = e_3$	3
$\mathfrak{g}_{4,3} = \mathfrak{r}_2 \oplus \mathbb{C}^2$	4	$[e_1, e_2] = e_1$	3
$\mathfrak{g}_{4,4} = \mathfrak{r}_3 \oplus \mathbb{C}^1$	4	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$	3
$\mathfrak{g}_{4,5} = \mathfrak{r}_{3,\lambda} \oplus \mathbb{C}^1$	4	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3,\ \lambda \in \mathbb{C}, 0 < \lvert\lambda\rvert \leq 1$	3
$\mathfrak{g}_{4,6} = \mathfrak{r}_2 \oplus \mathfrak{r}_2$	4	$[e_1, e_2] = e_1, [e_3, e_4] = e_3$	2
$\mathfrak{g}_{4,7} = \mathfrak{sl}_2 \oplus \mathbb{C}^1$	4	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$	2
$\mathfrak{g}_{4,8} = \mathfrak{n}_4$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$	3
$\mathfrak{g}_{4,9} = \mathfrak{r}_{4,\alpha}$	4	$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = \alpha e_4, \alpha \in \mathbb{C}^*$	3
$\mathfrak{g}_{4,10} = \mathfrak{r}_{4,\alpha,\beta}$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha e_2$ $-\beta e_3 + e_4, \alpha \in \mathbb{C}^*,\ \beta \in \mathbb{C}$ or $\alpha, \beta = 0$	3
$\mathfrak{g}_{4,11}^{\alpha}$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_4,$ $[e_1, e_4] = \alpha(e_2 + e_3), \alpha \in \mathbb{C}^*$	3
$\mathfrak{g}_{4,12}$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_2$	3
$\mathfrak{g}_{4,13}$	4	$[e_1, e_2] = \frac{1}{3}e_2 + e_3, [e_1, e_3] = \frac{1}{3}e_3, [e_1, e_4] = \frac{1}{3}e_4$	3
$\mathfrak{g}_{4,14}$	4	$[e_1, e_2] = e_2, [e_1, e_3] = e_3,$ $[e_1, e_4] = 2e_4, [e_2, e_3] = e_4$	2
$\mathfrak{g}_{4,15}$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_4$	2
$\mathfrak{g}_{4,16}^{\alpha}$	4	$[e_1, e_2] = e_3, [e_1, e_3] = -\alpha e_2 + e_3,$ $[e_1, e_4] = e_4, [e_2, e_3] = e_4, \alpha \in \mathbb{C}$	2

Table 4.5: 5-dimensional non-decomposable real solvable non-nilpotent Lie algebras

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{5,7}$	$[e_1,e_5]=e_1, [e_2,e_5]=\alpha e_2,$ $[e_3,e_5]=\beta e_3, [e_4,e_5]=\gamma e_4$	4	$-1\leq\gamma\leq\beta\leq\alpha\leq 1,$ $\alpha\beta\gamma\neq 0.$
$\mathfrak{g}_{5,8}$	$[e_2,e_5]=e_1, [e_3,e_5]=e_3, [e_4,e_5]=\gamma e_4,$	4	$0<\lvert\gamma\rvert\leq 1$
$\mathfrak{g}_{5,9}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_3,$ $[e_3,e_5]=\beta e_3, [e_4,e_5]=\gamma e_4$	4	$0\neq\gamma\leq\beta$
$\mathfrak{g}_{5,10}$	$[e_2,e_5]=e_1, [e_3,e_5]=e_2, [e_4,e_5]=e_4$	4	
$\mathfrak{g}_{5,11}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2,$ $[e_3,e_5]=e_2+e_3, [e_4,e_5]=\gamma e_4$	4	$\gamma\neq 0$
$\mathfrak{g}_{5,12}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2,$ $[e_3,e_5]=e_2+e_3, [e_4,e_5]=e_3+e_4$	4	
$\mathfrak{g}_{5,13}$	$[e_1,e_5]=e_1, [e_2,e_5]=\gamma e_2,$ $[e_3,e_5]=pe_3-se_4, [e_4,e_5]=se_3+pe_4$	4	$\gamma s\neq 0, \lvert\gamma\rvert\leq 1$
$\mathfrak{g}_{5,14}$	$[e_2,e_5]=e_1, [e_3,e_5]=pe_3-e_4,$ $[e_4,e_5]=e_3+pe_4$	4	
$\mathfrak{g}_{5,15}$	$[e_1,e_5]=e_1, [e_3,e_5]=\gamma e_3,$ $[e_2,e_5]=e_1+e_2, [e_4,e_5]=e_3+\gamma e_4$	4	$-1\leq\gamma\leq 1$
$\mathfrak{g}_{5,16}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2,$ $[e_3,e_5]=pe_3-se_4, [e_4,e_5]=se_3+pe_4$	4	$s\neq 0$
$\mathfrak{g}_{5,17}$	$[e_1,e_5]=pe_1-e_2, [e_2,e_5]=e_1+pe_2,$ $[e_3,e_5]=qe_3-se_4, [e_4,e_5]=se_3+qe_4$	4	$s\neq 0$
$\mathfrak{g}_{5,18}$	$[e_3,e_5]=e_1+pe_3-e_4, [e_2,e_5]=e_1+pe_2$ $[e_1,e_5]=pe_1-e_2, [e_4,e_5]=e_2+e_3-pe_4$	4	$p\geq 0$
$\mathfrak{g}_{5,19}$	$[e_2,e_3]=e_1, [e_1,e_5]=(1+\alpha)e_1,$ $[e_2,e_5]=e_2, [e_3,e_5]=\alpha e_3, [e_4,e_5]=\beta e_4$	3	$\beta\neq 0$
$\mathfrak{g}_{5,20}$	$[e_2,e_3]=e_1, [e_1,e_5]=(1+\alpha)e_2,$ $[e_2,e_5]=e_2, [e_3,e_5]=\alpha e_3,$ $[e_4,e_5]=e_1+(1+\alpha)e_4$	3	
$\mathfrak{g}_{5,21}$	$[e_2,e_3]=e_1, [e_1,e_5]=2e_1, [e_4,e_5]=e_4,$ $[e_2,e_5]=e_2+e_3, [e_3,e_5]=e_3+e_4$	3	
$\mathfrak{g}_{5,22}$	$[e_2,e_3]=e_1, [e_2,e_5]=e_3,$ $[e_4,e_5]=e_4$	3	
$\mathfrak{g}_{5,23}$	$[e_2,e_3]=e_1, [e_1,e_5]=2e_1, [e_3,e_5]=e_3,$ $[e_2,e_5]=e_2+e_3, [e_4,e_5]=\beta e_4$	3	$\beta\neq 0$
$\mathfrak{g}_{5,24}$	$[e_2,e_3]=e_1, [e_1,e_5]=2e_1, [e_3,e_5]=e_3,$ $[e_2,e_5]=e_2+e_3, [e_4,e_5]=\epsilon e_1+2e_4$	3	$\epsilon=\pm 1$
$\mathfrak{g}_{5,25}$	$[e_2,e_3]=e_1, [e_1,e_5]=2pe_1, [e_4,e_5]=\beta e_4,$ $[e_2,e_5]=pe_2+e_3, [e_3,e_5]=-e_2+pe_3$	3	$\beta\neq 0$
$\mathfrak{g}_{5,26}$	$[e_2,e_5]=pe_2+e_3, [e_1,e_5]=2pe_1,$ $[e_2,e_3]=e_1, [e_3,e_5]=-e_2+pe_3,$ $[e_4,e_5]=\epsilon e_1+2pe_4$	3	$\epsilon=\pm 1$
$\mathfrak{g}_{5,27}$	$[e_2,e_3]=e_1, [e_3,e_5]=e_3+e_4,$ $[e_1,e_5]=e_1, [e_4,e_5]=e_1+e_4$	3	
$\mathfrak{g}_{5,28}$	$[e_2,e_3]=e_1, [e_2,e_5]=\alpha e_2, [e_4,e_5]=e_4,$ $[e_1,e_5]=(1+\alpha)e_1, [e_3,e_5]=e_3+e_4$	3	
$\mathfrak{g}_{5,29}$	$[e_2,e_3]=e_1, [e_1,e_5]=e_1,$ $[e_2,e_5]=e_2, [e_3,e_5]=e_4$	3	
$\mathfrak{g}_{5,30}$	$[e_2,e_4]=e_1, [e_3,e_4]=e_2,$ $[e_1,e_5]=(2+h)e_1, [e_4,e_5]=e_4,$ $[e_2,e_5]=(1+h)e_2, [e_3,e_5]=he_3$	3	

Table 4.6: 5-dimensional non-decomposable real solvable non-nilpotent Lie algebras (II)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{5,31}$	$[e_2,e_4]=e_1, [e_3,e_4]=e_2,$ $[e_1,e_5]=3e_1, [e_3,e_5]=e_3,$ $[e_2,e_5]=2e_2, [e_4,e_5]=e_3+e_4$	3	
$\mathfrak{g}_{5,32}$	$[e_2,e_4]=e_1, [e_3,e_4]=e_2, [e_1,e_5]=e_1,$ $[e_2,e_5]=e_2, [e_3,e_5]=he_1+e_3$	3	
$\mathfrak{g}_{5,33}$	$[e_1,e_4]=e_1, [e_3,e_4]=\beta e_3,$ $[e_2,e_5]=e_2, [e_3,e_5]=\gamma e_3$	3	$\beta^2+\gamma^2\neq 0$
$\mathfrak{g}_{5,34}$	$[e_1,e_4]=\alpha e_1, [e_2,e_4]=e_2,$ $[e_3,e_4]=e_3, [e_1,e_5]=e_1, [e_3,e_5]=e_2$	3	
$\mathfrak{g}_{5,35}$	$[e_1,e_4]=he_1, [e_2,e_4]=e_2, [e_3,e_4]=e_3,$ $[e_2,e_5]=-e_3, [e_1,e_5]=\alpha e_1, [e_3,e_5]=e_2$	3	$h^2+\alpha^2\neq 0$
$\mathfrak{g}_{5,36}$	$[e_2,e_3]=e_1, [e_1,e_4]=e_1,$ $[e_2,e_4]=e_2, [e_3,e_5]=e_3, [e_2,e_5]=-e_2$	2	
$\mathfrak{g}_{5,37}$	$[e_2,e_3]=e_1, [e_1,e_4]=2e_1, [e_2,e_4]=e_2,$ $[e_3,e_4]=e_3, [e_2,e_5]=-e_3, [e_3,e_5]=e_2$	2	
$\mathfrak{g}_{5,38}$	$[e_1,e_4]=e_1, [e_2,e_5]=e_2, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{5,39}$	$[e_1,e_4]=e_1, [e_2,e_4]=e_2,$ $[e_1,e_5]=-e_2, [e_2,e_5]=e_1, [e_4,e_5]=e_3$	3	

Table 4.7: Cases when α and β invariants are different in non-decomposable real solvable Lie algebras of dimension 6

$\mathfrak{g}$	$\alpha(\mathfrak{g})$	$\beta(\mathfrak{g})$	Abelian ideals of dimension $\alpha(\mathfrak{g})$ over $\mathbb{C}$
$\mathfrak{g}_{6,j}$, for $j=\{73,74,77,78\}$	4	3	$\langle e_1,e_4,e_5,e_2\pm ie_3\rangle$, $\langle e_1,e_4,e_5,e_3\pm ie_2\rangle$
$\mathfrak{g}_{6,k}$, for $k=\{35,36,37,39,40\}$	3	2	$\langle e_3,e_6,e_4\pm ie_5\rangle$, $\langle e_3,e_6,e_5\pm ie_4\rangle$
$\mathfrak{g}_{6,129}$	3	2	$\langle e_1,e_2,e_3\pm ie_5\rangle$, $\langle e_1,e_2,e_5\pm ie_3\rangle$ $\langle e_1,e_4,e_3\pm ie_5\rangle$, $\langle e_1,e_4,e_5\pm ie_3\rangle$
$\mathfrak{g}_{6,130}$	3	2	$\langle e_1,e_2\pm e_4,e_3\pm ie_5\rangle$, $\langle e_1,e_2\pm e_4,e_5\pm ie_3\rangle$ $\langle e_1,e_4\pm e_2,e_3\pm ie_5\rangle$, $\langle e_1,e_4\pm e_2,e_5\pm ie_3\rangle$

Table 4.8: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,1}^{a,b,c,d}$	$[e_1,e_3]=ae_3,[e_1,e_4]=ce_4,[e_1,e_6]=e_6$ $[e_2,e_3]=be_3,[e_2,e_4]=de_4,[e_2,e_5]=e_5$	4	$ab\neq 0$ $c^2+d^2\neq 0$
$\mathfrak{g}_{6,2}^{a,b,c}$	$[e_1,e_3]=ae_3,[e_1,e_4]=e_4,[e_1,e_5]=e_6,$ $[e_2,e_6]=e_6,[e_2,e_3]=be_3,$ $[e_2,e_4]=ce_4,[e_2,e_5]=e_5$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,3}^{a}$	$[e_1,e_3]=e_3,[e_2,e_5]=e_5,[e_1,e_4]=e_4,$ $[e_1,e_5]=e_6,[e_2,e_3]=ae_3+e_4,$ $[e_2,e_4]=ae_4,[e_2,e_6]=e_6$	4	
$\mathfrak{g}_{6,4}^{a,b}$	$[e_1,e_3]=e_3,[e_1,e_4]=e_4,[e_1,e_5]=e_6,$ $[e_2,e_6]=ae_6,[e_2,e_3]=e_4,$ $[e_2,e_4]=-e_3,[e_2,e_5]=ae_5+be_6$	4	$a\neq 0$
$\mathfrak{g}_{6,5}^{a,b}$	$[e_1,e_3]=ae_3,[e_1,e_5]=e_5+e_6,[e_1,e_6]=e_6$ $[e_2,e_3]=be_3,[e_2,e_4]=e_4$	4	$a\neq 0$
$\mathfrak{g}_{6,6}^{a,b}$	$[e_1,e_3]=ae_3,[e_1,e_4]=ae_4,[e_1,e_5]=e_5+e_6$ $[e_1,e_6]=e_6,[e_2,e_3]=e_3+e_4,[e_2,e_4]=e_4$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,7}^{a,b,c}$	$[e_1,e_3]=ae_3,[e_1,e_4]=ae_4,[e_2,e_5]=be_6,$ $[e_1,e_6]=e_6,[e_2,e_3]=ce_3+e_4,$ $[e_2,e_4]=-e_3+ce_4,[e_1,e_5]=e_5+e_6$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,8}$	$[e_1,e_3]=e_3,[e_1,e_4]=e_6,[e_2,e_4]=e_4$ $[e_2,e_5]=e_5+e_6,[e_2,e_6]=e_6$	4	
$\mathfrak{g}_{6,9}$	$[e_1,e_3]=e_3,[e_1,e_4]=e_6,[e_2,e_6]=e_6$ $[e_2,e_4]=e_4+e_5,[e_2,e_5]=e_5+ae_6$	4	
$\mathfrak{g}_{6,10}^{a,b}$	$[e_1,e_3]=ae_3,[e_1,e_4]=e_3+be_6,[e_1,e_5]=e_5$ $[e_2,e_5]=e_6,[e_1,e_6]=e_6,$ $[e_2,e_3]=e_3,[e_2,e_4]=e_5$	4	
$\mathfrak{g}_{6,11}^{a}$	$[e_1,e_3]=e_4,[e_2,e_3]=e_3,[e_2,e_4]=e_4,$ $[e_1,e_6]=e_6,[e_2,e_5]=ae_5,$ $[e_1,e_5]=e_5+e_6,[e_2,e_6]=ae_6$	4	
$\mathfrak{g}_{6,12}^{a,b}$	$[e_1,e_3]=e_3+e_4,[e_1,e_5]=e_5+e_6$ $[e_1,e_4]=e_4,[e_1,e_6]=e_6,$ $[e_2,e_3]=ae_4+e_5-be_6,[e_2,e_4]=e_6$ $[e_2,e_5]=-e_3+be_4+ae_6,[e_2,e_6]==-e_4$	4	
$\mathfrak{g}_{6,13}^{a,b,c,d}$	$[e_1,e_3]=ae_3,[e_1,e_4]=ce_4,[e_1,e_5]=e_6,$ $[e_2,e_5]=e_5,[e_1,e_6]=-e_5,[e_2,e_3]=be_3,$ $[e_2,e_4]=de_4,[e_2,e_6]=e_6$	4	
$\mathfrak{g}_{6,14}^{a,b,c}$	$[e_1,e_5]=ce_5+e_6,[e_1,e_6]=-e_5+ce_6$ $[e_1,e_3]=ae_3,[e_2,e_3]=be_3,[e_2,e_4]=e_4$	4	$ab\neq 0$
$\mathfrak{g}_{6,15}^{a,b,c,d}$	$[e_1,e_4]=e_4,[e_1,e_5]=ae_5+be_6$ $[e_2,e_6]=de_6,[e_1,e_6]=-be_5+ae_6$ $[e_2,e_3]=ce_3+e_4,[e_2,e_5]=de_5,$ $[e_1,e_3]=e_3,[e_2,e_4]=-e_3+ce_4$	4	$b\neq 0$
$\mathfrak{g}_{6,16}^{a,b}$	$[e_1,e_3]=e_4,[e_1,e_5]=ae_5+e_6,[e_2,e_3]=e_3,$ $[e_1,e_6]=-e_5+ae_6,[e_2,e_4]=e_4,$ $[e_2,e_5]=be_5,[e_2,e_6]=be_6$	4	
$\mathfrak{g}_{6,17}^{a}$	$[e_1,e_3]=ae_3+e_4,[e_1,e_4]=ae_4,[e_1,e_5]=e_6$ $[e_1,e_6]=-e_5,[e_2,e_5]=e_5,[e_2,e_6]=e_6$	4	
$\mathfrak{g}_{6,18}^{a,b,c}$	$[e_1,e_3]=e_4,[e_1,e_5]=ae_5+be_6,$ $[e_1,e_4]=-e_3,[e_2,e_6]=de_6,$ $[e_2,e_4]=e_4,[e_1,e_6]=-be_5+ae_6,$ $[e_2,e_3]=e_3,[e_2,e_5]=ce_5$	4	$b\neq 0$

Table 4.9: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (II)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,19}$	$[e_1,e_3]=e_4+e_5, [e_1,e_6]=-e_5, [e_1,e_5]=e_6,$ $[e_2,e_5]=e_5, [e_1,e_4]=-e_3+e_6, [e_2,e_3]=e_3,$ $[e_2,e_4]=e_4, [e_2,e_6]=e_6$	4	
$\mathfrak{g}_{6,20}^{a,b}$	$[e_1,e_4]=ae_4, [e_1,e_6]=e_6, [e_2,e_4]=be_4$ $[e_2,e_5]=e_5, [e_1,e_2]=e_3$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,21}^{a}$	$[e_1,e_4]=e_4, [e_1,e_5]=e_6, [e_2,e_4]=ae_4$ $[e_2,e_5]=e_5, [e_2,e_6]=e_6, [e_1,e_2]=e_3$	4	
$\mathfrak{g}_{6,22}^{a,\epsilon}$	$[e_1,e_3]=e_3, [e_1,e_5]=e_6, [e_2,e_3]=ae_3$ $[e_2,e_4]=e_4, [e_1,e_2]=\epsilon e_5$	4	$a^2+\epsilon^2\neq 0$ $\epsilon=0,1$
$\mathfrak{g}_{6,23}^{a,\epsilon}$	$[e_1,e_3]=e_3, [e_1,e_4]=e_4, [e_1,e_5]=e_6,$ $[e_2,e_3]=e_4, [e_2,e_4]=-e_3,$ $[e_2,e_5]=ae_6, [e_1,e_2]=\epsilon e_5$	4	$\epsilon=0,1$
$\mathfrak{g}_{6,24}$	$[e_1,e_5]=e_5+e_6, [e_1,e_6]=e_6,$ $[e_2,e_4]=e_4, [e_1,e_2]=e_3$	4	
$\mathfrak{g}_{6,25}^{a,b}$	$[e_1,e_4]=ae_4, [e_1,e_5]=e_6, [e_1,e_6]=-e_5,$ $[e_2,e_5]=e_5, [e_2,e_6]=e_6,$ $[e_1,e_2]=e_3, [e_2,e_3]=be_4$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,26}^{a}$	$[e_1,e_5]=ae_5+e_6, [e_1,e_6]=-e_5+ae_6$ $[e_2,e_4]=e_4, [e_1,e_2]=e_3$	4	
$\mathfrak{g}_{6,27}^{\epsilon}$	$[e_1,e_3]=e_4, [e_1,e_5]=e_6, [e_1,e_6]=-e_5$ $[e_2,e_5]=e_5, [e_2,e_6]=e_6, [e_1,e_2]=\epsilon e_3$	4	$\epsilon=0,1$
$\mathfrak{g}_{6,28}$	$[e_1,e_3]=e_3, [e_4,e_6]=e_3, [e_1,e_5]=-e_5,$ $[e_1,e_6]=e_6, [e_2,e_4]=e_4, [e_2,e_5]=2e_5,$ $[e_2,e_6]=-e_6, [e_5,e_6]=e_4$	3	
$\mathfrak{g}_{6,29}^{a,b}$	$[e_1,e_3]=e_3, [e_4,e_5]=e_3, [e_1,e_4]=e_4,$ $[e_1,e_6]=ae_6, [e_2,e_3]=e_3,$ $[e_2,e_5]=e_5, [e_2,e_6]=be_6$	3	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,30}^{a}$	$[e_1,e_3]=2e_3, [e_1,e_5]=e_5, [e_2,e_4]=e_5,$ $[e_2,e_6]=e_6, [e_1,e_6]=ae_6,$ $[e_4,e_5]=e_3, [e_1,e_4]=e_4$	3	
$\mathfrak{g}_{6,31}$	$[e_1,e_4]=e_4, [e_1,e_5]=-e_5, [e_2,e_3]=e_3$ $[e_2,e_5]=e_5, [e_2,e_6]=e_3+e_6, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,32}^{a}$	$[e_1,e_4]=e_4, [e_1,e_5]=-e_5, [e_1,e_6]=e_3,$ $[e_2,e_6]=e_6, [e_2,e_3]=e_3,$ $[e_2,e_4]=ae_4, [e_2,e_5]=(1-a)e_5, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,33}$	$[e_1,e_3]=e_3, [e_1,e_4]=e_4, [e_2,e_3]=e_3$ $[e_2,e_5]=e_5+e_6, [e_2,e_6]=e_6, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,34}^{a}$	$[e_1,e_3]=e_3, [e_1,e_4]=e_4, [e_1,e_5]=e_6,$ $[e_2,e_6]=e_6, [e_2,e_3]=(1+a)e_3, [e_2,e_4]=ae_4,$ $[e_2,e_5]=e_5, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,35}^{a,b}$	$[e_1,e_4]=e_5, [e_1,e_6]=ae_6, [e_2,e_3]=2e_3,$ $[e_2,e_4]=e_4, [e_1,e_5]=-e_4, [e_2,e_5]=e_5,$ $[e_2,e_6]=be_6, [e_4,e_5]=e_3$	3	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,36}$	$[e_1,e_4]=e_5, [e_1,e_5]=-e_4, [e_2,e_3]=2e_3,$ $[e_4,e_5]=e_3, [e_2,e_4]=e_4,$ $[e_2,e_5]=e_5, [e_2,e_6]=e_3+2e_6$	3	
$\mathfrak{g}_{6,37}^{a}$	$[e_2,e_4]=e_4+ae_5, [e_1,e_4]=e_5, [e_1,e_5]=-e_4,$ $[e_1,e_6]=e_3, [e_2,e_3]=2e_3, [e_2,e_5]=-ae_4+e_5,$ $[e_2,e_6]=2e_6, [e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,38}$	$[e_1,e_3]=e_3, [e_1,e_4]=e_4, [e_2,e_3]=e_3$ $[e_2,e_5]=e_5, [e_1,e_2]=e_6, [e_4,e_5]=e_3$	3	

Table 4.10: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (III)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,39}$	$[e_1,e_2]=e_6,[e_2,e_5]=e_5,[e_1,e_4]=e_5,$ $[e_1,e_5]=-e_4,[e_2,e_3]=2e_3,$ $[e_2,e_4]=e_4,[e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,40}$	$[e_1,e_2]=e_3,[e_1,e_4]=e_5,[e_1,e_5]=-e_4$ $[e_2,e_6]=e_6,[e_4,e_5]=e_3$	3	
$\mathfrak{g}_{6,41}^{a,b,c,d}$	$[e_1,e_6]=e_1,[e_2,e_6]=ae_2,[e_3,e_6]=be_3$ $[e_4,e_6]=ce_4,[e_5,e_6]=de_5$	5	$0<\lvert d\rvert\le\lvert c\rvert$ $\le\lvert b\rvert\le\lvert a\rvert\le 1$
$\mathfrak{g}_{6,42}^{a,c,d}$	$[e_1,e_6]=ae_1,[e_2,e_6]=e_1+ae_2,[e_3,e_6]=e_3$ $[e_4,e_6]=ce_4,[e_5,e_6]=de_5$	5	$0<\lvert d\rvert\le\lvert c\rvert\le 1$
$\mathfrak{g}_{6,43}^{a,d}$	$[e_1,e_6]=ae_1,[e_2,e_6]=e_1+ae_2,[e_3,e_6]=e_2+ae_3$ $[e_4,e_6]=e_4,[e_5,e_6]=de_5$	5	$0<\lvert d\rvert\le 1$
$\mathfrak{g}_{6,44}^{a}$	$[e_1,e_6]=ae_1,[e_2,e_6]=e_1+ae_2,[e_3,e_6]=e_2+ae_3$ $[e_4,e_6]=e_3+ae_4,[e_5,e_6]=e_5$	5	
$\mathfrak{g}_{6,45}$	$[e_1,e_6]=e_1,[e_2,e_6]=e_1+e_2,[e_3,e_6]=e_2+e_3$ $[e_4,e_6]=e_3+e_4,[e_5,e_6]=e_4+e_5$	5	
$\mathfrak{g}_{6,46}^{s,h}$	$[e_1,e_6]=e_1,[e_2,e_6]=se_2,[e_3,e_6]=e_2+se_3$ $[e_4,e_6]=he_4,[e_5,e_6]=e_4+he_5$	5	$s\le h$
$\mathfrak{g}_{6,47}^{a,b}$	$[e_1,e_6]=ae_1,[e_2,e_6]=e_1+ae_2,[e_3,e_6]=e_2+ae_3$ $[e_4,e_6]=be_4,[e_5,e_6]=e_4+be_5$	5	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,48}^{a,b,c,p}$	$[e_1,e_6]=ae_1,[e_2,e_6]=be_2,[e_3,e_6]=ce_3$ $[e_4,e_6]=pe_4-e_5,[e_5,e_6]=e_4+pe_5$	5	$0<\lvert d\rvert\le\lvert b\rvert\le a$
$\mathfrak{g}_{6,49}^{a,b,p}$	$[e_1,e_6]=ae_1,[e_2,e_6]=be_2,[e_3,e_6]=e_2+be_3$ $[e_4,e_6]=pe_4-e_5,[e_5,e_6]=e_4+pe_5$	5	$a\neq 0$
$\mathfrak{g}_{6,50}^{a,p}$	$[e_1,e_6]=ae_1,[e_2,e_6]=e_1+ae_2,[e_3,e_6]=e_2+ae_3$ $[e_4,e_6]=pe_4-e_5,[e_5,e_6]=e_4+pe_5$	5	
$\mathfrak{g}_{6,51}^{a,p,s,q}$	$[e_1,e_6]=ae_1,[e_2,e_6]=pe_2-e_3,[e_3,e_6]=e_2+pe_3$ $[e_4,e_6]=qe_4-se_5,[e_5,e_6]=se_4+qe_5$	5	$as\neq 0$
$\mathfrak{g}_{6,52}^{a,p}$	$[e_1,e_6]=ae_1,[e_2,e_6]=pe_2-e_3,[e_3,e_6]=e_2+pe_3$ $[e_4,e_6]=e_2+pe_4-e_5,[e_5,e_6]=e_3+e_4+pe_5$	5	$a\neq 0$
$\mathfrak{g}_{6,53}^{a,b,h}$	$[e_2,e_3]=e_1,[e_1,e_6]=(a+b)e_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=be_3,[e_4,e_6]=e_4,[e_5,e_6]=he_5$	4	
$\mathfrak{g}_{6,54}^{a,b}$	$[e_2,e_3]=e_1,[e_1,e_6]=(a+b)e_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=be_3,[e_4,e_6]=e_4,[e_5,e_6]=e_1+(a+b)e_5$	4	
$\mathfrak{g}_{6,55}^{h}$	$[e_2,e_3]=e_1,[e_1,e_6]=(1+h)e_1,[e_2,e_6]=e_2+e_4,$ $[e_3,e_6]=he_3+e_5,[e_4,e_6]=e_4,[e_5,e_6]=he_5$	4	
$\mathfrak{g}_{6,56}$	$[e_2,e_3]=e_1,[e_1,e_6]=e_1,[e_2,e_6]=e_2+e_4,$ $[e_3,e_6]=e_5,[e_4,e_6]=e_1+e_4$	4	
$\mathfrak{g}_{6,57}^{a,\epsilon}$	$[e_2,e_3]=e_1,[e_1,e_6]=ae_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=e_4,[e_4,e_6]=\epsilon e_1,[e_5,e_6]=e_5$	4	$a^2+\epsilon^2\neq 0$ $a\epsilon\neq 0$
$\mathfrak{g}_{6,58}^{a,b}$	$[e_2,e_3]=e_1,[e_1,e_6]=(1+a)e_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=e_3+e_4,[e_4,e_6]=e_4,[e_5,e_6]=be_5$	4	$b\neq 0$
$\mathfrak{g}_{6,59}^{a}$	$[e_2,e_3]=e_1,[e_1,e_6]=(1+a)e_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=e_3+e_4,[e_4,e_6]=e_4,[e_5,e_6]=e_1+(a+1)e_5$	4	
$\mathfrak{g}_{6,60}^{b}$	$[e_2,e_3]=e_1,[e_1,e_6]=e_1,$ $[e_3,e_6]=e_3+e_4,[e_4,e_6]=e_1+e_4,[e_5,e_6]=be_5$	4	$b\neq 0$
$\mathfrak{g}_{6,61}^{a,h}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,[e_2,e_6]=ae_2+e_3,$ $[e_3,e_6]=ae_3,[e_4,e_6]=e_4,[e_5,e_6]=he_5$	4	$h\neq 0$

Table 4.11: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (IV)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,62}^{a}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,[e_2,e_6]=ae_2+e_3,$ $[e_3,e_6]=ae_3,[e_4,e_6]=e_4,[e_5,e_6]=e_1+2ae_5$	4	
$\mathfrak{g}_{6,63}^{a,\epsilon,h}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,$ $[e_2,e_6]=ae_2+e_3,[e_3,e_6]=ae_3+e_4,$ $[e_4,e_6]=ae_4,[e_5,e_6]=\epsilon e_1+(2a+h)e_5$	4	$\epsilon h=0$
$\mathfrak{g}_{6,64}^{a,h}$	$[e_2,e_3]=e_1,[e_2,e_6]=ae_3,$ $[e_3,e_6]=e_4,[e_4,e_6]=he_1,[e_5,e_6]=e_5$	4	
$\mathfrak{g}_{6,65}^{b,h}$	$[e_2,e_3]=e_1,[e_1,e_6]=(1+h)e_1,[e_2,e_6]=e_2,$ $[e_3,e_6]=he_3,[e_4,e_6]=be_4+e_5,[e_5,e_6]=be_5$	4	
$\mathfrak{g}_{6,66}^{h}$	$[e_2,e_3]=e_1,[e_1,e_6]=(1+h)e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=he_3,$ $[e_4,e_6]=(1+h)e_4+e_5,[e_5,e_6]=e_1+(1+h)e_5$	4	
$\mathfrak{g}_{6,67}^{a,b,\epsilon}$	$[e_2,e_3]=e_1,[e_1,e_6]=(a+b)e_1,$ $[e_2,e_6]=ae_2,[e_3,e_6]=be_3+e_4,$ $[e_4,e_6]=be_4+e_5,[e_5,e_6]=\epsilon e_1+be_5$	4	$a^2+b^2\neq 0$ $\epsilon a=0$
$\mathfrak{g}_{6,68}^{b}$	$[e_2,e_3]=e_1,[e_1,e_6]=2e_1,[e_2,e_6]=e_2+e_3,$ $[e_3,e_6]=e_3,[e_4,e_6]=be_4+e_5,[e_5,e_6]=be_5$	4	
$\mathfrak{g}_{6,69}$	$[e_2,e_3]=e_1,[e_1,e_6]=2e_1,[e_2,e_6]=e_2+e_3,$ $[e_3,e_6]=e_3,[e_4,e_6]=2e_4+e_5,[e_5,e_6]=e_1+2e_5$	4	
$\mathfrak{g}_{6,70}$	$[e_2,e_3]=e_1,[e_2,e_6]=e_3,$ $[e_4,e_6]=e_4+e_5,[e_5,e_6]=e_5$	4	
$\mathfrak{g}_{6,71}$	$[e_2,e_3]=e_1,[e_1,e_6]=2e_1,[e_2,e_6]=e_2+e_3,$ $[e_3,e_6]=e_3+e_4,[e_4,e_6]=e_4+e_5,[e_5,e_6]=e_5$	4	
$\mathfrak{g}_{6,72}^{a,c,h,\epsilon}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,$ $[e_2,e_6]=ae_2+e_3,[e_3,e_6]=-e_2+ae_3,$ $[e_4,e_6]=\epsilon e_1+(2a+h)e_4,[e_5,e_6]=ce_5$	4	$2a+h>c$ $\epsilon h=0$
$\mathfrak{g}_{6,73}^{a,b}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,[e_2,e_6]=ae_2+e_3,$ $[e_3,e_6]=-e_2+ae_3,[e_4,e_6]=be_4,[e_5,e_6]=2ae_5+e_1$	4	$b\leq 2a$
$\mathfrak{g}_{6,74}^{a,h,\epsilon}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,$ $[e_2,e_6]=ae_2+e_3,[e_3,e_6]=-e_2+ae_3,$ $[e_4,e_6]=(2a+h)e_4+e_5,[e_5,e_6]=(2a+h)e_5+\epsilon e_1$	4	$\epsilon h=0$
$\mathfrak{g}_{6,75}^{a,b,c}$	$[e_2,e_3]=e_1,[e_1,e_6]=(a+b)e_1,[e_2,e_6]=ae_2,$ $[e_3,e_6]=be_3,[e_4,e_6]=ce_4+e_5,[e_5,e_6]=-e_4+ce_5$	4	$a^2+b^2\neq 0$
$\mathfrak{g}_{6,76}^{a,c}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,[e_2,e_6]=ae_2+e_3,$ $[e_3,e_6]=ae_3,[e_4,e_6]=ce_4+e_5,[e_5,e_6]=-e_4+ce_5$	4	
$\mathfrak{g}_{6,77}^{a,b,s}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,$ $[e_2,e_6]=ae_2+e_3,[e_3,e_6]=-e_2+ae_3,$ $[e_4,e_6]=be_4+se_5,[e_5,e_6]=-se_4+be_5$	4	$s\neq 0$
$\mathfrak{g}_{6,78}^{a}$	$[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,$ $[e_2,e_6]=ae_2+e_3+e_4,[e_3,e_6]=-e_2+ae_3+e_5,$ $[e_4,e_6]=ae_4+e_5,[e_5,e_6]=-e_4+ae_5$	4	
$\mathfrak{g}_{6,79}^{c,h}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=(h+1)e_1,$ $[e_2,e_6]=(h+2)e_2,[e_3,e_6]=ce_3,$ $[e_4,e_6]=he_4,[e_5,e_6]=e_5$	4	$c\neq 0$
$\mathfrak{g}_{6,80}^{h}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=(h+1)e_1,$ $[e_2,e_6]=(h+2)e_2,[e_3,e_6]=e_2+(h+2)e_3,$ $[e_4,e_6]=he_4,[e_5,e_6]=e_5$	4	

Table 4.12: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (V)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,81}^{h}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=(h+1)e_1,$ $[e_2,e_6]=(h+2)e_2,[e_3,e_6]=he_3,$ $[e_4,e_6]=e_3+he_4,[e_5,e_6]=e_5$	4	
$\mathfrak{g}_{6,82}^{h}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=(h+1)e_1,$ $[e_2,e_6]=(h+2)e_2,[e_3,e_6]=e_3,$ $[e_4,e_6]=he_4,[e_5,e_6]=e_3+e_5$	4	
$\mathfrak{g}_{6,83}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_3,e_6]=e_2+e_3,$ $[e_2,e_6]=e_2,[e_4,e_6]=-e_4,[e_5,e_6]=e_3+e_5$	4	
$\mathfrak{g}_{6,84}^{c}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=2e_1,$ $[e_2,e_6]=3e_2,[e_3,e_6]=ce_3,$ $[e_4,e_6]=e_4,[e_5,e_6]=e_4+e_5$	4	$c\neq 0$
$\mathfrak{g}_{6,85}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=2e_1,$ $[e_2,e_6]=3e_2,[e_3,e_6]=e_2+3e_3,$ $[e_4,e_6]=e_4,[e_5,e_6]=e_4+e_5$	4	
$\mathfrak{g}_{6,86}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=2e_1,$ $[e_2,e_6]=3e_2,[e_3,e_6]=e_3,$ $[e_4,e_6]=e_3+e_4,[e_5,e_6]=e_4+e_5$	4	
$\mathfrak{g}_{6,87}^{c,\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=ce_3,[e_4,e_6]=\epsilon e_2+e_4$	4	$c\neq 0,$ $\epsilon=0,\pm 1$
$\mathfrak{g}_{6,88}^{c,\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=ce_2+e_3,[e_4,e_6]=\epsilon e_4$	4	
$\mathfrak{g}_{6,89}^{\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_4,e_6]=\epsilon e_2+e_4,[e_5,e_6]=e_3$	4	$\epsilon=0,\pm 1$
$\mathfrak{g}_{6,90}^{\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=\epsilon e_2+e_3,[e_4,e_6]=e_3+e_4$	4	$\epsilon=0,\pm 1$
$\mathfrak{g}_{6,91}^{\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,$ $[e_3,e_6]=e_3,[e_4,e_6]=\epsilon e_2$	4	$\epsilon=\pm 1$
$\mathfrak{g}_{6,92}^{\epsilon}$	$[e_1,e_5]=e_2,[e_4,e_5]=e_1,$ $[e_3,e_6]=e_3,[e_4,e_6]=\epsilon e_2,[e_5,e_6]=e_4$	4	$\epsilon=0,\pm 1$
$\mathfrak{g}_{6,93}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_3,e_6]=e_3,$ $[e_4,e_6]=e_4,[e_5,e_6]=-e_5$	4	
$\mathfrak{g}_{6,94}^{c,\lambda}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1,[e_2,e_6]=\lambda e_2,$ $[e_3,e_6]=(1-c)e_3,[e_4,e_6]=(\lambda-c)e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,95}^{c}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_2,e_6]=(1+c)e_2,[e_3,e_6]=(1-c)e_3$ $[e_4,e_6]=e_1+e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,96}^{c}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_2,e_6]=(1-c)e_2,[e_3,e_6]=e_2+(1-c)e_3$ $[e_4,e_6]=(1-2c)e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,97}^{c}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_2,e_6]=2ce_2,[e_3,e_6]=(1-c)e_3$ $[e_4,e_6]=ce_4,[e_5,e_6]=e_4+ce_5$	4	
$\mathfrak{g}_{6,98}^{\omega}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=3e_1$ $[e_2,e_6]=2e_2,[e_3,e_6]=e_2+2e_3$ $[e_4,e_6]=e_4,[e_5,e_6]=\omega e_4+e_5$	4	$\omega=0,1$
$\mathfrak{g}_{6,99}^{h}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_3,e_6]=e_3,[e_4,e_6]=e_2,[e_5,e_6]=he_4$	4	
$\mathfrak{g}_{6,100}^{\omega}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_2,e_6]=2e_2,[e_4,e_6]=e_1+e_4,[e_5,e_6]=\omega e_4+e_5$	4	$\omega=0,1$

Table 4.13: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (VI)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,101}^{\lambda}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=2e_1$ $[e_2,e_6]=2\lambda e_2,[e_3,e_6]=e_3$ $[e_4,e_6]=(2\lambda-1)e_4,[e_5,e_6]=e_3+e_5$	4	
$\mathfrak{g}_{6,102}^{\omega}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=2e_1$ $[e_2,e_6]=e_2,[e_3,e_6]=e_2+e_3,[e_5,e_6]=\omega e_3+e_5$	4	$\omega=0,1$
$\mathfrak{g}_{6,103}^{\lambda}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1$ $[e_2,e_6]=\lambda e_2,[e_3,e_6]=e_3,[e_4,e_6]=e_2+\lambda e_4$	4	
$\mathfrak{g}_{6,104}^{h}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1,[e_2,e_6]=e_2,$ $[e_3,e_6]=he_2+e_3,[e_4,e_6]=e_1+e_4$	4	
$\mathfrak{g}_{6,105}^{c,\lambda}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=\lambda e_1+e_2$ $[e_2,e_6]=\lambda e_2,[e_3,e_6]=(1-c)e_3+e_4$ $[e_4,e_6]=(\lambda-c)e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,106}^{c,\lambda}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=\lambda e_1+e_2$ $[e_2,e_6]=\lambda e_2,[e_3,e_6]=(1-c)e_3+e_4$ $[e_4,e_6]=(\lambda-c)e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,107}^{h}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=2e_1+e_2$ $[e_2,e_6]=2e_2,[e_3,e_6]=e_3+e_4$ $[e_4,e_6]=e_4,[e_5,e_6]=he_4+e_5$	4	
$\mathfrak{g}_{6,108}^{c}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1+e_2$ $[e_2,e_6]=e_2,[e_3,e_6]=e_3+e_4,[e_4,e_6]=ce_1+e_4$	4	
$\mathfrak{g}_{6,109}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1+e_2$ $[e_2,e_6]=e_2,[e_3,e_6]=e_3+e_4,[e_4,e_6]=e_2+e_4$	4	
$\mathfrak{g}_{6,110}^{p,c}$	$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=pe_1+e_2$ $[e_2,e_6]=-e_1+pe_2,[e_3,e_6]=(p-c)e_3+e_4$ $[e_4,e_6]=-e_3+(p-c)e_4,[e_5,e_6]=ce_5$	4	
$\mathfrak{g}_{6,111}^{h}$	$[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,$ $[e_1,e_6]=(h+3)e_1,[e_2,e_6]=(h+2)e_2,$ $[e_3,e_6]=(h+1)e_3,[e_4,e_6]=he_4,[e_5,e_6]=e_5$	4	
$\mathfrak{g}_{6,112}$	$[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,$ $[e_1,e_6]=4e_1,[e_2,e_6]=3e_2,[e_3,e_6]=2e_3,$ $[e_4,e_6]=e_4,[e_5,e_6]=e_4+e_5$	4	
$\mathfrak{g}_{6,113}^{\epsilon}$	$[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=\epsilon e_1+e_3,[e_4,e_6]=\epsilon e_2+e_4$	4	$\epsilon=\pm 1$
$\mathfrak{g}_{6,114}$	$[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=e_3,[e_4,e_6]=e_4$	4	
$\mathfrak{g}_{6,115}$	$[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=e_3,[e_4,e_6]=e_1+e_4$	4	
$\mathfrak{g}_{6,116}^{h}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,$ $[e_1,e_6]=(2h+1)e_1,[e_2,e_6]=(h+1)e_2,$ $[e_3,e_6]=(h+2)e_3,[e_4,e_6]=e_4,[e_5,e_6]=he_5$	3	
$\mathfrak{g}_{6,117}^{\epsilon}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1,$ $[e_2,e_6]=e_2,[e_3,e_6]=2e_3,[e_4,e_6]=\epsilon e_1+e_4$	3	$\epsilon=\pm 1$
$\mathfrak{g}_{6,118}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_3,e_6]=e_3,$ $[e_1,e_6]=-e_1,[e_4,e_6]=e_3+e_4,[e_5,e_6]=-e_5$	3	
$\mathfrak{g}_{6,119}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,$ $[e_1,e_6]=3e_1,[e_2,e_6]=2e_2,$ $[e_3,e_6]=e_1+3e_3,[e_4,e_6]=e_4+e_5,[e_5,e_6]=e_5$	3	

Table 4.14: 6-dimensional non-decomposable solvable non-nilpotent Lie algebras (VII)

$\mathfrak{g}$	Lie brackets	$\alpha(\mathfrak{g})$	Parameters
$\mathfrak{g}_{6,120}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_2,e_6]=e_2,$ $[e_1,e_6]=2e_1,[e_3,e_6]=e_3,[e_5,e_6]=e_5$	3	
$\mathfrak{g}_{6,121}^{\epsilon}$	$[e_2,e_4]=e_3,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_2,e_6]=e_2,$ $[e_1,e_6]=2e_1,[e_3,e_6]=e_3,[e_5,e_6]=\epsilon e_3+e_5$	3	$\epsilon=\pm1$
$\mathfrak{g}_{6,122}^{a,\lambda,\lambda_1}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=(\frac{a}{2}+\lambda)e_2,[e_3,e_6]=(\frac{a}{2}+\lambda_1)e_3,$ $[e_4,e_6]=(\frac{a}{2}-\lambda)e_4,[e_5,e_6]=(\frac{a}{2}-\lambda_1)e_5$	3	$a=0,2$
$\mathfrak{g}_{6,123}^{a,\lambda}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=(\frac{a}{2}+\lambda)e_2+e_3,[e_3,e_6]=(\frac{a}{2}+\lambda)e_3,$ $[e_4,e_6]=(\frac{a}{2}-\lambda)e_4,[e_5,e_6]=-e_4+(\frac{a}{2}-\lambda)e_5$	3	
$\mathfrak{g}_{6,124}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,$ $[e_2,e_6]=e_2,[e_4,e_6]=-e_4,[e_5,e_6]=e_3$	3	
$\mathfrak{g}_{6,125}^{\lambda}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=2e_1,[e_2,e_6]=(\lambda+1)e_2,$ $[e_3,e_6]=e_3,[e_4,e_6]=(1-\lambda)e_4,[e_5,e_6]=e_3+e_5$	3	
$\mathfrak{g}_{6,126}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=2e_1,[e_3,e_6]=e_3,$ $[e_2,e_6]=e_2+e_3,[e_4,e_6]=e_4,[e_5,e_6]=-e_4+e_5$	3	
$\mathfrak{g}_{6,127}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=2e_1,[e_4,e_6]=e_4,$ $[e_2,e_6]=e_2+e_5,[e_3,e_6]=e_3+e_4,[e_5,e_6]=e_3+e_5$	3	
$\mathfrak{g}_{6,128}^{a,\mu_0,\nu_0}$	$[e_2,e_4]=e_1,[e_3,e_6]=-\nu_0e_2+(\frac{a}{2}+\mu_0)e_3,[e_3,e_5]=e_1,$ $[e_1,e_6]=ae_1,[e_5,e_6]=-\nu_0e_4+(\frac{a}{2}-\mu_0)e_5,$ $[e_4,e_6]=(\frac{a}{2}-\mu_0)e_4+\nu_0e_5,[e_2,e_6]=(\frac{a}{2}+\mu_0)e_2+\nu_0e_3$	3	
$\mathfrak{g}_{6,129}^{s,a,\nu_0}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=(s+\frac{a}{2})e_2,[e_3,e_6]=\frac{a}{2}e_3+\nu_0e_5,$ $[e_4,e_6]=(\frac{a}{2}-s)e_4,[e_5,e_6]=-\nu_0e_3+\frac{a}{2}e_5$	3	
$\mathfrak{g}_{6,130}^{a,\nu_0}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=\frac{a}{2}e_2+e_4,[e_3,e_6]=\frac{a}{2}e_3+\nu_0e_5,$ $[e_4,e_6]=e_2+\frac{a}{2}e_4,[e_5,e_6]=-\nu_0e_3+\frac{a}{2}e_5$	3	$\nu_0\neq1$ if $a=0$
$\mathfrak{g}_{6,131}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_2,e_6]=e_4,$ $[e_3,e_6]=e_5,[e_4,e_6]=e_2,[e_5,e_6]=-e_3$	3	
$\mathfrak{g}_{6,132}^{a,\mu_0,\nu_0}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=\frac{a}{2}e_2+\nu_0e_3,[e_3,e_6]=\frac{a}{2}e_3-\mu_0e_2,$ $[e_4,e_6]=\mu_0e_5+\frac{a}{2}e_2,[e_5,e_6]=-\mu_0e_4+\frac{a}{2}e_5$	3	
$\mathfrak{g}_{6,133}^{a,\nu_0}$	$[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_1,e_6]=ae_1,$ $[e_2,e_6]=\frac{a}{2}e_2+e_4+\nu_0e_5,[e_3,e_6]=\frac{a}{2}e_3+\nu_0e_4,$ $[e_4,e_6]=e_2-\nu_0e_3+\frac{a}{2}e_4,[e_5,e_6]=-\nu_0e_2+\frac{a}{2}e_5$	3	
$\mathfrak{g}_{6,134}^{\lambda}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,$ $[e_1,e_6]=(\lambda+2)e_1,[e_2,e_6]=(\lambda+1)e_2,[e_3,e_6]=\lambda e_3,$ $[e_4,e_6]=2e_4,[e_5,e_6]=e_5$	3	
$\mathfrak{g}_{6,135}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_2,e_6]=e_2,$ $[e_1,e_6]=2e_1,[e_4,e_6]=e_1+2e_4,[e_5,e_6]=e_5$	3	
$\mathfrak{g}_{6,136}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_6]=e_2+2e_4,$ $[e_1,e_6]=3e_1,[e_2,e_6]=2e_2,[e_3,e_6]=e_3,[e_5,e_6]=e_3+e_5$	3	
$\mathfrak{g}_{6,137}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_1,e_6]=4e_1,$ $[e_2,e_6]=3e_2,[e_3,e_6]=2e_3+e_4,[e_4,e_6]=2e_4,[e_5,e_6]=e_5$	3	
$\mathfrak{g}_{6,138}^{h}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_1,e_6]=e_1,$ $[e_2,e_6]=he_1+e_2,[e_3,e_6]=e_3,[e_5,e_6]=he_4$	3	
$\mathfrak{g}_{6,139}$	$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,$ $[e_4,e_5]=e_3,[e_j,e_6]=(6-j)e_j,\forall 1\le j\le 5$	3	

Table 4.15: α invariant for nilpotent Lie algebras of dimension less than 6.

$\mathfrak{g}$	$\dim(\mathfrak{g})$	Lie brackets	$\alpha(\mathfrak{g})$
$\mathfrak{n}_3$	3	$[e_1,e_2]=e_3$	2
$\mathfrak{n}_4$	4	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4$	3
$\mathfrak{g}_{5,6}$	5	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5$	3
$\mathfrak{g}_{5,5}$	5	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5$	4
$\mathfrak{g}_{5,3}$	5	$[e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5$	3
$\mathfrak{g}_{5,4}$	5	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5$	3
$\mathfrak{g}_{5,2}$	5	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5$	4
$\mathfrak{g}_{5,1}$	5	$[e_1,e_3]=e_5,\ [e_2,e_4]=e_5$	3

Table 4.16: α invariant for 6-dimensional nilpotent Lie algebras

Magnin	de Graaf	Seeley	$\alpha(\mathfrak{g})$
$\mathfrak{g}_{6,20}$	$L_{6,14}$	12346_E	3
$\mathfrak{g}_{6,18}$	$L_{6,16}$	12346_C	3
$\mathfrak{g}_{6,19}$	$L_{6,15}$	12346_D	4
$\mathfrak{g}_{6,17}$	$L_{6,17}$	12346_B	4
$\mathfrak{g}_{6,15}$	$L_{6,21}(1)$	1346_C	4
$\mathfrak{g}_{6,13}$	$L_{6,13}$	1246	4
$\mathfrak{g}_{6,16}$	$L_{6,18}$	12346_A	5
$\mathfrak{g}_{6,14}$	$L_{6,21}(0)$	2346	4
$\mathfrak{g}_{6,9}$	$L_{6,19}(1)$	136_A	4
$\mathfrak{g}_{6,12}$	$L_{6,11}$	1346_B	4
$\mathfrak{g}_{5,6}\oplus\mathbb{C}$	$L_{6,6}$	$1+1235_B$	4
$\mathfrak{g}_{6,5}$	$L_{6,24}(1)$	246_E	4
$\mathfrak{g}_{6,10}$	$L_{6,20}$	136_B	4
$\mathfrak{g}_{6,11}$	$L_{6,12}$	1346_A	4
$\mathfrak{g}_{5,5}\oplus\mathbb{C}$	$L_{6,7}$	$1+1235_A$	5
$\mathfrak{g}_{6,8}$	$L_{6,24}(0)$	246_D	4
$\mathfrak{g}_{6,4}$	$L_{6,19}(0)$	246_B	4
$\mathfrak{g}_{6,7}$	$L_{6,23}$	246_C	4
$\mathfrak{g}_{6,2}$	$L_{6,10}$	146	4
$\mathfrak{g}_{6,6}$	$L_{6,25}$	246_A	5
$\mathfrak{g}_{5,4}\oplus\mathbb{C}$	$L_{6,9}$	$1+235$	4
$\mathfrak{g}_{5,3}\oplus\mathbb{C}$	$L_{6,5}$	$1+135$	4
$\mathfrak{n}_3\oplus\mathfrak{n}_3$	$L_{6,22}(1)$	$13+13$	4
$\mathfrak{n}_4\oplus\mathbb{C}^2$	$L_{6,3}$	$2+124$	5
$\mathfrak{g}_{6,1}$	$L_{6,22}(0)$	26	4
$\mathfrak{g}_{6,3}$	$L_{6,26}$	36	4
$\mathfrak{g}_{5,2}\oplus\mathbb{C}$	$L_{6,8}$	$1+25$	5
$\mathfrak{g}_{5,1}\oplus\mathbb{C}$	$L_{6,4}$	$1+15$	4
$\mathfrak{n}_3\oplus\mathbb{C}^3$	$L_{6,2}$	$3+13$	5
$\mathbb{C}^6$	$L_{6,1}$	0	6

Table 4.17: 6-dimensional non-decomposable nilpotent Lie algebras

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{6,1}$	$[e_1,e_2]=e_5,[e_1,e_4]=e_6,[e_2,e_3]=e_6.$
$\mathfrak{g}_{6,2}$	$[e_1,e_2]=e_5,[e_1,e_5]=e_6,[e_3,e_4]=e_6.$
$\mathfrak{g}_{6,3}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_2,e_3]=e_6.$
$\mathfrak{g}_{6,4}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_6,[e_2,e_4]=e_5.$
$\mathfrak{g}_{6,5}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_6,[e_2,e_4]=e_6.$
$\mathfrak{g}_{6,6}$	$[e_1,e_2]=e_4,[e_2,e_3]=e_6,[e_2,e_4]=e_5.$
$\mathfrak{g}_{6,7}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_2,e_3]=-e_6.$
$\mathfrak{g}_{6,8}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5,[e_2,e_4]=e_6.$
$\mathfrak{g}_{6,9}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_2,e_5]=e_6,[e_3,e_4]=e_6.$
$\mathfrak{g}_{6,10}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_3,e_5]=e_6.$
$\mathfrak{g}_{6,11}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_6.$
$\mathfrak{g}_{6,12}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_6,[e_2,e_4]=e_6.$
$\mathfrak{g}_{6,13}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_5,[e_3,e_4]=-e_6.$
$\mathfrak{g}_{6,14}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_6.$
$\mathfrak{g}_{6,15}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_5]=e_6,[e_2,e_3]=e_5,[e_2,e_4]=e_6.$
$\mathfrak{g}_{6,16}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6.$
$\mathfrak{g}_{6,17}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_6.$
$\mathfrak{g}_{6,18}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_5]=e_6,[e_3,e_4]=-e_6.$
$\mathfrak{g}_{6,19}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,$ $[e_1,e_5]=e_6,[e_2,e_3]=e_5,[e_2,e_4]=e_6.$
$\mathfrak{g}_{6,20}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,$ $[e_2,e_3]=e_5,[e_2,e_5]=e_6,[e_3,e_4]=-e_6.$

Table 4.18: 7-dimensional non-decomposable nilpotent Lie algebras (I)

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{7,0.1}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_4]=e_7,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,0.2}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_5+e_7,[e_2,e_4]=e_6,[e_2,e_5]=e_7.$
$\mathfrak{g}_{7,0.3}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_6+e_7,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,0.4(\lambda)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=\lambda e_7+e_6,[e_1,e_5]=e_7,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_5,[e_2,e_4]=e_7,[e_2,e_5]=e_6,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,0.5}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6+e_7,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_5,[e_2,e_5]=e_6,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,0.6}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_7,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,0.7}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_7,[e_1,e_5]=e_7,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_7,[e_2,e_5]=e_6,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,0.8}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_7,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_2,e_3]=e_6,[e_2,e_4]=e_6,[e_2,e_6]=e_7,[e_4,e_5]=-e_7.$
$\mathfrak{g}_{7,1.01(i)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5+e_7,[e_3,e_4]=-e_6,[e_3,e_5]=-e_7.$
$\mathfrak{g}_{7,1.01(ii)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_6+e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.02}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_5]=e_6,[e_2,e_3]=e_5,$ $[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_2,e_6]=e_7,[e_3,e_5]=-e_7.$

Table 4.19: 7-dimensional non-decomposable nilpotent Lie algebras (II)

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{7,1.03}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_4]=e_7,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.1(i_\lambda)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,$ $[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_5,$ $[e_2,e_4]=e_6,[e_2,e_5]=\lambda e_7,[e_3,e_4]=e_7(-\lambda+1).$
$\mathfrak{g}_{7,1.1(ii)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.1(iii)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_2,e_5]=-e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,1.1(iv)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,1.1(v)}$	$[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.1(vi)}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.2(i_\lambda)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_6,[e_1,e_4]=e_5,[e_1,e_5]=e_7,$ $[e_2,e_3]=\lambda e_5,[e_2,e_4]=e_6,[e_2,e_6]=e_7,[e_3,e_4]=e_7(-\lambda+1).$
$\mathfrak{g}_{7,1.2(ii)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_7,[e_1,e_6]=e_7$ $[e_2,e_3]=e_6,[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.2(iii)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_6,[e_1,e_4]=e_5,[e_1,e_5]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.2(iv)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_6,[e_1,e_5]=-e_7,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,1.3(i_\lambda)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_4]=\lambda e_7,[e_2,e_5]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.3(ii)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_4]=e_7,[e_2,e_5]=\frac{e_7}{2},[e_3,e_4]=\frac{-e_7}{2}.$
$\mathfrak{g}_{7,1.3(iii)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_4]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.3(iv)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_2,e_3]=e_6,[e_2,e_4]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.3(v)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_6$ $[e_2,e_4]=e_7,[e_3,e_4]=-e_7,[e_3,e_5]=-e_7.$
$\mathfrak{g}_{7,1.4}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6$ $[e_1,e_6]=e_7,[e_2,e_3]=e_6,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,1.5}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6$ $[e_2,e_3]=e_6,[e_2,e_5]=-e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,1.6}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,$ $[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_7.$
$\mathfrak{g}_{7,1.7}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_1,e_5]=e_7,[e_2,e_3]=e_6,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,1.8}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.9}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_6,[e_1,e_4]=e_5,$ $[e_1,e_5]=e_7,[e_2,e_3]=e_7,[e_2,e_4]=e_6.$
$\mathfrak{g}_{7,1.10}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_5,[e_2,e_5]=e_7.$
$\mathfrak{g}_{7,1.11}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7$ $[e_2,e_3]=e_6,[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,1.12}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7$ $[e_2,e_3]=e_7,[e_2,e_4]=e_6,[e_2,e_5]=e_7.$
$\mathfrak{g}_{7,1.13}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_1,e_5]=e_7$ $[e_1,e_6]=e_7,[e_2,e_3]=e_5,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,1.14}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_2,e_3]=e_5$ $[e_2,e_5]=-e_7,[e_2,e_6]=-e_7,[e_3,e_4]=e_7.$

Table 4.20: 7-dimensional non-decomposable nilpotent Lie algebras (III)

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{7,1.15}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_7,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,1.16}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_1,e_5]=e_7,[e_1,e_6]=e_7,[e_2,e_3]=e_7.$
$\mathfrak{g}_{7,1.17}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_5,$ $[e_2,e_5]=e_6,[e_2,e_6]=e_7,[e_3,e_4]=-e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.18}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_7,$ $[e_2,e_3]=e_6+e_7,[e_2,e_4]=e_6.$
$\mathfrak{g}_{7,1.19}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_7,$ $[e_1,e_5]=e_6,[e_2,e_4]=e_6,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,1.20}$	$[e_1,e_2]=e_3,[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_4,$ $[e_2,e_4]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,1.21}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_6,$ $[e_2,e_4]=e_6,[e_2,e_6]=e_7,[e_4,e_5]=-e_7.$
$\mathfrak{g}_{7,2.1(i_\lambda)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_5]=\lambda e_7,[e_3,e_4]=e_7(\lambda-1).$
$\mathfrak{g}_{7,2.1(ii)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.1(iii)}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,2.1(iv)}$	$[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.1(v)}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,$ $[e_2,e_3]=e_6,[e_2,e_5]=e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.2}$	$[e_1,e_2]=e_5,[e_1,e_3]=e_6,[e_1,e_4]=2e_7,[e_2,e_3]=e_4,$ $[e_2,e_6]=e_7,[e_3,e_5]=-e_7,[e_3,e_6]=e_7.$
$\mathfrak{g}_{7,2.3}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7.$
$\mathfrak{g}_{7,2.4}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,$ $[e_1,e_5]=e_6,[e_2,e_5]=-e_7,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.5}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.6}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,$ $[e_2,e_3]=e_5,[e_2,e_6]=e_7,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,2.7}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_5.$
$\mathfrak{g}_{7,2.8}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_1,e_5]=e_7,[e_2,e_3]=e_5,[e_2,e_4]=e_7.$
$\mathfrak{g}_{7,2.9}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_2,e_3]=e_5,[e_2,e_5]=e_7.$
$\mathfrak{g}_{7,2.10}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_7,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_6]=e_7,[e_4,e_5]=-e_7.$
$\mathfrak{g}_{7,2.11}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_7,[e_1,e_6]=e_7,[e_2,e_3]=e_6,[e_2,e_4]=e_6.$
$\mathfrak{g}_{7,2.12}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_2,e_4]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.13}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_2,e_3]=e_6,[e_2,e_6]=e_7,[e_4,e_5]=-e_7.$
$\mathfrak{g}_{7,2.14}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,$ $[e_2,e_3]=e_5,[e_3,e_4]=-e_6,[e_3,e_5]=-e_7.$
$\mathfrak{g}_{7,2.15}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,$ $[e_1,e_6]=e_7,[e_2,e_3]=e_6,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,2.16}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_7.$
$\mathfrak{g}_{7,2.17}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_7,[e_2,e_3]=e_5,[e_2,e_4]=e_6,[e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,2.18}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_7,[e_2,e_3]=e_7,[e_2,e_4]=e_6.$
$\mathfrak{g}_{7,2.19}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_6,[e_1,e_4]=e_5,[e_1,e_5]=e_7,[e_2,e_4]=e_6.$
$\mathfrak{g}_{7,2.20}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.21}$	$[e_1,e_2]=e_4,[e_1,e_3]=e_5,[e_1,e_4]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_7.$
$\mathfrak{g}_{7,2.22}$	$[e_1,e_3]=e_4,[e_1,e_4]=e_6,[e_1,e_6]=e_7,[e_2,e_3]=e_5,[e_3,e_4]=e_7.$

Table 4.21: 7-dimensional non-decomposable nilpotent Lie algebras (IV)

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{7,2.23}$	$[e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_5, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.24}$	$[e_1,e_2]=e_4, [e_1,e_4]=e_6, [e_1,e_5]=-e_7, [e_1,e_6]=e_7, [e_2,e_3]=e_5, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.25}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_6, [e_3,e_4]=-e_7, [e_3,e_5]=-e_7.$
$\mathfrak{g}_{7,2.26}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7, [e_3,e_5]=e_6.$
$\mathfrak{g}_{7,2.27}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_7, [e_1,e_5]=e_6, [e_2,e_4]=e_7, [e_2,e_5]=e_7.$
$\mathfrak{g}_{7,2.28}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_1,e_6]=e_7, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.29}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_2,e_3]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.30}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_1,e_6]=e_7, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.31}$	$[e_1,e_2]=e_4, [e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_5, [e_2,e_5]=e_7.$
$\mathfrak{g}_{7,2.32}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.33}$	$[e_1,e_2]=e_4, [e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_5, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.34}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_7, [e_2,e_4]=e_6, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.35}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_6, [e_2,e_4]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.36}$	$[e_1,e_3]=e_5, [e_1,e_4]=e_7, [e_2,e_3]=e_6, [e_2,e_4]=-e_5, [e_3,e_6]=-e_7.$
$\mathfrak{g}_{7,2.37}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_5, [e_1,e_6]=e_7,$ $[e_2,e_4]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,2.38}$	$[e_1,e_2]=e_3, [e_1,e_3]=e_6, [e_1,e_5]=e_7, [e_2,e_3]=e_7, [e_2,e_4]=e_6.$
$\mathfrak{g}_{7,2.39}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_1,e_5]=e_7, [e_2,e_3]=e_6.$
$\mathfrak{g}_{7,2.40}$	$[e_1,e_2]=e_3, [e_1,e_3]=e_5, [e_1,e_4]=e_7, [e_2,e_3]=e_6, [e_2,e_4]=e_5.$
$\mathfrak{g}_{7,2.41}$	$[e_1,e_2]=e_3, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_6, [e_2,e_5]=e_7.$
$\mathfrak{g}_{7,2.42}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_6, [e_2,e_3]=e_6, [e_2,e_4]=e_7.$
$\mathfrak{g}_{7,2.43}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_7, [e_1,e_5]=e_6, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,2.44}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_7, [e_2,e_3]=e_6, [e_2,e_4]=e_7.$
$\mathfrak{g}_{7,2.45}$	$[e_1,e_2]=e_5, [e_1,e_4]=e_7, [e_1,e_5]=e_6, [e_2,e_3]=e_7, [e_2,e_4]=e_6.$
$\mathfrak{g}_{7,3.1(i_\lambda)}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_6]=e_7,$ $[e_2,e_3]=e_6, [e_2,e_5]=\lambda e_7, [e_3,e_4]=e_7(\lambda-1).$
$\mathfrak{g}_{7,3.1(iii)}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_6]=e_7, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.2}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_1,e_6]=e_7.$
$\mathfrak{g}_{7,3.3}$	$[e_1,e_2]=e_4, [e_1,e_4]=e_6, [e_1,e_6]=e_7, [e_2,e_3]=e_5.$
$\mathfrak{g}_{7,3.4}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_2,e_4]=e_6, [e_3,e_5]=e_7$
$\mathfrak{g}_{7,3.5}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_2,e_4]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.6}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_7, [e_2,e_3]=e_6.$
$\mathfrak{g}_{7,3.7}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_2,e_4]=e_6, [e_3,e_4]=-e_7.$
$\mathfrak{g}_{7,3.8}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_1,e_5]=e_7, [e_2,e_4]=e_7.$
$\mathfrak{g}_{7,3.9}$	$[e_1,e_2]=e_5, [e_1,e_5]=-e_7, [e_2,e_3]=e_6, [e_2,e_4]=e_7.$
$\mathfrak{g}_{7,3.10}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_2,e_4]=e_6, [e_2,e_6]=e_7, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,3.11}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_1,e_5]=e_7, [e_2,e_4]=e_6.$
$\mathfrak{g}_{7,3.12}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_2,e_4]=e_6, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.13}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.14}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_1,e_6]=e_7, [e_2,e_4]=e_7.$
$\mathfrak{g}_{7,3.15}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.16}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_7, [e_3,e_4]=e_6, [e_3,e_6]=e_7.$
$\mathfrak{g}_{7,3.17}$	$[e_1,e_2]=e_5, [e_1,e_5]=e_6, [e_1,e_6]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.18}$	$[e_1,e_2]=e_6, [e_1,e_6]=e_7, [e_2,e_5]=e_7, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,3.19}$	$[e_1,e_2]=e_6, [e_1,e_3]=e_7, [e_3,e_4]=e_6, [e_4,e_5]=e_7.$
$\mathfrak{g}_{7,3.20}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_1,e_5]=e_7.$
$\mathfrak{g}_{7,3.21}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_4]=e_6, [e_3,e_5]=e_7.$
$\mathfrak{g}_{7,3.22}$	$[e_1,e_2]=e_4, [e_1,e_3]=e_5, [e_1,e_5]=e_7, [e_2,e_5]=e_6, [e_3,e_4]=e_6.$
$\mathfrak{g}_{7,3.23}$	$[e_1,e_2]=e_3, [e_1,e_3]=e_5, [e_1,e_4]=e_7, [e_2,e_3]=e_6.$
$\mathfrak{g}_{7,3.24}$	$[e_1,e_2]=e_5, [e_2,e_3]=e_6, [e_2,e_4]=e_7, [e_3,e_4]=e_5.$
$\mathfrak{g}_{7,4.1}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_3,e_4]=e_7.$
$\mathfrak{g}_{7,4.2}$	$[e_1,e_2]=e_5, [e_1,e_3]=e_6, [e_1,e_4]=e_7.$
$\mathfrak{g}_{7,4.3}$	$[e_1,e_2]=e_6, [e_3,e_5]=e_6, [e_4,e_5]=e_7.$
$\mathfrak{g}_{7,4.4}$	$[e_1,e_4]=e_7, [e_2,e_5]=e_7, [e_3,e_6]=e_7.$

4.3 Matrix representation of filiform Lie algebras

In this section we obtain a minimal faithful unitriangular matrix representation for each model filiform Lie algebra. Additionally, we also introduce a method for obtaining such representations for non-model filiform Lie algebras, debugging and improving the one given in [7]. In order to compute those representations, we have used the Lie algebras $\mathfrak{g}_n$ of $n \times n$ strictly upper-triangular matrices. Therefore, we have taken advantage of the previous calculations done in Chapter 3 to compare the abelian subalgebras and ideals in Lie algebras $\mathfrak{g}_n$ and the unique abelian ideal of maximal dimension for each filiform Lie algebra, as it was proved in Proposition 3.2. This can be seen in Example 4.3 and in Tables 4.22-4.24 where we have computed the minimal faithful unitriangular matrix representations of filiform Lie algebras with dimension less than 9.

Given a Lie algebra $\mathfrak{g}$, a *representation* of $\mathfrak{g}$ in $\mathbb{C}^n$ is a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(\mathbb{C}, n)$. The natural integer n is called the *dimension* of this representation. Ado's theorem states that every finite-dimensional Lie algebra over a field of characteristic zero (as in the case of $\mathbb{C}$) has a linear injective representation on a finite-dimensional vector space, that is, a *faithful representation.*

Usually, representations are defined by using an arbitrary n-dimensional vector space V (like in [43]) and homomorphisms of Lie algebras from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ of endomorphisms of V; that is, by using $\mathfrak{g}$-*modules.*

With respect to minimal representations of Lie algebras, Burde [15] introduced the following invariant value for an arbitrary Lie algebra $\mathfrak{g}$

$$\mu(\mathfrak{g}) = \min\{\dim(M) \mid M \text{ is a faithful } \mathfrak{g}\text{-module}\}.$$

The invariant $\mu(\mathfrak{g})$ plays an important role in the theory of affine flat manifolds and affine crystallographic groups, see [17]. Moreover, there exist several relations between this invariant and α invariant of Lie algebras. For example, if $\mathfrak{g}$ is a Lie algebra with $dim(\mathcal{D}(\mathfrak{g})) = 1$, then $\mu(\mathfrak{g}) > dim(\mathfrak{g}) - \alpha(\mathfrak{g}) + 1$. In this section, matrix representations of filiform Lie algebras are studied. Moreover, we are interested in minimal matrix representations of these algebras with a particular restriction: the representations have to be contained in $\mathfrak{g}_n$. In this way, given a filiform Lie algebra $\mathfrak{g}$, we want to compute the minimal value n such that $\mathfrak{g}_n$ contains a subalgebra isomorphic to $\mathfrak{g}$. This value is also an invariant of $\mathfrak{g}$ and its expression is given by

$$\bar{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \exists \text{ subalgebra of } \mathfrak{g}_n \text{ isomorphic to } \mathfrak{g}\}.$$

Let us note that the invariants $\mu(\mathfrak{g})$ and $\bar{\mu}(\mathfrak{g})$ can be different from each other.

Proposition 4.7. *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra. Then $\bar{\mu}(\mathfrak{g}) \geq n$.*

Proof. For a given n-dimensional filiform Lie algebra $\mathfrak{g}$, we have to prove that it is not possible to find a subalgebra of $\mathfrak{g}_{n-1}$ isomorphic to $\mathfrak{g}$.

First, we write the vectors of an adapted basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$ as linear combinations of the vectors in the basis $\mathcal{B}_{n-1}$ of $\mathfrak{g}_{n-1}$

$$e_k = \sum_{1 \leq i < j \leq n-1} \lambda_{i,j}^k X_{i,j}, \text{ for } 1 \leq k \leq n.$$

We will prove that all the coefficients $\lambda_{i,j}^2$ of $e_2 \in Z(\mathfrak{g})$ have to be zero.

From the equation $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$, the following relations are obtained

$$\left.\begin{array}{l} \lambda_{\beta,\beta+1}^{h-1} = 0, \\ \lambda_{\beta,\alpha_\beta}^{h-1} = \sum_{\beta<p<\alpha_\beta} (\lambda_{\beta,p}^1 \lambda_{p,\alpha_\beta}^h - \lambda_{p,\alpha_\beta}^1 \lambda_{\beta,p}^h), \end{array}\right\} \text{ for } \left\{\begin{array}{l} 1 \leq \beta \leq n-2; \\ \alpha_\beta \geq \beta + 2. \end{array}\right. \quad (4.1)$$

From the equality $[e_1, e_3] = e_2$, we can conclude that $\lambda_{\beta,\beta+1}^2 = 0$, for $1 \leq \beta \leq n-2$. Now, we have to prove that $\lambda_{l,\alpha_l}^2 = 0$, for $1 \leq l \leq n-3$. To do so, we are going to prove that $\lambda_{p,\alpha_\beta}^3 = \lambda_{\beta,p}^3 = 0$ in each case.

From the equality $[e_1, e_k] = e_{k-1}$, for $3 \leq k \leq n-1$, we can affirm that $\lambda_{\beta,\beta+1}^{k-1} = 0$, for $1 \leq \beta \leq n-2$. This implies that $\lambda_{p,q}^3 = 0$, when $q - p < n - 4$.

If we consider the bracket $[e_1, e_n] = e_{n-1}$, we conclude that $\lambda_{\beta,\beta+1}^{n-1} = 0$, and, therefore, $\lambda_{p,q}^3 = 0$, where $q - p = n - 3$. Consequently, all the coefficients of e_2 are null and this is a contradiction. □

4.3.1 Model filiform Lie algebras

The law of a fixed n-dimensional model filiform Lie algebra $\mathfrak{g}$ with an adapted basis $\{e_i\}_{i=1}^n$ is the following

$$[e_1, e_h] = e_{h-1}, \quad \text{for } 3 \leq h \leq n. \quad (4.2)$$

Now, we will construct the n-dimensional subalgebra $\mathfrak{f}_n'$ of $\mathfrak{g}_n$ and its law is exactly the same of the model filiform Lie algebra $\mathfrak{g}$. Just define a basis $\{e_i\}_{i=1}^n$ of this subalgebra as linear combinations of the vectors in the basis $\mathcal{B}_n$ of the Lie algebra $\mathfrak{g}_n$

$$e_1 = \sum_{i=1}^{n-2} X_{i,i+1}, \; e_2 = X_{1,n}, \; e_3 = X_{2,n}, \; \ldots, \; e_n = X_{n-1,n} \quad (4.3)$$

Consequently, we have defined the subalgebra $\mathfrak{f}'_n$ whose elements have the following form

$$f'_n(x_k) = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 & x_2 \\ 0 & 0 & x_1 & \cdots & 0 & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_1 & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & x_n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad (x_k \in \mathbb{C}, \text{ for } k = 1, \dots, n).$$

With respect to this faithful matrix representation, the adapted basis $\{e_i\}_{i=1}^n$ of $\mathfrak{f}'_n$ is given as follows

$$e_h = f'_n(x_k), \quad \text{with } x_k = \begin{cases} 1, & \text{if } k = h; \\ 0, & \text{if } k \neq h. \end{cases}$$

According to the reasoning just given above and Proposition 2, we can affirm the following result

Proposition 4.8. Every n-dimensional model filiform Lie algebra has an n-dimensional minimal faithful unitriangular matrix representation. Moreover, a representative is given by the Lie algebra $\mathfrak{f}'_n$ given in (4.3). □

4.3.2 Non-model filiform Lie algebras

If the filiform Lie algebra $\mathfrak{g}$ is non-model, then the invariants z_1 and z_2 exist. Hence, there exist some additional non-zero brackets to $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$. Consequently, non-model filiform Lie algebras cannot be represented by Lie algebras $\mathfrak{f}'_n$.

Now, we show an algorithmic method to compute minimal faithful unitriangular matrix representations for non-model filiform Lie algebras. These representations are minimal in the following sense: finding a faithful matrix representation of a given Lie algebra $\mathfrak{g}$ in $\mathfrak{g}_n$, but no representations of $\mathfrak{g}$ can be obtained in $\mathfrak{g}_{n-1}$.

To do so, we give a step-by-step explanation of the method used to determine these minimal representations for a given filiform Lie algebra $\mathfrak{g}$ of dimension $n > 4$.

Let us note that this algorithmic method debugs and improves the one introduced previously in [7] to be applied for any filiform Lie algebra, not necessarily model. When imposing the condition of being non-model, remarkable improvements have been achieved in its implementation with respect to computing time and memory usage.

1. According to Proposition 1.1, we have to compute the first natural integer l such that the lower central series of $\mathfrak{g}_l$ is compatible with the one associated with $\mathfrak{g}$. By bearing in mind Proposition 4.7, we can start considering $l = n$, for n-dimensional filiform Lie algebras. Hence, we have ruled out the Lie algebra $\mathfrak{g}_l$ with $l < n$.

2. Now, we search a subalgebra of $\mathfrak{g}_l$ isomorphic to $\mathfrak{g}$, where $l \geq n$ and l is as small as posible. To do so, an adapted basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$ is considered and its vectors are expressed as linear combinations of the basis $\mathcal{B}_l$

$$e_h = \sum_{1 \leq i < j \leq l} \lambda_{i,j}^h X_{i,j}, \quad \text{for } 1 \leq h \leq n.$$

3. The next step is to impose the brackets given in (4.2), obtaining again the equations shown in (4.1), but with respect to the algebra $\mathfrak{g}_l$ considered by the method.

4. After solving the system of equations resulting from the previous step, we solve a new system obtained by imposing the rest of the brackets in the law of $\mathfrak{g}$. In particular, we will compare the abelian subalgebras of maximal dimension of $\mathfrak{g}$ and $\mathfrak{g}_l$.

Obviously, the solutions of this system depend on the particular Lie algebra studied in each moment. Hence, we have generalized some cases by using invariants z_1 and z_2 in Section 4.3.4. The solutions of the system given in Step 4 have been computed by using the command `solve` of the symbolic computation package Maple 12. This command works efficiently with polynomial equations, receives as inputs the list of equations and the list of variables, and returns as output the algebraic expression of the solutions.

After checking the existence of representations in the Lie algebra $\mathfrak{g}_l$, we search a natural representative in the sense of considering the following conditions: $e_2 \in \langle X_{1,l} \rangle$ and there exist the greatest possible number of null-parameters. The first condition is due to the fact that $Z(\mathfrak{g}_l) = \langle X_{1,l} \rangle$ and is also in concordance with Proposition 2.

Another point to consider is the number of solutions for the system. This number can be studied by defining the set F of polynomial expressions and by using the command `is_finite`, which determines if the number of solutions is or not finite for the system defined by the input set F. Likewise, the Noether normalization lemma is also very useful to describe the elements in an algebraic variety.

Furthermore, in order to compute a particular solution of the previous system, we have searched one such that the number of null coefficients is as greater as possible. In this way, the coefficients could be assumed equal to zero when they do not appear in the relations obtained. This will be a natural representative of the Lie algebra $\mathfrak{g}$.

4.3.3 Examples of application

Next, we give two examples of computing minimal faithful matrix representations for filiform Lie algebras. The first is referred to a model filiform Lie algebra and, afterwards, we apply our algorithmic method (given in the previous section) to compute this type of representation for a non-model filiform Lie algebra.

Example 4.1. Consider model filiform Lie algebra $\mathfrak{f}_4^1$ generated by $\{e_i\}_{i=1}^4$ with law $[e_1, e_3] = e_2, \quad [e_1, e_4] = e_3$.

We can compute its minimal faithful unitriangular matrix representation as was shown in Subsection 4.3.1 by using the Lie algebra $\mathfrak{f}_4'$. However, although we can obtain directly a representative of such representation, we might be interested in determining how many representations the algebra $\mathfrak{f}_4^1$ has: a finite or infinite amount. To do so, it is necessary to apply the algorithm given for non-model filiform Lie algebras in Subsection 4.3.2, but directly over its dimension, 4, and only using Steps 3 and 4. In this way, we determine if there is a finite or infinite number of solution of the system obtained. First, we define the vectors

$$e_1 = \lambda_{1,2}^1 X_{1,2} + \lambda_{1,3}^1 X_{1,3} + \ldots + \lambda_{3,4}^1 X_{3,4},$$

$$e_2 = \lambda_{1,2}^2 X_{1,2} + \lambda_{1,3}^2 X_{1,3} + \ldots + \lambda_{3,4}^2 X_{3,4},$$

$$e_3 = \lambda_{1,2}^3 X_{1,2} + \lambda_{1,3}^3 X_{1,3} + \ldots + \lambda_{3,4}^3 X_{3,4},$$

$$e_4 = \lambda_{1,2}^4 X_{1,2} + \lambda_{1,3}^4 X_{1,3} + \ldots + \lambda_{3,4}^4 X_{3,4}.$$

Now, we introduce the following commands in MAPLE, taking into account that $\lambda_{i,j}^k$ is denoted by `cijk`

`> ec1:=c121*c233-c231*c123-c132:`	$[e_1, e_3] = e_2$
`> ec2:=c121*c243+c131*c343-c241*c123-c341*c133-c142:`	$[e_1, e_3] = e_2$
`> ec3:=c231*c343-c341*c233-c242:`	$[e_1, e_3] = e_2$
`> ec4:=c122:`	$[e_1, e_3] = e_2$
`> ec5:=c232:`	$[e_1, e_3] = e_2$
`> ec6:=c342:`	$[e_1, e_3] = e_2$
`> ec7:=c121*c234-c231*c124-c133:`	$[e_1, e_4] = e_3$
`> ec8:=c121*c244+c131*c344-c241*c124-c341*c134-c143:`	$[e_1, e_4] = e_3$
`> ec9:=c231*c344-c341*c234-c243:`	$[e_1, e_4] = e_3$
`> ec10:=c123:`	$[e_1, e_4] = e_3$
`> ec11:=c233:`	$[e_1, e_4] = e_3$
`> ec12:=c343:`	$[e_1, e_4] = e_3$
`> ec13:=c121*c232-c231*c122:`	$[e_1, e_2] = 0$
`> ec14:=c121*c242+c131*c342-c241*c122-c341*c132:`	$[e_1, e_2] = 0$
`> ec15:=c231*c342-c341*c232:`	$[e_1, e_2] = 0$
`> ec16:=c122*c233-c232*c123:`	$[e_2, e_3] = 0$
`> ec17:=c122*c243+c132*c343-c242*c123-c342*c133:`	$[e_2, e_3] = 0$
`> ec18:=c232*c343-c342*c233:`	$[e_2, e_3] = 0$
`> ec19:=c122*c234-c232*c124:`	$[e_2, e_4] = 0$
`> ec20:=c122*c244+c132*c344-c242*c124-c342*c134:`	$[e_2, e_4] = 0$
`> ec21:=c232*c344-c342*c234:`	$[e_2, e_4] = 0$
`> ec22:=c123*c234-c233*c124:`	$[e_3, e_4] = 0$
`> ec23:=c123*c244+c133*c344-c243*c124-c343*c134:`	$[e_3, e_4] = 0$
`> ec24:=c233*c344-c343*c234:`	$[e_3, e_4] = 0$

Note that the previous commands correspond to the six bracket products giving the law of the Lie algebra $\mathfrak{f}_4^1$ (including the non-zero ones), showing the corresponding bracket beside the commands. All these previous polynomial expressions form the system of equations to be solved to obtain the coefficients $\lambda_{i,j}^k$ determining the subalgebra in $\mathfrak{g}_4$ isomorphic to $\mathfrak{f}_4^1$; i.e. the minimal faithful representation. The system is the following

```
> F:=[ec1,ec2,ec3,ec4,ec5,ec6,ec7,ec8,ec9,ec10,ec11,ec12,
ec13,ec14,ec15,ec16,ec17,ec18,ec19,ec20,ec21,ec22,ec23,ec24];

>is_finite(F);
                              false
```

Consequently, there exist infinite solutions of this system and, hence, of possible minimal faithful representations. However, after applying Noether's normalization lemma and intersecting the previous set with the algebraic variety defined by the equations $\lambda_{1,2}^i = 0$, for $i = 1, 2, 3, 4$, we obtain a finite number of solutions. Effectively

```
eq1:=c121: eq2:=c122: eq3:=c123: eq4:=c124:

H:=[ec1,ec2,ec3,ec4,ec5,ec6,ec7,ec8,ec9,ec10,ec11,ec12,ec13,
ec14,ec15,ec16,ec17,ec18,ec19,ec20,ec21,ec22,ec23,ec24,eq1,eq2,
eq3,eq4]:

> is_finite(H);
                           true
```

However, both the original system and the one after applying Noether's normalization lemma do not impose the condition of "being linearly independent" necessary so that $\{e_i\}_{i=1}^4$ can be a basis of $\mathfrak{f}_4^1$. Hence, we must also consider the condition saved in `eq` corresponding to define a matrix given by the coordinates of e_i in the i^{th} row and then to impose the maximal rank for the matrix.

```
> with(LinearAlgebra):

> M:=Matrix([[c[1,2,1],c[1,3,1],c[1,4,1],c[2,3,1],c[2,4,1],
  c[3,4,1]],[c[1,2,2],c[1,3,2],c[1,4,2],c[2,3,2],c[2,4,2],c[3,4,2]],
  [c[1,2,3],c[1,3,3],c[1,4,3],c[2,3,3],c[2,4,3],c[3,4,3]],
  [c[1,2,4],c[1,3,4],c[1,4,4],c[2,3,4],c[2,4,4],c[3,4,4]]]):

> eq:=Rank(M)=4:
```

Note that this reasoning can be done for every model filiform Lie algebra.

Example 4.2. Consider Lie algebra $\mathfrak{f}_5^2$ generated by $\{e_i\}_{i=1}^5$ and with non-zero brackets: $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$; and $[e_4, e_5] = e_2$.

The lower central series of $\mathfrak{g}_5$ and $\mathfrak{f}_5^2$ are compatible

$$\mathcal{C}^2(\mathfrak{f}_5^2) = \langle e_2, e_3, e_4\rangle \subseteq \mathcal{C}^2(\mathfrak{g}_5) = \langle X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{3,5}\rangle$$

$$\mathcal{C}^3(\mathfrak{f}_5^2) = \langle e_2, e_3\rangle \subseteq \mathcal{C}^3(\mathfrak{g}_5) = \langle X_{1,4}, X_{1,5}, X_{2,5}\rangle$$

$$\mathcal{C}^4(\mathfrak{f}_5^2) = \langle e_2\rangle \subseteq \mathcal{C}^4(\mathfrak{g}_5) = \langle X_{1,5}\rangle$$

After applying Steps 2 and 3, the following basis vectors are obtained

$$e_1 = \lambda_{1,2}^1 X_{1,2} + \lambda_{1,3}^1 X_{1,3} + \ldots + \lambda_{3,5}^1 X_{3,5} + \lambda_{4,5}^1 X_{4,5},$$

$$e_2 = \lambda_{1,5}^2 X_{1,5}, \quad e_3 = \lambda_{1,4}^3 X_{1,4} + \lambda_{1,5}^3 X_{1,5} + \lambda_{2,5}^3 X_{2,5},$$

$$e_4 = \lambda_{1,3}^4 X_{1,3} + \lambda_{1,4}^4 X_{1,4} + \lambda_{1,5}^4 X_{1,5} + \lambda_{2,4}^4 X_{2,4} + \lambda_{2,5}^4 X_{2,5} + \lambda_{3,5}^4 X_{3,5},$$

$$e_5 = \lambda_{1,2}^5 X_{1,2} + \lambda_{1,3}^5 X_{1,3} + \ldots + \lambda_{3,5}^5 X_{3,5} + \lambda_{4,5}^5 X_{4,5},$$

restricted under the following constraints

$$\begin{cases} \lambda^4_{3,5} = & \lambda^1_{3,4}\lambda^5_{4,5} - \lambda^1_{4,5}\lambda^5_{3,4}, \\ \lambda^3_{2,5} = & \lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5} - 2\lambda^1_{4,5}\lambda^1_{2,3}\lambda^5_{3,4} + \lambda^1_{4,5}\lambda^1_{3,4}\lambda^5_{2,3}, \\ \lambda^4_{1,5} = & \lambda^1_{1,2}\lambda^5_{2,5} + \lambda^1_{1,3}\lambda^5_{3,5} + \lambda^1_{1,4}\lambda^5_{4,5} - \lambda^1_{4,5}\lambda^5_{1,4} - \lambda^1_{3,5}\lambda^5_{1,3} - \lambda^1_{2,5}\lambda^5_{1,2}, \\ \lambda^3_{1,4} = & \lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,4} - 2\lambda^1_{1,2}\lambda^1_{3,4}\lambda^5_{2,3} + \lambda^1_{3,4}\lambda^1_{2,3}\lambda^5_{1,2}, \\ \lambda^2_{1,5} = & \lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5} - 3\lambda^1_{4,5}\lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,4} + 3\lambda^1_{4,5}\lambda^1_{1,2}\lambda^1_{3,4}\lambda^5_{2,3} - \lambda^1_{4,5}\lambda^1_{3,4}\lambda^1_{2,3}\lambda^5_{1,2}, \\ \lambda^3_{1,5} = & \lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,5} + \lambda^1_{1,2}\lambda^1_{2,4}\lambda^5_{4,5} - 2\lambda^1_{4,5}\lambda^1_{1,2}\lambda^5_{2,4} - \lambda^1_{1,2}\lambda^1_{3,4}\lambda^5_{2,3} + \lambda^1_{1,3}\lambda^1_{3,4}\lambda^5_{4,5} \\ & -2\lambda^1_{1,3}\lambda^1_{4,5}\lambda^5_{3,4} + \lambda^1_{4,5}\lambda^1_{2,4}\lambda^5_{1,2} + \lambda^1_{4,5}\lambda^1_{3,4}\lambda^5_{1,3} - \lambda^1_{3,5}\lambda^1_{1,2}\lambda^5_{2,3} + \lambda^1_{3,5}\lambda^1_{2,3}\lambda^5_{1,2}, \\ \lambda^4_{1,3} = & \lambda^1_{1,2}\lambda^5_{2,3} - \lambda^1_{2,3}\lambda^5_{1,2}, \\ \lambda^4_{1,4} = & \lambda^1_{1,2}\lambda^5_{2,4} + \lambda^1_{1,3}\lambda^5_{3,4} - \lambda^1_{2,4}\lambda^5_{1,2} - \lambda^1_{3,4}\lambda^5_{1,3}, \\ \lambda^4_{2,5} = & \lambda^1_{2,3}\lambda^5_{3,5} + \lambda^1_{2,4}\lambda^5_{4,5} - \lambda^1_{4,5}\lambda^5_{2,4} - \lambda^1_{3,4}\lambda^5_{2,3}, \ \lambda^4_{2,4} = \lambda^1_{2,3}\lambda^5_{3,4} - \lambda^1_{3,4}\lambda^5_{2,3}. \end{cases}$$

Note that the previous system is of degree four, being well-known the existence of efficient methods in Algebraic Geometry to solve this type of systems. For a filiform Lie algebra of higher dimension, the degree of the corresponding system is always one unit less than the dimension of the algebra.

Now, by imposing the bracket $[e_4, e_5] = e_2$, the following vectors are obtained

$$e_1 = \lambda^1_{1,2}X_{1,2} + \lambda^1_{1,3}X_{1,3} + \ldots + \lambda^1_{3,5}X_{3,5} + \lambda^1_{4,5}X_{4,5},$$

$$e_2 = \lambda^2_{1,5}X_{1,5}, \quad e_3 = \lambda^3_{1,4}X_{1,4} + \lambda^3_{1,5}X_{1,5} + \lambda^3_{2,5}X_{2,5},$$

$$e_4 = \lambda^4_{1,3}X_{1,3} + \lambda^4_{1,4}X_{1,4} + \lambda^4_{1,5}X_{1,5} + \lambda^4_{2,5}X_{2,5} + \lambda^4_{3,5}X_{3,5},$$

$$e_5 = \lambda^5_{1,2}X_{1,2} + \lambda^5_{1,3}X_{1,3} + \ldots + \lambda^5_{1,5}X_{1,5} + \lambda^5_{2,4}X_{2,4} + \ldots + \lambda^5_{4,5}X_{4,5},$$

under these constraints

$$\begin{cases} 3\lambda^1_{4,5}\lambda^4_{1,4} = 3\lambda^1_{1,2}\lambda^5_{2,4}\lambda^1_{4,5} - \lambda^5_{1,3}\lambda^5_{3,5}, \ 3\lambda^1_{4,5}\lambda^3_{1,4} = -2\lambda^1_{1,2}\lambda^5_{3,5}\lambda^5_{2,3}, \\ 3\lambda^1_{4,5}\lambda^4_{2,5} = 3\lambda^1_{4,5}\lambda^1_{2,3}\lambda^5_{3,5} - 3(\lambda^1_{4,5})^2\lambda^5_{2,4} - \lambda^5_{3,5}\lambda^5_{2,3}, \ 3\lambda^3_{2,5} = \lambda^5_{3,5}, \\ \lambda^4_{1,5} = \lambda^1_{1,2}\lambda^5_{2,5} + \lambda^1_{1,3}\lambda^5_{3,5} - \lambda^1_{4,5}\lambda^5_{1,4} - \lambda^1_{3,5}\lambda^5_{1,3}, \\ 3\lambda^1_{4,5}\lambda^1_{3,4} = \lambda^5_{3,5}, \lambda_{1,34} = \lambda^1_{1,2}\lambda^5_{2,3}, \ \lambda^2_{1,5} = \lambda^5_{3,5}\lambda^1_{1,2}\lambda^5_{2,3}, \\ 3\lambda^1_{4,5}\lambda^4_{2,4} = -\lambda^5_{3,5}\lambda^5_{2,3}, \ 3\lambda^1_{4,5}\lambda^3_{1,5} = 3\lambda^1_{1,2}\lambda^1_{4,5}\lambda^1_{2,3}\lambda^5_{3,5} - \\ 6\lambda^1_{1,2}\lambda^5_{2,4}(\lambda^1_{4,5})^2 - \lambda^5_{3,5}\lambda^1_{1,2}\lambda^5_{2,3} + \lambda^1_{4,5}\lambda^5_{1,3}\lambda^5_{3,5} - 3\lambda^1_{3,5}\lambda^1_{1,2}\lambda^5_{2,3}\lambda^1_{4,5}. \end{cases}$$

Next, we show a particular solution to obtain a representative for Lie algebra $\mathfrak{f}^2_5$

$$e_1 = X_{1,2} + X_{2,3} + X_{3,4}, \quad e_2 = X_{1,5}, \quad e_3 = X_{2,5},$$

$$e_4 = X_{1,4} + X_{3,5}, \quad e_5 = X_{2,4} + X_{4,5}.$$

Example 4.3. *We consider the Lie algebra $\mathfrak{f}_6^4$ generated by $\{e_i\}_{i=1}^6$ and with non-zero brackets: $[e_1, e_h] = e_{h-1}$, for $3 \le h \le 6$; and $[e_4, e_5] = e_2, [e_4, e_6] = e_3, [e_5, e_6] = e_4$.*

The lower central series of $\mathfrak{g}_6$ and $\mathfrak{f}_6^4$ are compatible

$$\mathcal{C}^2(\mathfrak{f}_6^4) = \langle e_2, e_3, e_4, e_5 \rangle \subseteq \mathcal{C}^2(\mathfrak{g}_6) = \langle X_{1,3}, X_{1,4}, X_{1,5}, X_{1,6}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6} \rangle$$

$$\mathcal{C}^3(\mathfrak{f}_6^4) = \langle e_2, e_3, e_4 \rangle \subseteq \mathcal{C}^3(\mathfrak{g}_6) = \langle X_{1,4}, X_{1,5}, X_{1,6}, X_{2,5}, X_{2,6}, X_{3,6} \rangle$$

$$\mathcal{C}^4(\mathfrak{f}_6^4) = \langle e_2, e_3 \rangle \subseteq \mathcal{C}^4(\mathfrak{g}_6) = \langle X_{1,5}, X_{1,6}, X_{2,6} \rangle$$

$$\mathcal{C}^5(\mathfrak{f}_6^4) = \langle e_2 \rangle \subseteq \mathcal{C}^5(\mathfrak{g}_6) = \langle X_{1,6} \rangle$$

After that, we apply Steps 2 and 3, obtaining these vectors

$$e_1 = \lambda_{1,2}^1 X_{1,2} + \lambda_{1,3}^1 X_{1,3} + \lambda_{1,4}^1 X_{1,4} + \ldots + \lambda_{4,5}^1 X_{4,5} + \lambda_{4,6}^1 X_{4,6} + \lambda_{5,6}^1 X_{5,6},$$

$$e_2 = \lambda_{1,6}^2 X_{1,6}, e_3 = \lambda_{1,5}^3 X_{1,5} + \lambda_{1,6}^3 X_{1,6} + \lambda_{2,6}^3 X_{2,6},$$

$$e_4 = \lambda_{1,4}^4 X_{1,4} + \lambda_{1,5}^4 X_{1,5} + \lambda_{1,6}^4 X_{1,6} + \lambda_{2,5}^4 X_{2,5} + \lambda_{2,6}^4 X_{2,6} + \lambda_{3,6}^4 X_{3,6},$$

$$e_5 = \lambda_{1,3}^5 X_{1,3} + \ldots + \lambda_{1,6}^5 X_{1,6} + \lambda_{2,4}^5 X_{2,4} + \ldots + \lambda_{2,6}^5 X_{2,6} + \lambda_{3,5}^5 X_{3,5} + \lambda_{3,6}^5 X_{3,6} + \lambda_{4,6}^5 X_{4,6},$$

$$e_6 = \lambda_{1,2}^6 X_{1,2} + \lambda_{1,3}^6 X_{1,3} + \lambda_{1,4}^6 X_{1,4} + \ldots + \lambda_{4,5}^6 X_{4,5} + \lambda_{4,6}^6 X_{4,6} + \lambda_{5,6}^6 X_{5,6}.$$

Now, we impose the brackets $[e_4, e_5] = e_2, [e_4, e_6] = e_3, [e_5, e_6] = e_4$ and the fact that $\langle e_2, e_3, e_4 \rangle$ and $\langle X_{1,4}, X_{1,5}, X_{1,6}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,4}, X_{3,5}, X_{3,6} \rangle$ are abelian ideals of maximal dimension in $\mathfrak{f}_6^4$ and $\mathfrak{g}_6$, respectively. Under those conditions, we can obtain the following representative

$$e_1 = X_{1,2} + X_{2,3} + X_{3,4} + X_{4,5},\ e_2 = X_{1,6},\ e_3 = -\frac{1}{2}X_{1,5} + X_{2,6},$$

$$e_4 = \frac{1}{2}X_{1,4} + \frac{1}{3}X_{1,5} + X_{3,6},\ e_5 = -\frac{1}{2}X_{1,3} + \frac{1}{3}X_{2,5} + X_{4,6},$$

$$e_6 = -\frac{1}{2}X_{1,2} + \frac{1}{3}X_{1,3} - X_{2,3} + \frac{1}{3}X_{2,4} - X_{3,4} - \frac{1}{3}X_{3,5} - X_{4,5} + X_{5,6}.$$

4.3.4 Filiform Lie algebras of dimension less than 9

It is well-known that $\mu(\mathfrak{g}) = dim(\mathfrak{g})$ when $\mathfrak{g}$ is a filiform Lie algebra of dimension less than 10 (see [15]). Our main goal is to show explicit representatives for the minimal faithful unitriangular matrix representation of filiform Lie algebras with dimension less than 9 and the generalization for two families of n-dimensional filiform Lie algebras where $\mathcal{D}(\mathfrak{g})$ is abelian. In Tables 4.22–4.24, we write those representatives by using the classification given in [13]. Let us note that, for these algebras, we have only written down the non-zero brackets not given by filiformity; i.e. we are assuming implicitly Equation (4.2). In this way, we can state the following results

Proposition 4.9. If $\mathfrak{g}$ is a filiform Lie algebra of dimension $n < 9$, then $\bar{\mu}(\mathfrak{g}) = n$ (i.e. the minimal faithful matrix representation of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}_n$). Moreover, such representations can be obtained with a natural representative.

Proposition 4.10. *Let $\mathfrak{f}$ be an n-dimensional filiform Lie algebra verifying $z_1 = z_2 = n-1$. Then, $\bar{\mu}(\mathfrak{f}) = n$ and a natural representative of $\mathfrak{f}$ is determined by the following vectors*

$$e_1 = \sum_{i=1}^{n-2} X_{i,i+1}; \quad e_k = X_{k-1,n}, \, \forall\, 2 \leq k \leq n-2;$$

$$e_{n-1} = X_{1,n-1} + X_{n-2,n}; \quad e_n = X_{2,n-1} + X_{n-1,n}.$$

Proof. It suffices to apply our algorithmic procedure taking into consideration that non-zero brackets for the case $z_1 = z_2 = n-1$ are the following

$$[e_1, e_h] = e_{h-1}, \, \forall\, 3 \leq h \leq n; \quad [e_{n-1}, e_n] = e_2. \qquad \square$$

Proposition 4.11. *Let $\mathfrak{f}$ be an n-dimensional filiform Lie algebra verifying $z_1 = n-2$ and $z_2 = n-1$. Then, $\bar{\mu}(\mathfrak{f}) = n$ and a natural representative of $\mathfrak{f}$ is determined by the following vectors*

$$e_1 = \sum_{i=1}^{n-2} X_{i,i+1}; \quad e_k = X_{k-1,n}, \, \forall\, 2 \leq k \leq n-2; \quad e_{n-2} = X_{1,n-1} + X_{n-3,n};$$

$$e_{n-1} = X_{2,n-1} + X_{n-2,n}; \quad e_n = X_{3,n-1} + X_{n-1,n}.$$

Proof. Analogous to the previous proof, but with the following non-zero brackets

$$[e_1, e_h] = e_{h-1}, \, \forall\, 3 \leq h \leq n; \quad [e_{n-2}, e_n] = e_2; \quad [e_{n-1}, e_n] = e_3. \qquad \square$$

Table 4.22: Minimal faithful matrix representations for dimension ≤ 7

	Law	**Representation**
$\mathfrak{f}_3^1$		$e_1 = X_{1,2}, e_2 = X_{1,3}, e_3 = X_{2,3}$.
$\mathfrak{f}_4^1$		$e_1 = X_{1,2} + X_{2,3}, e_2 = X_{1,4}, e_3 = X_{2,4}, e_4 = X_{3,4}$.
$\mathfrak{f}_5^1$		$e_1 = X_{1,2} + X_{2,3} + X_{3,4}, e_2 = X_{1,5},$ $e_3 = X_{2,5}, e_4 = X_{3,5}, e_5 = X_{4,5}$.
$\mathfrak{f}_5^2$	$[e_4, e_5] = e_2$.	$e_1 = X_{1,2} + X_{2,3} + X_{3,4}, e_2 = X_{1,5}, e_3 = X_{2,5},$ $e_4 = X_{1,4} + X_{3,5}, e_5 = X_{2,4} + X_{4,5}$.
$\mathfrak{f}_6^1$		$e_1 = X_{1,2} + X_{2,3} + X_{3,4} + X_{4,5}, e_2 = X_{1,6}, e_3 = X_{2,6},$ $e_4 = X_{3,6}, e_5 = X_{4,6}, e_6 = X_{5,6}$.
$\mathfrak{f}_6^2$	$[e_5, e_6] = e_2$.	$e_1 = \sum_{i=1}^4 X_{i,i+1}, e_2 = X_{1,6}, e_3 = X_{2,6},$ $e_4 = X_{3,6}, e_5 = X_{1,5} + X_{4,6}, e_6 = X_{2,5} + X_{4,6}$.
$\mathfrak{f}_6^3$	$[e_4, e_6] = e_2,$ $[e_5, e_6] = e_3$.	$e_1 = \sum_{i=1}^4 X_{i,i+1}, e_2 = X_{1,6}, e_3 = X_{2,6},$ $e_4 = X_{3,6} + X_{1,5}, e_5 = X_{2,5} + X_{4,6}, e_6 = X_{3,5} + X_{5,6}$.
$\mathfrak{f}_6^4$	$[e_4, e_5] = e_2$ $[e_4, e_6] = e_3,$ $[e_5, e_6] = e_4$.	$e_1 = X_{1,2} + X_{2,3} + X_{3,4} + X_{4,5}, e_2 = X_{1,6},$ $e_3 = \frac{-1}{2} X_{1,5} + X_{2,6}, e_4 = \frac{1}{2} X_{1,4} + \frac{1}{3} X_{1,5} + X_{3,6},$ $e_5 = \frac{-1}{2} X_{1,3} + \frac{1}{3} X_{2,5} + X_{4,6}, e_6 = \frac{-1}{2} X_{1,2}+$ $\frac{1}{3} X_{1,3} - X_{2,3} + \frac{1}{3} X_{2,4} - X_{3,4} - \frac{1}{3} X_{3,5} - X_{4,5} + X_{5,6}$
$\mathfrak{f}_6^5$	$[e_4, e_5] = e_2$ $[e_4, e_6] = e_3 + e_2,$ $[e_5, e_6] = e_4 + e_3$.	$e_1 = X_{1,2} + X_{2,3} + X_{3,4} + X_{4,5}, e_2 = X_{1,6}, e_3 = \frac{-1}{2} X_{1,5}$ $+X_{2,6}, e_4 = \frac{1}{2} X_{1,4} + X_{3,6}, e_5 = -\frac{1}{2} X_{1,3} + X_{2,5} + X_{4,6},$ $e_6 = -\frac{1}{2} X_{1,2} - X_{1,3} - X_{2,3} - X_{3,4} - X_{4,5} + X_{5,6}$.
$\mathfrak{f}_7^1$		$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = X_{1,7}, e_3 = X_{2,7},$ $e_4 = X_{3,7}, e_5 = X_{4,7}, e_6 = X_{5,7}, e_7 = X_{6,7}$.
$\mathfrak{f}_7^2$	$[e_6, e_7] = e_2$	$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = X_{1,7}, e_3 = X_{2,7}, e_4 = X_{3,7},$ $e_5 = X_{4,7}, e_6 = X_{1,6} + X_{5,7}, e_7 = X_{2,6} + X_{6,7}$.
$\mathfrak{f}_7^3$	$[e_5, e_7] = e_2,$ $[e_6, e_7] = e_3$	$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = X_{1,7}, e_3 = X_{2,7}, e_4 = X_{3,7},$ $e_5 = X_{4,7}, e_6 = X_{5,7}, e_7 = -X_{1,4} - X_{2,5} - X_{3,6} + X_{6,7}$.
$\mathfrak{f}_7^4$	$[e_5, e_7] = e_2,$ $[e_6, e_7] = e_2 + e_3$	$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = -4X_{1,7}, e_3 = X_{1,6} - 18X_{1,7} - 3X_{2,7},$ $e_4 = 9X_{1,6} - 66X_{1,7} + X_{2,6} - 9X_{2,7} - 2X_{3,7}, e_5 = -3X_{1,5}+$ $44X_{1,6} + 6X_{2,6} - 22X_{2,7} + X_{3,6} - 3X_{3,7} - X_{4,7},$ $e_6 = -22X_{1,5} - 3X_{2,5} + 22X_{2,6} + 3X_{3,6} + X_{4,6},$ $e_7 = -X_{1,4} - 22X_{2,5} - 3X_{3,5} + X_{5,7}$.
$\mathfrak{f}_7^5$	$[e_4, e_7] = [e_5, e_7] = e_2$ $[e_5, e_6] = -e_2,$ $[e_6, e_7] = e_3$	$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = -5X_{1,7}, e_3 = X_{1,6} - 4X_{2,7}, e_4 = -X_{2,6}$ $-3X_{3,7}, e_5 = -\frac{5}{3} X_{1,5} + X_{3,6} - 2X_{4,7}, e_6 = \frac{5}{3} X_{1,4} - \frac{19}{6} X_{1,5} + X_{4,6}-$ $X_{4,7} - X_{5,7}, e_7 = -\frac{5}{3} X_{1,3} - \frac{5}{2} X_{1,4} + \frac{5}{2} X_{1,5} - 4X_{2,5} + X_{4,6} + X_{5,6}$.
$\mathfrak{f}_7^6$	$[e_4, e_7] = e_2,$ $[e_5, e_7] = e_3$ $[e_6, e_7] = e_2 + e_4$	$e_1 = \sum_{i=1}^5 X_{i,i+1}, e_2 = 2X_{1,7}, e_3 = -\frac{1}{2} X_{1,6} + \frac{3}{2} X_{2,7},$ $e_4 = X_{1,5} - 2X_{1,7} -! \frac{1}{2} X_{2,6} + X_{3,7}, e_5 = X_{1,6} + X_{2,5} - \frac{1}{2} X_{3,6}$ $-X_{4,7}, e_6 = X_{3,5} + X_{4,6}, e_7 = -X_{1,3} + \frac{1}{2} X_{2,4} + 2X_{2,5}-$ $2X_{4,5} + X_{5,7} + X_{2,6} - X_{3,5} - X_{4,6} - X_{5,6} + X_{6,7}$.
$\mathfrak{f}_7^7$	$[e_5, e_7] = e_3,$ $[e_4, e_7] = e_2$ $[e_6, e_7] = e_4$	$e_1 = X_{1,2} + X_{2,3} - \frac{1}{2} X_{3,4} + X_{6,7}, e_2 = 2X_{1,7}, e_3 = -\frac{1}{2} X_{1,6}$ $+\frac{3}{2} X_{2,7}, e_4 = X_{1,5} - \frac{1}{2} X_{2,6} + X_{3,7}, e_5 = X_{2,5} - \frac{1}{2} X_{3,6} - X_{4,7},$ $e_6 = X_{3,5} + X_{4,6}, e_7 = -X_{1,3} + \frac{1}{2} X_{2,4} - 2X_{4,5} + X_{5,7}$
$\mathfrak{f}_7^8$	$[e_4, e_7] = \alpha e_2,$ $[e_5, e_6] = e_2$ $[e_5, e_7] = (1+\alpha) e_3,$ $[e_6, e_7] = (1+\alpha) e_4$	$e_1 = X_{1,3} + X_{2,3} - X_{2,4} + X_{3,4} + \beta i X_{3,6} + X_{4,5} + i X_{5,6} + X_{6,7},$ $e_2 = -i X_{1,7}, e_3 = i X_{1,6} - i X_{1,7}, e_4 = -X_{1,5} + i X_{1,6}+$ $(\frac{1}{2} i\beta - 1 - \frac{1}{2} i) X_{1,7} + \frac{1}{2} i(\beta + 1 + 2\alpha) X_{2,7}, e_5 = X_{1,4} - X_{1,5}$ $+X_{1,6} - (1+\alpha) i X_{2,6} + \frac{1}{2} i(\beta + 3 + 2\alpha) X_{2,7} + \frac{1}{2} i(\beta - 1) X_{3,7},$ $e_6 = -X_{1,3} + X_{1,4} + i X_{1,5} + (\alpha - \beta + 1) X_{2,5} - i(1+\alpha) X_{2,6}+$ $(\beta + \frac{4}{3} \alpha i + 2 + i + \alpha + \frac{1}{3} i\alpha^2) X_{2,7} - i\beta X_{3,6} - \frac{1}{2} i(\beta+1) X_{4,7},$ $e_7 = (\beta - i) X_{1,4} + X_{1,2} + (2\beta - 1 - \alpha) X_{2,4} + (1+\alpha) X_{2,5}-$ $(\beta + \frac{4}{3} i\alpha - 2 - i - \alpha - \frac{1}{3} i\alpha^2) X_{2,6} + \beta X_{3,5} - \frac{1}{2} i(\beta+1) X_{5,7},$ β is a root of $3Z^2 - 2\alpha^2 - 5\alpha - 3 - 3\alpha Z$.

Table 4.23: Minimal faithful matrix representations for dimension 8 (I)

	Law	**Representation**
$\mathfrak{f}_8^1$		$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8},$ $e_5=X_{4,8}, e_6=X_{5,8}, e_7=X_{6,8}, e_8=X_{7,8}.$
$\mathfrak{f}_8^2$	$[e_7,e_8]=e_2$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8}, e_5=X_{4,8},$ $e_6=X_{5,8},\ e_7=X_{1,7}+X_{6,8}, e_8=X_{2,7}+X_{7,8}.$
$\mathfrak{f}_8^3$	$[e_6,e_8]=e_2,$ $[e_7,e_8]=e_3$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8}, e_5=X_{4,8},$ $e_6=X_{5,8},\ e_7=X_{6,8}, e_8=-X_{1,5}-X_{2,6}-X_{3,7}+X_{7,8}.$
$\mathfrak{f}_8^4$	$[e_6,e_8]=e_2,$ $[e_7,e_8]=e_2+e_3$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8}, e_5=X_{4,8},$ $e_6=X_{5,8}, e_7=X_{6,8},$ $e_8=-X_{1,5}-X_{1,6}-X_{2,6}-X_{2,7}-X_{3,7}+X_{7,8}.$
$\mathfrak{f}_8^5$	$[e_5,e_8]=e_2,$ $[e_6,e_8]=e_3$ $[e_7,e_8]=e_2+e_4$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8}, e_5=X_{4,8},$ $e_6=X_{5,8}, e_7=X_{6,8},$ $e_8=-X_{1,4}-X_{1,6}-X_{2,5}-X_{2,7}-X_{3,6}-X_{4,7}+X_{7,8}$
$\mathfrak{f}_8^6$	$[e_5,e_8]=e_2,$ $[e_6,e_8]=e_2+e_3$ $[e_7,e_8]=\alpha e_2+e_3+e_4$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8}, e_5=X_{4,8},$ $e_6=X_{5,8}, e_7=X_{6,8}, e_8=-X_{1,4}-X_{1,5}-\alpha X_{1,6}+X_{1,8}-$ $X_{2,5}-X_{2,6}-\alpha X_{2,7}-X_{3,6}-X_{3,7}-X_{4,7}+X_{7,8}$
$\mathfrak{f}_8^7$	$[e_5,e_8]=\alpha e_2,$ $[e_6,e_7]=e_2$ $[e_6,e_8]=(1+\alpha)e_3$ $[e_7,e_8]=(1+\alpha)e_4$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8},$ $e_5=-X_{1,7}+X_{4,8}, e_6=X_{1,6}+X_{5,8},$ $e_7=X_{2,6}+X_{3,7}+X_{6,8}, e_8=-(1+\alpha)X_{1,4}+X_{1,8}$ $-(1+\alpha)X_{2,5}-\alpha X_{3,6}+(1-\alpha)X_{4,7}+X_{7,8}$
$\mathfrak{f}_8^8$	$[e_5,e_8]=\alpha e_2,$ $[e_6,e_7]=e_2$ $[e_6,e_8]=(1+\alpha)e_3+e_2$ $[e_7,e_8]=(1+\alpha)e_4+e_3$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8},$ $e_5=-X_{1,7}+X_{4,8}, e_6=X_{1,6}+X_{5,8}, e_7=X_{2,6}+X_{3,7}+X_{6,8},$ $e_8=-(1+\alpha)X_{1,4}-X_{1,5}+X_{1,8}-(1+\alpha)X_{2,5}-X_{2,6}$ $-\alpha X_{3,6}-X_{3,7}+(1-\alpha)X_{4,7}+X_{7,8}.$
$\mathfrak{f}_8^9$	$[e_4,e_8]=e_2,$ $[e_5,e_8]=e_3$ $[e_6,e_8]=e_4,$ $[e_7,e_8]=e_5$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{1,7}+X_{3,8},$ $e_5=X_{2,7}+X_{4,8}, e_6=-\frac{1}{2}X_{1,5}-\frac{1}{2}X_{2,6}+\frac{1}{2}X_{3,7}+X_{5,8},$ $e_7=-\frac{1}{2}X_{2,5}-X_{3,6}-\frac{1}{2}X_{4,7}+X_{6,8},$ $e_8=-\frac{1}{2}X_{3,5}-\frac{3}{2}X_{4,6}-2X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{10}$	$[e_4,e_8]=e_2,$ $[e_5,e_8]=e_3$ $[e_6,e_8]=e_4,$ $[e_7,e_8]=e_2+e_5$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{1,7}+X_{3,8},$ $e_5=X_{2,7}+X_{4,8}, e_6=-\frac{1}{2}X_{1,5}-\frac{1}{2}X_{2,6}+\frac{1}{2}X_{3,7}+X_{5,8},$ $e_7=-\frac{1}{2}X_{2,5}-X_{3,6}-\frac{1}{2}X_{4,7}+X_{6,8},$ $e_8=-X_{1,6}-X_{2,7}-\frac{1}{2}X_{3,5}-\frac{3}{2}X_{4,6}-2X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{11}$	$[e_4,e_8]=e_2,$ $[e_5,e_8]=e_3$ $[e_6,e_8]=e_2+e_4,$ $[e_7,e_8]=\alpha e_2+e_3+e_5$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{1,7}+X_{3,8},$ $e_5=X_{2,7}+X_{4,8}, e_6=-\frac{1}{2}X_{1,5}-\frac{1}{2}X_{2,6}+\frac{1}{2}X_{3,7}+X_{5,8},$ $e_7=-\frac{1}{2}X_{2,5}-X_{3,6}-\frac{1}{2}X_{4,7}+X_{6,8}, e_8=-X_{1,5}-\alpha X_{1,6}$ $-X_{2,6}-\alpha X_{2,7}-\frac{1}{2}X_{3,5}-X_{3,7}-\frac{3}{2}X_{4,6}-2X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{12}$	$[e_4,e_8]=e_2,$ $[e_5,e_8]=e_3,$ $[e_6,e_7]=e_2,$ $[e_6,e_8]=\alpha e_2+e_3+e_4$ $[e_7,e_8]=\alpha e_3+e_4+e_5$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{1,7}+X_{3,8},$ $e_5=-2X_{1,7}+X_{2,7}+X_{4,8}, e_6=-\frac{1}{2}X_{1,5}+X_{1,6}-\frac{1}{2}X_{2,6}$ $-X_{2,7}+\frac{1}{2}X_{3,7}+X_{5,8}, e_7=-\frac{1}{2}X_{2,5}+X_{2,6}-X_{3,6}-$ $\frac{1}{2}X_{4,7}+X_{6,8}, e_8=-2X_{1,4}-\alpha X_{1,5}-2X_{2,5}-\alpha X_{2,6}$ $-\frac{1}{2}X_{3,5}-X_{3,6}-\alpha X_{3,7}-\frac{3}{2}X_{4,6}-X_{4,7}-2X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{13}$	$[e_4,e_8]=-e_2,$ $[e_6,e_8]=e_3+e_4,$ $[e_6,e_7]=e_2+e_3,$ $[e_5,e_7]=e_2,$ $[e_7,e_8]=e_4+e_5$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8}, e_4=X_{3,8},$ $e_5=X_{1,6}+X_{2,7}+X_{4,8},$ $e_6=-X_{1,5}+\frac{1}{2}X_{1,6}+\frac{1}{2}X_{2,7}+X_{3,7}+X_{5,8},$ $e_7=-\frac{1}{2}X_{1,5}-X_{2,5}-X_{3,6}+\frac{1}{2}X_{3,7}+X_{6,8},$ $e_8=X_{1,3}+X_{2,4}-\frac{1}{2}X_{2,5}-\frac{1}{2}X_{3,6}-X_{4,6}-X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{14}$	$[e_4,e_8]=\alpha e_2,$ $[e_5,e_7]=e_2$ $[e_5,e_8]=-e_3,$ $[e_6,e_7]=e_3$	$e_1=\sum_{i=1}^6 X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8},$ $e_4=-2X_{1,7}+X_{3,8}, e_5=X_{1,6}-X_{2,7}+X_{4,8},$ $e_6=X_{2,6}+X_{5,8}, e_7=X_{3,6}+X_{4,7}+X_{6,8},$ $e_8=X_{4,6}+2X_{5,7}+X_{7,8}.$

Table 4.24: Minimal faithful matrix representations for dimension 8 (II)

	Law	**Representation**
$\mathfrak{f}_8^{15}$	$[e_4,e_8]=-2e_2,$ $[e_5,e_7]=e_2$ $[e_5,e_8]=-e_3+e_2,$ $[e_6,e_7]=e_3$ $[e_6,e_8]=e_3,$ $[e_7,e_8]=e_4$	$e_1=\sum_{i=1}^{6} X_{i,i+1}, e_2=X_{1,8}, e_3=X_{2,8},$ $e_4=-2X_{1,7}+X_{3,8}, e_5=X_{1,6}+3X_{1,7}-$ $X_{2,7}+X_{4,8}, e_6=X_{2,6}+3X_{2,7}+X_{5,8},$ $e_7=X_{3,6}+X_{3,7}+X_{4,7}+X_{6,8},$ $e_8=2X_{1,4}+2X_{2,5}+2X_{3,6}+$ $X_{4,6}+5X_{4,7}+2X_{5,7}+X_{7,8}.$
$\mathfrak{f}_8^{16}$	$[e_4,e_7]=e_2,$ $[e_5,e_6]=-e_2$ $[e_4,e_8]=e_3$ $[e_5,e_8]=e_4,$ $[e_6,e_8]=e_5,$ $[e_7,e_8]=e_6.$	$e_1=\sum_{i=1}^{6} X_{i,i+1}, e_2=X_{1,8}, e_3=-\frac{1}{2}X_{1,7}+$ $X_{2,8}, e_4=\frac{1}{2}X_{1,6}+X_{3,8}, e_5=-\frac{1}{2}X_{1,5}+$ $X_{4,8}, e_6=\frac{1}{2}X_{1,4}+X_{5,8}, e_7=-\frac{1}{2}X_{1,3}$ $+X_{6,8}, e_8=-\frac{1}{2}X_{1,2}-X_{2,3}-X_{3,4}-X_{4,5}$ $-X_{5,6}-X_{6,7}+X_{7,8}.$
$\mathfrak{f}_8^{17}$	$[e_4,e_7]=e_2,$ $[e_4,e_8]=e_3$ $[e_5,e_6]=-e_2,$ $[e_5,e_8]=e_4,$ $[e_6,e_7]=e_2,$ $[e_6,e_8]=e_3+e_5$ $[e_7,e_8]=e_4+e_6.$	$e_1=\sum_{i=1}^{6} X_{i,i+1}, e_2=X_{1,8}, e_3=-\frac{1}{2}X_{1,7}$ $+X_{2,8}, e_4=\frac{1}{2}X_{1,6}+X_{3,8}, e_5=-\frac{1}{2}X_{1,5}-$ $3X_{1,7}+X_{4,8}, e_6=\frac{1}{2}X_{1,4}+X_{1,6}$ $-2X_{2,7}+X_{5,8}, e_7=-\frac{1}{2}X_{1,3}+X_{2,6}-X_{3,7}$ $+X_{6,8}, e_8=-3X_{1,4}-\frac{1}{2}X_{1,2}-X_{2,3}-$ $3X_{2,5}-X_{3,4}-2X_{3,6}-X_{4,5}-$ $3X_{4,7}-X_{5,6}-X_{6,7}+X_{7,8}.$
$\mathfrak{f}_8^{18}$	$[e_4,e_7]=e_2,$ $[e_4,e_8]=e_3$ $[e_5,e_6]=-e_2,$ $[e_5,e_8]=e_4,$ $[e_6,e_8]=e_2+e_5$ $[e_7,e_8]=e_3+e_6.$	$e_1=\sum_{i=1}^{6} X_{i,i+1}+\frac{3}{40}X_{7,8}, e_2=X_{1,8},$ $e_3=-\frac{10}{3}X_{1,7}+\frac{3}{4}X_{2,8}, e_4=\frac{4}{3}X_{1,6}-2X_{2,7}$ $+\frac{3}{5}X_{3,8}, e_5=-\frac{4}{3}X_{1,5}-2X_{3,7}+\frac{9}{20}X_{4,8},$ $e_6=\frac{4}{3}X_{1,4}-2X_{4,7}+\frac{3}{10}X_{5,8}, e_7=-\frac{4}{3}X_{1,3}$ $-2X_{5,7}+\frac{3}{20}X_{6,8}, e_8=\frac{1}{3}X_{1,2}$ $-5X_{1,5}-X_{2,3}-5X_{2,6}-X_{3,4}-5X_{3,7}$ $-X_{4,5}-\frac{3}{8}X_{4,8}-X_{5,6}-3X_{6,7}-\frac{3}{40}X_{7,8}.$
$\mathfrak{f}_8^{19}$	$[e_4,e_7]=e_2, [e_4,e_8]=e_3$ $[e_5,e_6]=-e_2, [e_5,e_8]=e_4,$ $[e_6,e_7]=e_2,$ $[e_6,e_8]=e_2+e_3+e_5$ $[e_7,e_8]=e_3+e_4+e_6.$	$e_1=X_{1,2}+X_{1,3}-X_{2,3}+164X_{2,5}-X_{3,4}-$ $164X_{3,6}+\frac{164}{3}X_{3,7}-164X_{4,5}+164X_{4,7}$ $-X_{5,6}-X_{6,7}-\frac{1}{328}X_{7,8}, e_2=X_{1,8}, e_3=$ $164X_{1,7}+\frac{1}{2}X_{2,8}, e_4=164X_{1,6}+164X_{1,7}-$ $\frac{1}{2}X_{3,8}, e_5=164X_{1,5}+164X_{1,6}-X_{1,8}+$ $\frac{1}{2}X_{4,8}, e_6=X_{1,4}+164X_{1,5}+164X_{1,6}-$ $\frac{164}{3}X_{1,7}-\frac{1}{3}X_{2,8}-\frac{1}{2}X_{3,8}-\frac{1}{328}X_{5,8},$ $e_7=X_{1,3}+X_{1,4}+\frac{820}{3}X_{1,6}-\frac{328}{3}X_{2,7}$ $+\frac{1}{328}X_{6,8}, e_8=X_{2,3}-\frac{164}{3}X_{2,6}+$ $X_{3,4}+328X_{3,6}+164X_{4,5}+X_{5,6}+X_{6,7}.$
$\mathfrak{f}_8^{20}$	$[e_4,e_7]=e_2, [e_4,e_8]=e_2+e_3$ $[e_5,e_6]=-e_2, [e_5,e_7]=-\frac{2}{5}e_2,$ $[e_5,e_8]=e_4+\frac{3}{5}e_3,$ $[e_6,e_7]=-\frac{2}{5}e_3,$ $[e_6,e_8]=e_5+\frac{1}{5}e_4,$ $[e_7,e_8]=e_6+\frac{1}{5}e_5.$	$e_1=X_{1,2}+X_{1,3}-X_{2,3}+\frac{3}{5}X_{2,4}-\frac{37}{25}X_{2,5}$ $-X_{3,4}+\frac{2}{5}X_{3,5}-\frac{2}{5}X_{3,6}+\frac{1}{25}X_{3,7}$ $-\frac{3}{250}X_{3,8}-X_{4,5}-\frac{2}{5}X_{4,6}-\frac{2}{25}X_{4,7}$ $-X_{5,6}-\frac{3}{5}X_{5,7}-X_{6,7}-\frac{1}{2}X_{7,8},$ $e_2=X_{1,8}, e_3=X_{1,7}-X_{1,8}+\frac{1}{2}X_{2,8},$ $e_4=X_{1,6}+\frac{3}{5}X_{1,8}-\frac{1}{2}X_{2,8}-\frac{1}{2}X_{3,8},$ $e_5=X_{1,5}+\frac{3}{5}X_{1,6}+\frac{1}{5}X_{1,7}-\frac{3}{25}X_{1,8}$ $-\frac{6}{25}X_{2,7}+\frac{3}{10}X_{2,8}+\frac{1}{5}X_{3,8}+\frac{1}{2}X_{4,8},$ $e_6=X_{1,4}+\frac{4}{5}X_{1,5}+\frac{1}{5}X_{1,6}+\frac{6}{25}X_{1,7}+$ $\frac{3}{125}X_{1,8}-\frac{3}{5}X_{2,6}-\frac{29}{25}X_{2,7}-\frac{1}{25}X_{2,8}+$ $\frac{3}{5}X_{3,7}-\frac{1}{5}X_{3,8}-\frac{1}{10}X_{4,8}-\frac{1}{2}X_{5,8},$ $e_7=X_{1,3}+X_{1,4}+\frac{7}{25}X_{1,6}+\frac{1}{5}X_{2,5}-$ $\frac{7}{5}X_{2,6}-\frac{2}{25}X_{2,7}+\frac{3}{250}X_{2,8}+\frac{4}{5}X_{3,6}+$ $\frac{1}{5}X_{4,7}+\frac{1}{2}X_{6,8}, e_8=X_{2,3}-\frac{1}{25}X_{2,6}$ $+X_{3,4}+\frac{8}{25}X_{3,6}+X_{4,5}+X_{5,6}+X_{6,7}.$

Conclusions

In this dissertation, we have dealt with abelian subalgebras and ideals of maximal dimension for arbitrary Lie algebras and, as a particular case, for solvable Lie algebras. We have also analyzed several special families of solvable Lie algebras in relation to this notion. We have studied this from both a theoretical view-point and an algorithmic approach. In this way, Chapter 2 constitutes a theoretical study of abelian subalgebras and ideals contained in Lie algebras. We have given several general properties and bounds for α and β invariants in cases of supersolvable, solvable and nilpotent Lie algebras. Then, we have focused on the cases of Lie algebras containing abelian subalgebras of codimension 1 and 2, and the case of codimension 3 for nilpotent Lie algebras.

In Chapter 3, we have shown algorithmic methods to compute abelian subalgebras and ideals of maximal dimension for the most important families of solvable Lie algebras: Lie algebra $\mathfrak{g}_n$, Lie algebra $\mathfrak{h}_n$ and Heisenberg algebra $\mathfrak{H}_k$. In order to conclude the chapter, we have studied filiform Lie algebras, proving that there exists a unique abelian ideal of maximal dimension for these algebras. In fact,we do not only have analyzed α and β invariants with respect to the dimension of a given Lie algebra, but we have also obtained a relation allowing us to determine the law of a general filiform Lie algebra. At this respect, we have given a characterization of the law of a general filiform Lie algebra and we have proved several result about the coefficients in those laws.

In Chapter 4, we have shown an algorithmic method to compute abelian subalgebras and ideals of any finite-dimensional Lie algebra, starting from the non-zero brackets in its law. Moreover, we included in this chapter two different applications of the previous results. The first was related to the computation of α and β invariants for low dimensional Lie algebras. The second application was the computation of minimal faithful unitriangular matrix representations of filiform Lie algebras and the use of these invariants to retrieve these representations.

In our opinion, the results obtained in this dissertation can be considered a new step forward in the study of Lie algebras in general. More concretely, we believe that our main contribution is related to the study of abelian subalgebras and ideals of maximal dimension in Lie algebras and more concretely on solvable ones. We have also determined laws for filiform Lie algebras starting from the value of the maximal dimension of their abelian ideals. Therefore, we think that this can be considered another step towards the problem of classifying filiform Lie algebras in general, which seems to be a very difficult task. Moreover, we have also given several algorithmic methods that we believe that can be very useful to compute abelian subalgebras and ideals in Lie algebras, as well as making easier the computation of matrix representations of filiform Lie algebras. In fact, besides explaining and implementing algorithms to compute the value of α and β invariants for any given Lie algebra, we have also applied them to several types of Lie algebras; namely: general, solvable and nilpotent for dimension less than 5, 7 and 8, respectively. Finally, we have also determined a representative for the matrix representation of general model filiform Lie algebras and, applying our general algorithmic method, for all filiform Lie algebras up to dimension 8.

However, there exist several open problems that we want to study in the near future. Regarding the second chapter, we have the following questions

1. Does Theorem 2.2 hold for supersolvable Lie algebras?
2. Let $\mathfrak{g}$ be a supersolvable/nilpotent Lie algebra with $\alpha(\mathfrak{g}) = n-k$ containing an abelian subalgebra $\mathfrak{a}$ of maximal dimension, and let I be a maximal subalgebra containing $\mathfrak{a}$ that is an ideal of $\mathfrak{g}$.
   (i) Is it true that $\dim Z(I) \geq n-2k+1$?
   (ii) Is it true that $\dim \mathcal{C}_2(I) \leq k-1$?
   (iii) If they are true, do 1 and 2 imply that $\beta(\mathfrak{g}) = n-k$?

In reference to Chapter 3, we would like to study the isomorphism classes of filiform Lie algebras starting from some relation over the coefficients α_i and α_j^{j-1} that were defined at the end of this chapter. Another open problem that we want to deal with is to compute the matrix representation of filiform Lie algebras of dimension greater than or equal to 9. We will also keep studying new methods to improve our representations and to obtain new ones for other types of Lie algebras such as solvable ones, for instance. Finally, we will try to look for some applications of all the results obtained in this dissertation to other fields or sciences like Applied Mathematics, Physics, etc.

We hope to continue this research in the future in order to solve all these open problems.

Bibliography

[1] R.K. Amayo. Quasi-ideals of Lie algebras II. *Proc. London Math. Soc. (3)* **33**:1 (1979), 37–64.

[2] G.F. Armstrong. Unimodal Betti numbers for a class of nilpotent Lie algebras. *Comm. Algebra* **25**:6 (1997), 1893–1915.

[3] J. Arthur, W. Schmid and P.E. Trapa (Eds.). *Representation theory of real reductive Lie groups. Papers from the AMS-IMS-SIAM Joint Summer Research Conference*. Contemp. Math. **472**. Providence: American Mathematical Society, 2006.

[4] L. Auslander and J. Brezin. Almost algebraic Lie algebras. *J. Algebra* **8** (1968), 295–313.

[5] D.W. Barnes and M.L. Newell. Some theorems on saturated homomorphs of soluble Lie algebras. *Math. Z.* **115** (1970), 179–187.

[6] J.C. Benjumea, F.J. Echarte, J. Núñez and A.F. Tenorio. An obstruction to represent abelian Lie algebras by unipotent matrices. *Extracta Math.* **19**:2 (2004), 269–277.

[7] J.C. Benjumea, F.J. Echarte, J. Núñez and A.F. Tenorio. A method to obtain the Lie group associated with a nilpotent Lie algebra. *Comput. Math. Appl.* **51**: 9–10 (2006), 1493–1506.

[8] J.C. Benjumea, J. Núñez and A.F. Tenorio. The maximal abelian dimension of linear algebras formed by strictly upper triangular matrices. *Theor. Math. Phys.* **152**:3 (2007), 1225–1233.

[9] A. Borel. Groupes linéaires algébriques. *Ann. of Math. (2)* **64**:1 (1956), 20–82.

[10] N. Bourbaki. *Elements of Mathematics. Lie groups and Lie algebras. Chapters 1-3*. Berlin: Springer, 1989.

[11] K. Bowman and D.A. Towers. On almost nilpotent-by-abelian Lie algebras. *Linear Algebra Appl.* **247** (1996), 159–167.

[12] K. Bowman, D.A. Towers and V.R. Varea. Two generator subalgebras of Lie algebras. *Linear Multilinear Algebra* **55**:5 (2007), 429–438.

[13] L. Boza, E.M. Fedriani and J. Núñez. A new method for classifying complex filiform Lie algebras. *Appl. Math. Comput.* **121**:2–3 (2001), 169-175.

[14] F. Bratzlavsky. Classification des algèbres de Lie nilpotentes de dimension n, de classe $n-1$, dont l'idèal dèrivè est commutatif. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **60** (1974), 858–865.

[15] D. Burde. On a refinement of Ado's theorem. *Arch. Math.(Basel)* **70**:2 (1998), 118–127.

[16] D. Burde. Affine cohomology classes for filiform Lie algebras. In P.Igodt, H. Abels, Y. Félix and F. Grunewald (Eds.). Crystallographic groups and their generalizations. Contemp. Math. **262** (pp. 159–170). Providence: American Mathematical Society, 2000.

[17] D. Burde. Left-symmetric algebras, or pre-Lie algebras in geometry and physics. *Cent. Eur. J. Math.* **4**:3 (2006), 323–357.

[18] D. Burde and M. Ceballos. Abelian ideals of maximal dimension for solvable Lie algebras. arXiv:0911.2995v1 [math.RA] (2009). To appear in Journal of Lie Theory.

[19] D. Burde and C. Steinhoff. Classification of orbit closures of 4–dimensional complex Lie algebras. *J. Algebra* **214**:2 (1999), 729–739.

[20] E. Cartan. *Sur la structure des groupes de tranformations finis et continus.* Ph.D. Thesis, Paris, Nony, 1894.

[21] M. Ceballos, J. Núñez and A.F. Tenorio. The computation of abelian subalgebras in the Lie algebra of upper-triangular matrices. *Analele Stiintifice ale Universitatii Ovidius Constanta* **16**:1 (2008), 59–66.

[22] M. Ceballos, J. Núñez and A.F. Tenorio. Algorithm to compute the maximal abelian dimension of Lie algebras. *Computing* **84**:3-4 (2009), 231–239.

[23] M. Ceballos, J. Núñez and A.F. Tenorio. Abelian subalgebras in some particular types of Lie algebras. *Nonlinear Anal.* **71**:12 (2009), e401–e408.

[24] M. Ceballos, J. Núñez and A.F. Tenorio. The computation of abelian subalgebras in low-dimensional solvable Lie algebras. *WSEAS Trans. Math.* **9**:1 (2010), 22–31.

[25] M. Ceballos, J. Núñez and A.F. Tenorio. Computing matrix representations of filiform Lie algebras. In V.P. Gerdt, W. Koepf, E.W. Mayr and E.V. Vorozhtsov (Eds.) *Computer algebra in scientific computing. 12th International Workshop, CASC 2010.* Lect. Notes Comput. Sc. **6244** (pp. 61–72). Berlin: Springer, 2010.

[26] M. Ceballos, J. Núñez and A.F. Tenorio. General laws of filiform Lie algebras using invariants. *Submitted.*

[27] M. Ceballos, J. Núñez and A.F. Tenorio. Algorithmic method to obtain abelian subalgebras and ideals in Lie algebras. *Submitted.*

[28] M. Ceballos, D. Towers. *On abelian subalgebras and ideals of maximal dimension in supersolvable lie algebras.* arXiv:1110.2389v1 [math.RA] (2011).

[29] P. Cellini, P.M. Frajria and P. Papi. Abelian subalgebra in $\mathbb{Z}_2$-graded Lie algebras and affine Weyl groups. *Int. Math. Res. Not.* **43** (2004), 2281–2304.

[30] P. Cellini and P. Papi. Ad-nilpotent ideals of a Borel subalgebra. *J. Algebra* **225**:1 (2000), 130–141.

[31] J. Cossey and S. Stonehewer. On the derived length of finite dinilpotent groups. *Bull. London Math. Soc.* **30**:3 (1998), 247–250.

[32] W.A. de Graaf. Classification of solvable Lie algebras. *Experiment. Math.* **14**:1 (2005), 15–25.

[33] W.A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra* **309**:2 (2007), 640–653.

[34] F.J. Echarte, J.R. Gómez and J. Núñez. Les algébres de Lie filiformes complexes derivées d'autres algébres de Lie. In M. Goze (Ed.). *Lois d'algèbres et variétés algébriques.* Travaux en Cours **50** (pp. 45–55). Paris: Hermann, 1996.

[35] F.J. Echarte, J. Núñez and F. Ramírez. Description of some families of filiform Lie algebras. *Houston J. Math.* **34**:1 (2008), 19–32.

[36] F.J. Echarte, J. Núñez and F. Ramírez. Relations among invariants of complex filiform Lie algebras. *Appl. Math. Comput.* **147**:2 (2004), 365–376.

[37] F.J. Echarte, J. Núñez and F. Ramírez. Study of two invariants in complex filiform Lie algebras. *Algebras Groups Geom.* **13**:1 (1996), 55–70.

[38] A.G. Elashvili and A.I. Ooms. On commutative polarizations. *J. Algebra* **264**:1 (2003), 129–154.

[39] A. Elduque. A note on noncentral simple minimal nonabelian Lie algebras. *Comm. Algebra* **15**:7 (1987), 1313–1318.

[40] A. Elduque and V.R. Varea. Lie algebras, all of whose subalgebras are supersolvable. In D.J. Britten, F.W. Lemire and R.V. Moody (Eds.). *Lie algebras and related topics.* CMS Conference Proceedings **5** (pp. 209–218). Providence: American Mathematical Society, 1986.

[41] R. Farnsteiner. On the structure of simple-semiabelian Lie algebras. *Pacific J. Math.* **111**:2 (1984), 287–299.

[42] R. Farnsteiner. Quaternionic Lie algebras. *Linear Algebra Appl.* **61** (1984), 225–231.

[43] W. Fulton and J. Harris. *Representation Theory. A First Course.* New York: Springer-Verlag, 1991.

[44] H. Garland and J. Lepowsky. Lie algebra homology and the Macdonald-Kac formulas. *Invent. Math.* **34**:1 (1976), 37–76.

[45] A.G. Gein. Minimal nonabelian Lie algebras of prime characteristic. *C. R. Acad. Bulgare Sci.* **37**:10 (1984), 1291–1293.

[46] A.G. Gein. Minimal noncommutative and minimal nonabelian algebras. *Comm. Algebra* **13**:2 (1985), 305–328.

[47] A.G. Gein and S.V. Kuznecov. Minimal non-nilpotent Lie algebras. *Ural. Gos. Univ. Mat. Zap.* **8** (1972), 18–27.

[48] V.V. Gorbatsevich. On the level of some solvable Lie algebras. *Siberian Math. J.* **39**:5 (1998), 872–883.

[49] F. Grunewald and J. O'Halloran. Varieties of nilpotent Lie algebras of dimension less than six. *J. Algebra* **112**:2 (1988), 315–325.

[50] S. Halperin. Le complexe de Koszul en algèbre et topologie. *Ann. Inst. Fourier (Grenoble)* **37**:4 (1987), 77–97.

[51] J. Heber. Noncompact homogeneous Einstein spaces. *Invent. Math.* **133**:2 (1998) 279–352.

[52] W. Heisenberg. Über quantentheoretische Umdeutung Kinematicher und mecanischer Beziehungen. *Zeits. f. Phys.* **33** (1925), 879-893.

[53] I. Hernández, C. Mateos, J. Núñez and A.F. Tenorio. Lie theory: applications for solving problems in mathematical finance and economics. *Appl. Math. Comput.* **208**:2 (2009), 446–452.

[54] S. Hervik. Einstein metrics: homogeneous solvmanifolds, generalised Heisenberg groups and black holes. *J. Geom. Phys.* **52**:3, 298–312 (2004).

[55] F. Iachello. *Lie algebras and Applications.* Lecture Notes in Physics **708**. Berlin: Springer, 2006.

[56] N. Ito. Über das produkt von zwei abelschen gruppen. *Math. Z.* **62** (1955), 400–401.

[57] N. Jacobson. Schur's theorem on commutative matrices. *Bull. Amer. Math. Soc.* **50** (1944), 431–436.

[58] N. Jacobson. *Lie algebras.* New York: Dover, 1979.

[59] W. Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen I. *Math. Ann.* **31** (1888), 252–290.

[60] W. Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen II. *Math. Ann.* **33** (1889), 1–48.

[61] W. Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen III. *Math. Ann.* **34** (1889), 57–122.

[62] W. Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen IV. *Math. Ann.* **36** (1890), 161–189.

[63] A. Kirillov. *Elements of the Theory of Representations.* Berlin-New York: Springer-Verlag, 1976.

[64] E.R. Kolchin. Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations. *Ann. of Math. (2)* **49**:1, (1948) 1–42.

[65] B. Kostant. Eigenvalues of the Laplacian and commutative Lie subalgebras. *Topology* **3**: suppl. 2 (1965), 147–159.

[66] B. Kostant. The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations. *Internat. Math. Res. Notices* **5** (1998), 225–252.

[67] B. Kostant. Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra. *Invent. Math.* **158**:1 (2004), 181–226.

[68] F. Kubo. Finiteness conditions for abelian ideals and nilpotent ideals in Lie algebras. *Hiroshima Math. J.* **8**:2 (1978), 301–303.

[69] E.E. Levi. Sulla struttura dei gruppi finiti e continui. *Atti. Accad. Sci. Torino* **40** (1905), 551–565.

[70] L. Luo. Abelian ideals and cohomology of symplectic type. *Proc. Amer. Math. Soc.* **137**:2 (2009), 479–485.

[71] L. Magnin. *Adjoint and trivial cohomology tables for indecomposable nilpotent Lie algebras of dimension* ≤ 7 *over* $\mathbb{C}$. Online e-book, 2nd Corrected Edition. Bourgogne: Université de Bourgogne, 2007.

[72] A. Malcev. Commutative subalgebras of semi-simple Lie algebras. *Bull. Acad. Sci. URSS Sér. Math.* **9** (1945), 291–300 (in Russian). Translated to English in A. Malcev. Commutative subalgebras of semi-simple Lie algebras. *Amer. Math. Soc. Translation* **40** (1951), 15pp.

[73] M.V. Milentyeva. On the dimensions of commutative subalgebras and subgroups. *J. Math. Sci. (N.Y.)* **149**: 2 (2008), 1135–1145.

[74] M.V. Milentyeva. On the dimensions of commutative subalgebras and subgroups of nilpotent algebras and Lie groups of class 2. *Comm. Algebra* **35**: 4 (2007), 1141–1154.

[75] G.M. Mubarakzyanov. Classification of real structures of Lie algebras of fifth order. *Izv. Vysš. Učebn. Zaved. Matematika* **3**: 34 (1963), 99–106.

[76] G.M. Mubarakzyanov. Classification of solvable Lie algebras of order 6 with one nilpotent basis element. *Izv. Vysš. Učebn. Zaved. Matematika* **4**: 35 (1963), 104–116.

[77] M. Nesterenko and R. Popovych. Contractions of low-dimensional Lie algebras. *J. Math. Phys.* **47**: 12 (2006), 123515, 45 pp.

[78] J. Núñez and A.F. Tenorio. A study of the maximal abelian dimension of Heisenberg algebras. In J. Bilbao (Ed.). *Proceedings of 2007 International Conference on Engineering and Mathematics (ENMA 2007)* (pp. 103–110). Bilbao: University of Bilbao, 2007.

[79] P. J. Olver. *Applications of Lie Groups to Differential Equations.* New York: Springer, 1986.

[80] L. Orsina and P. Papi. Enumeration of ad-nilpotent ideals of a Borel subalgebra in type A by class of nilpotence. *C. R. Acad. Sci. Paris Sér. I Math.* **330**:8 (2000), 651–655.

[81] D.I. Panyushev. Isotropy representations, eigenvalues of a Casimir element, and commutative Lie subalgebras. *J. London Math. Soc. (2)* **64**: 1 (2001), 61–80.

[82] D.I. Panyushev. Long abelian ideals. *Adv. Math.* **186**: 2 (2004), 307–316.

[83] D. Panyushev and G. Röhrle. Spherical orbits and abelian ideals. *Adv. Math.* **159**: 2 (2001), 229–246.

[84] A.P. Petravchuk. On the sum of an almost abelian Lie algebra and a Lie algebra finite-dimemsional over its center. *Ukrainian Math. J.* **51**: 5 (1999), 707–715.

[85] A.Z. Petrov. *Einstein spaces.* Oxford-Edinburgh-New York: Pergamon Press, 1969.

[86] H. Pouseele and P. Tirao. Constructing Lie algebra homology classes. *J. Algebra* **292**:2 (2005), 585–591.

[87] P.J. Olver. *Applications of Lie Groups to Differential Equations.* New York: Springer-Verlag, 1986

[88] D.M. Riley. How abelian is a finite-dimensional Lie algebra? *Forum Math.* **15**: 3 (2003), 455–463.

[89] N.S. Romanovskii and I.P. Shestakov. Noetherianness of wreath products of abelian Lie algebras with respect to equations of universal enveloping algebra. *Algebra Logic* **47**: 4 (2008), 269–278.

[90] R. Schimming. On constant solutions of the Yang-Mills equations. *Arch. Math. (Brno)* **24**: 2 (1988), 65–73.

[91] I. Schur. Zur theorie vertauschbarer matrizen. *J. Reine. Angew. Math.* **130** (1905), 66–76.

[92] C. Seeley. Degenerations of 6-dimensional nilpotent Lie algebras over $\mathbb{C}$. *Comm. Algebra* **18**: 10 (1990), 3493–3505.

[93] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt. *Exact solutions of Einstein's field equations.* Cambridge: Cambridge University Press, 2003.

[94] I. Stewart. Bounds for the dimensions of certain Lie algebras. *J. London Math. Soc. (2)* **3** (1971), 731–732.

[95] E.L. Stitzinger. Minimal nonnilpotent solvable Lie algebras. *Proc. Amer. Math. Soc.* **28** (1971), 47–49.

[96] D.A. Suprunenko and R.I. Tyshkevich. *Commutative matrices.* New York: Academic Press, 1968.

[97] R. Suter. Abelian ideals in a Borel subalgebra of a complex simple Lie algebra. *Invent. Math.* **156**: 1 (2004), 175–221.

[98] A.F. Tenorio. Solvable Lie algebras and maximal abelian dimensions. *Acta Math. Univ. Comenian. (N.S.)* **77**:1 (2008), 141–145.

[99] D.A. Towers. A Frattini theory for algebras. *Proc. London Math. Soc. (3)* **27** (1973), 440–462.

[100] D.A. Towers. Lie algebras, all of whose proper subalgebras are nilpotent. *Linear Algebra Appl.* **32** (1980), 61–73.

[101] D.A. Towers. Minimal non-supersolvable Lie algebras. *Algebras Groups Geom.* **2** (1985), 1–9.

[102] D.A. Towers. Almost nilpotent Lie algebras. *Glasgow Math. J.* **29**: 1 (1987), 7–11.

[103] D.A. Towers. The index complex of a maximal subalgebra of a Lie algebra. *Proc. Edinb. Math. Soc. (2)* **54**: 2 (2011), 531–542.

[104] P. Turkowski. Solvable Lie algebras of dimension six. *J. Math. Phys.* **31**: 6, 1990, pp. 1344–1350.

[105] V.S. Varadarajan. *Lie Groups, Lie Algebras and Their Representations.* New York: Springer-Verlag, 1984.

[106] V.R. Varea. Lie algebras none of whose Engel subalgebras are in intermediate position. *Comm. Algebra* **15**: 12 (1985), 1135–1150.

[107] M. Vergne. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algebres de Lie nilpotentes. *Bull. Soc. Math. France* **98** (1970), 81–116.

[108] H.S. Wilf. *Algorithms and complexity.* Englewood Cliffs: Prentice Hall, 1986.

[109] D.J. Winter. Cartan decompositions and Engel subalgebra triangulability. *J. Algebra* **62**: 2 (1980), 400–417.

[110] A.A. Zaicev. Positive polarizations and abelian ideals in Lie algebras. *Mat. Sb. (N.S.)* **112(154)**: 2(6) (1980), 242–255 (in Russian). Translated to English in A.A. Zaicev, Positive polarizations and abelian ideals in Lie algebras. *Math. USSR Sb.* **40**: 2 (1981), 227–240.

[111] P. Zusmanovich. ω-Lie algebras. *arXiv:0812.0080v2 [math.RA]* (2010).

List of Figures

List of Tables

List of authors