From symmetric functions to qubits

Lecture II

Symmetric functions and characters of the symmetric group

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Representations of finite group

Consider a finite group G and a finite dimension space V.

A **representation** of G in V is a morphism $\rho : G \to End(V)$

That is $\rho(x.y) = \rho(x)\rho(y)$

Equivalently, ρ defines an action of G on V $x.v = \rho(x)v$ ($v \in V$ $x \in G$)

The **dimension** of the representation ρ is the dimension of V.

Representations of finite groups: Example 1

Trivial representation

$$\rho : G \to \mathbb{C}$$

$$\rho(x) = 1$$

Alternated representation of $G = \mathfrak{S}_n$

$$\rho(x) = \mathrm{sign}(x)$$

Representations of finite groups: Example 2

Regular representation

We consider a space V whose a basis is indexed by the elements of G. The **regular representation** of G, is defined by

$$\rho(x)_{y,z} = \begin{cases} 1 & \text{if } yx = z \\ 0 & \text{otherwise} \end{cases}$$

Proof:

Consider the product of the two matrices

$$(\rho(x)\rho(x'))_{yz} = \sum_{y' \in G} \rho(x)_{yy'}\rho(x')_{y'z}$$

There is at most one term y' in the sum, and if it exists it verifies

$yx = y'$ and $y'x' = z$ or equivalently $y' = yx = zx'^{-1}$.

Hence, $yxx' = y'x' = z$

This show that ρ is a representation

Representations of finite groups: Example 3

Regular representation of $\mathfrak{S}_3 = [[1,2,3],[1,3,2],[3,1,2],[2,1,3],[2,3,1],[3,2,1]]$

It suffices to construct the representation for the two generators [213] and [132]

$$\rho(213) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\rho(132) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

One verifies that

$$\rho(231) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \rho(213)\rho(132)$$

Representations of finite groups: Example 4

Permutation representation

We suppose that the group G is a subgroup of a symmetric group and we replace each permutation by the associated permutation matrix.

For example consider the symmetric group

$$[[1,2,3],[1,3,2],[3,1,2],[2,1,3],[2,3,1],[3,2,1]]$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

A **permutation representation** of the cyclic group G={[123],[231],[312]} is given by the matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\qquad\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

[123] [231] [312]

Representations of finite groups: Example 5

Representations of dimension 1 of the cyclic groups

Consider the cyclic group

$$C_n = \{[12\ldots n], [23\ldots n1], \ldots, [n12\ldots n-1]\}$$

$$\rho([23\ldots n1]) = \exp\left\{\frac{2k\pi}{n}\right\}, k \in \mathbb{Z}$$

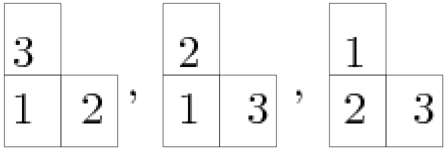
Representations of finite groups: Example 6 (1)

Specht modules

Consider a partition λ of n and the set of the set partitions of n with shape λ.

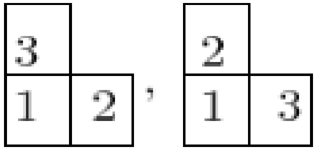
For example, if $\lambda=21$ one has $[\{1,2\},\{3\}],\ [\{1,3\},\{2\}],\ [\{2,3\},\{1\}]$

We to each set partition, one associates a tableau:



We consider the space spanned by these generators.

We select only the standard tableaux



Consider the natural action of $\mathfrak{S}_n$ on the integers {1,...,n}

Representations of finite groups: Example 6 (2)

Specht modules

For each tableau, we consider the permutation which does not change the set of elements of each column:

$$123.\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}, \; 321.\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 2 \\ \hline\end{array}.$$

And we sort the lines

$$123.\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}, \; 321.\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 \\ \hline\end{array}.$$

From a standard tableau, we construct the formal alternated sums of the tableaux obtained by this process

$$e_{\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}} = \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} - \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 \\ \hline\end{array} \qquad e_{\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array}} = \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array} - \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 \\ \hline\end{array}$$

Representations of finite groups: Example 6 (3)

Specht modules

The vector space spanned by these elements is a module for the symmetric group.

One has

$$132.e_{\begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{2}\end{array}} = \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array} - \begin{array}{l}\boxed{1}\\ \boxed{2}\boxed{3}\end{array} = e_{\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array}},$$

and,

$$213.e_{\begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{2}\end{array}} = -e_{\begin{array}{l}\boxed{2}\\ \boxed{3}\boxed{1}\end{array}}, \; 132.e_{\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array}} = e_{\begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{2}\end{array}}, \; 213.e_{\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array}} = e_{\begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{2}\end{array}} - e_{\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array}}$$

This defines a representation of $\mathfrak{S}_3$

$$\rho(132) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \rho(213) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

This is an example of a more general construction: **Specht representation**

Subrepresentations

Let (ρ,V) be a representation of a group G, a **subrepresentation** is given by a subspace V1 stable for the action of ρ.

Example:
Consider a permutation representation (ρ,V) of a finite group G.
The subspace generated by the vector whose each entry equals 1 is a subrepresentation of (ρ,V). Indeed, the action of an element of G permutes the line of the vectors.
Hence:

$$x.\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\rho(x)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

Maschke Theorem

Maschke Theorem

If V_1 is a subrepresentation of (ρ,V), then V can be decomposed as a direct sum

$$V \simeq V_1 \oplus V_2$$

where V_2 is a subrepresentation of (ρ,V).

Example
Consider a subgroup G of $\mathfrak{S}_5$. The subspace $\mathbb{C}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$ is a subrepresentation of the permutation representation.

The complementary space is the set of the vectors $\begin{pmatrix}a_1\\a_2\\a_3\\a_4\\a_5\end{pmatrix}$ verifying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

Remark that:

$$\sigma.\begin{pmatrix}a_1\\a_2\\a_3\\a_4\\a_5\end{pmatrix} = \begin{pmatrix}a_{\sigma(1)}\\a_{\sigma(2)}\\a_{\sigma(3)}\\a_{\sigma(4)}\\a_{\sigma(5)}\end{pmatrix} \in V_2$$

Reducible and irreducible representations

A representation having a proper subrepresentation is called **reducible** otherwise it is called **irreducible**.

Example

Permutation representations are reducible.
Regular representation are reducible.

Representations of deg*ee* 1 are irreducible.

The Specht module $S^{[2,1]}$ (see Example 6) is an irreducible representation of $\mathfrak{S}_3$

Indeed, suppose that $S^{[2,1]}$ has a proper subspace V, the dimension of V is 1. And V is an eigenvectors of both $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

But this is not possible, since the two matrices has different eigenvectors.

$$\text{eigenvect}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \text{eigenvect}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

More generaly Specht modules are the irreducible representations of the symmetric groups.

Decomposition of reducibles representation

Consequence of the Maschke theorem: Each reducible representation is a direct sum of irreducible representations

$$V \simeq \bigoplus_r \alpha_r V_r$$

Example:
Consider the permutation representation of the cyclic group $C_3 := \{123, 231, 312\}$

$$\rho(231) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Since this group has only one generator and the matrix has three eigenspaces of dimension 1

$$\text{eigenvects}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} e^{\frac{2i\pi}{3}} \\ e^{\frac{-2i\pi}{3}} \\ 1 \end{pmatrix}, \begin{pmatrix} e^{\frac{-2i\pi}{3}} \\ e^{\frac{2i\pi}{3}} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$e^{\frac{2i\pi}{3}} \qquad e^{\frac{-2i\pi}{3}} \qquad 1$$

One has the following decomposition

$$\mathbb{C}^3 = \mathbb{C}\begin{pmatrix} e^{\frac{2i\pi}{3}} \\ e^{\frac{-2i\pi}{3}} \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} e^{\frac{-2i\pi}{3}} \\ e^{\frac{2i\pi}{3}} \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Induced representations

Let H be a subgroup of G and set $g = \mathrm{Card}(G),\ h = \mathrm{Card}(H),\ k = \dfrac{g}{h}.$

Let ρ be a representation of H. How to construct a representation of G from ρ?

One has: $G = t_1 H \oplus t_2 H \oplus \cdots \oplus t_k H \quad t_1, \ldots, t_k \in G$

One can set $t_1 = 1$

The map defined by $\rho \uparrow_H^G (x) = \left(\widetilde{\rho}(t_i^{-1} x t_j)\right)_{i,j}$

with

$$\widetilde{\rho}(x) = \begin{cases} \rho(x) & \text{if } x \in H \\ 0_{m\times m} & \text{otherwise.} \end{cases}$$

is a representation of G called the **induced representation** of ρ.

Induced representations(2) Example

$$H = \{123, 321\} \subset G = \mathfrak{S}_3$$

$$\rho(321) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathfrak{S}_3 = \{123,\ 132,\ 213,\ 231,\ 312,\ 321\}$$

$$\mathfrak{S}_3 = H \oplus 132.H \oplus 213.H$$

$$\rho \uparrow_H^G (213) = \left(\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \mathbf{0} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \mathbf{0} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \mathbf{0} & \mathbf{0} \end{array}\right), \ \rho \uparrow_H^G (132) = \left(\begin{array}{c|c|c} \mathbf{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \mathbf{0} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right)$$

Characters of representations

The **character** χ(ρ) of a representation ρ is defined by

$$\chi(\rho) : \left\{ \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ x & \longrightarrow & \mathrm{tr}(\rho(x)) \end{array} \right.$$

Remark that

$$\chi(\rho)(yxy^{-1}) = \mathrm{tr}\left(\rho(yxy^{-1})\right) = \mathrm{tr}\left(\rho(y)\rho(x)\rho(y)^{-1}\right).$$

But

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$$

Hence, a character is invariant on the conjugacy class

$$\chi(\rho)(yxy^{-1}) = \mathrm{tr}(\rho(x)) = \chi(\rho(x)).$$

Induced characters

Induced character= character of induced representation

Let $H \subset G$

Consider the decomposition of G into cosets

$$G = t_1 H \oplus t_2 H \oplus \cdots \oplus t_k H$$

One has $$\chi(\rho \uparrow_H^G) = \sum_{i=1}^{m} \widetilde{\chi}(t_i^{-1} x t_i)$$

where

$$\chi(x) = \begin{cases} \chi(\rho(x)) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

Central functions (1) Definition and example

The **central functions** of a group G are the complex functions which are invariant on the conjugacy classes.

Examples

The characters are central functions

If $\{C_1, \ldots, C_p\}$ denotes the set of the conjugacy classes of G, the functions

$$b_i(x) = \begin{cases} \frac{\mathrm{Card} G}{\mathrm{Card} C_i} & \text{if } x \in C_i \\ 0 & \text{otherwise} \end{cases}$$

are central.

Central functions (2)
A scalar product and a natural basis

One defines a scalar product on the space of the central functions as the following summation over the orbit of G:

$$\langle f, g \rangle_G = \frac{1}{\mathrm{Card}G} \sum_{x \in G} f(x) g^*(x).$$

The functions

$$b_i(x) = \begin{cases} \frac{\mathrm{Card}G}{\mathrm{Card}C_i} & \text{if } x \in C_i \\ 0 & \text{otherwise} \end{cases}$$

are a natural orthogonal basis

$$\langle b_i, b_j \rangle_G = \frac{\mathrm{Card}G}{\mathrm{Card}C_i} \delta_{i,j}.$$

Orthogonality of irreducible characters

Irreducible characters = characters of irreducible representations.

We denote by Irrep G the set of all inequivalent irreductible representations of G, and by $\{C_1, \ldots, C_k\}$ the set of the conjucacy classes of G.

Two orthogonality relations:

1- If $\rho, \rho' \in \mathrm{Irrep}\ G$

$$\langle \chi^{\rho}, \chi^{\rho'} \rangle = \frac{1}{\mathrm{Card}\ G} \sum_{i=1}^{k} \mathrm{Card}(C_i) \chi_i^{\rho} (\chi_i^{\rho})^* = \delta_{\rho,\sigma}$$

The set of irreducible characters is an orthonormal basis with respect to the scalar product.

2- for each pair (i,j):

$$(C_i, C_j) := \frac{\mathrm{Card}\ C_i}{\mathrm{Card}\ G} \sum_{\rho \in \mathrm{Irrep}\ G} \chi_i^{\rho} (\chi_j^{\rho})^* = \delta_{i,j}.$$

Furthermore a representation ρ is irreducible iff $\langle \chi^{\rho}, \chi^{\rho} \rangle = 1$

Conjugacy classes in the symmetric group

A cycle c of length k is a permutation such there exist disctinct elements $a_0, \ldots, a_{k-1}$ such that

$$c^i(a_0) = a_i$$

Example: $521463 : 1 \to 5 \to 6 \to 3$

Each permutation is a product of disjoint cycles.

$$42156398 = (14563)(2)(89).$$

Cycle type: the partition of the lengths of the cycles

$$\mathrm{ct}(354178269) = [3321]$$

Fact:
Two permutations are conjugate iff they have the same cycle type.

Table of characters: an example

Consider the table of characters of the group $\mathfrak{S}_3$

	111	21	3
χ^1	1	1	1
χ^2	2	0	-1
χ^3	1	-1	1

Conjugacy classes

Trivial representation

Specht module 21 (Example 6)

Alternated representation

First orthogonality relations

$$\left\{\begin{aligned}
\langle\chi^1,\chi^1\rangle &= (1.1.1+3.1.1+2.1.1)/6=1\\
\langle\chi^2,\chi^1\rangle &= (1.2.1+3.0.1+2.(-1).1)/6=0\\
\langle\chi^3,\chi^1\rangle &= (1.1.1+3.(-1).1+2.11)/6=0\\
\langle\chi^2,\chi^2\rangle &= (1.2.2+3.0.0+2.(-1).(-1))/6=1\\
\langle\chi^3,\chi^2\rangle &= (1.1.2+3.(-1).0+2.1.(-1))/6=0\\
\langle\chi^3,\chi^3\rangle &= (1.2.1+3.(-1).(-1)+2.1.1)/6=1.
\end{aligned}\right.$$

Second orthogonality relations

$$\left\{\begin{aligned}
(111,111) &= 1.1+2.2+1.1=6\\
(21,21) &= 1.1+0.0+1.1=2\\
(3,3) &= 1.1+(-1).(-1)+1.1=3\\
(111,21) &= 1.1+2.0+1.(-1)=0\\
(111,3) &= 1.1+2.(-1)+1.1=0\\
(21,3) &= 1.1+0.-1+(-1).1=0
\end{aligned}\right.$$

Central functions of the symmetric group

Recall that the space of central functions is spanned by the functions:

$$b_i(x) = \begin{cases} \frac{\mathrm{Card} G}{\mathrm{Card} C_i} & \text{if } x \in C_i \\ 0 & \text{otherwise} \end{cases}$$

In the case of the symmetric group $\mathfrak{S}_n$ the conjugacy classes are indexed by the partitions of n.

The cardinal of the conjugacy classes $[\lambda]$ is known to be $\frac{\mathrm{Card}\ G}{z_\lambda}$

Recall that $z_\lambda = 1^{i_1} i_1! 2^{i_2} i_2! \dots k^{i_k} i_k! \dots$ if $\lambda = [\dots, k^{i_k}, \dots, 2^{u_2}, 1^{i_1}]$

.

$$z_{[4,4,2,2,2,1]} = 1.1!.2^3.3!3^0.0!4^2.2! = 1536.$$

Hence, the space of central functions of the symmetric group is spanned by

$$b_\lambda(\sigma) = \begin{cases} \frac{n!}{z_\lambda} & \text{if } \mathrm{ct}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$$

About symmetric functions

Compare $\langle b_\lambda, b_\mu\rangle = z_\lambda \delta_{\lambda,\mu}$ and $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda,\mu}$

where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ if $\lambda = [\lambda_1, \ldots, \lambda_k]$ and $p_m = x_1^m + x_2^m + \ldots$

denotes the multiplicative basis of power sums symmetric functions.

The polynomial p_λ encodes the function b_λ

Remark: $\chi(\mathbf{1}_{\mathfrak{S}_n}) = \sum_{\lambda \vdash n} z_\lambda^{-1} b_\lambda$

But $h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$

The complete function $h_n = \sum_\lambda m_\lambda$ encodes the trivial representation.

Young subgroups

The Young subgroup associated to the partition $\lambda = [\lambda_1, \ldots, \lambda_k] \vdash n$ is defined by

$$H_\lambda := \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \otimes \cdots \otimes \mathfrak{S}_{\{\lambda_1+\cdots+\lambda_{k-1}+1,\ldots,\lambda_1+\cdots+\lambda_k\}}$$

Example:

$$H_{3221} = \mathfrak{S}_{\{1,2,3\}} \otimes \mathfrak{S}_{\{4,5\}} \otimes \mathfrak{S}_{\{6,7\}} \otimes S_{\{8\}}$$

$$31254678 \in H_{3221}$$

$$31264587 \notin H_{3221}$$

Induced representations of Young subgroups

$$\mathbf{1}_\lambda := \chi(\mathbf{1} \uparrow_{H_\lambda}^{\mathfrak{S}_n}).$$

Example:

$$H_{21} = 123 + 213 \subset \mathfrak{S}_3$$

$$\mathfrak{S}_3 = H_{21} \oplus 132 H_{21} \oplus 231 H_{21}$$

$$\mathbf{1} \uparrow_{H_{21}}^{\mathfrak{S}_n} (213) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{1} \uparrow_{H_{21}}^{\mathfrak{S}_n} (132) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{1} \uparrow_{H_{21}}^{\mathfrak{S}_n} (231) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{1}_{21}([111]) = 3,\ \mathbf{1}_{21}([21]) = 1, \mathbf{1}_{21}([3]) = 0$$

Young subgroups and complete functions

$$\mathbf{1}_{21}([111]) = 3,\ \mathbf{1}_{21}([21]) = 1, \mathbf{1}_{21}([3]) = 0$$

Hence,

$$\mathbf{1}_{21} = \frac{1}{2}b_{21} + \frac{1}{2}b_{111}$$

Compare to

$$h_{2,1} = \frac{1}{2}p_2p_1 + \frac{1}{2}p_1^3$$

This is a general fact:

The multiplicative basis of the complete symmetric function encodes the (induced) character of the trivial representations of the Young subgroups.

h to p

Equivalently:

$$\boxed{h_\lambda = \sum_\mu z_\mu^{-1} \mathbf{1}_\lambda(\mu) p_{\mu_1} \dots p_{\mu_n}.}$$

By duality

$$p_\mu = \sum_\lambda \mathbf{1}_\lambda(\mu) m_\lambda$$

Orthogonalization and irreducible characters

One obtains the irreducible characters by orthogonalization of the $\mathbf{1}_\lambda := \chi(\mathbf{1} \uparrow_{H_\lambda}^{\mathfrak{S}_n})$.

$$\chi^4 = \mathbf{1}_[4]$$

$$\chi^{31} = \mathbf{1}_{[31]} \text{ - } \frac{\langle \chi^4, \mathbf{1}_{[31]} \rangle}{\langle \chi^4, \chi^4 \rangle} \chi^4 = \mathbf{1}_{[3,1]} - \mathbf{1}_{[4]}$$

$$\chi^{22} = \mathbf{1}_{[22]} \text{ - } \frac{\langle \chi^4, \mathbf{1}_{[22]} \rangle}{\langle \chi^4, \chi^4 \rangle} \chi^4 - \frac{\langle \chi^{31}, \mathbf{1}_{[22]} \rangle}{\langle \chi^{31}, \chi^{31} \rangle} \chi^{31} = \mathbf{1}_{[22]} - \mathbf{1}_{[3,1]}$$

$$\cdots$$

Compare to

$$S_4 := h_4$$

$$S_{3,1} := h_{3,1} - \frac{\langle S_4, h_{3,1} \rangle}{\langle S_4, S_4 \rangle} S_4 = h_{3,1} - h_4$$

$$S_{2,2} := h_{2,2} - \frac{\langle S_4, h_{2,2} \rangle}{\langle S_4, S_4 \rangle} S_4 - \frac{\langle S_{[3,1]}, h_{2,2} \rangle}{\langle S_{[3,1]}, S_{[3,1]} \rangle} S_{3,1} = h_{2,2} - h_{3,1}$$

Irreducible characters and Schur functions

The Schur basis encodes the irreducible representations of the symmetric group.

Frobenius-Schur character formula:

$$S_\lambda = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu) p^\mu$$

Or equivalently (by duality)

$$p^\mu = \sum_\lambda \chi^\lambda(\mu) S_\lambda$$

Conclusion

We have shown the following correspondancen: (Frobenius map)

Central functions		Symmetric functions
$b_\lambda(\sigma) = \begin{cases} \frac{n!}{z_\lambda} & \text{if } \mathrm{ct}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$	↔	$p_\lambda = p_{\lambda_1} \dots p_{\lambda_k}$
$\langle b_\lambda, b_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$	↔	$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$
$\mathbf{1}_\lambda := \chi(\mathbf{1} \uparrow_{H_\lambda}^{\mathfrak{S}_n}).$	↔	$h_\lambda = \sum_\mu z_\mu^{-1} \mathbf{1}_\lambda(\mu) p_{\mu_1} \dots p_{\mu_n}.$
Irreducible characters	↔	Schur functions

Other correspondances

Characters of induced of irreducible rep of Young subgroups	↔	Product of Schur functions
Kronecker product	↔	Inner product

Thank you