From symmetric functions to qubits

From symmetric functions to qubits

Our purpose:

Explain how to compute the Hilbert series of the algebras of covariants for (pure) qubits systems.

Why these algebras are relevant?

The (pure) qubit systems are regarded as multilinear forms on which acts a product of linear groups (SLOCC: Stochastic Local Operations and the Classical Communication).

The knowledge of the covariants allows:

- (In principle) to describe the structure of the orbits.
- Construct entanglement monotones and measure of entanglement.

From symmetric functions to qubits

Our plan:

Lecture I: Introduction to symmetric functions (Jean-Gabriel Luque)

We will present the main tool: the symmetric functions.

Lecture II: Symmetric functions and characters of the symmetric groups.(JGL)

We will explain why the symmetric functions encodes the characters of the symmetric groups.

Lecture III: Vertex operator and Kronecker coefficients (Jean-Yves Thibon)

We will explain the links between the Kronecker coefficients and the symmetric functions.

Lecture IV: Hilbert series of the algebras of invariants for a k-qubits (pure) system. (JYT)

We will explain how to use these tools to compute the Hilbert series.

From symmetric functions to qubits

Lecture I
Introduction to symmetric functions

Jean-Gabriel Luque

Symmetric functions

Definition:

Polynomials in several variables (alphabet X={x_1,...,x_n,...}) which are invariant under permutations of the variables.

Sym(X) : algebra of symmetric polynomials for the alphabet X

Example

The monomial functions:

$$\begin{aligned} m_{32}(x_1+x_2+x_3+x_4) &= {x_1}^3{x_2}^2+{x_1}^3{x_3}^2+{x_1}^3{x_4}^2+{x_2}^3{x_1}^2+{x_2}^3{x_3}^2+{x_2}^3{x_4}^2 \\ &+{x_3}^3{x_1}^2+{x_3}^3{x_2}^2+{x_3}^3{x_4}^2+{x_4}^3{x_1}^2+{x_4}^3{x_2}^2+{x_4}^3{x_3}^2 \end{aligned}$$

$$m_\lambda := \sum_{m \text{ is a monomial of multidegree } \lambda} m$$

Symmetric functions

Definition:

Polynomials in several variables (alphabet $X=\{x_1,\ldots,x_n,\ldots\}$ which are invariant under permutations of the variables.

Sym(X) : algebra of symmetric polynomials for the alphabet X

Example

The complete function h_n is the sum of all the monomials of degree n

$$h_5 = m_5 + m_{41} + m_{32} + m_{311} + m_{221} + m_{2111} + m_{11111}$$

Symmetric functions

Definition:

Polynomials in several variables (alphabet $X=\{x_1,...,x_n,...\}$ which are invariant under permutations of the variables.

Sym(X) : algebra of symmetric polynomials for the alphabet X

Example

The elementary functions

$$e_n := m_{\underbrace{1\ldots1}_{n\times}}$$

$$e_2(x_1+x_2+x_3+x_4) = m_{11} = x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$$

Symmetric functions

Definition:

Polynomials in several variables (alphabet X={x_1,...,x_n,...} which are invariants under permutation if the variables.

Sym(X) : algebra of symmetric polynomials for the alphabet X

Example

The power sums

$$p_n = m_n$$

$$p_2(x_1 + x_2 + x_3 + x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

Bases

Suppose that $X=\{x_1,...,x_n,...\}$ is an infinite alphabet.

Multiplicative bases:

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0)$

$h_\lambda := h_{\lambda_1} \dots h_{\lambda_n}$ Complete functions

$p^\lambda := p_{\lambda_1} \cdots p_{\lambda_n}$ Power sums

$e^\lambda := e_{\lambda_1} \dots e_{\lambda_n}$ Elementary functions

$$Sym(X)_{\mathbb{C}} = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2, \dots] = \mathbb{C}[e_1, e_2, \dots]$$

Non multiplicative bases:

m_λ Monomial functions

s_λ Schur functions

Generating functions

Elementary functions

$$\lambda_t(\mathbb{X}) := \sum_n e_n(\mathbb{X}) t^n$$

$$\lambda_t(\mathbb{X}) = \sum_n \left(\sum_{i_1,\ldots,i_n \text{ distinct}} x_{i_1} \ldots x_{i_n} \right) t^n$$

Equivalently

$$\lambda_t(\mathbb{X}) = \prod_{x \in \mathbb{X}} (1 + xt)$$

Generating functions

Complete functions (Cauchy function)

$$\sigma_t(\mathbb{X}) := \sum_n h_n(\mathbb{X}) t^n$$

$$\sigma_t(\mathbb{X}) = \sum_n \left(\sum_{m \text{ monomial of degree} n} m \right) t^n$$

Equivalently

$$\sigma_t(\mathbb{X}) = \prod_{x \in \mathbb{X}} \frac{1}{1 - xt}$$

Generating functions

Power sums

$$\phi_t(\mathbb{X}) := \log \sigma_t(\mathbb{X}) = \sum_x \log \frac{1}{1-xt}$$

But

$$\log \frac{1}{1-xt} = \sum_n \frac{x^n t^n}{n}$$

Hence,

$$\phi_t(\mathbb{X}) = \sum_n \frac{p_n(\mathbb{X})}{n} t^n$$

h to e

Write a complete function as a sum of elementary functions

$$\sigma_t(\mathbb{X}) = (\lambda_{-t}(\mathbb{X}))^{-1}$$

$$\sum_n h_n t^n = \left(1 - e_1 t + e_2 t^2 - e_3 t^3 + \dots\right)^{-1}$$

$$h_n = \sum_{i_1+\cdots+i_k=n} (-1)^{k+i_1+\cdots+i_k} e_{i_1} \dots e_{i_k}$$

$$h_n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1 \geq i_2 \geq \cdots \geq i_k}} \pm(\text{ a multinomial})\, e_{i_1} \dots e_{i_k}$$

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toe(h7);
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$$3\,e3\,e2^2 - 2\,e4\,e3 + e7 - 4\,e2^3\,e1 - 6\,e2\,e1^5 - 2\,e6\,e1 + 6\,e4\,e2\,e1 - 12\,e3\,e2\,e1^2 - 4\,e4\,e1^3 + 3\,e3^2\,e1 + 10\,e2^2\,e1^3 + 5\,e3\,e1^4 + e1^7 - 2\,e5\,e2 + 3\,e5\,e1^2$$

h to p

Write a complete function as a sum of power sums

$$\sigma_t(\mathbb{X}) = \exp\{\phi_t(\mathbb{X})\}$$

$$\sum_n h_n t^n = \exp\{\sum_n \frac{p_n(\mathbb{X})}{n} t^n\}$$

$$\lambda = [\ldots, k^{i_k}, \ldots, 2^{i_2}, 1^{i_1}]$$

$$\sigma_t(\mathbb{X}) = \sum_\lambda \frac{1}{z_\lambda} p^\lambda t^n$$

$$z_\lambda = 1^{i_1} i_1! 2^{i_2} i_2! \ldots k^{i_k} i_k! \ldots$$

$$z_{[4,4,2,2,2,1]} = 1.1!.2^3.3!3^0.0!4^2.2! = 1536.$$

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p^\lambda$$

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> top(h7);
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$$\frac{1}{5040} p1^7 + \frac{1}{240} p1^5 p2 + \frac{1}{48} p1^3 p2^2 + \frac{1}{48} p1\, p2^3 + \frac{1}{18} p1\, p3^2 + \frac{1}{10} p2\, p5 + \frac{1}{24} p2^2 p3 + \frac{1}{12} p3\, p4 + \frac{1}{8} p1\, p2\, p4 + \frac{1}{6} p1\, p6 + \frac{1}{10} p1^2 p5 + \frac{1}{24} p1^3 p4 + \frac{1}{72} p3\, p1^4 + \frac{1}{12} p1^2 p2\, p3 + \frac{1}{7} p7$$

Sum of alphabets

$\mathbb{X} = \{x_1, \ldots\}, \mathbb{Y} = \{y_1, \ldots\}$

$$\sigma_t(\mathbb{X} \cup \mathbb{Y}) = \prod_{x \in \mathbb{X}} \frac{1}{1 - xt} \prod_{Y \in \mathbb{Y}} \frac{1}{1 - yt} = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y})$$

$$\sigma_t(\mathbb{X} + \mathbb{Y}) := \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y})$$

Example:

$\mathbb{X} = \{a, b, c\}, Y = \{c, d\}$

$$\sigma_t(\mathbb{X} + \mathbb{Y}) = \frac{1}{1 - at} \frac{1}{1 - bt} \frac{1}{1 - ct}^{2} \frac{1}{1 - dt}$$

An alphabet: the formal sum of its letters.

Differences of alphabets

$$\sigma_t(2\mathbb{X}) = \sigma_t(\mathbb{X})^2 \qquad \sigma_t(\alpha\mathbb{X}) = \sigma_t(\mathbb{X})^\alpha$$

$$\sigma_t(-\mathbb{X}) = (\sigma_t(\mathbb{X}))^{-1} = \lambda_{-t}(\mathbb{X})$$

$$h_n(-\mathbb{X}) = (-1)^n e_n(\mathbb{X})$$

$$\sigma_t(\mathbb{X} - \mathbb{Y}) = \sigma_t(\mathbb{X})\lambda_{-t}(\mathbb{Y})$$

$$h_n(\mathbb{X} - \mathbb{Y}) = \sum_{k=0}^{n} (-1)^{n-k} h_k(\mathbb{X}) e_{n-k}(\mathbb{Y})$$

Product of alphabets

$$\mathbb{X} = x_1 + x_2 + \ldots, \mathbb{Y} = y_1 + y_2 + \ldots, \mathbb{X}\mathbb{Y} = \sum_{x \in \mathbb{X}, y \in \mathbb{Y}} xy$$

A power sum of a product of alphabets is the product of the power sums.

$$p_n(\mathbb{X}\mathbb{Y}) = \sum_{x,y} x^n y^n = \sum_x x^n \sum_y y^n = p_n(\mathbb{X}) p_n(\mathbb{Y})$$

Hence,

$$\sigma_1(\mathbb{X}\mathbb{Y}) = \sum_\lambda \frac{1}{z_\lambda} p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y})$$

Scalar product and reproducing Kernel (1)

Consider a scalar product: $\{\ ,\ \}$

With a pair of bases in dualit $\{B_\lambda, C_\mu\} = \delta_{\lambda,\mu}$

The reproducing kernel associated to { , } is a multivariate series on two alphabets defined by

$$K_{\{\ ,\ \}}(\mathbb{X},\mathbb{Y}) := \sum_\lambda B_\lambda(\mathbb{X})C_\lambda(\mathbb{Y})$$

Why this series is called a reproducing kernel?

$$\{K_{\{\ ,\ \}}(\mathbb{X},\mathbb{Y}), C_\mu(\mathbb{X})\}_{\mathbb{X}} = \sum_\lambda \{B_\lambda(\mathbb{X}), C_\mu(\mathbb{X})\}C_\mu(\mathbb{Y}) = C_\mu(\mathbb{Y})$$

Scalar product and reproducing Kernel (2)

As a consequence, if $P(\mathbb{X}) = \sum_{\lambda} \alpha_{\lambda} C_{\lambda}(\mathbb{X})$ one has

$$\{K_{\{\,,\,\}}(\mathbb{X},\mathbb{Y}), P(\mathbb{X})\}_{\mathbb{X}} = \sum_{\lambda} \alpha_{\lambda} C_{\lambda}(\mathbb{Y}) = P(\mathbb{Y}).$$

For any pair of bases in duality: $\{D_{\lambda}, E_{\mu}\} = \delta_{\lambda,\mu}$

$$K_{\{\,,\,\}}(\mathbb{X},\mathbb{Y}) = \sum_{\lambda} D_{\lambda}(\mathbb{X}) E_{\lambda}(\mathbb{Y})$$

Usual scalar product and inner product (1)

We set $K(\mathbb{X},\mathbb{Y}) = \sigma_1(\mathbb{X}\mathbb{Y})$

Since, $\sigma_1(\mathbb{X}\mathbb{Y}) = \sum_\lambda \frac{1}{z_\lambda} p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y})$

This is the reproducing kernel of the usual scalar product defined by

$$\langle \frac{1}{z_\lambda} p^\lambda, p^\mu \rangle = \delta_{\lambda,\mu}$$

Or equivalently, $\langle p^\lambda, p^\mu \rangle = z_\lambda \delta_{\lambda,\mu}$

The usual inner product is defined by $p^\lambda \star p^\mu = z_\lambda \delta_{\lambda,\mu} \ \ p^\lambda$

Usual scalar product and inner product (2)

Write

$$\sigma_1(\mathbb{XY}) = \prod_x \prod_y \frac{1}{1-xy} = \prod_x \sigma_x(\mathbb{Y})$$

Since $\sigma_x(\mathbb{Y}) = \sum_n x^n h_n(\mathbb{Y})$

The coefficient of $h_{32}(\mathbb{Y}) = h_3(\mathbb{Y})h_2(\mathbb{Y})$ in $\sigma_1(\mathbb{XY})$ is

$$\sum_{x_1, x_2} x_1^3 x_2^2 = m_{32}(\mathbb{X})$$

More generaly

$$\sigma_1(\mathbb{XY}) = \sum_\lambda m_\lambda(\mathbb{X}) h_\lambda(\mathbb{Y})$$

m_λ and h_λ are two bases in duality.

Orthogonalisation of complete functions

One apply the Gramm-Schmidt orthogonalisation process to the complete functions for the inverse order of dominance

$$[n],\ [n,1],\ \ldots,\ [1^n]$$

Example for the degree 4:

$$S_4 := h_4$$

$$S_{3,1} := h_{3,1} - \frac{\langle S_4, h_{3,1}\rangle}{\langle S_4, S_4\rangle} S_4 = h_{3,1} - h_4$$

$$S_{2,2} := h_{2,2} - \frac{\langle S_4, h_{2,2}\rangle}{\langle S_4, S_4\rangle} S_4 - \frac{\langle S_{[3,1]}, h_{2,2}\rangle}{\langle S_{[3,1]}, S_{[3,1]}\rangle} S_{3,1} = h_{2,2} - h_{3,1}$$

By definition the S_λ are orthogonal. This basis is called the Schur basis.

An orthonormal basis (1): Semi-standard tableaux

Semi-standard (Young) tableaux of shape $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$

We fill up the nodes of the shape with non-negative integers such that the entries are strictly increasing along each column and just non-degreasing along each row.

$\lambda = (3, 2, 2, 1)$

$$\begin{array}{llll}
\lambda_4 = 1 \longrightarrow & \star & & \\
\lambda_3 = 2 \longrightarrow & \star & \star & \\
\lambda_2 = 2 \longrightarrow & \star & \star & \\
\lambda_1 = 3 \longrightarrow & \star & \star & \star
\end{array}$$

$$\begin{array}{lll}
4 & & \\
2 & 2 & \\
2 & 4 & \\
1 & 3 & 3
\end{array}
\qquad\qquad
\begin{array}{lll}
4 & & \\
3 & 2 & \\
2 & 4 & \\
1 & 3 & 3
\end{array}
\qquad\qquad
\begin{array}{lll}
4 & & \\
3 & 5 & \\
2 & 4 & \\
1 & 3 & 3
\end{array}$$

no　　　　no　　　　ok

An orthonormal basis (2): Schensted algorithm

Schensted algorithm: construct a semi-standard tableau from a sequence of integers

Iteration of the insertion:

Insertion of an integer n into a tableau t

1) First try to insert on the first line.

......
......
...........m n

If m≤ n the result is obtain by glueing n at the end of the line
2) otherwise, let p the smallest integer of the line strictly greater that n

....
....
.......p........m n

Replace the first occurrence of p by n and try to add p into the next line

....
.... p
.......n........m

An orthonormal basis (3): Schensted algorithm, example

Consider the sequence 42214163311

4

	4
42	2

4
22

		4
4	42	2
221	12	12

4
2
124

4	4
2	2 2
1241	1 1 4

4
22
1146

4	4
22	224
11463	1136

4	4
224	2246
11363	1133

4	4	44
2246	22463	2236
11331	1113	1113

44	44	446
2236	22363	2233
11131	1111	1111

An orthonormal basis (4): Robinson-Schensted-Knuth correspondence

RSK: There is a bijection between the sets of biletters and the pairs of semi-standard tableaux with same shape. The correspondance is explicit and use the Schensted algorithm.

First one sorts the set following the order

$$\binom{1}{1} \leq \binom{1}{2} \leq \dots \binom{2}{1} \leq \binom{2}{2} \leq \dots \binom{3}{1} \leq \binom{3}{2} \leq \dots$$

And one applies Schensted for the sequence of the bottom numbers. One obtains the first tableau.

Hence, one repeats almost the same computation but following the order

$$\binom{1}{1} \leq \binom{2}{1} \leq \dots \binom{1}{2} \leq \binom{2}{2} \leq \dots \binom{1}{3} \leq \binom{3}{3} \leq \dots$$

And for the top integers. One obtains a second tableau (with the same shape than the first)

An orthonormal basis (5): Robinson-Schensted-Knuth Example

Consider the sequence $\binom{1}{2}\binom{3}{4}\binom{2}{1}\binom{1}{2}\binom{2}{3}\binom{4}{2}\binom{5}{1}$

First tableau

$$\binom{1}{2}\binom{1}{2}\binom{2}{1}\binom{2}{3}\binom{3}{4}\binom{4}{2}\binom{5}{1}$$

$$2213521 \xrightarrow{\text{Schensted}} \begin{array}{llll} 3 & & & \\ 2 & 2 & & \\ 1 & 1 & 2 & 4 \end{array}$$

Second tableau

$$\binom{2}{1}\binom{5}{1}\binom{1}{2}\binom{1}{2}\binom{4}{2}\binom{2}{3}\binom{3}{4}$$

$$2511423 \xrightarrow{\text{Schensted}} \begin{array}{llll} 5 & & & \\ 2 & 4 & & \\ 1 & 1 & 2 & 3 \end{array}$$

An orthonormal basis (6): constructing an orthonormal basis

$$K(\mathbb{X},\mathbb{Y}) = \sigma_1(\mathbb{X}\mathbb{Y}) = \sum_n h_n(\mathbb{X}\mathbb{Y}) = \sum_\lambda m_\lambda(\mathbb{X}\mathbb{Y}).$$

$$K(\mathbb{X},\mathbb{Y}) = \sum \text{all the monomials over the alphabet} \mathbb{X}\mathbb{Y}$$

$$x_i y_j \to \binom{i}{j}$$

$$x_{i_1} y_{j_1} \ldots x_{i_n} y_{j_n} \to \binom{i_1}{j_1} \ldots \binom{i_n}{j_n}$$

RSK

T1 T2

$$x^{T_1} = \prod_{i \text{ is an entry of } T_1} x_i \qquad y^{T_2} = \prod_{i \text{ is an entry of } T_2} x_i$$

$$K(\mathbb{X},\mathbb{Y}) = \sum_\lambda \sum_{T_1, T_2 \models \lambda} x^{T_1} y^{T_2} = \sum_\lambda b_\lambda(\mathbb{X}) b_\lambda(\mathbb{Y})$$

Where $b_\lambda(\mathbb{X}) = \sum_{T \models \lambda} x^T$ is orthonormal

An orthonormal basis (7): An example

$$x_1 y_2 x_1 y_2 x_2 y_1 x_2 y_3 x_4 y_2 x_5 y_1 \sim \binom{1}{2}\binom{1}{2}\binom{2}{1}\binom{2}{3}\binom{4}{2}\binom{5}{1}$$

RSK

5			
2	4		
1	1	2	3

3			
2	2		
1	1	2	4

$$x^{T_1} = x_1 x_1 x_2 x_2 x_4 x_5$$

$$y^{T_2} = y_2 y_2 y_1 y_3 y_2 y_1$$

An orthonormal basis (8): The functions b are symmetric

Suppose that the tableau

3
2 2
1 1 2 3

appears in the sum $b_\lambda(\mathbb{X}) = \sum_{T \models \lambda} x^T$

3 22 1123	3 22 1133	3 23 1112
$x_1^2 x_2^3 x_3^2$	$x_1^2 x_2^2 x_3^3$	$x_1^3 x_2^2 x_3^2$

The monomial m_{322} appears in the sum.

$b_\lambda(\mathbb{X}) = \sum_{T \models \lambda} x^T$ is symmetric

Schur functions are orthonormal

$$K(\mathbb{X},\mathbb{Y}) = \sum_{\lambda} b_\lambda(\mathbb{X}) b_\lambda(\mathbb{Y})$$

$$b_\lambda(\mathbb{X}) = m_\lambda + \sum_{\mu \leq \lambda} (*) m_\mu$$

Example: $\lambda = [4, 2, 2]$

33
22
1111

But by definition, the Schur basis is the unique orthogonal basis verifying

$$S_\lambda = m_\lambda + \sum_{\mu \leq \lambda} (*) m_\mu$$

Hence, $S_\lambda = b_\lambda$

Schur to m

$$S_\lambda(\mathbb{X}) = \sum_{T \models \lambda} x^T$$

Hence
$$S_\lambda(\mathbb{X}) = \sum_{\mu} K_{\lambda,\mu} m_\mu$$

where

$$K_{\lambda,\mu} = \text{ number of semi-standards tableaux of shape } \lambda \text{ with } \mu_1 \times 1, \mu_2 \times 2, \ldots$$

are the Kostka numbers

Example:

$$\begin{array}{rcl}
S_{31} & = & \\
m_{31} & \to & \begin{array}{lll} 2 & & \\ 1 & 1 & 1 \end{array} \\
m_{22} & \to & \begin{array}{lll} 2 & & \\ 1 & 1 & 2 \end{array} \\
2m_{211} & \to & \begin{array}{lll} 2 & & \\ 1 & 1 & 3 \end{array} + \begin{array}{lll} 3 & & \\ 1 & 1 & 2 \end{array} \\
3m_{1111} & \to & \begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array} + \begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array} + \begin{array}{lll} 2 & & \\ 1 & 3 & 4 \end{array}
\end{array}$$

h to Schur

$S_\lambda(\mathbb{X}) = \sum_\mu K_{\lambda,\mu} m_\mu \Rightarrow \langle S_\lambda, h_\mu \rangle = K_{\lambda,\mu}$ Since (h_λ, m_λ) are in duality

The Schur basis being orthonormal, one has

$\langle h_\mu, S_\lambda \rangle = \langle S_\lambda, h_\mu \rangle = K_{\lambda,\mu}$ implies $h_\mu = \sum K_{\lambda,\mu} S_\lambda$

$$S_{[3,2]} = \mathbf{1} m_{[3,2]} + \mathbf{1} m_{[3,1,1]} + \mathbf{2}\, m_{[2,2,1]} + \mathbf{3}\, m_{[2,1,1,1]} + \mathbf{5}\, m_{[1,1,1,1,1]}$$

$$h_{[3,2]} = S_{[5]} + S_{[4,1]} + \mathbf{1} S_{[3,2]}$$

$$h_{[3,1,1]} = S_{[5]} + 2\, S_{[4,1]} + \mathbf{1} S_{[3,2]} + S_{[3,1,1]}$$

$$h_{[2,2,1]} = S_{[5]} + 2\, S_{[4,1]} + \mathbf{2}\, S_{[3,2]} + S_{[3,1,1]} + S_{[2,2,1]}$$

$$h_{[2,1,1,1]} = S_{[5]} + 3\, S_{[4,1]} + \mathbf{3}\, S_{[3,2]} + 3\, S_{[3,1,1]} + 2\, S_{[2,2,1]} + S_{[2,1,1,1]}$$

$$h_{[1,1,1,1,1]} = S_{[5]} + 4\, S_{[4,1]} + \mathbf{5}\, S_{[3,2]} + 6\, S_{[3,11]} + 5\, S_{[2,2,1]} + 4\, S_{[2,1,1,1]} + S_{[1,1,1,1,1]}$$

Conclusion

+ We have shown how to use generating function for

    + Computing changes of bases.

$$\sigma_t(\mathbb{X}) = (\lambda_{-t}(\mathbb{X}))^{-1}$$

    + Defining the scalar standard scalar product and the standard inner product.

$$K(\mathbb{X}, \mathbb{Y}) = \sigma_1(\mathbb{X}\mathbb{Y})$$

+ We have defined Schur function by orthogonalizing the complete functions.

+ We have proved that the Schur functions are orthonormal for the standard scalar product.