Discrete tomography, RSK correspondence and Kronecker products

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Kronecker products

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• Given two partitions λ, μ of n, let $\chi^\lambda \otimes \chi^\mu$ denote the **Kronecker product** of χ^λ and χ^μ.

Kronecker coefficients

• The **Kronecker coefficient** $\mathsf{k}(\lambda, \mu, \nu)$ is the multiplicity of χ^ν in the product $\chi^\lambda \otimes \chi^\mu$. That is

$$\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash n} \mathsf{k}(\lambda, \mu, \nu) \chi^\nu .$$

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- Orthogonality relations imply

$$\mathsf{k}(\lambda, \mu, \nu) = \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle = \langle \chi^\lambda \otimes \chi^\mu \otimes \chi^\nu, \chi^{(n)} \rangle .$$

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Open problem

Find a combinatorial or geometric description of $\mathsf{k}(\lambda,\mu,\nu)$ as it is done with Kostka numbers or with Littlewood-Richardson coefficients.

Permutation characters

- Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a composition of n and let

$$\mathsf{S}_\lambda := \mathsf{S}_{\lambda_1} \times \cdots \times \mathsf{S}_{\lambda_p}$$

be the **Young subgroup** of S_n corresponding to λ.

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be the **Young subgroup** of S_n corresponding to λ.

• Then $\phi^\lambda := \mathrm{Ind}_{\mathsf{S}_\lambda}^{\mathsf{S}_n}(1_\lambda)$ is called a **permutation character** and the set

$$\{\phi^\lambda\}_{\lambda \vdash n}$$

is another basis for the character ring of S_n.

Kostka numbers

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- The numbers $K_{\gamma\nu}$ are called **Kostka numbers** and count the number of semistandard tableaux of shape γ and content ν.

- They have the following property:

$$K_{\gamma\nu} > 0 \Longleftrightarrow \gamma \succcurlyeq \nu \, ,$$

where $\succcurlyeq$ denotes the **dominance order** of partitions.

Matrices and characters I

- Given a matrix $A = (a_{ij})$ of size $p \times q$ we define
  - $\text{row}(A) := (r_1, \ldots, r_p)$, where $r_i = \sum_j a_{ij}$ and
  - $\text{col}(M) := (c_1, \ldots, c_q)$, where $c_j = \sum_i a_{ij}$.

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- The compositions $\text{row}(A)$ and $\text{col}(A)$ are called the **row sum** vector and **column sum** vector of A, respectively.
They are also called the **1-marginals** of A.

Matrices and characters II

- Given λ, μ compositions of n, we denote
  - by $\mathrm{M}(\lambda, \mu)$ the set of all matrices $A = (a_{ij})$ with nonnegative integer entries and 1-marginals λ, μ, and
  - by $\mathrm{m}(\lambda, \mu) := |\mathrm{M}(\lambda, \mu)|$ its cardinality.

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- We also denote
  - by $\mathrm{M}^*(\lambda, \mu)$ the set of all binary matrices $A = (a_{ij})$ with 1-marginals λ, μ, and
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Characters, matrices and the RSK correspondence I

- There is a well known formula

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• Expanding, by Young's rule, we get a formula for which the RSK correspondence is a combinatorial realization.

$$\mathrm{m}(\lambda, \mu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \langle \chi^{\alpha} \otimes \chi^{\beta}, \chi^{(n)} \rangle = \sum_{\sigma \vdash n} K_{\sigma\lambda} K_{\sigma\mu} \,.$$

Characters, matrices and the RSK correspondence II

- Similarly, one has

$$\mathsf{m}^*(\lambda, \mu) = \langle \phi^{\lambda} \otimes \phi^{\mu}, \chi^{(1^n)} \rangle \, .$$

Characters, matrices and the RSK correspondence II

- Similarly, one has

$$\mathrm{m}^*(\lambda, \mu) = \langle \phi^{\lambda} \otimes \phi^{\mu}, \chi^{(1^n)} \rangle \,.$$

- Again, expanding, by Young's rule, we get a formula for which the dual RSK correspondence is a combinatorial realization.

$$\mathrm{m}^*(\lambda, \mu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \langle \chi^{\alpha} \otimes \chi^{\beta}, \chi^{(1^n)} \rangle = \sum_{\sigma \vdash n} K_{\sigma\lambda} K_{\sigma'\mu} \,.$$

3-dimensional matrices I

Definition

The **1-marginals** of a 3-dimensional matrix $A = (a_{ijk})$ of size $p \times q \times r$ are the vectors $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, $\nu = (\nu_1, \ldots, \nu_r)$ defined by:

$$\sum_{j,k} a_{ijk} = \lambda_i \, , \quad \sum_{i,k} a_{ijk} = \mu_j \, , \quad \sum_{i,j} a_{ijk} = \nu_k \, .$$

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$$\sum_{j,k} a_{ijk} = \lambda_i \,, \quad \sum_{i,k} a_{ijk} = \mu_j \,, \quad \sum_{i,j} a_{ijk} = \nu_k \,.$$

Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has 1-marginals $\lambda = (9, 4, 2)$, $\mu = (8, 5, 2)$ and $\nu = (7, 4, 2, 2)$.

3-dimensional matrices II

- Given compositions λ, μ, ν of n we denote
  - by $\mathrm{M}(\lambda, \mu, \nu)$ is the set of all matrices $A = (a_{ijk})$ with nonnegative integer entries and 1-marginals λ, μ and ν, and
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• For 3-dimensional matrices there are formulas similar to those for 2-dimensional matrices.

Theorem (E. Snapper (1971))

Let λ, μ, ν be compositions of n. Then

$$\mathrm{m}(\lambda, \mu, \nu) = \langle \phi^{\lambda} \otimes \phi^{\mu} \otimes \phi^{\nu}, \chi^{(n)} \rangle$$

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Snapper's theorem

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$$\mathsf{m}^*(\lambda, \mu, \nu) = \langle \phi^{\lambda} \otimes \phi^{\mu} \otimes \phi^{\nu}, \chi^{(1^n)} \rangle .$$

• Now we apply Young's rule to each of the permutation characters.

- Triple application of Young's formula yields

$$\mathsf{m}(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta,\, \gamma \vdash n} K_{\alpha\lambda} K_{\beta\mu} K_{\gamma\nu}\, \mathsf{k}(\alpha, \beta, \gamma)\,.$$

and

$$\mathsf{m}^*(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta,\, \gamma \vdash n} K_{\alpha\lambda} K_{\beta\mu} K_{\gamma\nu}\, \mathsf{k}(\alpha, \beta, \gamma')\,.$$

Matrices and Kronecker coefficients

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- These formulas indicate how extensions of the RSK correspondence and its dual should be.

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- In particular these extensions would contain combinatorial descriptions of Kronecker coefficients.

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- These formulas indicate how extensions of the RSK correspondence and its dual should be.

- In particular these extensions would contain combinatorial descriptions of Kronecker coefficients.
- A similar observation was done by F. Caselli (2009).

Matrices and Littlewood-Richardson multitableaux

• We follow a more modest, but more realistic approach: We apply Young's rule *only* two times. Thus, we get:

$$\mathrm{m}(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \langle \chi^{\alpha} \otimes \chi^{\beta} \otimes \phi^{\nu}, \chi^{(n)} \rangle$$

and

$$\mathrm{m}^{*}(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \langle \chi^{\alpha} \otimes \chi^{\beta} \otimes \phi^{\nu}, \chi^{(1^n)} \rangle \, .$$

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• The point is that the inner products on the right hand side have combinatorial descriptions in terms of Littlewood-Richardson coefficients.

Littlewood-Richardson multitableaux I. Definition

Definition

Let $\alpha \vdash n$ and $\nu = (\nu_1, \ldots, \nu_r)$ be a composition of n, then a sequence $T = (T_1, \ldots, T_r)$ of tableaux is called a **Littlewood-Richardson multitableau** of **shape** α and **type** ν if there exists a sequence of partitions

$$\emptyset = \alpha(0) \subseteq \alpha(1) \subseteq \cdots \subseteq \alpha(r) = \alpha$$

such that T_i is a Littlewood-Richardson tableau of shape $\alpha(i)/\alpha(i-1)$ and size ν_i for all $i \in [\, r\,]$.
If each T_i has content $\rho(i)$, then we say that T has **content** $(\rho(1), \ldots, \rho(r))$.

Littlewood-Richardson multitableaux II. Example

Example

The LR multitableau

1	**1**	**1**	**1**	*1*	*1*	*1*	1	1	1
2	**2**	**2**	**2**	*2*	2	2	2		
3	**3**	*2*	*2*	*3*					
3	3								

has:
shape $(10, 8, 5, 2)$,
content $((4, 4, 2), (3, 3, 2), (3, 3, 1))$ and
type $(10, 8, 7)$.

Definition

Given partitions α, β of n and ν a composition of n, we denote

- by $\mathrm{LR}(\alpha, \beta; \nu)$ the set of all pairs (T, S) of Littlewood-Richardson multitableaux of shape (α, β) and type ν such that S and T have the same content and
- by $\mathrm{lr}(\alpha, \beta; \nu) := |\mathrm{LR}(\alpha, \beta; \nu)|$ its cardinality.

Littlewood-Richardson multitableaux III. Notation

Definition

Given partitions α, β of n and ν a composition of n, we denote

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- by $\mathrm{lr}(\alpha, \beta; \nu) := |\mathrm{LR}(\alpha, \beta; \nu)|$ its cardinality.

Similarly, we denote

- by $\mathrm{LR}^*(\alpha, \beta; \nu)$ the set of all pairs (T, S) of Littlewood-Richardson multitableaux of shape (α, β), type ν and **conjugate** content, that is, if T has content $(\rho(1), \ldots, \rho(r))$, then S has content $(\rho(1)', \ldots, \rho(r)')$ and
- by $\mathrm{lr}^*(\alpha, \beta; \nu) := |\mathrm{LR}^*(\alpha, \beta; \nu)|$ its cardinality.

Littlewood-Richardson multitableaux IV. Example

Let $T =$

1	**1**	**1**	**1**	*1*	*1*	*1*	1	1	1
2	**2**	**2**	**2**	*2*	2	2	2		
3	**3**	*2*	*2*	*3*					
3	3								

and $S =$

1	**1**	**1**	*1*	*1*	*1*	1	1	1
2	**2**	**2**	*2*	*2*	*2*	2		
3	**3**	*3*	2	3				
4	**4**	3						
3								

.

Then (T, S) is a pair of Littlewood-Richardson multitableaux in $\mathrm{LR}^*((10,8,5,2),(9,7,5,3,1);(10,8,7))$.
T has content $((4,4,2),(3,3,2),(3,3,1))$ and S has content $((3,3,2,2),(3,3,2),(3,2,2))$.

Littlewood-Richardson multitableaux V

Lemma

Let α*,* β *be partitions of* n *and let* ν *be a composition of* n*. Then*
(1) $\mathrm{lr}(\alpha, \beta; \nu) = \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^\nu, \chi^{(n)} \rangle$*.*
(2) $\mathrm{lr}^*(\alpha, \beta; \nu) = \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^\nu, \chi^{(1^n)} \rangle$*.*

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(2) $\mathrm{lr}^*(\alpha, \beta; \nu) = \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^\nu, \chi^{(1^n)} \rangle$.

- Therefore we have

$$\mathrm{m}(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu}\, \mathrm{lr}(\alpha, \beta; \nu)$$

and

$$\mathrm{m}^*(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu}\, \mathrm{lr}^*(\alpha, \beta; \nu)\,.$$

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- Therefore we have

$$\mathrm{m}(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \, \mathrm{lr}(\alpha, \beta; \nu)$$

and

$$\mathrm{m}^*(\lambda, \mu, \nu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \, \mathrm{lr}^*(\alpha, \beta; \nu)\,.$$

- We will give one-to-one correspondences that realize these identities.

Littlewood-Richardson multitableaux VI

- Another application of Young's rule yields

$$\mathsf{lr}(\alpha,\beta;\nu) = \sum_{\gamma\vdash n} K_{\gamma\nu}\,\mathsf{k}(\alpha,\beta,\gamma)$$

and

$$\mathsf{lr}^*(\alpha,\beta;\nu') = \sum_{\gamma\vdash n} K_{\gamma'\nu'}\,\mathsf{k}(\alpha,\beta,\gamma)$$

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• Therefore we can think of $\mathsf{lr}(\alpha,\beta;\nu)$ and of $\mathsf{lr}^*(\alpha,\beta;\nu')$ as combinatorial approximations of Kronecker coefficients.

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• Therefore we can think of $\mathrm{lr}(\alpha, \beta; \nu)$ and of $\mathrm{lr}^*(\alpha, \beta; \nu')$ as combinatorial approximations of Kronecker coefficients.

• In fact, we will next see that in some cases these numbers coincide.

Extremal components

Definition

A component χ^ν of $\chi^\alpha \otimes \chi^\beta$ is called

- **maximal** if for all $\gamma \succ \nu$ one has $\mathsf{k}(\alpha, \beta, \gamma) = 0$,
- **minimal** if for all $\gamma \prec \nu$ one has $\mathsf{k}(\alpha, \beta, \gamma) = 0$.

Lemma

Let χ^ν be a component of $\chi^\alpha \otimes \chi^\beta$. Then
(1) χ^ν is a maximal component of $\chi^\alpha \otimes \chi^\beta$ if and only if
$\mathsf{k}(\alpha, \beta, \nu) = \mathsf{lr}(\alpha, \beta; \nu)$.
(2) χ^ν is a minimal component of $\chi^\alpha \otimes \chi^\beta$ if and only if
$\mathsf{k}(\alpha, \beta, \nu) = \mathsf{lr}^*(\alpha, \beta; \nu')$.

Main theorems

• Let λ, μ, ν be compositions of n.
• For any partition α of n, let $K_{\alpha\lambda}$ denote the set of all semistandard tableaux of shape α and content λ.

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- Let λ, μ, ν be compositions of n.
- For any partition α of n, let $\mathsf{K}_{\alpha\lambda}$ denote the set of all semistandard tableaux of shape α and content λ.

Theorem

There is a one-to-one correspondence between the set $\mathsf{M}(\lambda, \mu, \nu)$ *of 3-dimensional matrices with nonnegative integer coefficients that have 1-marginals* λ, μ, ν *and the set of triples* $\coprod_{\alpha,\, \beta \vdash n} \mathsf{K}_{\alpha\lambda} \times \mathsf{K}_{\beta\mu} \times \mathsf{LR}(\alpha, \beta; \nu)$.

Main theorems

- Let λ, μ, ν be compositions of n.
- For any partition α of n, let $\mathsf{K}_{\alpha\lambda}$ denote the set of all semistandard tableaux of shape α and content λ.

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Theorem

There is a one-to-one correspondence between the set $\mathsf{M}^*(\lambda, \mu, \nu)$ *of 3-dimensional binary matrices that have 1-marginals* λ, μ, ν *and the set of triples* $\coprod_{\alpha,\, \beta \vdash n} \mathsf{K}_{\alpha\lambda} \times \mathsf{K}_{\beta\mu} \times \mathsf{LR}^*(\alpha, \beta; \nu)$.

Sketch of proof I

First bijection (tautological)

Let λ, μ, ν be compositions of size p, q, r, respectively. There is a one-to-one correspondence

$$A = (a_{ijk}) \in \mathrm{M}(\lambda, \mu, \nu) \leftrightarrow (A^{(1)} = (a^{(1)}_{ij}), \ldots, A^{(r)} = (a^{(r)}_{ij}))\,,$$

where each $A^{(k)}$ has size $p \times q$ and

$$\sum_{k=1}^{r} \mathrm{row}(A_k) = \lambda, \qquad \sum_{k=1}^{r} \mathrm{col}(A_k) = \mu,$$

$$\text{sum of the entries of } A_k = \nu_k,\ \ k \in [\,r\,].$$

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- The correspondence is $a_{ij}^{(k)} = a_{ijk}$.

Sketch of proof II

Second bijection

There is a one-to-one correspondence

$$(A_1, \ldots, A_r) \leftrightarrow ((P_1, \ldots, P_r), (Q_1, \ldots, Q_r)),$$

where each matrix A_k satisfies the conditions from previous slide and on the right hand side we have r-tuples of semistandard tableaux such that

$$\sum_{k=1}^{r} \mathrm{cont}(Q_k) = \lambda, \qquad \sum_{k=1}^{r} \mathrm{cont}(P_k) = \mu,$$

$$\mathrm{sh}(P_k) = \mathrm{sh}(Q_k) \text{ and } |\mathrm{sh}(P_k)| = \nu_k, \; k \in [\, r \,].$$

Sketch of proof II

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$$\mathsf{sh}(P_k) = \mathsf{sh}(Q_k) \text{ and } |\mathsf{sh}(P_k)| = \nu_k, \; k \in [\, r\,].$$

• The correspondence follows applying the RSK correspondence on each level matrix.

Sketch of proof III

Third bijection

There is a one-to-one correspondence

$$((P_1, \dots, P_r), (Q_1, \dots, Q_r)) \leftrightarrow \coprod_{\alpha,\, \beta \vdash n} \mathrm{K}_{\alpha\lambda} \times \mathrm{K}_{\beta\mu} \times \mathrm{LR}(\alpha, \beta; \nu)\,,$$

where the pair of r-tuples of semistandard tableaux satisify the conditions of the previous slide.

Sketch of proof III

Third bijection

There is a one-to-one correspondence

$$((P_1,\ldots,P_r),(Q_1,\ldots,Q_r)) \leftrightarrow \coprod_{\alpha,\,\beta\vdash n} \mathsf{K}_{\alpha\lambda}\times \mathsf{K}_{\beta\mu}\times \mathsf{LR}(\alpha,\beta;\nu)\,,$$

where the pair of r-tuples of semistandard tableaux satisfy the conditions of the previous slide.

- It is a consequence of the next theorem:

Sketch of proof IV

Theorem (G.P. Thomas (1974))

There is a one-to-one correspondence between the set of all r-tuples $(P_1, \ldots, P_r)$ of semistandard tableaux and the set of pairs (P, S) such that P is a semistandard tableau and S is a Littlewood-Richardson multitableau of shape $\mathrm{sh}(P)$*. Moreover, under this correspondence*

$$\mathrm{cont}(P) = \sum_{k=1}^{r} \mathrm{cont}(P_k) \quad \text{and} \quad \mathrm{cont}(S) = (\mathrm{sh}(P_1), \ldots, \mathrm{sh}(P_r)) .$$

Remark

$P = P_r \cdots P_1$ is a product of tableaux, and $S = (S_1, \ldots, S_r)$ is a list of recording tableaux, one for each for each factor.

Discrete Tomography I. Motivation

• In the 2-dimensional case there are well known conditions for existence and uniqueness. The first one is due to D. Gale and H. Ryser (1957). The second is folklore.

$$\mathsf{m}^*(\lambda, \mu) > 0 \Longleftrightarrow \lambda' \succcurlyeq \mu \,,$$
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$$\mathsf{m}^*(\lambda,\mu) = 1 \Longleftrightarrow \lambda' = \mu\,.$$

• We are interested in a similar condition for $\mathsf{m}^*(\lambda,\mu,\nu) = 1$.

• In the next slides we show a condition that involves 2-dimensional matrices with nonnegative integer entries.

Let $A \in \mathsf{M}(\lambda, \mu)$.

- The **π-sequence** of A, denoted by $\pi(A)$, is the decreasing sequence of its entries.

Discrete Tomography II. Definitions

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- A is called **π-unique** if there is no other matrix in $\mathrm{M}(\lambda, \mu)$ with the same π-sequence.

Discrete Tomography II. Definitions

Let $A \in \mathrm{M}(\lambda, \mu)$.

- The $\boldsymbol{\pi}$**-sequence** of A, denoted by $\pi(A)$, is the decreasing sequence of its entries.
- A is called $\boldsymbol{\pi}$**-unique** if there is no other matrix in $\mathrm{M}(\lambda, \mu)$ with the same π-sequence.
- A is called **minimal** if there is no other matrix $B \in \mathrm{M}(\lambda, \mu)$ with $\pi(B) \prec \pi(A)$.

Example

Let $\lambda = \mu = (3, 3)$ and let

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then

Discrete Tomography III. Example

Example

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Then

- $\mathrm{M}(\lambda, \mu) = \{A, B, C, D\}$.

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Then

- $\mathrm{M}(\lambda, \mu) = \{A, B, C, D\}$.
- $\pi(B) = \pi(C) = (2^2, 1^2)$ and $\pi(A) = \pi(D) = (3^2)$.

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- B and C are minimal, while A and D are not.

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Then

- $\mathrm{M}(\lambda, \mu) = \{A, B, C, D\}$.
- $\pi(B) = \pi(C) = (2^2, 1^2)$ and $\pi(A) = \pi(D) = (3^2)$.
- B and C are minimal, while A and D are not.
- Neither of them is π-unique.

Discrete Tomography IV. Another example

Example

Let

$$A = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

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- Then both matrices have 1-marginals $(9, 4, 2)$ and $(8, 5, 2)$.

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- Moreover $\pi(B) \prec \pi(A)$. This means that B is *flatter* than A.

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- Then both matrices have 1-marginals $(9, 4, 2)$ and $(8, 5, 2)$.
- Moreover $\pi(B) \prec \pi(A)$. This means that B is *flatter* than A.
- A matrix is minimal when it cannot be made flatter without changing the 1-marginals.

Discrete Tomography V. The graph of a matrix

- We denote
  - by $\mathsf{M}_\nu(\lambda, \mu)$ the set of matrices in $\mathsf{M}(\lambda, \mu)$ with π-sequence ν,
  - by $\mathsf{m}_\nu(\lambda, \mu) := |\mathsf{M}_\nu(\lambda, \mu)|$ its cardinality.

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- To each matrix $A = (a_{ij})$ in $\mathrm{M}_\nu(\lambda,\mu)$ we associate a 3-dimensional matrix $\mathrm{G}(A) = (a_{ijk})$ by

$$a_{ijk} = \begin{cases} 1 & \text{if } a_{ij} \le k, \\ 0 & \text{otherwise.} \end{cases}$$

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$$a_{ijk} = \begin{cases} 1 & \text{if } a_{ij} \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

- The correspondence $A \mapsto \mathrm{G}(A)$ defines an injective map

$$G_{\lambda,\mu,\nu} : \mathrm{M}_\nu(\lambda,\mu) \longrightarrow \mathrm{M}^*(\lambda,\mu,\nu').$$

Discrete Tomography VI. Example

Example

Let

$$A = \begin{bmatrix} 5 & 5 & 5 & 4 & 4 \\ 5 & 5 & 5 & 3 & 3 \\ 3 & 3 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the graph of A is:

Discrete Tomography VII. Minimality

Note that if there is a minimal matrix in $\mathsf{M}_\nu(\lambda, \mu)$, then all matrices in $\mathsf{M}_\nu(\lambda, \mu)$ are minimal.

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Definition

We say that ν is **minimal for** $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ if there is a minimal matrix in $\mathsf{M}_\nu(\lambda, \mu)$.

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We say that ν is **minimal for (λ, μ)** if there is a minimal matrix in $\mathsf{M}_\nu(\lambda, \mu)$.

Proposition (E.V. (2000, 2007))

Let λ, μ, ν be partitions of n. Then ν is minimal for (λ, μ) if and only if $G_{\lambda,\mu,\nu}$ is bijective.

Discrete Tomography VIII. Uniqueness

Theorem (A. Torres, E.V (1998))

Let λ, μ, ν *be partitions of n. Then*

- $\mathsf{m}^*(\lambda, \mu, \nu) = 1 \iff$ *there is a matrix* $A \in \mathsf{M}_{\nu'}(\lambda, \mu)$ *that is minimal and* π*-unique.*
- *If* $\mathsf{m}^*(\lambda, \mu, \nu) = 1$, *the unique matrix* $A \in \mathsf{M}_{\nu'}(\lambda, \mu)$ *is a plane partition.*

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• Due to the formula

$$\mathsf{m}^*(\lambda,\mu,\nu) = \sum_{\alpha\succcurlyeq\lambda,\,\beta\succcurlyeq\mu,\,\gamma\succcurlyeq\nu} K_{\alpha\lambda} K_{\beta\mu} K_{\gamma\nu}\,\mathsf{k}(\alpha,\beta,\gamma')\,,$$

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• Due to the formula

$$\mathrm{m}^*(\lambda,\mu,\nu) = \sum_{\alpha\succcurlyeq\lambda,\, \beta\succcurlyeq\mu,\, \gamma\succcurlyeq\nu} K_{\alpha\lambda} K_{\beta\mu} K_{\gamma\nu}\, \mathrm{k}(\alpha,\beta,\gamma')\,,$$

uniqueness implies the vanishing of several Kronecker coefficients.
• In fact minimality by itself will yield useful information on Kronecker coefficients.

Minimal matrices and Kronecker coefficients I

Proposition (E.V. (2000))

If ν is minimal for (λ, μ). Then
(1) $\mathsf{k}(\alpha, \beta, \gamma) = 0$ *for all* $\alpha \succcurlyeq \lambda$, $\beta \succcurlyeq \mu$, $\gamma \prec \nu$.
(2) $\mathsf{k}(\alpha, \beta, \nu) = \mathsf{lr}^*(\alpha, \beta; \nu')$ *for all* $\alpha \succcurlyeq \lambda$, $\beta \succcurlyeq \mu$.
In particular, for any pair of partitions (α, β) *such that* $\alpha \succcurlyeq \lambda$ *and* $\beta \succcurlyeq \mu$ *we have that* χ^ν *is a minimal component of* $\chi^\alpha \otimes \chi^\beta$ *if and only if* $\mathsf{lr}^*(\alpha, \beta; \nu')$ *is positive.*

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Proposition (E.V. (2000))

If ν is minimal for (λ, μ). Then

$$\mathsf{m}_\nu(\lambda, \mu) = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \, \mathsf{lr}^*(\alpha, \beta; \nu') = \sum_{\alpha,\, \beta \vdash n} K_{\alpha\lambda} K_{\beta\mu} \, \mathsf{k}(\alpha, \beta, \nu) \, .$$

Minimal matrices and Kronecker coefficients II

- Let

$$\Phi^*\colon \mathsf{M}^*(\lambda,\mu,\nu') \longrightarrow \coprod_{\alpha,\,\beta \vdash n} \mathsf{K}_{\alpha\lambda} \times \mathsf{K}_{\beta\mu} \times \mathsf{LR}^*(\alpha,\beta;\nu')\,.$$

denote the bijection that we get from the main theorem, when we apply in each level the dual RSK-correspondence.

Minimal matrices and Kronecker coefficients II

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- Then the composition

$$\Phi^*\circ G_{\lambda,\mu,\nu}\colon \mathsf{M}_\nu(\lambda,\mu) \longrightarrow \coprod_{\alpha,\,\beta\vdash n} \mathsf{K}_{\alpha\lambda}\times \mathsf{K}_{\beta\mu}\times \mathsf{LR}^*(\alpha,\beta;\nu')\,.$$

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is injective.

- If ν is minimal for (λ,μ), then $\Phi^*\circ G_{\lambda,\mu,\nu}$ is bijective. So, this map is a combinatorial realization of the identity that relates the number of minimal matrices to sums of Kronecker coefficients.

Minimal matrices and Kronecker coefficients III

Let f, g, h denote the components of $\Phi^* \circ G_{\lambda,\mu,\nu}$, that is, for any $A \in \mathsf{M}_\nu(\lambda,\mu)$ we have $\Phi^* \circ G_{\lambda,\mu,\nu}(A) = (f(A), g(A), h(A))$.

Theorem

Suppose ν is minimal for (λ,μ). Let P be a semistandard tableau of shape α and content λ, and Q be a semistandard tableau of shape β and content μ. Then

$$\mathsf{k}(\alpha,\beta,\nu) = \#\{A \in \mathsf{M}_\nu(\lambda,\mu) \mid f(A) = P \text{ and } g(A) = Q\}\,.$$

Moreover, if $\mathsf{k}(\alpha,\beta,\nu) > 0$, then χ^ν is a minimal component of $\chi^\alpha \otimes \chi^\beta$.

Minimal matrices and Kronecker coefficients IV

Example

Let $\lambda = (6,6)$, $\mu = (3,3,3,3)$. Then, there are six minimal matrices in $\mathrm{M}(\lambda, \mu)$, namely

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

Minimal matrices and Kronecker coefficients V

Example

Let $\nu = (2^4, 1^4)$ be the common π-sequence of the six matrices. After computing $\Phi^* \circ G_{\lambda,\mu,\nu}$ for each matrix we get

$$f(A) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} \end{array} \quad \text{and} \quad g(A) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline \end{array}.$$

$$f(B) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \cline{1-5} \end{array} \quad \text{and} \quad g(B) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 & 4 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array}.$$

Minimal matrices and Kronecker coefficients VI

Example

$$f(C) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \cline{1-5}\end{array} \quad \text{and} \quad g(C) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 4 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array} \; .$$

$$f(D) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \cline{1-5}\end{array} \quad \text{and} \quad g(D) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array} \; .$$

Minimal matrices and Kronecker coefficients VII

Example

$$f(E) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \quad \text{and} \quad g(E) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array}\ .$$

$$f(F) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \quad \text{and} \quad g(F) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array}\ .$$

Minimal matrices and Kronecker coefficients VIII

Example

- Recall, $\lambda = (6,6)$, $\mu = (3,3,3,3)$ and $\nu = (2^4, 1^4)$.

Minimal matrices and Kronecker coefficients VIII

Example

- Recall, $\lambda = (6,6)$, $\mu = (3,3,3,3)$ and $\nu = (2^4, 1^4)$.
- Let $\alpha = \mathrm{sh}(f(D)) = (7,5)$ and $\gamma = \mathrm{sh}(f(A)) = (8,4)$.
- $\beta = \mathrm{sh}(g(B)) = (4,3,3,2)$ and $\delta = \mathrm{sh}(g(E)) = (4,4,2,2)$.

Example

- Recall, $\lambda = (6,6)$, $\mu = (3,3,3,3)$ and $\nu = (2^4, 1^4)$.
- Let $\alpha = \mathrm{sh}(f(D)) = (7,5)$ and $\gamma = \mathrm{sh}(f(A)) = (8,4)$.
- $\beta = \mathrm{sh}(g(B)) = (4,3,3,2)$ and $\delta = \mathrm{sh}(g(E)) = (4,4,2,2)$.
- Thus, we obtain that
- $\mathrm{k}(\gamma, \mu, \nu) = 1$,
- $\mathrm{k}(\alpha, \beta, \nu) = 1$,
- $\mathrm{k}(\lambda, \delta, \nu) = 1$.

Minimal matrices and Kronecker coefficients VIII

Example

- Recall, $\lambda = (6,6)$, $\mu = (3,3,3,3)$ and $\nu = (2^4, 1^4)$.
- Let $\alpha = \mathrm{sh}(f(D)) = (7,5)$ and $\gamma = \mathrm{sh}(f(A)) = (8,4)$.
- $\beta = \mathrm{sh}(g(B)) = (4,3,3,2)$ and $\delta = \mathrm{sh}(g(E)) = (4,4,2,2)$.
- Thus, we obtain that
- $\mathrm{k}(\gamma, \mu, \nu) = 1$,
- $\mathrm{k}(\alpha, \beta, \nu) = 1$,
- $\mathrm{k}(\lambda, \delta, \nu) = 1$.
- Many other Kronecker coefficients are zero.

$$\mathrm{m}_\nu(\lambda, \mu) = \sum_{\alpha \succcurlyeq \lambda,\, \beta \succcurlyeq \mu} K_{\alpha\lambda} K_{\beta\mu} \, \mathrm{lr}^*(\alpha, \beta; \nu') = \sum_{\alpha \succcurlyeq \lambda,\, \beta \succcurlyeq \mu} K_{\alpha\lambda} K_{\beta\mu} \, \mathrm{k}(\alpha, \beta, \nu)\,.$$

Additive and minimal matrices

Definition

A matrix $A = (a_{ij})$ of size $p \times q$ with nonnegative integer entries is called **additive** if there exists real numbers $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ such that the condition

$$a_{ij} > a_{kl} \Longrightarrow x_i + y_j > x_k + y_l$$

holds for all i, j, k, l.

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Theorem (E.V. (2002, 2005))

Any additive matrix is minimal and π-unique.

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Any additive matrix is minimal and π-unique.

Corollary

Let $A \in \mathrm{M}_\nu(\lambda, \mu)$. If A is additive, then $\mathrm{k}(\lambda, \mu, \nu) = 1$.

Quadratic optimization and minimal matrices

Theorem (S. Onn, E.V. (2006))

Let A^* be an optimal solution to the problem

$$\begin{array}{ll} \min & \sum_{i,j} {x_{ij}}^2 \\ \text{subject to} & (x_{ij}) \in \mathsf{M}(\lambda, \mu). \end{array}$$

Then A^* is minimal.

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Then A^* is additive.

¡Gracias!